Solution to 9.3.4

Suppose n is a positive integer for which there exists an x such that $x^2+1 \equiv 0 \pmod{n}$. Then n can be written as a sum a^2+b^2, where a and b are relatively prime. To see this, note first that if $x^2+1 \equiv 0 \pmod{n}$ and d divides n, then the same is true mod d. Since no solution to this equation exists if $d = 4$, n can't be divisible by 4. Similarly, n can't be divisible by any prime congruent to 3 mod 4. Let's factor n into primes: $n = \prod p^{e_p}$. Then $e_p = 0$ if $p \equiv 3 \pmod{4}$ and $e_2 \leq 1$. If p is odd and $e_p \neq 0$, $p \equiv 1 \pmod{1}$, so we have $p = \alpha_p \overline{\alpha}_p$, where α_p is irreducible and α_p and α_p are not associate. Let $\alpha_2 := 1+i$, and let $\beta := \prod_p \alpha_p^{e_p}$. Let's check that β is not divisible by any odd prime q of $\mathbf{Z}$. If $q \equiv 3 \pmod{4}$, then q is prime in $\mathbf{Z}[i]$, and since q does not divide any α_p, q does not divide the product. If $q \equiv 1 \pmod{4}$, then q has a prime factorization $q = \alpha_q \overline{\alpha}_q$, and we see that β is divisible by at most one of α_q and $\overline{\alpha}_q$, but not by both. Since $e_2 < 2$, β is not divisible by 2. Now write $\beta = a + ib$. Since β is not divisible by any prime in $\mathbf{Z}$, a and b are relatively prime. But $n = a^2 + b^2$.