Math 1A — UCB, Spring 2010 — A. Ogus
Solutions[1] for Problem Set 12

§5.3 # 9. Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of $g(y) = \int_2^y t^2 sint dt$.
Solution.
By Part 1 of the Fundamental Theorem of Calculus, $g'(y) = y^2 siny$.

□

§5.3 # 16. Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of $y = \int_1^{cosx} (1+v^2)^{10} dv$.
Solution.
Let $g(x) = \int_1^x (1+v^2)^{10} dv$, then $y = g(cosx)$, apply the chain rule, $y' = g'(cosx)(-sinx)$. By Part 1 of the Fundamental Theorem of Calculus, $g'(cosx) = (1+cos^2x)^{10}$. Finally,

$$y' = g'(cosx)(-sinx) = (1+cos^2x)^{10}(-sinx)$$

□

§5.3 # 23. Evaluate $\int_0^1 x^{4/5} dx$.
Solution.
By Part 2 of the Fundamental Theorem of Calculus, $\int_0^1 x^{4/5} dx = \frac{5}{9}x^{9/5}]_0^1 = \frac{5}{9}$.

□

§5.3 # 31. Evaluate $\int_0^{\pi/4} sec^2 t dt$.
Solution.
By Part 2 of the Fundamental Theorem of Calculus, $\int_0^{\pi/4} sec^2 t dt = tanx]_0^{\pi/4} = tan(\pi/4) - tan0 = 1$.

□

§5.3 # 44. What is wrong withe equation $\int_{-1}^2 \frac{4}{x^3} dx = -\frac{2}{x^2}]_{-1}^2 = \frac{3}{2}$.
Solution.
$\frac{4}{x^3}$ is not defined (not integrable, not continuous) over [-1,2]. So one can not apply Part 2 of the Fundamental Theorem of Calculus.

□

§5.3 # 51 Evaluate the integral $\int_{-1}^2 x^3 dx$ and interpret it as a difference of areas. Illustrate with a graph.
Solution.
Apply Part 2 of the Fundamental Theorem of Calculus, $\int_{-1}^2 x^3 dx = \frac{1}{4}x^4]_{-1}^2 = \frac{1}{4}2^4$ - $\frac{1}{4}(-1)^4 = 4$-$\frac{1}{4}$ = $\frac{15}{4}$.

□

§5.3 # 53. Find the derivative of $g(x) = \int_{2x}^{3x} \frac{u^2-1}{u^2+1} du$.

Solution: Using the substitutions $s = 2x$ and $t = 3x$, we get that $\frac{d}{dx}g(x) = \frac{d}{dx}\int_{2x}^0 \frac{u^2-1}{u^2+1} du + \frac{d}{dx}\int_0^{3x} \frac{u^2-1}{u^2+1} du = \frac{d}{dx}(-\int_0^s \frac{u^2-1}{u^2+1} du) + \frac{d}{dx}\int_0^t \frac{u^2-1}{u^2+1} du = -\frac{d}{ds}\frac{u^2-1}{u^2+1} du \frac{ds}{dx} + \frac{d}{dx}\int_0^t \frac{u^2-1}{u^2+1} du \frac{dt}{dx} = -\frac{s^2-1}{s^2+1}2 + \frac{t^2-1}{t^2+1}3 = \frac{-2(4x^2-1)}{4x^2+1} + \frac{-3(9x^2-1)}{9x^2+1}$

§5.3 # 56. Find the derivative of $y = \int_{\cos x}^{5x} \cos(u^2)du$.

Solution: Using the substitutions $s = \cos x$ and $t = 5x$, we get that $\frac{dy}{dx} = \frac{d}{dx}(-\int_0^s \cos(u^2)du) + \frac{d}{dx}(\int_0^t \cos(u^2)du) = \frac{d}{ds}(-\int_0^s \cos(u^2)du)\frac{ds}{dx} + \frac{d}{dt}(\int_0^t \cos(u^2)du)\frac{dt}{dx} = \cos(s^2)\sin(x) + \cos(t^2)5 = \cos(\cos^2 x) + 5\cos(25x^2)$

§5.3 # 63. Let $g(x) = \int_0^x f(t)dt$, where f is the function whose graph is shown (page 389).
(a) At what avlues of x do the local maximum and minimum values of g occur?
(b) Where does g attain its abolute maximum?
(c) On what intervals is g concave downward?
(d) Sketch thegraph of g

