GLOBAL C^∞ IRREGULARITY OF THE $\bar\partial$–NEUMANN PROBLEM FOR WORM DOMAINS

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0. Introduction.

Let $\Omega \subset \mathbf{C}^n$ be a bounded, pseudoconvex domain with C^∞ boundary, equipped with the standard Hermitian metric inherited from $\mathbf{C}^n$. The $\bar\partial$–Neumann problem for (p,q) forms in Ω is the boundary value problem

$$\begin{cases} \Box u = f & \text{in } \Omega \\ u \,\lrcorner\, \bar\partial\rho = 0 & \text{on } \partial\Omega \\ \bar\partial u \,\lrcorner\, \bar\partial\rho = 0 & \text{on } \partial\Omega \end{cases}$$

where ρ is a defining function for Ω, $\Box = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$, u, f are (p,q) forms, and $\lrcorner$ denotes the interior product of forms. Under the stated hypotheses on Ω, this problem is uniquely solvable for every $f \in L^2(\Omega)$. The Neumann operator N, mapping f to the solution u, is continuous on $L^2(\Omega)$. The Bergman projection B is the orthogonal projection of $L^2(\Omega)$ onto the closed subspace of L^2 holomorphic functions on Ω, and is related to N by $B = I - \bar\partial^* N \bar\partial$.

N and B are C^∞ pseudolocal if Ω is strictly pseudoconvex, or more generally, is of finite type [Ca1]. Both preserve $C^\infty(\overline{\Omega})$ under certain weaker hypotheses [BS2],[Ca2]. For any pseudoconvex, smoothly bounded Ω and any finite exponent s, there exists a strictly positive weight $w \in C^\infty(\overline{\Omega})$ such that the Neumann operator and Bergman projection with respect to the Hilbert space $L^2(\Omega, w(x)dx)$ map the Sobolev space $H^t(\Omega)$ boundedly to $H^t(\Omega)$, for all $0 \le t \le s$ [K1]. It has remained an open question whether N and B, defined with respect to the standard metric, preserve $C^\infty(\overline{\Omega})$ without further hypotheses on Ω. An affirmative answer would have significant consequences [BL].

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Theorem. *There exist pseudoconvex, smoothly bounded domains $\Omega \Subset \mathbf{C}^2$ for which the Neumann operator on $(0,1)$ forms and Bergman projection fail to preserve $C^\infty(\overline{\Omega})$.*

Examples are the worm domains, originally introduced by Diederich and Fornæss [DF] for another purpose[1] but long considered likely candidates for N and B to fail to be globally regular in C^∞.

The proof depends on the observation of Barrett [B] proved that for each worm domain $\mathcal{W}$, for all sufficiently large s, N and B fail to map $H^s(\mathcal{W})$ boundedly to $H^s(\mathcal{W})$. We establish for each worm domain an *a priori* inequality of the form $\|Nf\|_{H^s} \le C_s\|f\|_{H^s}$, valid for all $f \in C^\infty(\overline{\mathcal{W}})$ such that $Nf \in C^\infty(\overline{\mathcal{W}})$, for a sequence of exponents s tending to ∞. If N were to preserve $C^\infty(\overline{\mathcal{W}})$, then since it is a bounded linear operator on $L^2(\mathcal{W})$ and since $C^\infty(\overline{\mathcal{W}})$ is dense in $H^s(\mathcal{W})$, it would follow that N maps $H^s(\mathcal{W})$ boundedly to itself, for a sequence of values of s tending to ∞, a contradiction. More accurately, our inequality is valid only for certain subspaces of $L^2(\mathcal{W})$ preserved by $\Box$, but this still suffices to contradict the theorem of Barrett.

An analogous counterexample in the real analytic context is already known [Ch1]: there exists a bounded, pseudoconvex domain $\Omega \subset \mathbf{C}^2$ having real analytic boundary, such that the Szegö projection fails to preserve $C^\omega(\partial\Omega)$. That result and its proof are however not closely related to the C^∞ case.

§§1 through 3 review material on worm domains and the $\bar\partial$–Neumann problem, and present some routine but tedious reductions. §4 formalizes a class of two-dimensional problems subsuming those to which the reductions lead. The analysis of those problems is contained in §§5 and 6.

1. Reduction to the Boundary.

The $\bar\partial$–Neumann problem is a boundary value problem for an elliptic partial differential equation, and as such is amenable to treatment by the method of reduction to a pseudodifferential equation on the boundary. This reduction has been carried out in detail for domains in $\mathbf{C}^2$ by Chang, Nagel and Stein [CNS]. We review here certain of their computations and direct consequences thereof.

Assume $\Omega \subset \mathbf{C}^2$ to be a smoothly bounded domain. The equation $\Box u = f$ on Ω for $(0,1)$ forms in $C^\infty(\overline{\Omega})$ is equivalent to an equation $\Box^+ v = g$ on $\partial\Omega$, where v, g are sections of a certain complex line bundle[2] $\mathcal{B}^{0,1}$. Let ρ be a smooth defining function for Ω and define $\bar\omega_2 = \bar\partial\rho$, and $\bar\omega_1 = (\partial\rho/\partial z_2)d\bar z_1 - (\partial\rho/\partial z_1)d\bar z_2$.

v is related to u by $u = Pv + Gf$, where P, G are respectively Poisson and Green operators for the elliptic system $\Box u = f$ with Dirichlet boundary conditions. In particular, if $f \in C^\infty(\overline{\Omega})$, then $u \in C^\infty(\overline{\Omega})$ if and only if the same holds for v. More precisely, G maps

[1] Some but not all worm domains have nontrivial Nebenhülle[FS, p. 111], [BF, Theorem 5.1] whereas all worm domains are counterexamples to global regularity.

[2] $\mathcal{B}^{0,1}$ is defined to be the quotient of the restriction to $\partial\Omega$ of $T^{0,1}\mathbf{C}^2$, modulo the span of $\bar\partial\rho$. Sections of $\mathcal{B}^{0,1}$ may be identified with scalar-valued functions times $\bar\omega_1$, hence with scalar-valued functions.

$H^s(\Omega)$ to $H^{s+2}(\Omega)$, while P maps $H^s(\partial\Omega)$ to $H^{s+1/2}(\Omega)$, for each $s \geq 0$. Thus if $f \in H^s$, in order to conclude that $u \in H^s$ it suffices to know that $v \in H^{s-1/2}$.

On the other hand, $g = (\bar\partial G f \lrcorner \bar\partial\rho)$, restricted to $\partial\Omega$. If $f \in H^s$ then $\bar\partial G f \in H^{s+1}(\Omega)$, so its restriction to the boundary belongs to $H^{s+1/2}(\partial\Omega)$. Thus in order to show that N preserves $H^s(\Omega)$ it suffices to show that if $\Box^+ v \in H^{s+1/2}(\partial\Omega)$, then $v \in H^{s-1/2}(\partial\Omega)$, assuming always that $s > 1/2$.

On $\partial\Omega$ a Cauchy-Riemann operator is the complex vector field $\bar L = (\partial_{\bar z_1}\rho)\partial_{\bar z_2} - (\partial_{\bar z_2}\rho)\partial_{\bar z_1}$. Define L to be the complex conjugate of $\bar L$. The characteristic variety[3] of $\bar L$ is a real line bundle Γ. Assuming Ω to be pseudoconvex and the set of strictly pseudoconvex points to be dense in $\partial\Omega$, Γ splits smoothly and uniquely as $\Gamma^+ \cup \Gamma^-$, where each fiber of $\Gamma^\pm$ is a single ray, and where Γ^+ is distinguished from Γ^- by the requirement that the principal symbol of $[\bar L, L]$ is nonpositive on Γ^+, modulo terms spanned by the symbols of the real and imaginary parts of $\bar L$ and a term of order 0. Equivalently, the principal symbol of $[\bar L, \bar L^*]$ is nonnegative on Γ^+, modulo the same kinds of error terms.

