Homework Assignment #4: Presentations, Tensor products, and Hom

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Recall that an abelian category is an additive category which admits finite limits and colimits and with the additional property that for every homomorphism θ, the canonical map from the coimage of θ to the image of θ is an isomorphism.

Let R be a commutative ring. Recall that a *presentation* of an R-module E is an exact sequence of the form

$$R^{(T)} \xrightarrow{\Phi} R^{(S)} \to E \to 0.$$

(Here the notation $R^{(X)}$ means the free R-module with basis X.) Thus E is isomorphic to the cokernel of Φ. We know that any E admits such a presentation. If there exists a presentation of E with S finite, E is said to be *finitely generated*, and if there exists a presentation with both S and T finite, E is said to be *finite presented*.

1. Let α and β be a pair of composable homomorphisms in an abelian category such that $\beta \circ \alpha = 0$. Assume that α is a monomorphism and that β is an epimorphism.

   (a) Prove that α is the kernel of β if and only if β is the cokernel of α.

(b) Suppose we are given a commutative square

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "b"'] & B \arrow[d, "c"] \\ C \arrow[r, "d"] & D \end{tikzcd}$$

Such a square is said to be cartesian if (A, a, b) is the limit (fiber product) of the maps (c, d) and is said to be cocartesian if (c, d, D) is the colimit (cofibered sum) of the maps (a, b). Prove that if c is an epimorphism and the square is Cartesian, then it is cocartesian.

2. Let A be the ring of polynomials in the variables x, y, z with coefficients in a field k. Find a presentation $A \to A^2$ of the ideal (x, y) generated by x and y, starting with these two generators. Then find a presentation $\Phi\colon A^m \to A^3$ of the ideal (x, y, z) generated by x, y, z, starting with these three generators, with m as small as possible. (Prove that your result really is a presentation, but do not try to prove that your m is as small as possible.) Finally, find a presentation for the kernel of the map Φ

3. Let E be an R-module, and let I be the category whose objects are the finitely generated submodules of E and whose morphisms are the inclusions. Show that I is filtering, and conclude that E is a direct limit of finitely generated modules. Now modify the proof to show that in fact E is a direct limit of finitely presented modules.

4. Let R be a commutative ring and E and R-module. so that $h^E\colon M \to \mathrm{Hom}_R(E, M)$ is a functor from the category of R-modules to itself. Prove that h^E commutes with filtering colimits (*i.e.*, direct limits) if E is finitely presented. Prove that the converse is also true.

5. Let E and F be R-modules. Show that there is a unique homomorphism $\mathrm{Hom}(E, R) \otimes F \to \mathrm{Hom}(E, F)$ with the property that $\phi \otimes f$ is the map h sending e to $\phi(e)f$. Show that this homomorphism is an isomorphism if E is projective and either F or E is finitely presented.

6. Let R be a commutative ring and let S be a multiplicative subset of R. Prove that the functor $M \mapsto M_S$ commutes with colimits and with finite inverse limits. Deduce that it commutes with direct sums. Give an example to show that it does not commute with infinite products in general, and hence that it does not commute with limits in general.