Homework Assignment #2: Functors, Equations, and Categories

January 31, 2016

1. Let A be a commative ring and let $S := \mathrm{Spec}(A)$, endowed with the Zariski topology. If $a \in A$, let $D(a)$ denote the set of prime ideals of A which do not contain a. Prove that $D(a)$ is open in S, and then that the family of all sets of the form $D(a)$ for $a \in A$ forms a basis for the topology, closed under finite intersections.

2. A morphism $f\colon A \to B$ in a category $\mathcal{C}$ is said to be an *epimorphism* (resp. *monomorphism*) if $g \circ f = g' \circ f$ (resp. $f \circ g = f \circ g'$) implies $g = g'$. Prove that in the category of commutative rings, the inclusion from the ring of integers to the ring of rational numbers is an epimorphism. Prove that in the category of groups, on the other hand, every epimorphism is surjective. This is easy if the groups are commutative. The noncommutative case is more difficult. Here is a hint. It is enough to show (why?) that if H is a proper subgroup of G, then there exists two different homomorphisms $\alpha, \beta\colon G \to G'$ with the same restriction to H. This is easy (why?) if H is normal and in particular if H has index two in G. Otherwise there exist elements g_1 and g_2 not in H such that the right cosets Hg_1 and Hg_2 are different. There is an obvious bijection $Hg_1 \to Hg_2$, and one can extend this to a bijection τ from the set G to itself by having it act as the identity on the complement of $Hg_1 \cup Hg_2$. Thus τ is an element of the group of permutations G' of the set G. Recall (Cayley) that if $g \in G$, left translation σ_g by g defines a permutation of this set, and $g \to \sigma_g$ is a homomorphism from G to G'. Compare σ with σ', where $\sigma'_g := \tau\sigma_g\tau^{-1}$.

3. Let R be a commutative ring and let $\mathcal{A}_R$ be the category of R-algebras.

Recall that if A is an object of $\mathcal{A}_R$, h^A is the functor from $\mathcal{A}_R$ to the category of sets sending B to $\mathrm{Mor}_{\mathcal{A}_R}(A, B)$. Let I be an ideal of A and let $Z_I(B)$ denote the set of all $\theta \in h^A(B)$ such that $\theta(I) = 0$. Show that there is a universal such element. Let $U_I(B)$ denote the set of all $\theta \in h^A(B)$ such that $\theta(I)$ generates the unit ideal of B. Show that U_I defines a subfunctor of h^A. Show that if B is a field, $h^A(B)$ is the disjoint union of $U_I(B)$ and of $Z_I(B)$, but that this need not be the case if B is an integral domain.

4. Let $\mathcal{C}$ and $\mathcal{C}'$ be categories. Then an *isomorphism* $\mathcal{C} \to \mathcal{C}'$ is a functor F which admits an inverse, that is, a functor $G: \mathcal{C}' \to \mathcal{C}$ such that $G \circ F$ is the identity functor of $\mathcal{C}$ and $F \circ G$ is the identity functor of $\mathcal{C}'$. Although such isomorphisms do exist (the identity functor is an isomorphism), they are not so common, and there is a weaker notion that is more useful in practice. A functor $F: \mathcal{C} \to \mathcal{C}'$ is called an *equivalence of categories* if it is fully faithful and if in addtion for every object C' of $\mathcal{C}'$, there is an object C of $\mathcal{C}$ such that $F(C)$ is isomorphic to C'. Now prove the following statement.

   A functor $F: \mathcal{C} \to \mathcal{C}'$ is an equivalence if and only if there exists a functor $G: \mathcal{C}' \to \mathcal{C}$ such that $G \circ F$ is *isomorphic* to the identity functor $\mathrm{id}_{\mathcal{C}}$ and $F \circ G$ is *isomorphic* to the identity functor $\mathrm{id}_{\mathcal{C}'}$

5. Let $\mathcal{C}$ be the category whose objects are the natural numbers, where the morphisms from n to m are the $m \times n$ matrices, and where composition is defined by matrix multiplication. Let k be a field. For each n, let $V(n)$ be the vector space k^n and for each $m \times n$ matrix A let $V(A): V(n) \to V(m)$ denote the linear transformation $V(n) \to V(m)$ defined by multiplication by A. Show that the functor thus defined is an equivalence from $\mathcal{C}$ to the category of finite dimensional k-vector spaces.

6. Let $\mathcal{E}ns$ denote the category of sets. Our aim is to construct a presheaf on $\mathcal{E}ns$ which is not representable, *i.e.*, not isomorphism to h_X for any set X. For each natural number n, let $S_n := \{0, 1, \cdots, n-1\}$ and let $S := \cup S_n$ Then the inclusions $S_{i-1} \to S_i$ induce maps of presheaves $h_{S_{i-1}} \to h_{S_i}$, and in fact $h_{S_{i-1}}(T) \subseteq h_{S_i}(T)$ for every T. Let $F(T) := \cup_i h_{S_i}(T)$. This is just the set of all functions from T to S whose image is finite. Show that F is indeed a presheaf but that it is not representable. (Hint: If X is a set, what does h_X do with colimits?)