$\Box^+$ is a classical pseudodifferential operator of order $+1$. Its principal symbol vanishes everywhere on Γ^+ but nowhere else. Microlocally in a conic neighborhood of Γ^+, $\Box^+$ takes the form

$$\Box^+ = Q\bar L L + F_1 \bar L + F_2 L + F_3$$

where Q is an elliptic pseudodifferential operator of order -1, and each F_j is a pseudodifferential operator of order less than or equal to -1. Since $\Box^+$ is elliptic except on Γ^+, for any pseudodifferential operator G of order zero whose symbol vanishes identically in some neighborhood of Γ^+, one has for all $u \in C^\infty$ and all $N < \infty$

$$\|Gu\|_{H^{t+1}(\partial\Omega)} \leq C\|\Box^+ u\|_{H^t(\partial\Omega)} + C_N\|u\|_{H^{-N}(\partial\Omega)}. \tag{1.1}$$

Let A be an elliptic pseudodifferential operator of order $+1$ such that $A \circ Q$ equals the identity on $L^2(\partial\Omega)$, modulo an operator smoothing of infinite order. Composing on the left with A, the equation $\Box^+ v = g$ may be rewritten as $\mathfrak{L} v = \tilde g$ microlocally in a conic neighborhood of Γ^+, where

$$\mathfrak{L} = \bar L L + B_1 \bar L + B_2 L + B_3, \tag{1.2}$$

$\|\tilde g\|_{H^t} \leq C\|g\|_{H^{t+1}} + C_N\|v\|_{H^{-N}}$ for any finite N, and each B_j is an operator of order less than or equal to zero. Therefore in order to show that the Neumann operator satisfies an *a priori* inequality of the form $\|Nf\|_{H^s(\Omega)} \leq C\|f\|_{H^s(\Omega)}$ for all $f \in C^\infty(\overline\Omega)$ such that $Nf \in C^\infty(\overline\Omega)$, it suffices to establish an *priori inequality* for all $v \in C^\infty(\partial\Omega)$ of the form

$$\|v\|_{H^t} \leq C\|\mathfrak{L}v\|_{H^t} + C\|v\|_{H^{t'}} + C\|\tilde Q v\|_{H^{t+2}} \tag{1.3}$$

where $t = s - 1/2$, for some $t' < t$ and some pseudodifferential operator $\tilde Q$ of order zero whose symbol vanishes identically in some neighborhood of Γ^+.

[3] By the characteristic variety of a pseudodifferential operator we mean the conic subset of the cotangent bundle on which its principal symbol vanishes.

2. Worm Domains.

A worm domain in $\mathbf{C}^2$ is an open set of the form

$$\mathcal{W} = \{z : |z_1 + e^{i \log |z_2|^2}|^2 < 1 - \phi(\log |z_2|^2)\}$$

where the function ϕ vanishes identically on some interval $[-r, r]$ of positive length, and is constructed [DF] so as to guarantee that $\mathcal{W}$ will be pseudoconvex with C^∞ boundary, and will be strictly pseudoconvex at every boundary point except those on the exceptional annulus $\mathcal{A} \subset \partial\mathcal{W}$ defined as

$$\mathcal{A} = \{z : z_1 = 0 \text{ and } |\log |z_2|^2| \le r\}.$$

The circle group acts as a group of automorphisms of $\mathcal{W}$ by $z \mapsto R_\theta z = (z_1, e^{i\theta} z_2)$. It acts on functions by $R_\theta f(z) = f(R_\theta z)$, and on $(0,1)$ forms by $R_\theta(f_1 d\bar{z}_1 + f_2 d\bar{z}_2) = (R_\theta f_1) d\bar{z}_1 + (R_\theta f_2) e^{-i\theta} d\bar{z}_2$. The Hilbert space $L^2_{(0,k)}(\mathcal{W})$ of square integrable $(0,k)$ forms decomposes as the orthogonal direct sum $\oplus_{j \in \mathbf{Z}} \mathcal{H}_j^k$ where $\mathcal{H}_j^k$ is the set of all $(0,k)$ forms f satisfying $R_\theta f \equiv e^{ij\theta} f$. $\bar\partial$ is an unbounded linear operator from $\mathcal{H}_j^k$ to $\mathcal{H}_j^{k+1}$, B maps $\mathcal{H}_j^0$ to itself, and the Neumann operator N maps $\mathcal{H}_j^1$ to $\mathcal{H}_j^1$ boundedly, for each j.

For each k and each $s \ge 0$, the Sobolev space $H^s(\mathcal{W})$ likewise decomposes as an orthogonal direct sum of subspaces H_j^s. It is known that for any smoothly bounded, pseudoconvex domain $\Omega \subset \mathbf{C}^2$, for any exponent $s \ge 0$, if N maps $H^s(\Omega)$ boundedly to itself, then B also maps $H^s(\Omega)$ boundedly to itself [BS1], where H^s denotes in the first instance a space of one forms, and in the second, a space of functions. Because N, B preserve the summands $\mathcal{H}_j$, the same proof shows[4] that for any fixed j, if N maps the space H_j^s of $(0,1)$ forms boundedly to itself, then B maps the space H_j^s of functions boundedly to itself. Barrett [B] that for each worm domain, for all sufficiently large s, for all j, B fails to map H_j^s boundedly to itself. Therefore in order to prove that N, acting on $(0,1)$ forms, fails to preserve $C^\infty(\overline{\mathcal{W}})$, it suffices to establish the following result for a single index j.

Proposition 1. *For each worm domain there exists a discrete subset $S \subset \mathbf{R}^+$ such that for each $s \notin S$ and each $j \in \mathbf{Z}$ there exists $C_{s,j} < \infty$ such that for every $(0,1)$ form $u \in \mathcal{H}_j^1 \cap C^\infty(\overline{\mathcal{W}})$ such that $Nu \in C^\infty(\overline{\mathcal{W}})$,*

$$\|Nu\|_{H^s(\mathcal{W})} \le C_{s,j} \|u\|_{H^s(\mathcal{W})}. \tag{2.1}$$

The defining function $\rho = 1 - \phi(\log |z_2|^2) - |z_1 + e^{i \log |z_2|^2}|^2$ for $\mathcal{W}$ is invariant under R_θ, as is the $(0,1)$ form $\bar\omega_2$ defined above. $\bar\omega_1$ satisfies $R_\alpha \bar\omega_1 = \exp(-i\alpha)\bar\omega_1$ for all α, but it may also be made invariant by multiplying it by the function $(z_1, re^{i\theta}) \mapsto e^{i\theta}$, which is smooth in a neighborhood of $\overline{\mathcal{W}}$. We work henceforth with this modified $\bar\omega_1$.

[4] This follows from the argument of Boas and Straube [BS1] because all elements of their proof may be chosen to be invariant under the automorphisms R_θ.

$\Box^+$ commutes with R_θ for all θ. Indeed, $\Box^+ v = \bar\partial P v \lrcorner \bar\partial\rho$ [CNS]. $\Box$ commutes with R_θ, hence so must P. $\bar\partial$ commutes with R_θ, and the Hermitian metric on $\mathbf{C}^2$ and $\bar\partial\rho$ are likewise R_θ–invariant. Thus all ingredients in the above expression for $\Box^+$ are invariant, hence so is $\Box^+$ itself.

