WKB ASYMPTOTICS OF GENERALIZED EIGENFUNCTIONS OF ONE-DIMENSIONAL SCHRÖDINGER OPERATORS

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ABSTRACT. We prove the WKB asymptotic behavior of solutions of the differential equation $-d^2u/dx^2 + V(x)u = k^2u$ in two cases. First, for a.e. k^2 when $V \in L^p(\mathbb{R})$, where $1 \le p < 2$. Second, for a.e. $k^2 > A$ when $V \in L^\infty(\mathbb{R})$ and $V' \in L^p(\mathbb{R})$, $1 \le p < 2$, where $A = \limsup_{x\to\infty} V(x)$. These results imply that Schrödinger operators with such potentials have absolutely continuous spectrum on $(0,\infty)$ ((A,∞) in the second case). We also establish WKB asymptotic behavior of solutions for some energy-dependent potentials.

1. INTRODUCTION

Let $D = d/dx$, where $x \in \mathbb{R}^1$. In this paper we establish criteria for the solutions of the ordinary differential equation

$$(-D^2 + V - k^2)u = 0 \tag{1.1}$$

to be globally bounded functions for almost every k, with respect to Lebesgue measure, and for the Schrödinger operator $-D^2 + V(x)$ to have absolutely continuous spectrum on the positive semi-axis. Our main interest here is in potentials V that are either decaying or have decaying derivative. We say that a function f belongs to the space $\ell^p(L^q)(\mathbb{R})$ if

$$\|f\|_{\ell^p(L^q)(\mathbb{R})} = \left(\sum_{n=-\infty}^{\infty}\left(\int_n^{n+1}|f(x)|^q\,dx\right)^{p/q}\right)^{1/p} < \infty.$$

Denote by $\tilde{u}_\pm(x,k)$ the unique solutions of (1.1) satisfying $\tilde{u}_\pm(0,k) = 1$ and $d\tilde{u}_\pm/dx\,(0,k) = \pm ik$. Our first main result is

Theorem 1.1. *Let $1 \le p < 2$, and let $V \in \ell^p(L^1)(\mathbb{R})$. Then for almost every energy k^2 with $k \in \mathbb{R}$, all solutions of $(-D^2+V-k^2)u = 0$ are bounded functions on the real line. For almost every k^2, there exist solutions with the WKB asymptotic behavior*

$$u_\pm(x,k) - e^{\pm ikx \mp \frac{i}{2k}\int_0^x V(y)\,dy} \to 0 \qquad \text{as } x \to +\infty. \tag{1.2}$$

Moreover

$$\int_a^b \log(1 + \sup_x|\tilde{u}_\pm(x,k)|)\,dk < \infty \tag{1.3}$$

for any $0 < a < b < \infty$.

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"Almost every" means with respect to Lebesgue measure. The same conclusion holds of course as $x \to -\infty$.

Remarks. 1. The usual Lebesgue space $L^r(\mathbb{R})$ is contained in $\ell^p(L^1)(\mathbb{R})$ for $1 \le r \le p$.
2. Implicit in our proofs is an extension of the estimate (1.3) to

$$\int_a^b \log(1 + \sup_x \|T(x,k)\|)\, dk < \infty,$$

where $T(x,k)$ is the 2×2 transfer matrix of the corresponding first order system (2.1). This matrix relates the vector $(w(x), w'(x))$ to $(w(0), w'(0))$, for an arbitrary solution $w(x)$ of (2.1).

Christ and Kiselev [3], and independently Remling [15], had earlier proved that if $V(x) = O(|x|^{-r})$ with $r > 1/2$ as $|x| \to \infty$, then almost every generalized eigenfunction is bounded. Moreover, the same conclusion holds [3] if there exists $\delta > 0$ such that $(1+|x|)^\delta V \in L^1 + L^2$. The motivation for those investigations was a corollary: every Schrödinger operator with such potential consequently has nonempty absolutely continuous spectrum, on any Borel set of positive Lebesgue measure in $(0, \infty)$. The result is rather sharp, in the sense that there exist potentials having purely singular spectrum, which are bounded by $C|x|^{-1/2}$ as $|x| \to \infty$.

