Mathematics 250A | Professor A. Ogus
December, 2010

Algebra Final Exam Problems—take 2

1. Let G be a nonabelian group of order 12. Assume G has a normal subgroup of order 4.

   (a) How many 2-Sylow subgroups does G have? Explain.

   (b) How many 3-Sylow subgroups does G have? Explain.

   (c) Prove that G is isomorphic to A_4.

2. Permutations

   (a) What is the definition of the group S_n?

   (b) Write each of the following elements of S_9 as a product of disjoint cycles, say whether it is even or odd, and compute its order. Then compute the number of its conjugates and describe its centralizer in S_9

   i. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 7 & 8 & 9 & 6 & 4 & 2 & 1 & 5 \end{pmatrix}$

   ii. $(5\ \ 1)(3\ \ 7)(1\ \ 3\ \ 5)(2\ \ 6)(4\ \ 8)(4\ \ 9)$

3. Prove that every automorphism of S_3 is inner. Define all these terms.

4. Let H and K be subgroups of a finite group G, and let HK denote the set of all elements of G which can be written as a product hk with $h \in H$ and $k \in K$. Without assuming anything about the normalizers of H and K, show that $|HK||H \cap K| = |H||K|$.

5. Cosets

   (a) If H is a subgroup of G, what is the definition of a *left coset of H*?

   (b) Let S be a set, let G be the group of permutations of S, let t be an element of S, and let $H := \{h \in G : h(t) = t\}$. Prove that H is a subgroup of G.

(c) With the notation of the previous problem, is H necessarily normal? Give a proof or counterexample.

(d) Continuing with the notation of the previous part, show that there is a well-defined map η from the set G/H of left cosets of H to S with the property that $\eta(A) = g(t)$ for every $g \in A$ and every left coset $A \in G/H$.

(e) Now show that the map η in the previous part is bijective.

6. Give a list of all isomorphism classes of abelian groups of order 48, with each isomorphism class occuring exactly once in your list.

7. Let k be a field and let A be a finite dimensional k-algebra.

   (a) What does it mean for A to be *separable* over k?

   (b) Let X denote the set of homomorphisms of k-algebras from A to an algebraic closure K of k. Assume A is separable. Prove that A is a a field if and only the action of $Aut(K/k)$ on X is transitive. More generally, show that the orbits of this action can be naturally identified with the set of prime ideals of A.

8. Explain Yoneda's theorem. Give an example of a functor from the category of groups to the category of sets which is not representable (with proof!).

9. Let G be a group and S_G the category of G-sets. Let $\Phi \colon S_G \to S$ be the forgetful functor. What is the group of automorphisms of Φ?

10. Let E/k be a finite Galois extension, and let $S(E/k)$ be the category of finite separable extensions F/k which admit a map to E/k. Let X be the functor from $S(E/k)$ to S which takes F/k to the set of all maps $F/k \to E/k$. What is the group of automorphisms of X?

11. Suppose that N and K are subgroups of a finite group G. Does $\#NK$ necessarily divide $\#G$? Give proof, counterexample, and/or a correct statement and proof.

12. Find the Sylow subgroups of the following groups (or similar ones):

(a) A finitely generated abelian group, given in terms of generators and realtions

(b) The automorphism group of a small set.

13. State and prove the Sylow theorems for finite groups. Use them to determine the isomorphism classes for groups of order (e.g.) 15, 45, etc.

14. State and prove the class formula for the action of a group on a finite set. For the size of the conjugacy classes of a finite group. Find all groups, up to isomorphism, of order 121.

15. Prove that every p-group has a nontrivial center.

16. Prove that S_n has a unique subgroup of index 2, if $n > 1$.

17. Find the conjugacy classes in S_n and A_n.

18. Show that the group A_n is simple if $n \geq 5$. Prove that every simple group of order 60 is isomorphic to A_5. Prove that every group of order less than 60 is solvable.

19. Let K and N be groups and $\alpha\colon K \to \mathrm{Aut}(N)$ a homomorphism. Define the semi-direct product of K and N with respect to α. Describe the dihedral group of order $2n$ as a semi-direct product.

20. Suppose that G is a finite simple group, and let H be a subgroup of index i. Prove that the order of G divides $i!/2$.

21. Give at least two equivalent characterizations of the concepts of primitivity and transitivity for G-sets.

22. Define what it means for a morphism in a category to be an epimorphism. Prove that every epimorphism in the category of groups is surjective. Prove that this is not true in the category of commutative rings.

23. Define what is meant by a inverse (resp. diret) limit of a sequence of abelian groups. Prove or disprove that formation of each of these limits preserves exactness of short exact sequences.

24. What is meant by a free object in the category of abelian groups? In the category of groups? What is a notion which is a useful replacement for freeness in an additive category?

25. Compute the automorphism group A of $\mathbf{Z}/p^e\mathbf{Z}$. Hint: If $e = 1$, it is cyclic because it is the group of units in a field. If $e > 1$, show that the map $A \to Aut(\mathbf{Z}/p\mathbf{Z})$ is surjective, and let K be the kernel. If p is odd, show that the kernel is cyclic, generated by $1 + p$. If $p = 2$, consider instead that map $A \to Aut(\mathbf{Z}/4\mathbf{Z})$, and analyze the kernel and the extension class.