Identify square integrable sections of $\mathcal{B}^{0,1}$ with scalar-valued L^2 functions as above, and decompose $L^2(\partial\mathcal{W}) = \oplus\mathcal{H}_j(\partial\mathcal{W})$ where $\mathcal{H}_j$ is the subspace of those functions satisfying $R_\theta f \equiv e^{ij\theta} f$. Then $\Box^+$ maps $\mathcal{H}_j(\partial\mathcal{W}) \cap C^\infty$ to $\mathcal{H}_j(\partial\mathcal{W})$. We have seen in §1 that Proposition 1 would be a consequence of the validity of (1.3) for all $v \in C^\infty(\partial\mathcal{W}) \cap \mathcal{H}_j$, for $t = s - 1/2$.

Fix j and assume henceforth that u belongs to $\mathcal{H}^1_j(\mathcal{W})$ and to C^∞. Then the associated boundary function v belongs to $\mathcal{H}_j \cap C^\infty(\partial\mathcal{W})$. Henceforth we work exclusively on the boundary, and simplify notation by writing simply $\mathcal{H}_j$ rather than $\mathcal{H}_j(\partial\mathcal{W})$.

Note that $\bar L, L$ take $\mathcal{H}_j \cap C^\infty$ to $\mathcal{H}_{j+1}$ and to $\mathcal{H}_{j-1}$, respectively. The operator A introduced after (1.1) may be constructed to be R_θ-invariant, for both $\Box^+$ and $\bar L \circ L$ are invariant while $\bar L, L$ are automorphic of certain degrees, so that averaging the equation $\Box^+ = Q\bar L L + F_1\bar L + F_2 L + F_3$ with respect to $R_\theta d\theta$ produces an invariant Q and F_3, and operators F_1, F_2 automorphic of the appropriate degrees. Thus $(\bar L L + B_1\bar L + B_2 L + B_3)v \in H^t$ microlocally near Γ^+, where B_1, B_2, B_3 map $\mathcal{H}_j$ to $\mathcal{H}_i$ for $i = j-1, j+1, j$ respectively.

Since $\mathcal{W}$ is strictly pseudoconvex at all points not in $\mathcal{A}$, it follows as in [K2] that on the complement of any neighborhood of $\mathcal{A}$, the H^{s+1} norm of v is majorized by $C\|\Box^+ v\|_{H^{t+1}(\partial\mathcal{W})} + C\|v\|_{H^{-N}}$, hence by $C\|\mathfrak{L}v\|_{H^s} + C\|v\|_{H^{-N}} + C\|\tilde Q v\|_{H^{s+2}}$ with $\tilde Q$ as in (1.3). This estimate is one derivative stronger than that which we seek. In particular, it now suffices to control the H^s norm of v in an arbitrarily small neighborhood U of $\mathcal{A}$, and to do so microlocally near Γ^+.

Fix a C^∞ cutoff function φ supported in a small neighborhood of $\mathcal{A}$ but identically equal to 1 in a smaller neighborhood, and fix an open set V disjoint from a neighborhood of $\mathcal{A}$ such that $\nabla\varphi$ is supported in V. By Leibniz's rule and the pseudolocality of pseudodifferential operators, the H^s norm of $\mathfrak{L}(\varphi v - v)$ is majorized by $C\|v\|_{H^{s+1}(V)} + C\|v\|_{H^{-N}}$ for any $N < \infty$. Thus by replacing v with φv we may reduce matters to the case where v is supported in an arbitrarily small neighborhood W of $\mathcal{A}$. Therefore it suffices to prove the existence of some W and an exponent $s' < s$ such that (1.3) holds (with t replaced by s) for all $v \in C^\infty_0$ supported in W.

In a neighborhood of $\mathcal{A}$ in $\partial\mathcal{W}$ introduce coordinates (x, θ, t) where

$$z_2 = e^{x+i\theta}, \qquad z_1 = e^{i2x}(e^{it}(1 - \phi(2x)) - 1)$$

with $2|x| < r + \delta$ and $|t| < \delta$ for some small $\delta > 0$. In these coordinates $\mathcal{A} = \{t = 0,\ |x| \le r/2\}$. Setting

$$\gamma(x,t) = 2\big[e^{-it} - 1 + \phi(2x) - i\phi'(2x)\big]/\big[1 - \phi(2x)\big],$$

the vector field $\bar L = \partial_x + i\partial_\theta + \gamma\partial_t$ annihilates both z_1 and z_2. Hence it differs from what was previously denoted as $\bar L$ by multiplication on the left by a nonvanishing factor, which may be verified to have the form $b(x,t)\exp(i\theta)$. For $|x| \le r/2$, $\phi(x) \equiv 0$ and consequently $\gamma(x,t) \equiv 2(e^{-it} - 1)$. Therefore[5]

$$\bar{L} = \partial_x + i\partial_\theta + it\alpha(t)\partial_t \qquad \text{where } |x| \le r/2, \text{ with } \operatorname{Re}\alpha(0) \neq 0. \tag{2.2}$$

The representation $\mathfrak{L} = \bar{L}L + B_1\bar{L} + B_2L + B_3$ in terms of the new $\bar{L}$ remains valid with modified coefficients B_i that now preserve each $\mathcal{H}_j$, once the operator formerly denoted by $\mathfrak{L}$ is multiplied by $|b|^{-2}$.

There exists a unique C^∞ real-valued function μ, independent of θ, such that $[\bar{L}, L] = i\mu(x,t)\partial_t$ modulo the span of the real and imaginary parts of $\bar{L}$. More precisely, since the coefficients of ∂_θ in $[\bar{L}, L]$ and in $\operatorname{Re}\bar{L}$ vanish while the coefficient in $\operatorname{Im}\bar{L}$ is nowhere zero,

$$[\bar{L}, L] = i\mu(x,t)\partial_t + i\nu(x,t)\operatorname{Re}\bar{L} \tag{2.3}$$

for unique real-valued, C^∞ coefficients μ, ν. The pseudoconvexity of $\partial\mathcal{W}$ means that μ does not change sign. Replacing t by $-t$ if necessary, we may assume that $\mu \ge 0$.

Fix an integer k. We identify functions of $(x,t) \in \mathbf{R}^2$ with elements of $\mathcal{H}_k$ via the correspondence $u(x,t) \mapsto u(x,t)e^{ik\theta}$; ∂_θ then becomes multiplication by ik. For the remainder of the paper we work in $\mathbf{R}^2$. Define

$$\mathcal{D} = \{(x,t) : |x| \le r/2 \text{ and } t = 0\}.$$

By incorporating ik into the B_j we may further rewrite $\mathfrak{L}$, when restricted to $\mathcal{H}_k$, as

$$\mathcal{L}_0 = \bar{\ell}\ell + B_1\bar{\ell} + B_2\ell + B_3$$

where $\bar{\ell}$ is a complex vector field in $\mathbf{R}^2$ which, for $|x| \le r/2$, takes the form

$$\bar{\ell} = \partial_x + it\alpha(t)\partial_t.$$

Here ℓ denotes the conjugate of $\bar{\ell}$, each B_j is a pseudodifferential operator of order ≤ 0 in $\mathbf{R}^2$, and $\operatorname{Re}\alpha(0) \neq 0$. Note that the commutator of the real and imaginary parts of $\bar{\ell}$ is not forced to vanish identically, because α is not real-valued. We have $[\bar{\ell}, \ell] = i\mu(x,t)\partial_t + i\nu(x,t)\operatorname{Re}\bar{\ell}$ with the same coefficients as in (2.3).

