# Polynomials and polynomial functions

Let $\mathbf{F}$ be a field. Recall that $\mathbf{F}^{\mathbf{N}}$ is the vector space of all sequences $a := (a_0, a_1, \ldots)$ with $a_i \in \mathbf{F}$, with addition and scalar multiplication defined coordinatewise. For each natural number $d$, let $\mathcal{P}_d$ denote the set of all sequences $a$ such that $a_i = 0$ for $i > d$. Check that $\mathcal{P}_d$ is a linear subspace of $\mathbf{F}^{\mathbf{N}}$, and that $\mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \cdots$. Now let $\mathcal{P} := \cup\{\mathcal{P}_d : d \in \mathbf{N}\}$, *i.e.*, $\mathcal{P}$ is the set of all $a$ such that there exists some $d$ (possibly depending on $a$) such that $a_i = 0$ for all $i > d$. Check that $\mathcal{P}$ is also a linear subspace of $\mathbf{F}^{\mathbf{N}}$.

We can think of a sequence $(a_0, a_1, \cdots)$ in $\mathcal{P}$ as specifying the coefficients of a polynomial (with coefficients in the field $\mathbf{F}$). More specifically:

**Definition:** Let $a$ be an element of $\mathcal{P}$, say $a$ in $\mathcal{P}_d$. Then $\tilde{a}$ is the function $\mathbf{F} \to \mathbf{F}$ such that $\tilde{a}(x) = a_0 + a_1 x + a_2 x^2 + \cdots a_d x^d$ for all $x \in \mathbf{F}$.

Note that if $d' \geq d$, then $a$ also belongs to $\mathcal{P}_{d'}$, and

$$a_0 + a_1 x + a_2 x^2 + \cdots a_d x^d = a_0 + a_1 x + a_2 x^2 + \cdots a_d x^d + \cdots + a_{d'} x^{d'},$$

since $a_i = 0$ for all $i > d$. Thus $\tilde{a}$ is independent of the choice of $d$. This means that we have a well-defined mapping

$$\epsilon \colon \mathcal{P} \to \mathbf{F}^{\mathbf{F}}, \text{ sending each } a \text{ to } \epsilon(a) := \tilde{a}.$$

An element $a \in \mathcal{P}$ is called a *polynomial*; really then a polynomial is just a list of coefficients, all but finitely many of which are zero. Associated to such a list $a$ we have formed a function $\tilde{a}$, which we call a *polynomial function.* Now recall that we also have operations on $\mathbf{F}^{\mathbf{F}}$ which make it into a vector space over $\mathbf{F}$. Fortunately, the next proposition tells us that these operations are compatible with the operations.

**Proposition:** If $a, b \in \mathcal{P}$ and $c \in \mathbf{F}$, then $\epsilon(a + b) = \epsilon(a) + \epsilon(b)$ and $\epsilon(ca) = c\epsilon(a)$.

Write out the proof of this proposition. It says that addition of polynomial functions can be computed by adding the corresponding coefficients, and similary for multiplication of a polynomial function by a constant.

Axler in his book doesn't really distinguish between the sequence of coefficients $a$ and the corresponding polynomial function $\tilde{a}$. This is because for him, $\mathbf{F}$ is always either $\mathbf{R}$ or $\mathbf{C}$, in which case one can recover the coeffients from the polynomial function. However for the field $\mathbf{F}_2$, this is not the case. For example, the polynomial $a := (0, 1, 1, 0, 0, \cdots,)$ gives rise to the same function as the polynomial $b := (0, 0, 0, \cdots)$. This is because $\tilde{a}(x) = x + x^2$, which is 0 if $x$ is any element of $\mathbf{F}_2$.