Determinants of operators and matrices

Let V be a finite dimensional $\mathbf{C}$-vector space and let T be an operator on V. Recall:

The characteristic polynomial f_T is the polynomial

$$f_T(t) : \prod_{\lambda} (t-\lambda)^{d_\lambda} \text{ where } d_\lambda - \dim GE_\lambda.$$

The Cayley-Hamilton theorem:

$$f_T(T) = 0 \text{ as an operator on } V.$$

Goal: compute f_T, without actually computing the generalized eigenspaces, or even the eigenvalues, just from $M_{\mathcal{B}}(T)$ for any arbtirary basis $\mathcal{B}$ for V.

$$f_T(t) := \prod_{\lambda} (t-\lambda)^{d_\lambda} = t^n + a_1 t^{n-1} + \cdots a_n,$$

for some list of complex numbers a_i.

Definition 1 *If $T \in \mathcal{L}(V)$ and*

$$f_T(t) = t^n + a_1 t^{n-1} + \cdots a_n,$$

is its characteristic polynomial, then

$$\operatorname{trace}(T) := -a_1 \text{ and}$$

$$\det(T) := (-1)^n a_n$$

Last time:
If $A := M_{\mathcal{B}}(T)$, then

$$\operatorname{trace}(T) = \operatorname{trace}(A) := \sum a_{i,i}$$

Outline:

1. Define the determinant of a matrix, $\det(A)$.
2. Check that if A is uppertriangular,
$$\det(A) = \prod a_{ii}.$$
3. Show that if A and B are $n \times n$ matrices,
$$\det(AB) = \det(A)\det(B).$$
4. Conclude that $\det(S^{-1}AS) = \det A$ for every invertible S.
5. Conclude that if $A = M_{\mathcal{B}}(T)$, then $\det(A) = \det(T)$.

Step (1) is probably the hardest. How to find the definition? Many approaches. I'll follow the book, more or less. Let's look at cyclic spaces.

Theorem 2 *Suppose that V is T-cyclic, so that there is a $v \in V$ with*
$$V = \operatorname{span}(v, Tv, T^2v, \dots T^{n-1}v).$$
Then $(v, Tv, \dots, T^{n-1}v)$ is a basis of V, and
$$T^n v = c_0 v + c_1 Tv + \cdots + c_{n-1}T^{n-1}v$$
for a unique list $(c_0, \dots, c_{n-1})$ in $\mathbf{C}$. Then the characteristic polynomial of T is
$$p(t) = t^n - c_{n-1}t^{n-1} - \cdots c_1 t - c_0.$$

Example 3 Last time we looked at $T(x_1, x_2) := (x_2, x_1)$. Let $v := (1, 0)$. Then (v, Tv) is a basis for V, and $T^2v = v$. Thus $c_0 = 1$ and $c_1 = 0$, the $p(t) = t^2 - 1$.

Proof: The equation for $T^n v$ says that
$$p(T)(v) = 0.$$
It follows that
$$p(T)(T^i v) = T^i p(T)(v) = 0 \text{ for all } i$$
and since the $T^i v$'s span V $p(T) = 0$. Since the list $(v, Tv, \dots T^{n-1}v)$ is indepenent, there is no polynomial of smaller degree that annihilates T. Thus p is the minimal polynomial of T, and since its degree is the dimension of V p is also the characteristic polyonmial of T. □

Corollary 4 *In the cyclic case above,* $\det(T) = (-1)^{n+1}c_0$.

Example 5 Let $\mathcal{B} = (v_1, \dots, v_n)$ and let T be the operator sending v_1 to v_2, v_2 to v_3, and so on, but then v_n to v_1. Then the characterisictic polynomial of T is
$$f_T(t) = t^n - 1 \text{ and } \det(T) = (-1)^{n+1}$$

In this example, our linear transformation just *permutes* the basis. Our next step is to discuss more general cases of this.

Permutations

Definition 6 *A permutation of the set* $1, \ldots, n$ *is a bijective function* σ *from the set* $\{1, \ldots, n\}$ *to itself. Equivalently, it is a list* $(\sigma(1), \ldots, \sigma(n))$ *such that each element of* $\{1, \ldots, n\}$ *occurs exactly once. The set of all permutations of length* n *is denoted by* S_n.

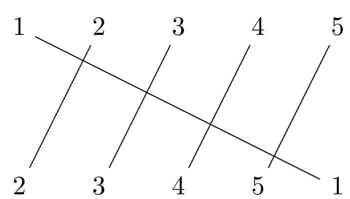
Examples in S_5 :
$(2, 3, 4, 5, 1)$
$(2, 4, 3, 1, 5)$
$(2, 4, 5, 1, 3)$
The first of these is cycle of length 5. Note that the second doesn't move 3 or 5, and can be viewed as a cyclic permutation of the set $\{1, 2, 4\}$. The last permuation can be viewed as a the product (composition) of a cyclic permutation of $\{1, 2, 4\}$ and a cyclic permutation of $\{3, 5\}$.

Definition 7 *The sign of a permuation* σ *is* $(-1)^m$ *where* m *is the number of pairs* (i, j) *where*

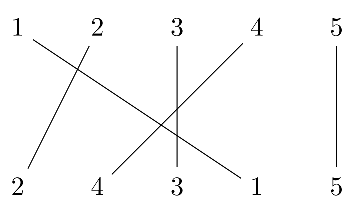
$$1 \leq i < j \leq n \text{ but } \sigma(i) > \sigma(j)$$

Here's an easy way to count: Arrange $(1, 2, \ldots, n)$ in one row, and $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ in a row below. Draw lines connecting i in the first row to i in the second. Then m is the number of crosses.

Examples:



so $m = 4$ and sgn $= +1$.



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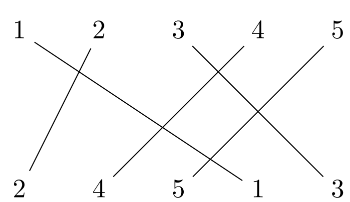

so $m = 5$ and sgn $= -1$.

Example 8 A cycle of length n has $n-1$ crossings, and so its sign is $(-1)^{n-1}$. Note that this is the same as the determinant of the corresponding linear transfomration. d

Theorem 9 *If $\sigma, \tau \in S_n$, then*

$$\operatorname{sgn}(\sigma\tau) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)$$

Omit the proof, at least for now.

Definition 10 *Let A be an $n \times n$ matrix. Then*

$$\det A := \sum \{\operatorname{sgn}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)} : \sigma \in S_n\}$$

Example 11 When $n = 2$ there are two permuations, and we get

$$\det(A) = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}.$$

Proposition 12 *Let A and B be $n \times n$ matrices.*

1. *If A is upper triangular,* $\det(A) = \prod_i a_{i,i}$.
2. *If B is obtained from A by interchanging two columns, then* $\det(B) = -\det(A)$.
3. *If two columns of A are equal,* $\det(A) = 0$.
4. $\det(A)$ *is a linear function of each column, (when all the other columns are fixed.)*
5. $\det(AB) = \det(A)\det(B)$.