Letting (ξ, τ) be Fourier variables dual to (x,t), define $\tilde{\Gamma} = \{(x,t,\xi,\tau) : (x,t) \in \mathcal{D} \text{ and } \xi = 0\}$. $\mathcal{L}_0$ is not elliptic at points of $\mathcal{D}$, but is elliptic at most points in its complement; $\tilde{\Gamma}$ is the intersection of the characteristic variety of $\mathcal{L}_0$ with $\{(x,t,\xi,\tau) : (x,t) \in \mathcal{D}\}$. Decompose $\tilde{\Gamma} = \tilde{\Gamma}^+ \cup \tilde{\Gamma}^-$ where $\tilde{\Gamma}^+ = \tilde{\Gamma} \cap \{\tau > 0\}$. Thus the principal symbol of $i\mu\partial_t$, namely $-\mu\tau$, is nonpositive in a conic neighborhood of $\tilde{\Gamma}^+$.

[5] Boas and Straube [BS2] have shown the $\bar{\partial}$–Neumann problem to be globally C^∞ hypoelliptic whenever there exists a real vector field on the boundary that is transverse to the complex tangent space and has a certain favorable commutation property. If $\operatorname{Re}\alpha(0)$ were to vanish then ∂_t would be such a vector field. Thus nonvanishing of $\operatorname{Re}\alpha(0)$ is for our purpose an essential feature of worm domains.

3. Two Pseudodifferential Manipulations.

By an operator we will always mean a classical pseudodifferential operator, that is, one whose symbol admits a full asymptotic expansion in homogeneous terms of integral degrees. $\sigma_j(T)$ denotes the j-th order symbol of T (in the Kohn-Nirenberg calculus), always with respect to the fixed coordinate system (x, t, ξ, τ). Henceforth we work under the convention that A, B, E denote operators whose orders are less than or equal to $0, 0, -1$ respectively, whose meanings are permitted to change freely from one occurrence to the next, even within the same line. $\mathcal{A}$ denotes always an operator having the additional property that $\sigma_0(\mathcal{A})(x, t, \xi, \tau) \equiv 0$ for all $(x, t) \in \mathcal{D}$. Any operator of type E may be regarded as one of type $\mathcal{A}$. Two operators are said to agree microlocally in some conic open set if the full symbol of their difference vanishes identically there.

Any operator B may be written in the form $B = \beta(x) + E \circ \ell + \mathcal{A}$ microlocally in a conic neighborhood of $\tilde{\Gamma}^+$, where β denotes both a C^∞ function and the operator defined by multiplication by that function. Indeed, $\sigma_0(B)(x, 0, 0, \tau)$ depends only on $(x, \mathrm{sgn}(\tau))$ and we define $\beta(x)$ to be this quantity for $\tau > 0$. Then where $\tau > 0$ and $|x| \le r/2$, $\sigma_0(B)(x, 0, \xi, \tau)$ is divisible by $\xi = -i\sigma_1(\ell)(x, 0, \xi, \tau)$. $\sigma_{-1}(E)(x, 0, \xi, \tau)$ is then uniquely determined for such x, τ by the equation $\sigma_0(B) = \beta(x) + \sigma_1(\ell) \cdot \sigma_{-1}(E)$. Define E to be any operator of order -1 whose principal symbol satisfies this equation when restricted to $(x, t) \in \mathcal{D}$ and to a conic neighborhood of $\tilde{\Gamma}^+$. Then simply define $\mathcal{A} = B - \beta - E \circ \ell$.

Writing $B_1 = \beta_1(x) + E_1 \circ \ell + \mathcal{A}_1$ and similarly $B_2 = \beta_2(x) + E_2 \circ \bar{\ell} + \mathcal{A}_2$, and expressing $E\ell\bar{\ell} = E\bar{\ell}\ell$ plus an operator of order ≤ 0, we obtain $\mathcal{L}_0 = (I+E)\bar{\ell}\ell + (\beta_1+\mathcal{A})\bar{\ell} + (\beta_2+\mathcal{A})\ell + B$. Composing both sides with a parametrix for $I + E$ and modifying the definition of $\mathcal{L}_0$ to include this factor, we have $\mathcal{L}_0 = \bar{\ell}\ell + (\beta_1 + \mathcal{A})\bar{\ell} + (\beta_2 + \mathcal{A})\ell + B$. Writing finally $B = \beta_3(x) + E \circ \ell + \mathcal{A}$ results in

$$\mathcal{L}_0 = \bar{\ell}\ell + (\beta_1 + \mathcal{A})\bar{\ell} + (\beta_2 + \mathcal{A})\ell + (\beta_3 + \mathcal{A}) \tag{3.1}$$

where the β_j are C^∞ functions depending only on x. $\square$

We next reduce the question of *a priori* H^s inequalities to L^2, simultaneously for all s. Fix an operator Q of order 0 that is elliptic in some conic neighborhood of $\tilde{\Gamma}^-$, whose symbol vanishes identically in some conic neighborhood of $\tilde{\Gamma}^+$. Fix an exponent $s > 0$, for which we seek an *a priori* inequality for all $v \in C^\infty$ of the form

$$\|v\|_{H^s} \le C\|\mathcal{L}_0 v\|_{H^s} + C\|v\|_{H^{s'}} + C\|Qv\|_{H^{s+2}} \tag{3.2}$$

for some $s' < s$. Having such an inequality for a sequence of exponents s tending to $+\infty$ would imply Proposition 1 by the preceding discussion. In particular, the H^{s+2} norm of Qv is already under control by virtue of (1.1), while the H^0 norm of v is harmless because the Neumann operator is bounded on L^2, and $\|v\|_{H^{s'}} \le \varepsilon\|v\|_{H^s} + C_{\varepsilon,N}\|v\|_{H^{-N}}$ for any $\varepsilon > 0$ and $N < \infty$.

Fix a C^∞, strictly positive function $m = m(\xi, \tau)$, homogeneous of degree 1 for large $|(\xi, \tau)|$ and identically equal to $(1 + \tau^2)^{1/2}$ in a conic neighborhood of $\{\xi = 0\}$. Define Λ^s to be the Fourier multiplier operator on $\mathbf{R}^2$ with symbol $m(\xi, \tau)^s$.

Substituting $v = \Lambda^{-s}u$ and $g = \Lambda^{-s}f$, estimation of the H^s norm of v, modulo a lower order norm, in terms of that of g is equivalent to estimation of the H^0 norm of u in terms of that of f, modulo a negative order norm of u. The equation $\mathcal{L}_0 v = g$ becomes $\Lambda^{-s}\mathcal{L}_0\Lambda^s u = f$. Write $\Lambda^{-s}\bar{\ell}\ell\Lambda^s = \Lambda^{-s}\bar{\ell}\Lambda^s \circ \Lambda^{-s}\ell\Lambda^s$ and similarly for other terms, and note that $\Lambda^{-s}(\beta_j + A)\Lambda^s = \beta_j + A$ with the same function β_j.