Deift and Killip [6] subsequently derived the same conclusion regarding the spectrum under the still sharper hypothesis $V \in L^1 + L^2$, via a remarkable inequality stemming from inverse scattering theory. In our context, this inequality can be rephrased in a less precise form as

$$\sup_x \int_a^b \log(1 + |\tilde{u}_\pm(x,k)|)\, dk < \infty \ . \tag{1.4}$$

The bound (1.3) is interesting because it provides an improvement of the estimate (1.4) for the potentials we consider. Although the inequality (1.4) yields more precise information in one respect, it fails to imply boundedness of the generalized eigenfunctions or provide information on their asymptotic behavior. It still remains an open question whether almost every eigenfunction is bounded for every $V \in L^2$. This, as our analysis will make clear, would be a nonlinear analogue of Carleson's theorem [2] on the almost everywhere convergence of Fourier series.

Our second main result in this work is devoted to potentials with decaying derivative.

Theorem 1.2. *Let $V = V_1 + V_2$, where $V_1 \in L^1(\mathbb{R})$, while $dV_2/dx \in \ell^p(L^1)(\mathbb{R})$, with $1 \le p < 2$, and $V_2 \in L^\infty(\mathbb{R})$. Then for almost every energy $k^2 > A = \limsup_{|x|\to\infty} V_2(x)$, all solutions of $(-D^2 + V - k^2)u = 0$ are bounded functions on the real line. Moreover, for almost every $k^2 > A$, there exist solutions with the WKB asymptotic behavior*

$$u_\pm(x,k) - (k^2 - V_2(x))^{-1/4} e^{\pm i \int_0^x \sqrt{k^2 - V_2(y)}\, dy} \to 0 \qquad \textit{as } x \to +\infty. \tag{1.5}$$

Remark. One scenario where Theorem 1.2 applies is $V_2 \in L^r(\mathbb{R})$ for any $r < \infty$, $V_2' \in L^p$, $p < 2$. Then $V_2 \to 0$ as $|x| \to \infty$, and there are solutions with WKB asymptotic behavior for any $k^2 > 0$.

The proof of Theorems 1.1 and 1.2 uses techniques introduced in our earlier work [3], but differs markedly from that work in one respect. In [3], solutions of (1.1) were represented as sums of infinite series of multilinear operators. We were able to estimate each term of these series, but were unable to sum the bounds. Instead, the extra fractional power $|x|^\delta$ allowed us to show, essentially, that the remainder after summing sufficiently many terms of the series belonged to L^1, and hence could be handled in an alternative way, by means of a classical theorem of Levinson. In the present paper, we employ a simpler representation for the solutions and do sum the resulting infinite series. We also hope that the simplicity of the series will allow future applications to even larger classes of potentials.

An additional motivation for our work is the desire to have an approach some of whose elements might have a chance of being extended to higher dimensions. While the one-dimensional structure is still essential for our results, an abolition of the use of Levinson's theorem appears to be a small step forward in this respect.

Potentially another such small step is the derivation of a representation for $u_\pm(x,k)$ directly from a resolvent-based expansion, making no use of reduction to a first-order system. We have succeeded in doing this, obtaining a representation resembling, though more complicated than, the one employed here. Its derivation was not completely straightforward; initially we obtained infinitely many terms which fail to satisfy the desired estimates, but eventually discovered an algebraic identity which demonstrates that all the undesirable terms sum to exactly zero. This derivation has at present no concrete applications, so is not included in this article.

The existence of solutions with the asymptotic behavior (1.2), (1.5) for almost every k^2 has direct spectral consequences for Schrödinger operators defined on the half-axis (with some self-adjoint condition at the boundary) or on the whole axis. It is well known [18, 16, 10] that boundedness of solutions of (1.1) at some energy implies positivity of the derivative of the spectral measure. For a more recent alternative approach, see [5]. Hence the following theorems are direct corollaries of the above results.

