The derivative

February 19

Definition: Let f be a real valued function with domain D and let a be an element of D. Then the *derivative* of f at a is

$$f'(a) := \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{h\to a} \frac{f(a+h)-f(a)}{h}.$$

The *tangent line to* f at a is the line passing through the point $(a, f(a))$ and whose slope is $f'(a)$.)

It follows from the point-slope formula for the equation of a line that the equation for the function ℓ_a defining the tangent line is

$$\ell_a(x) = f'(a)(x-a) + f(a).$$

(If you have trouble with this formula, just check that this is a line with slope $f'(a)$ and that it passses through the point $(a, f(a))$, because $\ell_a(a) = f(a)$.

Philosophy: The function ℓ_a is the best possible straight line approximation to the function f near $x = a$.

I hope to return to this later. In the meantime, let's discuss the following theorem.

Theorem: Assume that $f'(a)$ exists. Then f is continuous at a.

The proof in the book uses the limit theorems for the proof. Let's review this. We begin with a simple algebraic manipulation.

$$f(x) = \frac{f(x)-f(a)}{x-a}(x-a) + f(a).$$

Now we apply the limit laws (for sums and products):

$$\begin{aligned} \lim_{x\to a} f(x) &= \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right)\lim_{x\to a}(x-a) + \lim_{x\to a} f(a) \\ &= f'(a)0 + f(a) \\ &= f(a) \end{aligned}$$

This tells us exactly that f is continuous at a, as desired.

This situation is so important that it is worthwhile to give an ϵ-δ proof, which gives more information which can be quite useful.

We want to prove that given any $\epsilon > 0$, there is $\delta > 0$ such that

$$|f(x) - f(a)| < \epsilon \text{ for every } x \text{ such that } |x - a| < \delta.$$

What we know looks quite different: for any $\epsilon' > 0$, there is a $\delta' > 0$ such that

$$\left|\frac{f(x) - f(a)}{x - a} - f'(a)\right| < \epsilon' \text{ for every } x \text{ such that } 0 < |x - a| < \delta'|$$

What is the relationship between ϵ' and ϵ? Really there isn't one: ϵ is out of our control, and we get to choose any ϵ' that is convenient. In class I (as usual) chose $\epsilon' = 1$. Here let it just be any positive constant. If we multiply both sides of the equation above by the positive number $|x - a|$, we get

$$|f(x) - f(a) - f'(a)(x - a)| \leq \epsilon'|x - a|$$

for every x such that $0 < |x - a| < \delta'|$. Here I have written $\leq$ instead of $<$, because then the statement will be true even if $|x - a| = 0$ (just check both sides). Using the equation of the tangent line, we see the important equation

$$|f(x) - \ell_a(x)| \leq \epsilon'(x - a|.$$

If $\epsilon' < 1$ this tells us that the difference between the function f and its tangent line is smaller (by a factor of ϵ') than the difference between x and a.

Let's now return to our orginal problem. Let $m := f'(a)$. Given $\epsilon > 0$, choose any ϵ' you like, then choose δ' as above, and then choose

$$\delta := \min\left(\delta', \frac{\epsilon}{(|m| + \epsilon'}\right).$$

Suppose that $|x - a| < \delta$. Then $|x - a| < \delta'$, and we can use the equations above to conclude that

$$|f(x) - f(a) - m(x - a)| \leq \epsilon'|x - a|.$$

We compute

$$\begin{aligned}|f(x)-f(a)| &= |f(x)-f(a)-m(x-a)+m(x-a)| \\ &\leq |f(x)-f(a)-m(x-a)|+|m(x-a)| \\ &\leq \epsilon'|x-a|+|m||x-a| \\ &\leq |x-a|(\epsilon'+|m|) \\ &< \delta(\epsilon'+|m|)\end{aligned}$$

Now we use the fact that $\delta \leq \epsilon/(\epsilon'+|m|)$ to conclude that $|f(x)-f(a)| < \epsilon$, as desired. Note that once we have found δ', it is very easy to find δ.