$\Lambda^{\pm s}$ commutes with ∂_x and with ∂_t, so $\Lambda^{-s}(\partial_x + i\alpha t\partial_t)\Lambda^s = \partial_x + \Lambda^{-s}t\Lambda^s \circ \Lambda^{-s}\alpha\Lambda^s\partial_t$. Applying the Fourier transform gives immediately $\Lambda^{-s}[t,\Lambda^s]\partial_t = -s + E$, microlocally in some conic neighborhood of $\tilde{\Gamma}^+$, modulo operators smoothing there of infinite order. Since $\Lambda^{-s}[\alpha,\Lambda^s]\partial_t$ is of order 0,

$$\begin{aligned}\Lambda^{-s}t\Lambda^s \circ \Lambda^{-s}\alpha\Lambda^s\partial_t &= \alpha t\partial_t + \Lambda^{-s}[t,\Lambda^s]\alpha\partial_t + t \circ \Lambda^{-s}[\alpha,\Lambda^s]\circ\partial_t + E \\ &= \alpha \circ (t\partial_t - s) + tB + E,\end{aligned}$$

microlocally near $\tilde{\Gamma}^+$. Thus microlocally near $\tilde{\Gamma}^+$, $\Lambda^{-s}\mathcal{L}_0\Lambda^s = \mathcal{L}_s$ becomes

$$\mathcal{L}_s = \bar{\ell}_s\ell_s + (\beta_1 + A)\bar{\ell}_s + (\beta_2 + A)\ell_s + (\beta_3 + A)$$

where $\bar{\ell}_s, \ell_s$ are first-order differential operators differing from $\bar{\ell}, \ell$ respectively by terms of order zero, and taking the forms $\bar{\ell}_s = \partial_x + i\alpha(t\partial_t - s)$, $\ell_s = \partial_x - i\bar{\alpha}(t\partial_t - s)$ for $|x| \le r/2$, where α depends only on t and the β_i only on x.

To see that $\bar{\ell}_s$ does take the form claimed for $|x| \le r/2$, express $\bar{\ell} = \partial_x + i\alpha t\partial_t$ modulo terms $\gamma(x,t)\partial_x$ and $\gamma(x,t)\partial_t$ where $\gamma \equiv 0$ for $|x| \le r/2$. Then $\Lambda^{-s}[\gamma(x,t)\partial_x, \Lambda^s]$ and $\Lambda^{-s}[\gamma(x,t)\partial_t, \Lambda^s]$ are operators of the type A, since they have nonpositive orders and their symbols of order zero vanish identically for $|x| < r/2$. $\square$

4. A Two-Dimensional Problem And Preliminary Inequalities.

The remainder of the paper consists of a self-contained analysis of a special class of pseudodifferential equations in a real two-dimensional region. We begin by describing the equations in question and fixing notation, which in some respects differs from that of preceding sections.

Fix an interval $I = [-r, r] \subset \mathbf{R}$. Denote by $(x,t) \in \mathbf{R}^2$ coordinates in a neighborhood U of $\mathcal{D} = I \times \{0\}$. The interval $\mathcal{D}$ corresponds to the degenerate annulus embedded in the boundary of the worm domain, and will be the focus of attention. The convention concerning the symbols A, B, E introduced at the outset of §3 remains in force.

Consider a one parameter family of pseudodifferential operators of the form

$$\mathcal{L}_s = \bar{L}L + (\beta_1(x) + A)\bar{L} + (\beta_2(x) + A)L + (\beta_3(x) + A) \tag{4.1}$$

where the β_j are C^∞ functions. Suppose that $\bar{L}$, L are first-order differential operators depending on the real parameter s, and that $-L$ is the formal adjoint of $\bar{L}$, modulo an operator of order zero. Suppose that where $|x| \le r$, they take the special forms[6]

$$\bar{L} = \partial_x + ia(x)(t\partial_t + s) + O(t^2)\partial_t$$
$$L = \partial_x - i\bar{a}(x)(t\partial_t + s) + O(t^2)\partial_t.$$

Here $O(t^2)$ denotes multiplication by a smooth function divisible by t^2 on the region U. a and the coefficients β_j are assumed independent of s, but A and the terms $O(t^2)\partial_t$ are permitted to depend on s.

Assume that

$$\operatorname{Re} a(x) \neq 0 \qquad \text{for all } x \in I, \tag{4.2}$$

and that there exist smooth real-valued coefficients μ, ν such that $[\bar{L}, L] = i\mu(x,t)\partial_t + i\nu(x,t)\operatorname{Re}\bar{L}$, satisfying

$$\mu \ge 0 \text{ at every point of } U. \tag{4.3}$$

Because $\bar{L}^* = -L$ modulo a term of order zero, L has the same real part as $\bar{L}$. A change of variables of the form $(x,t) \mapsto (x, h(x,t))$, with $h(x,0) \equiv 0$ where $|x| \le r$, therefore reduces matters to the case where the real parts of both $\bar{L}$ and L are everywhere parallel to ∂_x, and $\bar{L} = \partial_x + i\tilde{a}(x,t)(t\partial_t + s) + O(t^2)\partial_t$ on $I \times \mathbf{R}$, with $\tilde{a}$ real-valued and nonvanishing. Rewrite $\tilde{a}(x,t) = a(x) + O(t)$, and incorporate the contribution of $O(t)$ into the various terms $O(t^2)\partial_t$ and A. (4.3) is invariant under diffeomorphism and hence μ cannot change sign, so the coefficient of t in the Taylor expansion of $\mu(x,t)$ about $t = 0$ must vanish identically, for $|x| \le r$. This forces $\partial_x a(x) \equiv 0$ there. Thus

$$\bar{L} = \partial_x + ia(t\partial_t + s) + O(t^2)\partial_t \tag{4.4}$$

for $|x| \le r$, where a is a nonzero real constant. Moreover, ∂_x may be expressed in U as a nonvanishing scalar multiple of $\bar{L} + L$, modulo an operator of order zero. From now on we work in these new coordinates.

Define $\Gamma = \{(x,t,\xi,\tau) : (x,t) \in \mathcal{D} \text{ and } \xi = 0\}$. Decompose $\Gamma = \Gamma^+ \cup \Gamma^-$ where $\Gamma^+ = \{\tau > 0\} \cap \Gamma$. Then by (4.3), in some conic neighborhood of Γ^+ the principal symbol of $[\bar{L}, L]$ equals a nonpositive symbol, modulo terms in the span of the symbols of $\bar{L}, L$ and a term of order zero.

The symbol $\|\cdot\|$, with no subscript, denotes the norm in $L^2(U)$, while $\|\cdot\|_t$ denotes any fixed norm for the Sobolev space H^t of functions having t derivatives in L^2 and supported in U. The goal of the remainder of the paper is the following *a priori* estimate.

[6] No assumption is now made on the vanishing or nonvanishing of the coefficient of ∂_t in $\bar{L}$ where $|x| > r$, but the strict pseudoconvexity of $\mathcal{W}$ outside the exceptional annulus was used to reduce Proposition 1 to Proposition 2 below.

Proposition 2. *Let $\{\mathcal{L}_s\}$ be a family of operators of the form (4.1) satisfying all of the hypotheses introduced above. Then there exist a discrete exceptional set $S \subset [0, \infty)$ such that for any $s \notin S$ and any pseudodifferential operator Q of order 0 whose principal symbol is nonzero in some conic neighborhood of Γ^-, there exist $C < \infty$ and a neighborhood W of $\mathcal{D}$ such that for every C^∞ function u supported in W,*

$$\|u\| + \|\bar{L}u\| + \|Lu\| \leq C \cdot (\|\mathcal{L}_s u\| + \|u\|_{-1} + \|Qu\|_1) .$$

The key conclusions are that there is no loss of derivatives in estimating u in terms of $\mathcal{L}_s u$, and that this holds for a sequence of values of s tending to $+\infty$. The assumption that $u \in C^\infty$ is essential. All hypotheses of §4 are satisfied by the family of operators $\mathcal{L}_s$ derived in §2 and §3. Proposition 2 thus implies the validity of (3.2), and hence of Proposition 1, which in turn implies our Theorem.