26. Let R be a commutative ring. Define prime ideals, maximal ideals, etc. Prove that every nonzero ring contains a maximal ideal. Prove that the intersection of all prime ideals in R is just the set of nilpotent elements.

27. Suppose that A is a commutative ring with identity, and $(m_1, \dots m_n)$ is a finite set of distinct maximal ideals. Prove that there is a natural isomorphism:

$$A/(m_1 \dots m_n) \cong A/m_1 \times \cdots A/m_n$$

Here $m_1 \dots m_n$ is the product of the ideals and $A/m_1 \times \cdots A/m_n$ is the Cartesian product in the category of rings.

28. Let M be a monoid What is meant by the monoid algebra $\mathbf{Z}[M]$? Describe its defining universal property.

29. Let R be a commutative ring, let E be a subset of the polynomial ring $R[x_1, \dots, x_n]$. What is meant by the universal solution to the family of equations E? When is this solution trivial?

30. Let R be a ring, let f be an element of the polynomial ring $R[x_1, \dots, x_n]$.

    (a) Explain how f defines a function $f_R \colon R^n \to R$.

    (b) Explain how the above construciton defines a homomorphism of R-algebras from the polynomial ring $R[x_1, \dots, x_n]$ to the algebra of functions $R^n \to R$. If R is a finite field, is this homomorphism injective? surjective?

(c) Generalize the above construction to define a function $f_A\colon A^n \to A$ for every R-algebra A. Show that the collection of all f_A defines a natural transformation $\tau_f\colon F^n \to F$, where F is the forgetful functor. Show that the map from the polynomial ring $R[x_1, \ldots, x_n]$ to the set of natural transformation $F^n \to F$ is bijective.

31. What is meant by an algebraic extension of fields? Prove that if E/k and K/E are algebraic, then K/k is algebraic. Prove that if E/k is an arbitrary extension of k, then the set of all elements of E which are algebraic over k is a subfield of E.

32. If E/k is a field extension and $e \in E$, what does it mean for e to be separable over k? Prove that the set of such e is a subfield of E. Give an example of an element which is not separable.

33. Suppose that E/k and F/k are Galois extensions of k, both contained in some algebraic closure K of k. Prove that there is an exact sequence of groups:

$$1 \to Gal(EF/F) \to Gal(E/k) \to Gal(E \cap F/k) \to 1$$

Conclude that $[E : k][F : k] = [EF : k][E \cap F : k]$. Give an example showing that this equation is false without the assumption that the extensions be Galois.

34. Prove that a finite separable extension is simple

35. State and prove the theorem on the independence of characters

36. Prove that every Galois extension which is cyclic of order m of a field k with m m^{th} roots of unity is obtained by extracting the m^{th} root of some element of k.

37. Compute the splitting fields and Galois groups of the following polynomials.....(e.g. $X^4 + 2$ over $\mathbf{Q}$, or $X^3 + t^2x + t$ over $\mathbf{Q}(t)$).

38. Let K/k be a finite Galois extension with group G. State the normal basis theorem, using the language of group algebras and also explicitly. Find a normal basis for the splitting field of the polynomial $x^3 - x - 1$ over the finite field with 3 elements.

39. Prove that every finite subgroup of the multiplcative group of a field is cyclic.

40. Show that every field extension of finite fields is Galois and compute its Galois group.

41. State and prove the Gauss lemma for polynomials with coefficients in a UFD.

42. What is the definition of an *ideal* in a ring R (not necessarily commutative)? If S is a subset of R, what is the *ideal* I *generated by* S, and what is the universal mapping property of the canonical homomorphism $\pi\colon R \to R/I$?

43. Let M be the abelian group with generators e_1, e_2, e_3, e_4 and relations $2e_1 + 6e_2 = 0$ and $2e_1 + 9e_2 + 6e_3 + 12e_4 = 0$. Write M as a product of cyclic groups. Is its torsion subgroup cyclic?

44. Find all the solutions of the following equations: in each case, explain why there are no other solutions.

    (a) $x^{10} = -1$ in the ring $\mathbf{Z}/5\mathbf{Z}$.

    (b) $x^6 = 1$ in the ring $\mathbf{Z}/91\mathbf{Z}$

    (c) $x^3 = 2$ in the ring $\mathbf{Z}[t]/(t^3 - 2)$.

45. Let k be a field and let m be an integer not divisible by the characterstic of k. Let a be an element of k and let K be a splitting field of the polynomial $t^m - a$. Let G be the Galois group of K/k.

    (a) Define what is meant by the cyclotomic character χ of G.

    (b) Let k_m be the subfield of K generated by the group μ_m of mth roots of unity in K. Show that there is an injective homomorphism $\tau\colon G(K/k_m) \to \mu_m$.

    (c) Describe the image of the cyclotomic character when $k = \mathbf{Q}$ and when k is a finite field with q elements.

    (d) Factor the polynomial $X^{48} - Y^{48}$ into irreducible factors in the ring $\mathbf{Q}[X, Y]$. Why are the factors irreducible?

46. (a) Let G be a group and N a normal abelian subgroup of G. Show that the action of G on N by conjugation defines a homomorphism

$$\alpha: G/N \to Aut(N).$$

(b) Apply the above construction when $N = G(K/k_m)$ and $G = Gal(K/k)$. Identify the action α of G/N on N via χ and τ.