Math 1A — UCB, Spring 2010 — A. Ogus
Solutions[1] for Problem Set 7

§3.4 # 5. Write the function $y(x) = e^{\sqrt{x}}$ in composite form $f(g(x))$, and find $\frac{dy}{dx}$.
Solution. $y(x) = f(g(x))$ where $g(x) = \sqrt{x} = x^{1/2}$, and $f(v) = e^v$. Since $g'(x) = \frac{1}{2}x^{-1/2}$ and $\frac{df}{dv} = e^v$,

$$\frac{dy}{dx} = f'(g(x) \cdot g'(x) = e^{\sqrt{x}} \cdot \tfrac{1}{2}x^{-1/2} = \tfrac{1}{2}x^{-1/2}e^{\sqrt{x}}.$$

□

§3.4 #13. Find the derivative of $y = \cos(a^3 + x^3)$.
Solution. (Here a is a constant.)

$$\frac{dy}{dx} = -\sin(a^3 + x^3) \cdot \frac{d}{dx}(a^3 + x^3) = -\sin(a^3 + x^3) \cdot 3x^2.$$

□

§3.4 #29. Find the derivative of $y = \sin(\tan 2x)$.
Solution. By the chain rule, $\frac{d}{dx}(\sin(\tan 2x)) = \cos(\tan 2x) \cdot \frac{d}{dx}(\tan 2x) = \cos(\tan 2x) \cdot \sec^2 2x \cdot \frac{d}{dx}2x = \cos(\tan 2x) \cdot \sec^2 2x \cdot 2$

§3.4 #31. Find the derivative of $y = 2^{\sin \pi x}$.
Solution. By the chain rule, $\frac{d}{dx}2^{\sin \pi x} = \frac{d}{dx}e^{\ln 2 \sin \pi x} = \ln 2 \cdot e^{\ln 2 \sin \pi x} \cdot \frac{d}{dx}\sin \pi x = \ln 2 \cdot 2^{\sin \pi x} \cdot \cos \pi x \cdot \frac{d}{dx}\pi x = \ln 2 \cdot 2^{\sin \pi x} \cdot \cos \pi x \cdot \pi$

§3.4 #42. Find the derivative of $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$
Solution. $\frac{d}{dx}(\sqrt{x + \sqrt{x + \sqrt{x}}}) = \frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}\frac{d}{dx}(x + \sqrt{x + \sqrt{x}}) = \frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}(1 + \frac{d}{dx}(\sqrt{x + \sqrt{x}})) = \frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}(1 + \frac{1}{2\sqrt{x+\sqrt{x}}}\frac{d}{dx}(x + \sqrt{x})) = \frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}(1 + \frac{1}{2\sqrt{x+\sqrt{x}}}(1 + \frac{1}{2\sqrt{x}}))$

§3.4 #65. Given are the graphs of functions f and g. Let $u(x) = f(g(x))$, $v(x) = g(f(x))$ and $w(x) = g(g(x))$. Find the derivative at 1 of each of u, v and w. If it does not exist, explain why.
Solution. (a) By the chain rule, $u'(1) = f'(g(1))g'(1)$. From the graph we see that $g(1) = 3$, $g'(1) = -3$ and $f'(g(1)) = f'(3) = -\frac{1}{4}$. So we get $u'(1) = \frac{3}{4}$.
(b) By the chain rule, $v'(1) = g'(f(1))f'(1)$. From the graph we see that $f(1) = 2$, $f'(1) = 2$ and $g'(f(1)) = g'(2)$ does not exist. Hence $v'(1)$ also does not exist.
(c) By the chain rule, $w'(1) = g'(g(1))g'(1)$. From the graph we see that $g(1) = 3$, $g'(1) = -3$ and $g'(g(1)) = g(3) = \frac{2}{3}$. So we get $w'(1) = -2$.

§3.4 #82a,b. Under certain circumstances a rumor spreads according to the equation $p = \frac{1}{1+ae^{-kt}}$, where $p(t)$ is the proportion of the population that knows the rumor at time t and a, k are positive constants.
(a) Find $\lim_{t\to\infty} p(t)$.
(b) Find the rate of spread of the rumor.