Our first preliminary estimate is a standard one valid for all $s \in \mathbf{R}$.

Lemma 1. *For each exponent s and each Q there exists $C < \infty$ such that*

$$\|\partial_x u\| \leq C\|u\| + C\|\mathcal{L}_s u\| + C\|Qu\|_1$$

for every $u \in C_0^\infty(U)$.

Proof. For $(x, t) \in \mathcal{D}$, $\sigma_2(\mathcal{L}_s)(x, t, \xi, \tau) = 0$ if and only if $(x, t, \xi, \tau) \in \Gamma$. Therefore the characteristic variety of $\mathcal{L}_s$ in T^*W is contained in an arbitrarily small conic neighborhood of Γ as $\delta \to 0$. Consequently there exists an operator $\tilde{Q}$ of order zero such that firstly, T^*W is contained in the union of the two regions where $\tilde{Q}$ is elliptic and $\tau > 0$, and secondly, the symbol of $\tilde{Q}$ is supported in the union of the two regions where $\mathcal{L}_s$ is elliptic, and where Q is elliptic.

Since $\mathcal{L}_s$ is elliptic outside a small conic neighborhood of Γ, the H^2 norm of u is majorized away from Γ by $\|\mathcal{L}_s u\| + \|u\|_{-1}$, while in a conic neighborhood of Γ^- the H^1 norm of u is majorized by $\|Qu\|_1 + \|u\|_{-1}$.

Write $\langle f,\, g\rangle = \int_U f\bar{g}\,dx\,dt$. By Gårding's inequality and the fact that $i\mu \cdot i\tau \leq 0$ in the support of the symbol of $\tilde{Q}$,

$$-\operatorname{Re}\left(\langle \bar{L}Lu,\, u\rangle\right) \geq c\|Lu\|^2 + c\|\bar{L}u\|^2 - C\|\bar{L}u\| \cdot \|u\| - C\|Lu\| \cdot \|u\| - C\|u\|^2 - C\|\tilde{Q}u\|_1^2.$$

The second condition imposed on $\tilde{Q}$ ensures that

$$\|\tilde{Q}u\|_1 \leq C\|\mathcal{L}_s u\| + C\|Qu\|_1 + C\|u\|_{-1}.$$

Estimating $\langle (\mathcal{L}_s - \bar{L}L)u,\, u\rangle$ by Cauchy-Schwarz thus leads to

$$\|\bar{L}u\| + \|Lu\| \leq C\|\mathcal{L}_s u\| + C\|u\| + C\|Qu\|_1.$$

But ∂_x may be expressed as a linear combination of $\bar{L}$ and of L modulo an operator of order 0. $\square$

Lemma 2. *There exists $C < \infty$ such that for every $f \in C^1(\mathbf{R})$ and every $\varepsilon > 0$,*

$$\|f\|_{L^2[\varepsilon,2\varepsilon]} \le C\|f\|_{L^2[-2\varepsilon,-\varepsilon]} + C\varepsilon\|\partial_x f\|_{L^2(\mathbf{R})}.$$

Likewise

$$|f(0) - f(-\varepsilon)| \le C\varepsilon^{1/2}\|\partial_x f\|_{L^2}.$$

The conclusions are invariant under translation, and the lemma will be invoked in that more general form.

Proof. For each $x \in (\varepsilon, 2\varepsilon)$, $|f(x) - f(x-3\varepsilon)| \le \int_{-2\varepsilon}^{2\varepsilon} |\partial_x f(y)|\, dy$ and both conclusions follow from the triangle and Cauchy-Schwarz inequalities. □

To simplify notation define

$$\mathfrak{B} = \|\mathcal{L}_s u\| + \|u\|_{-1} + \|Qu\|_1.$$

Let $\delta > 0$ be a small constant to be chosen in §6, and assume u to be supported in

$$W \subset \{(x,t) : |t| < \delta,\ |x| < r + \delta\}.$$

Applying Lemma 2 to the function $x \mapsto u(x,t)$ for each t and applying Lemma 1 gives the following estimate, under the hypotheses of Lemma 1.

Lemma 3.

$$\|\partial_x u\| + \|u\| \le C\|u\|_{L^2(I\times(-\delta,\delta))} + C\mathfrak{B}.$$

5. Limiting Operators and Mellin Transform.

Let a be a nonvanishing C^∞, real-valued function. For $\zeta \in \mathbf{C}$ define the ordinary differential operator

$$H_\zeta = (\partial_x + i\zeta a(x))(\partial_x - i\zeta a(x)) + \beta_1(x)(\partial_x + i\zeta a(x)) + \beta_2(x)(\partial_x - i\zeta a(x)) + \beta_3(x),$$

acting on functions of $x \in I$. Only the case of constant a will be needed in this paper, but the general case arises in another problem and hence merits discussion.

Definition 1. *$\mathfrak{S}$ is defined to be the set of all $\zeta \in \mathbf{C}$ such that there exists a solution g of $H_\zeta g \equiv 0$ on I, satisfying $g(-r) = g(r) = 0$.*

For any complex number w we write $\langle w\rangle = (1 + |w|^2)^{1/2}$.

Lemma 4. *$\mathfrak{S}$ is a discrete subset of $\mathbf{C}$, and for any compact subset K of $[0,\infty)$, the set of all $\zeta \in \mathfrak{S}$ having real part in K is finite. For each s such that $\mathfrak{S}\cap(s+i\mathbf{R}) = \emptyset$ there exists $C < \infty$ such that for all $\zeta \in s+i\mathbf{R}$, for all $f,\varphi,\psi \in C^\infty(I)$ satisfying $H_\zeta f = \varphi + \partial_x\psi$, one has*

$$\begin{aligned} \|f\|_{L^2(I)} &+ \langle\zeta\rangle^{-1}\|\partial_x f\|_{L^2(I)} \\ &\le C\langle\zeta\rangle^{-1/2}\big(|f(-r)|+|f(r)|\big) + C\langle\zeta\rangle^{-2}\|\varphi\|_{L^2(I)} + C\langle\zeta\rangle^{-1}\|\psi\|_{L^2(I)}. \end{aligned} \tag{5.1}$$

Proof. Throughout this proof, all norms without subscripts denote L^2 norms. The selfadjoint part of H_ζ, applied to f, equals $(\partial_x-\gamma a(x))(\partial_x+\gamma a(x))f$, modulo $O(\langle\gamma\rangle\|f\|+\|\partial_x f\|)$, in the $L^2(I)$ norm. Therefore for s in any fixed compact subset of $\mathbf{R}$ and any $\zeta = s+i\gamma \in s+i\mathbf{R}$, for any f vanishing at both endpoints of I,

$$-\operatorname{Re}\langle H_\zeta f,\, f\rangle \ge \|\partial_x f\|^2 + \gamma^2\int_I |f|^2|a|^2 - O(\langle\gamma\rangle\|f\|^2 + \|f\|\cdot\|\partial_x f\|).$$

The coefficient a vanishes nowhere, while

$$|\langle H_\zeta f,\, f\rangle| = |\langle \varphi+\partial_x\psi,\, f\rangle| \le \|f\|\cdot\|\varphi\| + \|\partial_x f\|\cdot\|\psi\|.$$

Combining the last two inequalities and invoking the Cauchy-Schwarz inequality and small constant – large constant trick, one obtains

$$\gamma^2\|f\| + |\gamma|\cdot\|\partial_x f\| \le C\|\varphi\| + C|\gamma|\cdot\|\psi\| \tag{5.2}$$

for all sufficiently large $|\gamma|$, under the additional hypothesis that f vanishes at both endpoints of I.

