Linear Algebra Final Exam Solutions, December 13, 2008

Write clearly, with complete sentences, explaining your work. You will be graded on clarity, style, and brevity. If you add false statements to a correct argument, you will lose points.

1		20
2		20
3		20
4		20
5		20
Total		100

1. Let F be a field and let V be a vector space over F.

   (a) Let $(v_1, v_2, \ldots, v_n)$ be a list in V. Define what it means for $(v_1, v_2, \ldots, v_n)$ to be a *basis* of V. Define the *dimension* of V, explaining when it exists (and is finite).

   (b) Prove that if W_1 and W_2 are linear subspaces of a finite dimensional space V, then

   $$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

   (c) Prove that, with the notation of the previous part,

   $$\dim(W_1 \cap W_2) \geq \dim(W_1) + \dim(W_2) - \dim V.$$

Solutions:

(a) The list is a basis for V if and only if every element of V can be written uniquely as a sum $\sum a_i v_i$, or, equivalently, if the list is independent and spans V. If such a basis exists, then any two bases have the same length, and this length is the dimension of V.

(b) Choose a basis for $W_1 \cap W_2$ and prolong it to bases for W_1 and W_2. Thus we find a list $(v_1, \ldots v_n)$ such that $(v_1, \ldots v_d)$ is a basis for $W_1 \cap W_2$, $(v_1, \ldots v_{d+d_1}$ is a basis for W_1, and $(v_1, \ldots, v_d, v_{d+d_1+1}, \ldots v_{d+d_1+d_2})$ is a basis for W_2. It is easy to see that this list spans $W_1 + W_2$. Let us check that it is independent. If $\sum a_i v_i = 0$, then $a_1 v_1 + \cdots a_{d+d_1} = -a_{d+d_1+1 - \cdots - a_{d+d_1+d_2}}$. The vector on the left belongs to V_1 and the vector on the right belongs to V_2, hence both belong to $V_1 \cap V_2$. Hence $a_1 v_1 + \cdots a_{d+d_1} = b_1 v_1 + \cdots v_d v_d$ for some b_i, and since $(v_1, \ldots, v_{d+d_1})$ is independent, $a_i = 0$ for $d < i \leq d + d_1$. But then we have a linear combination adding up to zero involving only the vectors in the basis for W_2 so all the coefficients are zero.

(c) Since $W_1 + W_2$ is contained in V, its dimension is less than or equal to the dimension of V. Hence

$$\dim(W_1 \cap W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 + W_2) \geq \dim(W_1) + \dim(W_2) - \dim V$$

2. Let T be a linear operator on a finite dimensional vector space V over a field F.

   (a) If $\lambda \in F$, what is meant by the λ-*eigenspace* $E_\lambda(T)$ of T?

   (b) If $\lambda \in F$, what is meant by the *generalized* λ-*eigenspace* $GE_\lambda(T)$ of T?

   (c) Let $(v_1, \ldots, v_n)$ be a list of nonzero generalized eigenvectorscorresponding to distinct eigenvalues $(\lambda_1, \ldots, \lambda_n)$. Prove that $(v_1, \ldots, v_n)$ is linearly independent. (Remark: if you have trouble with this, you can do it for eigenvalues instead of generalized eigenvalues for partial credit.)

Solutions:

(a) The λ-eigenspace of T is the kernel of $T - \lambda I$.

(b) The generalized λ-eigensapce of T is the set of all vectors v such that $(T - \lambda I)^k v = 0$ for some k.

(c) We prove this by induction on n, the cases of $n = 0, 1$ being trivial. First we recall that if N is a nilpotent operator and a is not zero, then $a + N$ is invertible. Furthermore, GE_i is invariant under $T - \lambda_n I$ and is equal to $T - \lambda_i + a_i)$, where $a_i = \lambda_i - \lambda_n$. Thus if $i < n$, $T - \lambda_n$ is invertible on GE_i, and hence so is each of its powers. Now suppose that $\sum a_i v_i = 0$. Apply $(T - \lambda_n)^k$ to conclude that
$$0 = \sum_{i=1}^{n} a_i (T - \lambda_n)^k v_i.$$
Let $v_i' := (T - \lambda_n)^k v_i$. For k sufficiently large $v_n' = 0$, so we get
$$0 = \sum_{i=1}^{n-1} a_i v_i' = 0.$$
But v_i' is a nonzero element of GE_i, and the induction assumption tells us that $a_i = 0$ for all $i < n$. Since $v_n \neq 0$ and now $a_n v_n - 0$, $a_n = 0$ also.