Solution. (a) $\lim_{t\to\infty} p(t) = \lim_{t\to\infty} \frac{1}{1+ae^{-kt}} = \frac{1}{1+\lim_{t\to\infty} ae^{-kt}} = \frac{1}{1+0}$ since k is positive and thus the exponent tends to negative infinity.
(b) The rate of spread of the rumor is the derivative of $p(t)$. $p'(t) = \frac{-(-kae^{-kt})}{(1+ae^{-kt})^2} = \frac{kae^{-kt}}{(1+ae^{-kt})^2}$

§3.4 #83. A particle moves along a straight line with displacement $s(t)$, velocity $v(t)$ and acceleration $a(t)$. Show that $a(t) = v(t)\frac{dv}{ds}$. Explain the difference between the meanings of the derivatives $\frac{dv}{ds}$ and $\frac{dv}{dt}$.
Solution. We know that the acceleration is the second derivative of $s(t)$. So, $a(t) = s''(t) = (v(t))' = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = \frac{ds}{dt}v(t)$ by the chain rule. $\frac{dv}{dt}$ is the rate of change of the velocity relative to time. $\frac{dv}{ds}$ is the rate of change of the velocity relative to the displacement.

§3.4 #95. If $y = f(u)$ and $u = g(x)$, where f and g are twice differentiable functions, show that $\frac{d^2y}{dx^2} = \frac{d^2y}{du^2}(\frac{du}{dx})^2 + \frac{dy}{du}\frac{d^2u}{dx^2}$
Solution. We will first use the product rule, then the chain rule and then another application of the product rule. $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{d}{dx}(f(g(x)))) = \frac{d}{dx}(f'(g(x))g'(x)) = g'(x)\frac{d}{dx}(f'(g(x))) + f'(g(x))\frac{d}{dx}(g'(x)) = g'(x)f''(g(x))g'(x) + f'(g(x))g''(x) = \frac{d^2y}{du^2}(\frac{du}{dx})^2 + \frac{dy}{du}\frac{d^2u}{dx^2}$

§3.5 #11. Find y' by implicit differentiation. $x^2y^2 + x\sin y = 4$
Solution. $\frac{d}{dx}(x^2y^2 + x\sin y) = \frac{d}{dx}4$
$2xy^2 + x^2\frac{dy^2}{dx} + \sin y + x\frac{d}{dx}(\sin y) = 0$ by the product rule.
$2xy^2 + 2x^2yy' + \sin y + xy'\cos y = 0$.
Solving for y' gives $y' = \frac{-(\sin y + 2xy^2)}{2x^2y + x\cos y}$.

§3.5 #18. Find y' by implicit differentiation. $\tan(x-y) = \frac{y}{1+x^2}$
Solution. $\tan(x-y)\cdot(1+x^2) = y$
$\frac{d}{dx}(\tan(x-y)\cdot(1+x^2)) = \frac{dy}{dx}$.
$(1+x^2)\sec^2(x-y)\cdot(1-y') + 2x\tan(x-y) = y'$
$y' = \frac{2x\tan(x-y)+(1+x^2)\sec^2(x-y)}{1+(1+x^2)\sec^2(x-y)}$

§3.5 #27. Use implicit differentiation to find an equation of the tangent line to $x^2 + y^2 = (2x^2 + 2y^2 - x)^2$ at $(0, 0.5)$
Solution. $2x + 2yy' = 2(2x^2 + 2y^2 - x)(4x + 4yy' - 1)$
$y' = \frac{2(2x^2+2y^2-x)(4x-1)-2x}{2y-8y(2x^2+2y^2-x)}$
y' at $(0, 0.5)$ is $\frac{2(0+0.5-0)\cdot(-1)-0}{1-4(0.5)} = 1$
The tangent line hence has equation $y = x + b$ for some $b \in \mathbb{R}$. Since $(0, 0.5)$ must lie on this line, we see that $b = 0.5$. So $y = x + 0.5$

§3.5 #43. Show, using implicit differentiation, that any tangent line at a point P to a circle with center O is perpendicular to the radius OP.
Solution. A circle has equation $x^2 + y^2 = 1$. Implicit differentiation yields $2x + 2yy' = 0$. Assume first that neither x nor y is zero. So we get $y' = -\frac{x}{y}$. The slope of the radius OP is $\frac{x}{y}$. Note that any two lines of slopes a and b are perpendicular if and only if $ab = -1$. Thus the tangent line and the radius are perpendicular. If x or y happen to be zero, we can argue that the same holds by rotational symmetry, since angles are preserved under rotations.

§3.5 #54. Find the derivative of $y = arctan(\sqrt{\frac{1-x}{1+x}})$.

