Linear Algebra Midterm Exam Solutions October 3, 2008

Write clearly, with complete sentences, explaining your work. You will be graded on clarity, style, and brevity. If you add false statements to a correct argument, you will lose points. Be sure to put your name on every page.

1. Let V be a vector space over a field F and let $\mathcal{L} := (v_1, v_2, \ldots, v_n)$ be a list in V.

   (a) (5 pts) What is the definition of the *span* of $\mathcal{L}$?
   The span of $\mathcal{L}$ is the set of all vectors v in V which can be written $v = a_1v_1 + \cdots a_nv_n$ for some choice of a_i in the field. Equivalently, it is the smallest linear subpace of V containing all the v_i's.

   (b) (5 pts) What does it mean to say that $\mathcal{L}$ is *linearly independent*?
   This means that whenever $a_1v_1 + \cdots a_nv_n = 0$, each $a_i = 0$.

   (c) (15 pts) Suppose that the sublist $(v_1, \ldots, v_k)$ of $\mathcal{L}$ is linearly independent but $\mathcal{L}$ is linearly dependent. State and prove the lemma on linear dependence.
   The lemma asserts then that there is some $j > k$ such that v_j belongs to the span of $(v_1, v_2, \cdots v_j)$, and further that the span of the list $\mathcal{L}'$ with v_j omitted is the same as the span of $\mathcal{L}$. To prove this, observe that since the list $\mathcal{L}$ is dependent, there exist $(a_1, \ldots a_n)$, not all zero, such that $a_1v_1 + \cdots + a_nv_n = 0$. Let j be the largest i such that $a_i \neq 0$. Then $a_1v_1 + \cdots a_jv_j = 0$. Since the sublist is linearly independent, $j > k$. Furthtermore, this equation can be solved for v_j, so that v_j belongs to the span of the list $\mathcal{L}'$. Then the span of $\mathcal{L}'$ is a linear subspace of V which contains all the vectors in $\mathcal{L}$, and hence is the same as the span of $\mathcal{L}$.

2. Let V be a finite dimensional vector space and let S and T be linear transformations from V to V.

   (a) (15 pts) If $S \circ T$ is surjective, does it follow that S and T are surjective? If so, explain why. If not, give a counterexample.
   Indeed, this is true. Since V is finite dimensional, any surjective map from V to itself is invertible. Since $R := S \circ T$ is surjective, S is surjective, hence invertible, and hence $T = S^{-1}R$ is also surjective (in fact invertible).

   Remark: Here is an alternative proof. If T were not surjective, then the dimension of it range would be less than the dimension V. But the dimension of the range of $S \circ T$ must be less than or equal to the dimension of the range of T, and this would be a contradiction.

   Some students argued that since the composition of S and T is bijective, it automatically follows that each of S and T is bijective. But this is not the case, as the example given below in the next part shows.

   (b) (10 pts) What happens in the previous problem if V is not finite dimensional?
   In this case S is necessarily surjective but T need not be. For example, consider the space V of all sequences $(a_0, a_1, a_2, \cdots)$ and let S be the shift left operator and T the shift right operator. Then ST is the identity, but T is not surjective.

3. Let $\mathcal{P}_3$ denote the vector space of real polynomials of degree less than or equal to 3 and let $L(\mathcal{P}_3)$ denote the vector space of linear transformations from $\mathcal{P}_3$ to $\mathcal{P}_3$.

   (a) (10 pts) What is the dimension of $\mathcal{P}_3$? What is the dimension of $L(\mathcal{P}_3)$?
   The space $\mathcal{P}_3$ has dimension 4, with basis $(1, x, x^2, x^3)$. It follows that the space $L(\mathcal{P}_3)$ has dimension 16.

   (b) (15 pts) Show that the list $(\mathrm{id}, D, D^2, D^3)$ is linearly independent in $L(\mathcal{P}_3)$.
   Suppose that $a_1\mathrm{id} + a_2D + a_3D^2 + a_4D^4 = 0$. Then for every polynomial α, $a_1\alpha + a_2D\alpha + a_3D^2\alpha + a_4D^4\alpha = 0$. For example, if we take $\alpha = 1$, we find that $a_1 = 0$. If we take $\alpha = x$, we then find that $a_2 = 0$. Continuing, we see that all $a_i = 0$. As a matter of fact, it suffices to look at $\alpha = x^3$, since then we find that

$$0 = a_1x^3 + 3a_2x^2 + 6a_3x + 6a_4 = 0,$$

   which is enough to imply that each $a_i = 0$.

4. Let V be a vector space of dimension n, let W be a vector space of dimension m, and let $L(V, W)$ denote the space of linear transformations from V to W. Fix a nonzero vector v of V. Answer the following questions with brief explanations; complete proofs are not required.

   (a) (10 pts) Show that the set L' of linear transformations from V to W such that $T(v) = 0$ is a linear subspace of $L(V, W)$.
   First of all, the zero transformation belongs to L'. Next, if T and T' belongs to L' and a is a scalar, $(aT + T')(v) = aT(v) + T'(v) = a0 + 0 = 0$, so $aT + T'$ belongs to L'. This proves that L' is a subspace.

   This problem caused a surprising amount of difficulty. Some people tried to look at $T(v + v')$ for some reason.

   (b) (5 pts) What is the dimension of the space L'?
   Its dimension is $nm - m$, as can be seen by looking at matrix representatives. (Choose a basis for V which contains v; then L' corresponds to $m \times n$ matrices whose first column is zero.

   Again this problem caused confusion. Some people seemed to think L' was the set of all T such that $T(v) = 0$ for all v, not some fixed v as specified. But such a T is just the zero transformation, not at all what was meant (or said).

   (c) (10 pts) If T is an element of L', what is the maximum possible dimension of the range of T?
   In fact we can choose a linear subspace U of V such that $V = U \oplus span(v)$. Then an element T of L' restricts to a map $T' : U \to W$, and T and T' will have the same range. The dimenison of the range can't be any more than the dimension of U or the dimension of W. Thus the maximum possible dimension is the maximum of $n - 1$ and m.