Midterm II, April 2, 2010

Work each problem on a separate sheet of paper. Be sure to put your name, your section number, and your GSI's name on each sheet of paper. Also, at the top of the page, in the center, write the problem number, and be sure to put the pages in order. Write clearly: explanations (with complete sentences when appropriate) will help us understand what you are doing.

1. Compute the derivatives:

   (a) (5 pts) $f(x) = (\sin x)^3$ for every x.
   Solution: By the chain rule, or the generalized power rule:
   $f'(x) = 3(\sin x)^2 \cos x$.

   (b) (5pts) $f(x) = \sin(x^3)$ for every x.
   By the chain rule: $f'(x) = \cos(x^3)3x^2$.

   (c) (5 pts) $f(x) = \sqrt[x]{x}$ for every $x > 0$.
   Using logarithmic differentiation:
   $\ln f(x) = \dfrac{\ln x}{x}$, so

   $$\frac{f'(x)}{f(x)} = \frac{x \ln' x - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

   Hence

   $$f'(x) = \frac{1 - \ln x}{x^2} \sqrt[x]{x}$$

   (d) (5 pts) The derivative of $y = f(x)$ at the point $(1, 2)$ assuming that $y = f(x)$ is a function satisifying the equation:

   $$y - 2x^3 = e^{2x} - e^y.$$

   Solution: Using implicit differentiation, we get

   $$\begin{aligned}
   y - 2x^3 &= e^{2x} - e^y \\
   y' - 6x^2 &= 2e^{2x} - y'e^y \\
   y'(1 + e^y) &= 6x^2 + 2e^{2x}
   \end{aligned}$$

   If $x = 1$ and $y = 2$, we find $y'(1 + e^2) = 6 + 2e^2$, so

   $$y' = (6 + 2e^2)/(1 + e^2).$$

2. Your job is to rapidly give good approximations to cube roots of numbers near 27.

   (a) (10 pts) Find the linear function $\ell(x)$ which is closest to $\sqrt[3]{x}$ for x near 27.
   Solution: This is the tangent line to f at $(27, 3)$. The slope of this line is the derivative of f, which is $(1/3)x^{-2/3}$ evaluated at $x = 27$, which is $1/27$. Since the line passes through $(3, 27)$, its formula is

   $$\ell(x) = (1/27)(x - 27) + 3 = x/27 + 2.$$

(b) (10 pts) Use your expression to estimate $\sqrt[3]{27.3}$.
Solution: The formula says that

$$\ell(27.3) = 27.3/27 + 2 = 1 + .3/27 + 2 = 3\frac{1}{90}.$$

(c) (10 pts) Assuming that x is near 27, which is bigger, $\ell(x)$ or $\sqrt[3]{x}$? Use calculus to explain how you know.
Solution: The function $\sqrt[3]{x}$ is concave down, since its second derivative $f''(x) = (-2/9)x^{-5/3}$ is negative. This means that $\ell(x)$ is bigger than $\sqrt[3]{x}$.

3. Your job is to find the point or points in the interval $[-3, 3]$ where the function $f(x) := \dfrac{x^4}{4} - x^3 + x^2$ achieves its minimum value.

(a) (5 pts) Explain how we know this is possible.
Solution: Our function is continuous on the closed interval $[-3, 3]$, and the extreme value theorem applies.

(b) (25 pts) Find all the local (relative) minima and maxima of f on this interval. Explain whether each of these is a local minimum or maximum. Show your reasoning and intermediate steps. You will not be given full credit if you just write the answers. This problem has lots of parts and partial credit, but only if you organize your answer well.
Solution: We calculate the derivative and second derivative of f:

$$\begin{aligned}
f(x) &:= & \frac{x^4}{4} - x^3 + x^2 \\
f'(x) &= & x^3 - 3x^2 + 2x \\
f''(x) &= & 3x^2 - 6x + 2
\end{aligned}$$

To find where $f'(x)$ has critical points, we factor $f'(x)$

$$f'(x) = x(x^2 - 3x + 2) = x(x-1)(x-2)$$

Thus the critical points of f are at $x = 0, 1$, and 2.
We classify these by calculating the second derivatives:

- At 0, $f''(x) = 2 > 0$ and $f(0) = 0$, so $(0,0)$ is a local minimum.
- At 1, $f''(1) = -1 < 0$ and $f(1) = 1/4$, so $(1, 1/4)$ is a local maximum.
- At 2, $f''(2) = 2 > 0$ and $f(2) = 0$, $(2,0)$ is another local minimum.

(10 pts for finding f' and the critical points, 5 pts for classifying each of them.)

(c) (5 pts) Now answer the original question. In fact the minimum values of f is 0, which is achieves at $x = 0$ and $x = 2$.

4. (15 pts) A particle in the plane is constrained to move along the curve $y^2 = x^3 + x + 1$. Suppose that at a given time t, the particle passes

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through the point $(0, -1)$ and that its velocity in the x-direction (that is, $x'(t)$), is 1 centimeter per second. What is the veloocity in the y-direction at that time?

Solution: Using implicit differentiation we find that $2yy' = 3x^2x' + x'$. If $x = 0$ and $y = -1$, it follows that $-2y' = x'$. Thus $y' = -.5$ centimeters per second.