Solution. $\tan^2 y = \frac{1-x}{1+x}$

Use implicit differentiation: $2\tan(y) \cdot \sec^2 y \cdot y' = \frac{-2}{(1+x)^2}$

$y' = \frac{-1}{(1+x)^2 \tan y \sec^2 y}$

§3.5 #67. Suppose that f is a one-to-one differentiable function and its inverse function f^{-1} is also differentiable. Show $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$.

Solution. Since f and f^{-1} are inverses, we have that $f \circ f^{-1}(x) = x$. Differentiating both sides with respect to x yields $f'(f^{-1}(x)) \cdot (f^{-1})'(x) = 1$. Hence $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$. Using the given values, we get $(f^{-1})'(5) = \frac{1}{f'(f^{-1}(5))} = \frac{1}{f'(4)} = \frac{3}{2}$ since $f(4) = 5$ implies that $f^{-1}(5) = 4$.

§3.6 #9. Differentiate $y = \sin x \ln(5x)$

Solution. $\frac{d}{dx}(\sin x \ln(5x)) = \cos x \ln(5x) + \sin x \cdot \frac{1}{5x} \cdot 5$

§3.6 #13. Differentiate $y = \ln(x\sqrt{x^2 - 1})$

Solution. $\frac{d}{dx}(\ln(x\sqrt{x^2-1})) = \frac{1}{x\sqrt{x^2-1}} \frac{d}{dx}(x\sqrt{x^2-1}) = \frac{1}{x\sqrt{x^2-1}}(\sqrt{x^2-1} + x \cdot \frac{1}{2\sqrt{x^2-1}} \cdot 2x) = \frac{1}{x\sqrt{x^2-1}}(\sqrt{x^2-1} + \frac{x^2}{\sqrt{x^2-1}})\frac{\sqrt{x^2-1}}{\sqrt{x^2-1}} = \frac{2x^2-1}{x(x^2-1)}$

§3.6 #27. Differentiate $y = \frac{x}{1-\ln(x-1)}$ and find the domain.

Solution. $y' = \frac{1-\ln(x-1)+x(\frac{1}{x-1})}{(1-\ln(x-1))^2} = \frac{2x-1-(x-1)\ln(x-1)}{(x-1)(1-\ln(x-1))^2}$. For the domain, we need to ensure that the argument of the ln is positive, i.e. we must have $x - 1 > 0$, i.e. $x > 1$. Moreover, we must ensure that the denominator is not zero, i.e. $x - 1 \neq 0$ and $1 - \ln(x-1) \neq 0$, i.e. $x \neq 1$ and $\ln(x-1) \neq 1$, i.e. $x \neq e + 1, 1$. Thus the domain is $\{x : x > 1 \text{ and } x \neq e + 1\}$

§3.6 #45. Use logarithmic differentiation to find the derivative of $y = (\cos x)^x$

Solution. $\ln y = \ln((\cos x)^x) = x \ln \cos x$

Implicit differentiation yields $\frac{y'}{y} = \ln \cos x + x\frac{1}{\cos x}(-\sin x)$. So $y' = (\cos x)^x(\ln \cos x - x \tan x)$

§3.7 #5. Given are graphs of the velocity of two particles. When is each particle speeding up? Slowing down?

Solution. If the velocity is positive, then each particle is speeding up whenever the graph is increasing and slowing down whenever the graphs are decreasing. If the velocity is negative, the particle is moving backwards and so it is speeding up whenever the graph is decreasing.

§3.7 #10. If a ball is thrown vertically upward with a velocity of 80ft/s, then its height after t seconds is $s = 80t - 16t^2$. What is the maximum height reached? What is the velocity of the ball when it is 96ft above the ground on its way up and on its way down.

Solution. The maximum height occurs when the ball turns from going up to going down. This happens when the velocity is zero. $v(t) = \frac{ds}{dt} = 80 - 32t$. This is zero if $t = \frac{5}{2}$. So the maximum height is $s(\frac{5}{2}) = 200 - 100 = 100$, i.e. $100ft$. If the ball is 96ft above the ground we have that $96 = s = 80t - 16t^2$, i.e. $6 = 5t - t^2$ i.e. $t = 2, 3$. So the velocity of the ball at 96ft above ground when it is on its way up if $v(2) = 80 - 32 \cdot 2 = 16$, i.e. $16\frac{ft}{s}$. On it's way down it has velocity $v(3) = 80 - 32 \cdot 3 = -16$, i.e. $-16\frac{ft}{s}$.