Homework Assignment #11:

April 14, 2016

1. Let A be a commutative ring and let E and E' be A-modules. Show that

$$supp(E \otimes_A E') \subseteq supp(E) \cap supp(E'),$$

with equality if E and E' are finitely generated. Give an example to show that equality need not hold if E or E' is not finitely generated.

2. Let R be a noetherian local ring and let E be a finitely generated R-module. Recall that $depth(E)$ is the maximal length of an E-regular sequence contained in the maximal ideal m of R. Our work on dimension theory showed that $depth(E) \leq \dim(E)$. Prove this directly, using induction on $depth(E)$ and the theory of associated primes.

3. Let G be an $\mathbf{N}$-graded R-algebra. The geometric meaning of the grading on G can be expressed by saying that $\operatorname{Spec} G$ is (in a weak sense) a cone. To make this precise, we consider the functor h^G from the category $\mathcal{A}$ of R-algebras to sets. Recall that the functor $h^{R[t]}$ takes $A \in \mathcal{A}$ to the underlying set of A. It has a natural structure of a monoid, given by the multipicative monoid structure of A. We write A for this monoid-valued functor, and think of it as "the affine line" Now we claim that the grading on G defines an action of the monoid-valued functor A on the set-valued functor h^G, that is, a natural transformation $\mathrm{A} \times h^G \to h^G$ satisfying the usual rules of a monoid action. Namely, if $\theta\colon G \to A$ is an R-algebra homomorphism and $a \in A$, define

$$a\theta : G \to A\colon g \mapsto \sum_i a^i \theta(g_i).$$

Check that this really is a homomorphism of R-algebras and that it really does define an action of A on h^G.

4. Continuing with the theme of the previous problem, show that if G is any R-algebra, an action of A on h^G defines an $\mathbf{N}$-grading on G. (Hint: Use the fact that $\mathrm{A} \times h^G$ is representable.)

5. Continuing with the them of problem 1, show that

$$Ann(E) + Ann(E') \subseteq Ann(E \otimes_A E')$$

but that equality need not hold even if E and E' are finitely generated.