Homework Assignment #8:

March 26, 2016

1. Let A be a ring and let M and N be A-modules. Suppose that N is faithfully flat and that $M \otimes N$ is flat. Prove that M is also flat.

2. Let $A \to B$ be a homomorphism of rings. We say that $A \to B$ is flat (resp. faithfully flat) if the underlying A-module of B is flat (resp. faithfully flat). Suppose that $A \to B$ and $B \to C$ are flat. Prove that $A \to C$ is flat. Suppose that $A \to C$ is flat and $B \to C$ is faithfully flat. Prove that $A \to B$ is flat. Suppose that $A \to B$ is flat (resp. faithfully flat) and $A \to A'$ is arbitrary. Prove that $A' \to A' \otimes_A B$ is flat (resp. faithfully flat) and that the converse holds if $A \to A'$ is faithfully flat.

3. Let A be a noetherian local ring with maximal ideal m and let E be a finitely generated A-module. For each prime ideal P of A let $k(P)$ be the residue field of A at P and let $E(P) := E \otimes_A k(P)$, a vector space over $k(P)$. For each P, let $d(P)$ denote the dimension of $E(P)$ over $k(P)$.

   (a) If E can be generated by d elements, then $d(P) \leq d$ for all P.

   (b) E can be generated by $d(m)$ elements.

   (c) $d(P) \leq d(m)$ for all P, with equality if E is free.

   (d) If $d(P) = d(M)$ for all P and A is reduced, then E is free.

4. Let k be a field, let $k[x_1, x_2, \cdots]$ be the polynomial ring on infinitely many variables, and let A be the quotient of this ring by the ideal generated by $\{x_i x_j - x_i \delta_{i,j}\}$. Let $A \to k$ be the homomorphism sending the image of each x_i to zero. Recall from a prevous exercise that this homomorphism is flat. It is of finite type but not of of finite presentation. Show that the image of $\operatorname{Spec}(k)$ in $\operatorname{Spec} A)$ is not open.

5. Let A be a ring and P a prime ideal of A. Show that P contains a minimal prime Q, without using any noetherian assumption on A. (Hint: Zorn's lemma)