Mathematics 250A

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Algebra Midterm Exam

Note: There are three problems. Write your answers on the exam, using both sides of the page if necessary. Use complete sentences and correct punctuation. You lose points for extraneous statements, especially if they are incorrect.

1. Let G be a nonabelian group of order 12. Assume G has a normal subgroup of order 4.

   (a) How many 2-Sylow subgroups does G have? Explain.
   Solution: The normal subgroup N of order 4 is a 2-Sylow subgroup. Since all such subgroups are conjugate, there is only one.

   (b) How many 3-Sylow subgroups does G have? Explain.
   Solution: The number of such groups divides $12/3 = 4$ and is congruent to 1 mod 3, hence is 1 or 4. If there were just one such group H, it would be normal, and then the elements of H and N would commute. But both H and N are commutative and $G = HN$, so G would also be commutative.

   (c) Prove that G is isomorphic to A_4.
   Solution: The action of G on the set of 3-Sylow subgroups defines a homomophism $\theta: G \to S_4$. The kernel of θ is a normal subgroup of G and is contained in the intersection of all the normalizers of the 3-Sylow subgroups, and hence is trivial. This gives an injection $G \to S_4$. But A_4 is the unique subgroup of A_4 whose order is 12, so $G \cong A_4$. Alternatively, we could consider a 3-Sylow subgroup H and let G act on G/H by translation. This also gives a homomorphism $G \to S_4$, since the index of H is 4. The kernel of this homomorphism is a normal subgroup G contained in H, hence is trivial, so the same argument applies. Another possible approach is to use the fact that G is a semidirect product of N and H with respect to a nontrivial homomorphism $\alpha: H \to Aut(N)$.

If N were cyclic, $Aut(N)$ would be μ_2 and there would be no such α. Hence $N = \mu_2 \times \mu_2$, and $Aut(N) \cong S_3$. Then α is given by an element of order 3, and any two are conjugate. The same argument applies to A_4 (after some computation of its Sylow structure), so this could be used to give a proof.

2. Permutations

   (a) What is the definition of the group S_n?
   Solution: This is the set of bijections from the set $\{1, \ldots, n\}$ to itself, with composition as the group law.

   (b) Write each of the following elements of S_9 as a product of disjoint cycles, say whether it is even or odd, and compute its order. Then compute the number of its conjugates and describe its centralizer in S_9

      i. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 7 & 8 & 9 & 6 & 4 & 2 & 1 & 5 \end{pmatrix}$

      Solution: This is $(1\ 3\ 8)(2\ 7)(4\ 9\ 5\ 6)$. Its order is 12 and it is even. It has $9!/3 \cdot 2 \cdot 4$ conjugates in S_9, and hence its centralizer has order $3 \cdot 2 \cdot 4$. It follows that the centralizer is the product of the cyclic groups generated by each of the cycles appearing in its decomposition:

      $$\langle(1\ 3\ 8)\rangle\langle(2\ 7)\rangle\langle(4\ 9\ 5\ 6)\rangle.$$

      ii. $(5\ \ 1)(3\ \ 7)(1\ \ 3\ \ 5)(2\ \ 6)(4\ \ 8)(4\ \ 9)$

      Solution: This is $(1\ \ 7\ \ 3)(2\ \ 6)(4\ \ 9\ \ 8)$. Its order is 6 and it is odd. It has $9!/3 \cdot 3 \cdot 2 \cdot 2$ conjugates, and its centralizer C has order 36. The centralizer contains the product of the cyclic groups generated by the cycles in the decomposition, which has order 18. Thus the centralizer is generated by these cycles as well as one additional element, for example $(1\ 4)(7\ 9)(3\ 8)$.

3. Let H and K be subgroups of a finite group G, and let HK denote the set of all elements of G which can be written as a product hk with $h \in H$ and $k \in K$. Without assuming anything about the normalizers of H and K, show that $|HK||H \cap K| = |H||K|$.

   Solution: We have a surjective map $H \times K \to HK$, so it suffices to prove that the inverse image of every $g \in HK$ has cardinality $|H \cap K|$. In fact $h_1k_1 = h_2k_2$ iff there exists an element $g \in H \cap K$ such that $h_2 = h_1g^{-1}$ and $k_2 = gk_1$, furthermore such an element is unique if it exists. This shows what we want. A better way to say this: The group $H \cap K$ acts on the left on the set $H \times K$ by $g(h, k) := (hg^{-1}, gk)$, and the fibers of the multiplication map $H \times K \to HK$ are the orbits for this action. Furthermore, the stabilizer subgroups are all trivial, so the cardinality of the orbits is the cardinality of $H \cap K$. Yet another way: Let S denote the set of cosets of K of the form $hK; h \in H$. Then HK is the union of the elements in S. This is a disjoint union of sets all of the same cardinality $|K|$, so $|HK| = |S||K|$. Moreover, the group H acts transitively on the set S, so $|S| = |H|/|H_K|$, where $H_K := \{h \in H : hK = K\}$. Then $H_K = H \cap K$, and the result follows.