Traces of operators and matrices

Let V be a finite dimensional $\mathbf{C}$-vector space and let T be an operator on V. Recall:

$$V = \oplus_\lambda GE_\lambda(T),$$

where $GE_\lambda(T)$ is the generalized λ-eigenspace of T:

$$GE_\lambda(T) = Ker((T - \lambda I)^n \text{ where } n = \dim V.$$

The characteristic polynomial f_T is the polynomial

$$f_T(t) : \prod_\lambda (t - \lambda)^{d_\lambda} \text{ where } d_\lambda - \dim GE_\lambda.$$

The Cayley-Hamilton theorem:

$$f_T(T) = 0 \text{ as an operator on } V.$$

Goal: compute f_T, without actually computing the generalized eigenspaces, or even the eigenvalues.

We know that $f_T(t)$ is monic:

$$f_T(t) = \prod_\lambda (t - \lambda)^{d_\lambda} = t^n + a_1 t^{n-1} + \cdots a_n,$$

for some list of complex numbers a_i.

Turns out: a_1 is easy to compute. a_n is harder but still possible.

Definition 1 *If $T \in \mathcal{L}(V)$ and*

$$f_T(t) = t^n + a_1 t^{n-1} + \cdots a_n,$$

is its characteristic polynomial, then

$$\text{trace}(T) := -a_1 \textit{ and}$$

$$\det(T) := (-1)^n a_n$$

A partial answer Theorem 8.10:

If $\mathcal{B}$ is a basis for V such that $A := M_{\mathcal{B}}(T)$ is upper triangular,

$$f_T(t) = \prod_i (t - a_{ii}).$$

However to compute $\mathcal{B}$ we must compute the eigenvalues, which is already difficult.

How does $M_{\mathcal{B}}(T)$ depends on $\mathcal{B}$?

Let $\mathcal{B} := (v_1, \ldots v_n)$, $\mathcal{B}' := (v'_1, \ldots v'_n)$ and $\mathcal{B}'' := (v''_1, \ldots v''_n)$ be bases for V. If T_1 and T_2 are operators on V, we had the formula

$$M^{\mathcal{B}}_{\mathcal{B}''}(T_1 T_2) = M^{\mathcal{B}'}_{\mathcal{B}''}(T_1) M^{\mathcal{B}}_{\mathcal{B}'}(T_2).$$

Let $S := M^{\mathcal{B}'}_{\mathcal{B}}(I)$. (The jth column $C_j(S)$ of S is $M_{\mathcal{B}}(v'_j)$.) Apply the formula to see that

$$M^{\mathcal{B}}_{\mathcal{B}'}(I) M^{\mathcal{B}'}_{\mathcal{B}}(I) = M^{\mathcal{B}'}_{\mathcal{B}'}(I) = I$$

Thus S is invertible and $S^{-1} = M^{\mathcal{B}}_{\mathcal{B}'}(I)$. Furthermore:

$$M^{\mathcal{B}'}_{\mathcal{B}'}(T) = M^{\mathcal{B}}_{\mathcal{B}'}(I) M^{\mathcal{B}}_{\mathcal{B}}(T) M^{\mathcal{B}'}_{\mathcal{B}}(I) = S^{-1} M^{\mathcal{B}}_{\mathcal{B}}(T) S.$$

Example Let $T\colon \mathbf{C}^2 \to \mathbf{C}^2$ be $T(x_1, x_2) : (x_2, x_1)$. Then in the standard basis $\mathcal{B} = (v_1, v_2)$, $T(v_1) = v_2$ and $T(v_2) = v_1$, so

$$M_{\mathcal{B}}(T) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

We also have an eigenbasis $\mathcal{B}'$, with $v'_1 = (1, 1)$ and $v'_2 = (1, -1)$, so that

$$M_{\mathcal{B}'}(T) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Note that

$$S = M^{\mathcal{B}}_{\mathcal{B}'}(I) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$S^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

The trace

Recall that $\text{trace}(T) = -a_1$, where

$$f_T(t) = \prod_\lambda (t-\lambda)^{d_\lambda}$$

Multiply this out to get:

$$a_1 = -\sum_\lambda d_\lambda \lambda$$

Definition 2 *If A is an $n \times n$ matrix,*

$$\text{trace}(A) = \sum_i a_{ii}$$

Theorem 3 *If T is an operator on V and $\mathcal{B}$ is any basis for V,*

$$\text{trace}(T) = \text{trace}(M_\mathcal{B}(T)).$$

Note this is true in the example above, since $1 + -1 = 0 + 0$.

Proposition 4 *If A and B are $n \times n$ matrices,*

$$\text{trace}(AB) = \text{trace}(BA)$$

Proof: Let $C := AB$ and $C' := BA$. Then for each i,

$$c_{ii} = \sum_k a_{ik} b_{ki}$$

$$\text{trace}C = \sum_i c_{ii} = \sum_i \sum_k a_{ik} b_{ki}$$

$$c'_{ii} = \sum_k b_{ik} a_{ki} = \sum_k a_{ki} b_{ik}$$

$$\text{trace}C' = \sum_i c'_{ii} = \sum_i \sum_k a_{ki} b_{ik}$$

$$= \sum_k \sum_i a_{ik} b_{ki} = \text{trace}C.$$

□

Corollary 5 *If S is invertible,*

$$\text{trace}(S^{-1}AS) = \text{trace}(ASS^{-1}) = \text{trace}(A)$$

Corollary 6 *If $\mathcal{B}$ and $\mathcal{B}'$ are two bases for V,*

$$\text{trace}M_\mathcal{B}(T) = \text{trace}M_{\mathcal{B}'}(T)$$

This proves the theorem: Let $\mathcal{B}'$ be a basis of generalized eigenvalues.