There exists a unique solution ϕ_ζ of $H_\zeta\phi_\zeta = 0$ on I, satisfying $\phi_\zeta(-r) = 0$, $\partial_x\phi_\zeta(-r) = 1$. Then $\phi_\zeta(r)$ is an entire holomorphic function of ζ, and $\zeta \in \mathfrak{S} \Leftrightarrow \phi_\zeta(r) = 0$. We have seen that $\zeta \notin \mathfrak{S}$ provided that the imaginary part of ζ is sufficiently large, when the real part stays in a bounded set. Thus $\phi_\zeta(r)$ is nonconstant, so has discrete zeros.[7]

To prove (5.1) let $\zeta = s+i\gamma$ and $f \in C^2(I)$ be given, and decompose $f = g+h$ where $H_\zeta g \equiv 0$ and h vanishes at the endpoints of I. The hypothesis $\mathfrak{S}\cap(s+i\mathbf{R}) = \emptyset$ means that the Dirichlet nullspace of H_ζ is $\{0\}$, so by elementary reasoning we conclude that for each γ there exists $C < \infty$ such that $\|h\| + \|\partial_x h\| \le C\|\varphi\| + \|\psi\|$, since $H_\zeta h = H_\zeta f = \varphi + \partial_x\psi$. Moreover, since H_ζ depends continuously on ζ, C may be taken to be independent of ζ in any compact subset of $\mathbf{C}\backslash\mathfrak{S}$. When $|\gamma|$ is sufficiently large, on the other hand, (5.2) implies $\|h\| + \langle\zeta\rangle^{-1}\|\partial_x h\| \le C\langle\zeta\rangle^{-2}\|\varphi\| + C\langle\zeta\rangle^{-1}\|\psi\|$. Thus the component h of f satisfies (5.1).

[7] An alternative method of proof would be to combine (5.2) with general results from the perturbation theory of linear operators [Ka], utilizing again the holomorphic dependence of H_ζ on ζ.

We have $H_\zeta g = 0$, so that clearly $\|g\|$ and $\|\partial_x g\|$ are majorized by $C|f(r)| + C|f(-r)|$, uniformly for ζ in any compact set disjoint from $\mathfrak{S}$. Assuming henceforth that $|\gamma|$ is large, the equation gives the inequality

$$\|\partial_x^2 g\| \le C\gamma^2 \|g\| + C|\gamma| \cdot \|\partial_x g\|.$$

Integrating by parts as in the proof of (5.2) yields

$$\|\partial_x g\|^2 + \gamma^2 \|g\|^2 \le C|g(-r)\partial_x g(-r)| + C|g(r)\partial_x g(r)|. \tag{5.3}$$

To control the right hand side we use the bound

$$|\partial_x g(-r)| + |\partial_x g(r)| \le C|\gamma|^{1/2} \|\partial_x g\| + C|\gamma|^{-1/2} \|\partial_x^2 g\|.$$

Indeed, setting $v = \partial_x g$, for any $r' \in [r - |\gamma|^{-1}, r]$

$$|v(r) - v(r')| \le C \int_{r'}^{r} |\partial_x v| \le C|\gamma|^{-1/2} \|\partial_x v\|_{L^2}.$$

Then

$$|v(r)| \le |\gamma| \int_{r-|\gamma|^{-1}}^{r} |v(r) - v(r')| \, dr' + |\gamma| \int_{r-|\gamma|^{-1}}^{r} |v(r')| \, dr'$$

and the desired bound follows by Cauchy-Schwarz.

Putting this into (5.3), introducing a parameter $\lambda \in \mathbf{R}^+$ and applying Cauchy-Schwarz yields

$$\begin{aligned}
&\gamma^2\|g\|^2 + \|\partial_x g\|^2 \\
&\qquad \le C\lambda|g(-r)|^2 + C\lambda|g(r)|^2 + C\lambda^{-1}|\gamma| \cdot \|\partial_x g\|^2 + C\lambda^{-1}|\gamma|^{-1}\|\partial_x^2 g\|^2 \\
&\qquad \le C\lambda|g(-r)|^2 + C\lambda|g(r)|^2 + C\lambda^{-1}|\gamma| \cdot \|\partial_x g\|^2 + C\lambda^{-1}|\gamma|^{-1}\left(\gamma^4\|g\|^2 + \gamma^2\|\partial_x g\|^2\right) \\
&\qquad \le C\lambda|g(-r)|^2 + C\lambda|g(r)|^2 + C\lambda^{-1}|\gamma| \cdot \|\partial_x g\|^2 + C\lambda^{-1}|\gamma|^3\|g\|^2.
\end{aligned}$$

Choose λ to be a large constant times $|\gamma|$. Then the last two terms on the right-hand side may be absorbed into the left, leaving

$$\gamma^2\|g\|^2 + \|\partial_x g\|^2 \le C|\gamma| \cdot |g(-r)|^2 + C|\gamma| \cdot |g(r)|^2.$$

Since $g = f$ at the endpoints of I, this is the desired inequality for g. Adding it to that for h concludes the proof. $\square$

Definition 2.

$$S = \{s \in [0,\infty) : \textit{ there exists } \gamma \in \mathbf{R} \textit{ such that } s - \tfrac{1}{2} + i\gamma \in \mathfrak{S}\}.$$

Lemma 4 guarantees that S is discrete.

Specialize now to the case where $a(x) \equiv a$, the real constant in (4.4). Define

$$\begin{aligned}\mathbb{L}_s =&(\partial_x + ia(t\partial_t + s)) \circ (\partial_x - ia(t\partial_t + s)) \\ &\quad + \beta_1(x)(\partial_x + ia(t\partial_t + s)) + \beta_2(x)(\partial_x - ia(t\partial_t + s)) + \beta_3(x).\end{aligned}$$

Expanding the last term in the expression $\mathbb{L}_s = \mathcal{L}_s + (\mathbb{L}_s - \mathcal{L}_s)$ gives

$$\mathbb{L}_s u = \Phi + \partial_x \Psi$$

where

$$\begin{aligned}\Phi &= \mathcal{L}_s u + (t\partial_t)^2 Au + t\partial_t Au + Au \\ \Psi &= t\partial_t Au + Au\end{aligned} \tag{5.4}$$

To reach (5.4) we may for instance express $t\partial_t \circ O(t^2)\partial_t$ as $(t\partial_t)^2 A + t\partial_t A + A$, since multiplication by t is an operator of the type A. Likewise $[A, t\partial_t] = t[A,\partial_t] + [A,t]\partial_t$ is an operator of type A, because $\sigma_{-1}([A,t]) = c\partial_\tau \sigma_0(A)$ vanishes identically for $(x,t) \in \mathcal{D}$ since $\sigma_0(A)$ itself vanishes there.

