Homework Assignment #6:

March 3, 2016

1. Let R be a commutative ring with identity.

   (a) Let Z be a subset of Spec(R) and let I be the intersection of all the prime ideals of R which belong to Z. Show that $\sqrt{I} = I$, where $\sqrt{I} := \{r \in R : r^n \in I \text{ for some } n\}$. Show that $Z(I)$ is the closure of Z in the Zariski topology.

   (b) Suppose that Z_1 and Z_2 are two closed subsets of Spec R, and let I_i be the intersection of the elements of Z_i (as above). Show that Z_1 and Z_2 are disjoint iff $I_1 + I_2 = R$, and show that $Z_1 \cup Z_2 = \text{Spec}(R)$ iff $I_1 \cap I_2 = \sqrt{0}$.

   (c) Supose that Spec(R) is disconnected. Show that then R contains a nontrivial idempotent element. (Hint: Use the previous result to find $a, b \in R$ such that $1 = a + b$ and such that ab is nilpotent. Then take a large power of both sides.

2. Let M be an R-module and I a nilpotent ideal (that is, $I^n = 0$ for some n). Show that if $M \otimes R/I = 0$, then $M = 0$.

3. Let $A \to B$ be a flat and local homomorphism of local rings. Suppose that I is an nilpotent ideal of A such that the induced map $A/I \to B/IB$ is an isomorphism. Prove that $A \to B$ is an isomorphism.

4. (Taken partly from Lang X, §3). Let k be a field, let R be the polynomial ring $k[t]$, let B be the polynomial ring $R[x]$, and let A be the subring of B consisting of those polynomials $f = r_0 + r_1 x + r_2 x^2 + \cdots$ such that r_1 is divisble by t. Let $P := A \cap (x)B$, a prime ideal of A, and let M be the A-module A/P^2. Find the associated primes of

M. Conclude that P^2 is not a primary ideal of A, *i.e.*, that M is not coprimary. Write P^2 as an intersection of two primary ideals in A.