Lecture 1—Sets and Functions

January 20

Theme: Mathematics is about families and relationships, not things.

1 Families

Definition: A *set* is a group (or family) of things. Two sets are equal if they contain the same elements.

Examples:

- $S := \{1, 2\}$ means the set containing just the two numbers 1 and 2.
- $T := \{1, 2, 3\}$ means the set containing the three numbers 1, 2 and 3.
- $\mathbf{N} := \{0, 1, 2, 3, \ldots\}$ is the (infinite) set of all natural numbers.

Notation:

- We write $\{a, b, c \ldots\}$ to mean the set containing the given elements.
- We write $\{a : a$ has some property P$\}$ for the set of all elements having the property P.
- $a \in A$ means that the element a belongs to the set A.
- $A \subseteq B$ means that every element of A belongs to the set B. Thus $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.
- $A \cap B$ is the set of elements which belong both to A and to B.
- $A \cup B$ is the set of elements which belong to either A or to B.
- (a, b) stands for the ordered pair whose first element is a and whose second element is b. Note that $\{1, 2\}$ is the same set as the set $\{2, 1\}$, but $(1, 2)$ is not the same ordered pair as $(2, 1)$.
- If A and B are sets, then $A \times B$ is the set of ordered pairs $(a, b) : a \in A$ and $b \in B$.

2 Relations

Definition: Let A and B be sets. A *relation* from A to B is a subset R of $A \times B$. Think of R as a list or table showing which elements of A are related to which elements of B. We say that a is related to b (by R) if the pair (a, b) belongs to the set R. A is called the *domain* of R and B is called the *codomain* of R. For example, a phone can be thought of as a relation from the set of people to the set of numbers.

3 Functions

Definition: A *function* from A to B is a relation R from A to B such that every element a of A is related to *exactly one* element of B. This means that the set R satisfies the *vertical slice test*: For every $a \in A$, the set $(\{a\} \times B) \cap R$ contains exactly one element.

Thus: a function is a *rule* which assigns to each element a of A a specific element of B—the only element of B which is related to a by f.

Notation: We usually denote a function by a small letter, such as f. We write $f\colon A \to B$ to mean that f is a function from A to B. We write $b = f(a)$ if b is related to a by f.

Functions can be specified in the following ways:

1. A table: This is really just our definition: the table *is* just the set of pairs (a, b) in f.

2. A graph: This is just a "picture" of the table, displayed with the elements A horizontally and the elements of B vertically.

3. A formula: sometimes it is possible to describe the relation by a simple formula for f. So we might be able to write $f := \{(a, b) : b = formula(a)\}$, or just $f(a) = formula(a)$. For example, $f(x) = x^2$ defines a function from $\mathbf{R}$ to $\mathbf{R}$ which takes a number to its square.

4. English sentences: sometimes one needs to use the complexity of the English language to describe how a formula works. For example, this can happen when cases have to be considered. For example, the absolute value function is given by:

$$| \ |\colon \mathbf{R} \to \mathbf{R} \quad |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

Warning: When I write $f\colon A \to B$ I mean that A is the domain of f and B is the codomain of f. This means that f accepts elements of A as inputs and provides elements of B as outputs. The book uses the word *range* for the "possible outputs" of f. This is a little bit ambiguous: does it mean the set of all outputs that actually occur or just those that might conceivably occur? Most likely it is the latter, which is not the same as my meaning of the word "codomain." In fact it can be very difficult to figure out what the "range" of a function really is.

When is a relation a function? This question can be answered in terms of the graph by means of the "vertical slice test." If $a \in A$, then the set $\{a\} \times B$ is a vertical slice of $A \times B$: it is the set of all pairs in $A \times B$ in which the first element is A. Then a relation $R \subseteq A \times B$ is (the graph of) a function if and only if every vertical slice meets R exactly once, that is, if and only if $(\{a\} \times B) \cap R$ has exactly one element, for every $a \in A$.