3. Let V be an inner product space over $\mathbf{C}$ and let T be a linear operator on V. Do not assume that V is finite dimensional.

   (a) Let W be a finite dimensional subspace of V. State the theorem on orthogonal projection to W and use it to prove that $(W^\perp)^\perp = W$.

   (b) Show that there is at most one operator T^* such that $(Tv|w) = (v|T^*w)$ for all v and w. If this operator exists, it is called the *adjoint* of T.

   (c) Suppose that T^* exists. Prove that $Ker(T) = Im(T^*)^\perp$. Prove that if $Im(T^*)$ is finite dimensional, then $Im(T^*) = Ker(T)^\perp$.

   (d) Is this same conclusion true assuming that $Ker(T)$ is finite dimensional?

Solutions:

(a) The theorem says that if W is finite dimensional, then $V = W \oplus W^\perp$. Say $w \in W$ and $v \in W^\perp$. Then $(v|w) = 0$, so certainly $w \in (W^\perp)^\perp$. On the other hand, say $v \in (W^\perp)^\perp$. Then v can be written as a sum $v = w + v'$, where $w \in W$ and $v' \in W^\perp$. Then $(v|v') = 0$ and $(w|v') = 0$, hence $(v'|v') = 0$ so $v' = 0$ and $v \in W$

(b) If T' and T'' both satisfy this equation, let $S := T' - T''$. Then $(v|Sw) = 0$ for all v and w. In particular $(Sw|Sw) = 0$ for all w, hence $Sw = 0$ for all w, hence $S = 0$ and $T' = T''$.

(c) If $v \in KerT$, then $(v|T^*w) = (Tv|w) = 0$ for all w, so $v \in (ImT^*)^\perp$. Conversely, if $v \in (ImT^*)^\perp$, then $(Tv|Tv) = (v|T^*Tv) = 0$ so $Tv = 0$ and $v \in KerT$. We know that $Ker(T) = Im(T^*)^\perp$, hence by part (a), $Ker(T)^\perp = Im(T^*)$ if $Im(T^*)$ is finite dimensional.

4. Say whether each of the following is true or false. If true, give a brief explanation. If false, give a counterexample, and if possible, a corrected version of the statement, for example by adding a missing hypothesis. (You need not prove the corrected statement in this case.)Here V is a finite dimensional complex vector space and T is a linear operator on V.

   (a) The characteristic polynomial $f_T(t)$ and the minimal polynomial of $p_T(t)$ have the same roots with the same multiplicities.

   (b) The operator T is diagonalizable if and only if $f_T(t)$ has distinct roots.

   (c) If V is an inner product space (still finite dimensional over $\mathbf{C}$) and W is T-invariant, then $W^\perp$ is also T-invariant.

   (d) The real vector space $\mathbf{R}^3$ has an orthogonal basis consisting of eigenvectors for the operator corresponding to the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$.

Solutions:

(a) This is not true. For example, the 0 operator on a two-dimensional vector space has characteristic polynomial t^2 but minimal polynomial t. It is true that f_T and p_T have the same roots, but the multiplicities of the roots of p_T may be less than the multiplicities of f_T.

(b) The example above is a counterexample. The correct statement is that T is diagonalizable if and only if p_T has distinct roots.

(c) This is not true: for example $T(x_1, x_2) = (x_2, 0)$. It is true if T is self-adjoint, or, more generally, normal.

(d) This is true since the operator is self adjoint.

5. Let T be a nilpotent operator on a vector space V. Let W be a T-invariant subspace of V.

   (a) Prove that if $Ker(T) \cap W = 0$, then $W = 0$.

   (b) Prove that if $Im(T) + W = V$, then $W = V$.

   Hint: If you are completely stuck, try this for small values of r, e.g. $r = 1$ or 2.

Solution

(a) Since W is T-invariant, we can consider the restriction T_W of T to W to get an operator on W. Since $KerT \cap W = 0$, T_W is injective. On the other hand, $T^n = 0$ for some n, and hence $T_W^n = 0$ also, Since T_W^n is again injective, $W = 0$.

(b) Say $v \in V$. We can write $v = w_1 + Tv_1$ for some $w_1 \in W$, $v_1 \in V$. Now apply the same reasoning to write $v_1 = w' + Tv_2$ and hence $v = w_1 + Tw' + T^2 v_2$. Since W is invariant $Tw' \in W$ and hence $w_2 := w_1 + Tw' \in W$. Thus we have written $v = w_2 + T^2 v_2$. Continuing in this way, we find some $w_n \in W$, $v_n \in V$ with $v = w_n + T^n v_n$ for all n, Since $T^n = 0$ for some n, we see that $v \in W$.