Solutions for some homework problems

7.2.4: Let p be a prime number and let C be a cyclic subgroup of order p in S_p. Compute the order of the normalizer $N(C)$ of C.

Solution: Let σ be a generator of C and write σ as a product $\gamma_1 \cdots \gamma_r$ of disjoint cycles. The order of σ is the least common multiple of the lengths of these cycles, so they each have length p, Since we are the S_p, we must have $r = 1$. Write $\sigma = (a_1\ a_2 \cdots\ a_p)$. There are $p!$ such expressions, but each cycle can be written p ways as such an expression. This gives us $(p-1)!$ p-cycles in S_p, and we know they are all conjugate. Each of these cycles generates a group of order p, and each such group has $p-1$ generators. Thus there are $(p-2)!$ cyclic subgroups of order p in S_p, all conjugate. Hence the normalizer of any one of them has index $(p-2)!$ and hence has order $p(p-1)$.

7.3.8: Let G be a finite p-group and let H be a proper subgroup. Show that there is an element $g \in G \setminus H$ such that $gHg^{-1} = H$.
Solution: Let us consider the action of G on the set $S := G/H$ of left cosets of H. Restrict this to an action of H on S: $H \times S \to S$. Note that H is a p-group and that S has $[G : H]$ elements; this number is a positive power of p. Then Lemma 7.3.7 implies that $|S^H|$ is divisible by p. Now $H \in |S^H|$, so $|S^H| \geq 1$, hence in fact $|S^H| \geq p$. Thus there exists some coset, call it gH, with $g \notin G$, such that $gH \in S^H$. Then for any $h \in H$, $hgH = gH$, *i.e.*$g^{-1}hgH = H$, so $g^{-1}hg \in H$.

7.2.18: Compute the conjugacy classes of A_5 and use the result to show that A_5 is a simple group.

Recall that in class we showed that two elements σ and σ' of S_n are conjugate if and only if their respective cycle decompositions:

$$\sigma = \gamma_1 \cdots \gamma_r, \sigma' = \gamma'_1 \cdots \gamma'_{r'}$$

have the same "shape"; *i.e.*, $r = r'$ and $length(\gamma_i) = length(\gamma'_i)$ for all i (after reordering if necessary). This is not quite true in A_n, but it not hard to see what is happening there.

Lemma. Let α be an element of A_n, let $Z_\alpha := \{\sigma : \alpha = \alpha^\sigma : \sigma \in S_n\}$ be its centralizer, and let $C(\alpha) := \{\alpha^\sigma : \sigma \in S_n\}$ be its S_n-conjugacy class. On the other hand, let $C'(\alpha) := \{\alpha^\sigma : \sigma \in A_n\}$ be the A_n conjugacy class of α. Evidently $C'(\alpha) \subseteq C(\alpha)$, and we want to know how and when these differ. We have isomorphisms: $C(\alpha) \cong S_n/Z_\alpha$ (of S_n-sets) and $C'(\alpha) \cong A_n/(Z_\alpha \cap A_n)$ (of A_n-sets). There are two cases:

Case 1: $Z_\alpha \subseteq A_n$. In this case $Z_\alpha \cap A_n = Z_\alpha$, and since A_n has just half as many elements as S_n, we have that $C'(\alpha) \cong A_n/Z_\alpha$ has half as many elements as $C(\alpha) \cong S_n/Z_\alpha$.

Case 2: $Z_\alpha \not\subseteq A_n$. In this case it follows that $C'(\alpha) = C(\alpha)$. Indeed, by assumption there is some odd element τ of Z_α. Then if σ is any odd element of S_n,

$$\alpha^\sigma = (\alpha^\tau)^\sigma = \alpha^{\sigma\tau} \in C'(\alpha)$$

since $\sigma\tau \in A_n$. If σ is even, α^σ was already in $C'(\alpha)$, so this shows that $C(\alpha) = C'(\alpha)$.

Now let us look at the various possibilities:

3-cycles, e.g. $\alpha = (1\ 2\ 3)$. There are 20 of these, all conjugate in S_5, and $(4\ 5)$ is an odd element of the centralizer of $(1\ 2\ 3)$, so they are also conjugate in A_5 (Case 1).

5-cycles:, e.g. $\alpha = (1\ 2\ 3\ 4\ 5)$. There are 24 of these, all conjugate in S_5. It follows that the index of the centralizer of Z_α in S_5 is 5, so $Z_\alpha = \langle(\alpha)\rangle$ is contained in A_5 (Case 2). Thus this conjugacy class splits into two pieces, each of size 12. (For example, $(2\ 1\ 3\ 4\ 5)$ is not in the A_n-conjugacy class of α.)

Products of 2 2-cycles, e.g. $\alpha = (1\ 2)(3\ 4)$. There are $(5 \cdot 4 \cdot 3 \cdot 2)/2 \cdot 2 \cdot 2 = 15$ of these, all conjugate in S_n. In fact $(1\ 2)$ is an odd element of the centralizer of α. so we are in Case 1 and they are all conjugate in A_n.

The identity element. This is fixed by conjugation.

Thus A_n has the following conjugacy classes:

- $C(1\ 2\ 3)$, with 20 elements.
- $C(1\ 2\ 3\ 4\ 5)$, with 12 elements.
- $C(2\ 1\ 3\ 4\ 5)$, with 12 elements.
- $C((1\ 2)(3\ 4))$ with 15 elements.
- $C(e)$ with 1 element.

  Note that $60 = 20+12+12+15+1$, as it should. Now any normal subgroup is invariant under conjugation, and hence must be a union of conjugacy classes, and must contain e. So the number of elements in such a group is a sum of the some of the above numbers, including 1. Furthermore, this number divides 60. But the only numbers of this form are 1 and 60. Hence A_n has no proper normal subgroups, and hence it is simple.

7.3.13ab Show that S_n acts transitively on the set $\{1, \cdots, n\}$, and that A_n also does if $n > 2$. Indeed, for S_n just have to show that given any i, j, there exists an element $\sigma(i) = j$. This is trivial if $i = j$, and if not we can take the transposition $(i\ j)$. To do this for A_n, we must use the fact that $n > 2$, so there is some k with $1 \leq k \leq n$ and $k \neq i, j$. Then the 3-cycle $(i\ j\ k)$ is even and takes i to j.