Mathematics 250A Professor A. Ogus
November 17, 2010

Algebra Midterm Exam II

Note: There are four problems. Write your answers on the exam, using both sides of the page if necessary. Use complete sentences and correct punctuation. You lose points for extraneous statements, especially if they are incorrect.

1. What is the definition of an *ideal* in a ring R (not necessarily commutative)? If S is a subset of R, what is the *ideal Ithe generated by S*, and what is the universal mapping property of the canonical homomorphism $\pi \colon R \to R/I$?

   Solution

   (a) An ideal of R is a subset I which is a subgroup of the group $(R, +)$ and such that rx and xr belong to I whenever $r \in R$ and $x \in I$.

   (b) The ideal I_S generated by S is the smallest ideal of R containing S, that is, the set of all elements of R which can be written as a finite sum: $\sum_{s \in S} a_s s b_s$ with $a_s, b_s \in R$, $s \in S$.

   (c) This the universal homomorphism from R to a ring under which all the elements of S map to 0.

2. If R is a commutative ring, show that the intersection of all the prime ideals of R is exactly the set of nilpotent elements of R. (Do not attempt to write too much here.)

   Solution: If P is a prime ideal and $r \in R$ is nilpotent, then r must be contained in P, since $r^n = 0$, by induction on n. If r is not nilpotent, then the multiplicative subset S of R generated by r does not contain the zero element, and it follows that the localization R_S of R by S is not the zero ring. Hence it contains a maximal ideal, and the inverse image of this ideal in R is a prime ideal not containing r.

3. Let M be the abelian group with generators e_1, e_2, e_3, e_4 and relations $2e_1 + 6e_2 = 0$ and $2e_1 + 9e_2 + 6e_3 + 12e_4 = 0$. Is its torsion subgroup cyclic?
Solution: The group M is the cokernel of the map $\mathbf{Z}^2 \to \mathbf{Z}^4$ whose matrix is $\begin{pmatrix} 2 & 2 \\ 6 & 9 \\ 0 & 6 \\ 0 & 12 \end{pmatrix}$. We can use elementary row and column operations. to obtain another presentation of M:

$$\begin{pmatrix} 2 & 2 \\ 6 & 9 \\ 0 & 6 \\ 0 & 12 \end{pmatrix} \to \begin{pmatrix} 2 & 0 \\ 6 & 3 \\ 0 & 6 \\ 0 & 12 \end{pmatrix} \to \begin{pmatrix} 2 & 0 \\ 0 & 3 \\ 0 & 6 \\ 0 & 12 \end{pmatrix} \to \begin{pmatrix} 2 & 0 \\ 0 & 3 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Thus M is isomorphic to $\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/3\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$. Its torsion subgroup is isomorphic to $\mathbf{Z}/6\mathbf{Z}$, which is cyclic.

4. Find all the solutions of the following equations: in each case, explain why there are no other solutions.

   (a) $x^{10} = -1$ in the ring $\mathbf{Z}/5\mathbf{Z}$.
   (b) $x^6 = 1$ in the ring $\mathbf{Z}/91\mathbf{Z}$
   This problem might take too long. I will give full credit if you just give the number of solutions, with a proof that this number is correct. Extra credit if you find the solutions more or less explicitly.

Solutions:

(a) In the ring $\mathbf{Z}/5\mathbf{Z}$, $a = a^5$, so $a^{10} = a^2$. Thus we are looking for solutions to the equation $x^2 + 1 = 0$. Since our ring is an integral domain, there are at most 2 solutions, In fact the classes of 2 and 3 are solutions.

(b) This ring R is not an integral domain. In fact $91 = 13 \times 7$, so $R = \mathbf{Z}/13\mathbf{Z} \times \mathbf{Z}/7\mathbf{Z}$. The group of units of $\mathbf{Z}/13\mathbf{Z}$ has order 12, and the set of elements of order 6 is cyclic, generated by the class of 4. The group of units of $\mathbf{Z}/7\mathbf{Z}$ has order 6, so every element satisfies this equation. Thus there are 36 solutions in the original ring R. Explcitly, take $x = 14 \times 4^i - 13j \bmod 91$ for $1 \leq i, j \leq 6$ to get all solutions.