Solution: (a) By the fundamental theorem of Calculus, f is the derivative of g, hence local maxima/minima require a point to be critical, i.e. the derivative to be undefined or zero. So only $1, 3, 5, 7$ are possible values for local extrema. At 1 and 5, the function g changes from being increasing to being decreasing, hence we have local maxima. At 3 and 7 the function g changes from being decreasing to being increasing, hence we have local minima.
(b) The absolute maximum can either be achieved at one of the endpoints or at one of the local maxima. $g(0) = \int_0^0 f(t)dt = 0$. The value $g(1)$ is positive, since it is the area between the graph and the x-axis integrated from 0 to 1. We have that $g(5) > g(1)$, since the area between the graph and the x-axis from 1 to 3 is smaller than the area from 3 to 5, so the area we added is bigger than the area we subtracted when taking the integral. Similarly $g(9) > g(5)$, which makes $x = 9$ the absolute maximum.
(c) The graph of g is concave downwards, whenever the graph of its derivative, i.e. of f, is decreasing. Thus on $(0.5, 2)$ and $(4, 6)$ and $(8, 9)$.
(d) See page A90

§5.3 # 65. Evaluate the limit by recognizing the sum as a Riemann sum for a function defined on $[0, 1]$. $lim_{n\to\infty}\Sigma_{i=1}^n \frac{i^3}{n^4}$.

Solution: $\Delta x = \frac{1}{n}$. Thus $R_n = \Sigma_{i=1}^n \frac{i^3}{n^4} = \frac{1}{n}\Sigma_{i=1}^n (\frac{i}{n})^3$. So the function is $f(x) = x^3$. So the limit is $lim_{n\to\infty}\Sigma_{i=1}^n \frac{i^3}{n^4} = \int_0^1 x^3 = \frac{1}{4}x^4]_0^1 = \frac{1}{4}$

§5.3 # 66. Evaluate the limit by recognizing the sum as a Riemann sum for a function defined on $[0, 1]$. $lim_{n\to\infty}\frac{1}{n}(\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + ... + \sqrt{\frac{1}{n}})$.

Solution: $\Delta x = \frac{1}{n}$. So $f(x) = \sqrt{x}$ and the limit is just $\int_0^1 \sqrt{x} = \frac{2}{3}x^{\frac{3}{2}}]_0^1 = \frac{2}{3}$

§5.3 # 68. If f is continuous and g and h are differentiable functions, find a formula for $\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)dt$

Solution: $\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)dt = \frac{d}{dx} - \int_0^{g(x)} f(t)dt + \frac{d}{dx}\int_0^{h(x)} f(t)dt = -f(g(x))g'(x) + f(h(x))h'(x)$

§5.3 # 74. The area B ios three times the area labeled A. Express b in terms of a.

Solution: We have that $3\int_0^a e^x dx = 3A = B = \int_0^a e^x dx$. Thus, $3(e^a - e^0) = e^b - e^0$ by the fundamental theorem of calculus. Thus, $e^b = 3e^a - 2$. Thus $b = \ln(3e^a - 2)$.

§5.4 # 1. Verify by differentiation that $\int \frac{x}{\sqrt{x^2+1}}dx = \sqrt{x^2+1} + C$.

Solution.

$$\frac{d}{dx}\sqrt{x^2+1} + C = \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \cdot \frac{d}{dx}(x^2+1) = \frac{1}{2\sqrt{x^2+1}} \cdot 2x = \frac{x}{\sqrt{x^2+1}}$$

$\square$

§5.4 # 11. Find $\int \frac{x^3+2\sqrt{x}}{x}dx$.
Solution.

$$\int \frac{x^3+2\sqrt{x}}{x}dx = \int x^2 + \frac{2}{\sqrt{x}}dx = \frac{x^3}{3} + 2\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C = \frac{x^3}{3} + 4\sqrt{x} + C.$$

$\square$

§5.4 # 16. Find $\int \sec x(\sec x + \tan x)dx$. **Solution.**

$$\int \sec x(\sec x + \tan x)dx = \int \sec^2 x + \sec x \tan x dx = \int \sec^2 x dx + \int \sec x \tan x dx = \tan x + \sec x + C$$