The partial Mellin transform of f with respect to the t variable is defined to be

$$\hat{f}(x,\gamma) = \int_0^\infty f(x,t) t^{-i\gamma}\, t^{-1} dt,$$

provided that the integral converges. If $f(x,\cdot) \in C^\infty[0,\infty)$ has bounded support for each x, then the integral defining $\hat{f}(x,\gamma)$ converges absolutely whenever γ has strictly positive imaginary part, and $\hat{f}(x,\gamma)$ extends to a meromorphic function of $\gamma \in \mathbf{C}$, whose only possible poles are at $\gamma = 0, -i, -2i, \dots$. Clearly

$$(t\partial_t f)\widehat{\ }(x,\gamma) = i\gamma \hat{f}(x,\gamma)$$

for all such f. Consequently

$$(\mathbb{L}_s u)\widehat{\ }(x,\gamma) = H_{s+i\gamma}\hat{u}(x,\gamma) \qquad \text{for all } \gamma \in \mathbf{C}\backslash\{0,-i,-2i,\dots\}.$$

The Mellin inversion and Plancherel formulas read

$$f(x,t) = c\int_{\mathbf{R}} \hat{f}(x,\gamma)\, t^{i\gamma}\, d\gamma, \qquad \int_0^\infty |f(x,t)|^2\, t^{-1} dt = c' \int_{\mathbf{R}} |\hat{f}(x,\gamma)|^2\, d\gamma.$$

It follows directly from the definitions that $(t^{1/2} f)\widehat{\ }(x,\gamma) = \hat{f}(x,\gamma + \frac{i}{2})$ for all $\gamma \in \mathbf{R}$. Thus the Plancherel identity may be rewritten as

$$\int_0^\infty |f(x,t)|^2\, dt = \int_0^\infty |t^{1/2} f(x,t)|^2\, t^{-1} dt = c' \int_{\mathbf{R}} |\hat{f}(x,\gamma + \tfrac{i}{2})|^2\, d\gamma.$$

6. Proof of the Main Estimate.

We may now estimate u in terms of $\mathcal{L}_s u$. To begin,

$$\iint_{I\times[0,\delta)} |u(x,t)|^2\,dx\,dt = c' \iint_{I\times\mathbf{R}} |\hat{u}(x,\gamma+\tfrac{i}{2})|^2\,d\gamma\,dx.$$

Assume that $s\notin S$, and write $\zeta = s-\frac{1}{2}+i\gamma$. With Φ,Ψ defined as in (5.4), $H_\zeta\hat{u}(x,\gamma+\frac{i}{2}) = \hat{\Phi}(x,\gamma+\frac{i}{2}) + \partial_x\hat{\Psi}(x,\gamma+\frac{i}{2})$. Applying Lemma 4 on I yields for each $\gamma\in\mathbf{R}$

$$\begin{aligned}\int_I \hat{u}(x,\gamma+\tfrac{i}{2})\,dx \le C\int_I |\hat{\Phi}(x,\gamma+\tfrac{i}{2})|^2\langle\gamma\rangle^{-4}\,dx + C\int_I |\hat{\Psi}(x,\gamma+\tfrac{i}{2})|^2\langle\gamma\rangle^{-2}\,dx \\ + C|\hat{u}(-r,\gamma+\tfrac{i}{2})|^2 + C|\hat{u}(r,\gamma+\tfrac{i}{2})|^2.\end{aligned} \tag{6.1}$$

Lemma 5. *Assume $W\subset\{(x,t): |t|<\delta,\ |x|<r+\delta\}$. Then there exists $C<\infty$ such that*

$$\iint_{I\times\mathbf{R}} |\hat{\Phi}(x,\gamma+\tfrac{i}{2})|^2\langle\gamma\rangle^{-4}\,d\gamma\,dx \le C\delta^2\|u\|^2 + C\mathfrak{B}^2$$

and

$$\iint_{I\times\mathbf{R}} |\hat{\Psi}(x,\gamma+\tfrac{i}{2})|^2\langle\gamma\rangle^{-2}\,d\gamma\,dx \le C\delta^2\|\partial_x u\|^2 + C\mathfrak{B}^2.$$

Granting the lemma, we conclude from (6.1) that

$$\|u\|^2_{L^2(I\times[0,\delta))} \le C\delta^2\|u\|^2 + C\delta^2\|\partial_x u\|^2 + C\mathfrak{B}^2 + C\int_{\mathbf{R}}|u(r,t)|^2\,dt + C\int_{\mathbf{R}}|u(-r,t)|^2\,dt.$$

But by Lemma 2 and the assumption that $u(x,t)\equiv 0$ for $|x|>r+\delta$, this last term is dominated by $C\delta\|\partial_x u\|^2$. Thus

$$\|u\|^2_{L^2(I\times[0,\delta))} \le C\delta^2\|u\|^2 + C\delta\|\partial_x u\|^2 + C\mathfrak{B}^2.$$

All the same reasoning applies on the region $I\times(-\delta,0]$, after the change of variables $t\mapsto -t$. Thus

$$\|u\|_{L^2(I\times(-\delta,\delta))} \le C\delta^{1/2}(\|u\|+\|\partial_x u\|) + C\mathfrak{B}.$$

Combining this with Lemma 3 gives

$$\|u\| + \|\partial_x u\| \le C\delta^{1/2}(\|u\|+\|\partial_x u\|) + C\mathfrak{B},$$

so choosing δ to be sufficiently small gives $\|u\|\le C\mathfrak{B}$, concluding the proof. $\square$

Proof of Lemma 5. The principal term in the double integral of the lemma for Φ is of course the contribution of $\mathcal{L}_s u$:

$$\iint_{I\times\mathbf{R}} \left|(\mathcal{L}_s u)\hat{}(x,\gamma+\tfrac{i}{2})\right|^2 \langle\gamma\rangle^{-4}\,d\gamma\,dx$$
$$\leq \iint_{I\times\mathbf{R}} \left|(\mathcal{L}_s u)\hat{}(x,\gamma+\tfrac{i}{2})\right|^2 d\gamma\,dx \quad \leq \quad C\|\mathcal{L}_s u\|^2,$$

as desired.

Any operator A of order ≤ 0 satisfying $\sigma_0(A)(x,t,\xi,\tau)\equiv 0$ for $(x,t)\in\mathcal{D}$ satisfies

$$\|Au\|^2 \leq C\delta^2\|u\|^2 + C\|u\|_{-1}^2$$

for all u supported in W, as $\delta\to 0$. A typical term of Φ resulting from $\mathbb{L}_s u - \mathcal{L}_s u$ is $(t\partial_t)^2 Au$. Its contribution to the first double integral in Lemma 5 is

$$\begin{aligned}\iint_{I\times\mathbf{R}} \left|((t\partial_t)^2 Au)\hat{}(x,\gamma+\tfrac{i}{2})\right|^2 \langle\gamma\rangle^{-4}\,d\gamma\,dx &\leq C\iint_{I\times\mathbf{R}} \left|(Au)\hat{}(x,\gamma+\tfrac{i}{2})\right|^2 d\gamma\,dx\\ &= C\iint_{I\times[0,\delta)} |Au(x,t)|^2\,dx\,dt\\ &\leq C\delta^2\|u\|^2 + C\|u\|_{-1}^2.\end{aligned}$$

A typical constituent of Ψ is the term $t\partial_t Au$. Its contribution is dominated by

$$C\iint_{I\times\mathbf{R}} \left|(t\partial_t Au)\hat{}(x,\gamma+\tfrac{i}{2})\right|^2 \langle\gamma\rangle^{-2}\,d\gamma\,dx \leq C\iint_{I\times\mathbf{R}} \left|(Au)\hat{}(x,\gamma+\tfrac{i}{2})\right|^2 d\gamma\,dx$$

and the remainder of the calculation is as above. □

COMMENTS. The author can advance no reason why the method of reduction to the boundary should be essential to this analysis. Working directly on $\overline{\mathcal{W}}$ might well result in a shorter proof. On the other hand, the analysis in §§4-6 applies with minor modification to a broader class of equations unconnected with the $\bar\partial$–Neumann problem.

Some refinements of this analysis and related observations will appear in [Ch2].

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