Homework Assignment #5:

February 18, 2016

Let A be a commutative ring. An *idempotent* of A is an element e such that $e^2 = e$.

1. Show that if e_1 is an idempotent of A, then $e_2 := 1 - e_1$ is another, and that $e_1 e_2 = 0$. Let A_i be the ideal of A generated by e_i, that A_i becomes a ring with identity element e_i, and that the natural map $A/A_1 \to A_2$ is an isomorphism. Show that the natural map $A \to A_1 \times A_2$ is an isomorphism, where $A_1 \times A_2$ is the product in the category of rings.

2. With the notation of the previous problem, show that A_1 is projective as an A-module. Show that if neither e_1 nor e_2 vanishes, then A_1 is not a free A-module.

3. Let k be a field and let A be the ring obtained by dividing the polynomial ring $k[x_1, x_2, \cdots ...]$ by the ideal generated by the polynomials $x_i^2 - x_i$ and $x_i x_j$ for $i \neq j$. The quotient of A by the ideal generated by the images of all the x_i is just k. Show that the A-module k is flat and finitely generated but not projective.

4. In the category of R-modules, prove that if I is injective and F is flat, then $\mathrm{Hom}_R(F, I)$ is again injective.

5. Let M and M' be R-modules. Construct an isomorphism $\mathrm{Tor}(M, M') \cong \mathrm{Tor}(M', M)$, using the definition in class of $\mathrm{Tor}(M', M)$ as the kernel of the map $K \otimes M' \to F \otimes M'$, where $0 \to K \to F \to M \to 0$ is any exact sequence, with F free. Hint: Start with an exact sequence $0 \to K' \to F' \to M'$, take a lot of tensor products to make a 3×3 square diagram, and then chase it.