$\square$

§5.4 # 26. Evaluate $\int_0^4 (2v+5)(3v-1)dv$.
Solution.

$$\int_0^4 (2v+5)(3v-1)dv = \int_0^4 (6v^2+13v-5)dv = 2v^3 + \frac{13}{2}v^2 - 5v \,|_0^4 = (128+104-20) - (0) = 212.$$

$\square$

§5.4 # 36. Evaluate $\int_{\pi/4}^{\pi/3} \sec\theta \tan\theta d\theta$.
Solution.

$$\int_{\pi/4}^{\pi/3} \sec\theta \tan\theta d\theta = \sec\theta \,|_{\pi/4}^{\pi/3} = \sqrt{2} - 2$$

$\square$

§5.4 # 44. Evaluate $\int_0^{3\pi/2} |\sin x| dx$.
Solution. Since $\sin x$ is positive for x between 0 and π, and negative for x between π and $3\pi/2$,

$$\int_0^{3\pi/2} |\sin x| dx = \int_0^{\pi} \sin x dx + \int_{\pi}^{3\pi/2} -\sin x dx = -\cos x \,|_0^{\pi} + \cos x \,|_{\pi}^{3\pi/2} = (1)-(-1)+(0)-(-1) = 3.$$

$\square$

§5.4 # 47. Find the area of the region pictured by evaluating $\int_0^2 (2y - y^2)dy$.
Solution.

$$\int_0^2 (2y-y^2)dy = y^2 - \frac{y^3}{3} \,|_0^2 = (4 - \frac{8}{3}) - (0) = \frac{4}{3}.$$

$\square$

§5.4 # 51. If oil leaks from a tank at a rate of $r(t)$ gallons per minute at time t, what does $\int_0^{120} r(t)dt$ represent?
Solution. $\int_0^{120} r(t)dt$ is the total amount of oil that has leaked from the tank in the first 120 minutes, given in gallons.

$\square$

§5.4 # 52. A honeybee population starts with 100 bees and increases at a rate of $n'(t)$ bees per week, what does $100 + \int_0^{15} n'(t)dt$ represent?
Solution. The total number of bees after 15 weeks.

$\square$

§5.4 # 59. $a(t) = t + 4, v(0) = 5, 0 \leq t \leq 10$. Find (a) the velocity at time t and (b) the distance travelled.

Solution.

(a) Since velocity is the antiderivative of acceleration, $v(t) = \int a(t)dt = \int (t+4)dt = \frac{t^2}{2} + 4t + C$. We can solve for C using the fact that $v(0) = 5$,

$$5 = v(0) = 0^2/2 + 4 \cdot 0 + C,$$

so $C = 5$ and $v(t) = \frac{t^2}{2} + 4t + 5$ (in m/s).

(b)The total distance travelled is given by $\int_0^{10} |v(t)|dt$. Since $v(t)$ is positive on this interval, we get

$$d = \int_0^{10} (\frac{t^2}{2} + 4t + 5)dt = (\frac{t^3}{6} + 2t^2 + 5t) \mid_0^{10} = (\frac{1000}{3} + 2(100) + 5(10)) - (0) = 583\frac{1}{3}$$

(in m).

□

§5.4 # 64. Suppose a volcano is erupting and the readings of the rate $r(t)$ at which solid materials are spewed into the atmosphere are given by the table. t is in seconds and $r(t)$ is in tonnes/sec. (a) give upper and lower estimates for the total quantity $Q(6)$ of erupted material after 6 seconds, and (b) use the midpoint rule to estimate $Q(6)$.

Solution.

(a)Since the numbers for $r(t)$ are increasing with t, the upper estimate will be given by using right endpoints in a Riemann Sum, and the lower estimate will be given by left endpoints. There are 6 intervals, and each is 1 second long, so the upper estimate is $10 + 24 + 36 + 46 + 54 + 60 = 220$ tonnes. The lower estimate is $2 + 10 + 24 + 36 + 46 + 54 = 172$ tonnes.

(b)To use the midpoint rule, we need to know the values of $r(t)$ at the midpoints of our subintervals. Thus the subintervals cannot be 1 second long (since we do not know the value at the half-seconds), so we should divide into three 2 second long intervals. Then the Riemann Sum is $r(1) \cdot 2 + r(3) \cdot 2 + r(5) \cdot 2 = 20 + 72 + 108 = 200$.