Math 1A — UCB, Fall 2010 — A. Ogus
Solutions[1] for Problem Set 3

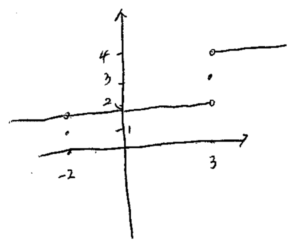
§2.2 # 1. Explain in your own words what it meant by the equation $\lim_{x\to 2} f(x) = 5$. Is it possible for this statement to be true and yet $f(2) = 3$? Explain.
Solution. $\lim_{x\to 2} f(x) = 5$ means that $f(x)$ can made arbitrarily close to 5 by making x sufficiently close to 2. It is still possible that $f(2) = 3$ because the limit depends on function values near 2 but not at 2. $\square$

§2.2 # 7.
Solution.(a) -1
(b) -2
(c) The limit does not exist, because the right hand limit is not equal to the left hand limit.
(d) 2
(e) 0
(f) The limit does not exist, because the right hand limit $\neq$ the left hand limit.
(g) 1
(h) 3 $\square$

§2.2 # 9.
Solution.
(a) $-\infty$
(b) ∞
(c) ∞
(d) $-\infty$
(e) ∞
(f) $x = -7$, $x = -3$, $x = 0$, and $x = 6$. $\square$

§2.2 # 15.

Solution to §2.2 15



$\square$

$\square$

§2.2 # 25. Determine the infinite limit $\lim_{x\to -3^+} \frac{x+2}{x+3}$.

Solution. As $x \to -3^+$, $x+3 \to \infty$, and $x+2 \to -1$. Therefore $\lim_{x\to -3^+} \frac{x+2}{x+3} = -\infty$.

Another way to do this is to write $\frac{x+2}{x+3} = 1 - \frac{1}{x+3}$. Since $\frac{1}{x+3} \to \infty$ as $x \to -3^+$, $1 - \frac{1}{x+3} \to -\infty$. Algebraic simplifications are often helpful in simplifying the calculation of limits. $\square$

§2.2 # 28. Determine the infinite limit $\lim_{x\to 5^-} \frac{e^x}{(x-5)^3}$.

Solution. $-\infty$. $\square$

§2.2 # 35. (a) Estimate the value of the limit $\lim_{x\to 0}$ to five decimal places. Does this number look familiar?

Solution. Plug in $x = 0.000001$ one gets $(1+1.000001)^{1000000} \approx 2.71828$. This number is close to e. $\square$

§2.2 # 40. In the theory of relativity, the mass of a particle with velocity v is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the mass of the particle at rest and c is the speed of the light. What happens as $v \to c^-$?

Solution. The mass of the particle satisfies $m \to \infty$. For as $v \to c$ from the left, $v^2/c^2 \to 1$ from below. Thus $1 - v^2/c^2 \to 0$ from above, so $\sqrt{1 - v^2/c^2}$ also tends to 0 from above, so $\frac{1}{\sqrt{1-v^2/c^2}} \to \infty$. $\square$

§2.3 # 2.

Solution. (a) 2+0=2

(b) The limit does not exist, because the right hand limit of g at $x = 1$ is 1 but the left hand limit of g at $x = 1$ is 2.

(c) 0*g(0)=0 (whatever g(0) is)

(d) The limit does not exist, because the right hand limit $-\infty$ but the left hand limit is ∞.
(e) $2^3 * 2$=16
(f) 2 $\square$

§2.3 # 5. Evaluate the limit $\lim_{x\to 8}(1+\sqrt[3]{x})(2-6x^2+x^3)$ and justify each step by indicating the appropriate Limit Laws.

Solution. First by the Product Law, $\lim_{x\to 8}(1+\sqrt[3]{x})(2-6x^2+x^3) = \lim_{x\to 8}(1+\sqrt[3]{x}) \lim_{x\to 8}(2-6x^2+x^3)$. Then, by the Sum Law, Limit Law 7, and Limit Law 10, $\lim_{x\to 8}(1+\sqrt[3]{x}) = \lim_{x\to 8} 1 + \lim_{x\to 8}\sqrt[3]{x} = 1+\sqrt[3]{8} = 3$. Similarly, by the Sum Law, Limit Law 7, and Limit Law 10, $\lim_{x\to 8}(2-6x^2+x^3) = \lim_{x\to 8} 2 + \lim_{x\to 8} -6x^2 + \lim_{x\to 8} x^3 = 2-6*8^2+8^3 = 130$. So $\lim_{x\to 8}(1+\sqrt[3]{x})(2-6x^2+x^3)$ = 3*130 =390. $\square$

§2.3 # 11. Evaluating $\lim_{x\to 2}\frac{x^2+x-6}{x-2}$.

Solution. $\lim_{x\to 2}\frac{x^2+x-6}{x-2} = \lim_{x\to 2}\frac{(x-2)(x+3)}{x-2} = \lim_{x\to 2} x+3 = 5$.

§2.3 # 22. Evaluate, if it exists: $\displaystyle\lim_{h\to 0}\frac{\sqrt{1+h}-1}{h}$

Solution.

Rationalize the difference of radicals:

$$\lim_{h\to 0}\frac{\sqrt{1+h}-1}{h} = \lim_{h\to 0}\frac{\sqrt{1+h}-1}{h}\cdot\frac{\sqrt{1+h}+1}{\sqrt{1+h}+1} = \lim_{h\to 0}\frac{h}{h(\sqrt{1+h}+1)} = \lim_{h\to 0}\frac{1}{\sqrt{1+h}+1} = \frac{1}{2}$$

$\square$

§2.4 # 30. Prove using the ε,δ definition of limit: $\displaystyle\lim_{x\to 3} x^2+x-4=8$

(Note: This proof *reads* logically from top to bottom, but as usual, it was *discovered* partially via "working backwards". First, read it forwards to see that the logic really does flow from start to finish, in the order written. Then start reading it backwards to get a feel for how the proof was actually discovered.)

Solution.

Suppose $\varepsilon > 0$, and $0 < |x-3| < \delta = \min\{6, \frac{\varepsilon}{13}\}$.

Then $|x-3| < 6$, $\therefore x \in (-3, 9)$, $\therefore x+4 \in (1, 13)$, $\therefore |x+4| = x+4 < 13$. Hence

$$\frac{\varepsilon}{13} < \frac{\varepsilon}{|x+4|}$$

But $|x-3| < \dfrac{\varepsilon}{13}$, hence

$$\begin{aligned} |x-3| &< \frac{\varepsilon}{|x+4|} \\ \Rightarrow |x-3|\cdot|x+4| &< \varepsilon \\ \Rightarrow |x^2+x-12| &< \varepsilon \\ \Rightarrow |(x^2+x-4)-8| &< \varepsilon \end{aligned}$$

Thus, for every $\varepsilon > 0$ there exists a $\delta > 0$ meeting the limit constraint, so $\displaystyle\lim_{x\to 3} x^2+x-4=8$.

$\square$