Theorem 1.3. *If $V \in \ell^p(L^1)(\mathbb{R})$ for some $1 \le p < 2$, the positive semi-axis $\mathbb{R}^+ = (0,\infty)$ is an essential support of the absolutely continuous spectrum of the operator $-D^2 + V$.*

The last conclusion means simply that the absolutely continuous component of the spectral measure of the operator gives positive weight to any Borel subset of $\mathbb{R}^+$ of positive Lebesgue measure, and zero weight to the negative semi-axis $\mathbb{R}^-$.

Theorem 1.4. *Assume $V = V_1 + V_2$ where $V_1 \in L^1(\mathbb{R})$, $dV_2/dx \in \ell^p(L^1)$, and V_2 tends to zero at infinity. Then $\mathbb{R}^+$ is an essential support of the absolutely continuous spectrum of the operator $-D^2 + V$. If we only assume that V_2 is bounded, and $A = \limsup_{|x|\to\infty} V_2(x)$, then $[A,\infty)$ is part of an essential support of the absolutely continuous spectrum of $-D^2 + V(x)$.*

Remarks. 1. A classical result of Weidmann [20] says that if $V' \in L^1$, then the spectrum on $\mathbb{R}^+$ is purely absolutely continuous. Our theorem shows that if $V' \in L^p$ with any $p < 2$, and $V \to 0$ as $x \to \infty$, then the absolutely continuous spectrum still fills $\mathbb{R}^+$ (but may not be pure anymore).
2. We did not distinguish in the above theorems between half-line and whole line operators. In the case of whole line operators we can say in addition that the multiplicity of the absolutely continuous spectrum is two [16]. In the half-line case, the hypothesis of course only imposed as $x \to +\infty$.
3. Theorem 1.4 is rather sharp in the following sense:
a. We can have an example of $V \in L^p$, $V' \in L^r$ with any $p > 2$, $r > 2$ and purely singular spectrum on $\mathbb{R}^+$. Existence of such examples follows in a direct manner from the results of [11].
b. The potentials $V(x) = \cos x^\alpha$, $0 < \alpha < 1$ are known to give rise to Schrödinger operators with absolutely continuous spectrum in $(1, \infty)$, but with pure point spectrum in $(-1, 1)$ [1, 12, 17, 19].

The last theorem we prove in this paper concerns energy-dependent potentials. This extension is not directly related to spectral theory, but is a natural ODE application of our methods.

Theorem 1.5. *Suppose that $p < 2$ and that $W(x,k)$ is real-valued, and that*

$$\partial^j W(x,k)/\partial k^j \in L^p(\mathbb{R})$$

uniformly in $k \in J$ for $j = 0, 1$. Suppose further that the derivatives $\partial^j W(x,k)/\partial k^j \to 0$ as $|x| \to \infty$, uniformly in $k \in J$, for $j = 2, 3$. Then for almost every $k \in J$, there exist linearly independent, bounded solutions $u_\pm(x,k)$ of

$$u''(x) = W(x,k) - k^2 u$$

with WKB asymptotic behavior as $x \to +\infty$.

In the hypotheses of this theorem we have made no effort to economize on derivatives with respect to k.

Related results have been announced by R. Killip [7], who establishes stability of the absolutely continuous spectrum under perturbation of certain classes of background potentials by functions V whose Fourier transforms are locally in L^2, and which satisfy mild additional decay hypotheses.

2. The Solution Series

Let us rewrite the equation (1.1) as a system

$$y' = \begin{pmatrix} 0 & 1 \\ V - k^2 & 0 \end{pmatrix} y \tag{2.1}$$

with $y = \begin{pmatrix} u \\ u' \end{pmatrix}$. Apply a variation of parameters transformation

$$y = \begin{pmatrix} e^{ikx} & e^{-ikx} \\ ike^{ikx} & -ike^{-ikx} \end{pmatrix} z,$$

to get

$$z' = \frac{i}{2k}\begin{pmatrix} -V(x) & -V(x)e^{-2ikx} \\ V(x)e^{2ikx} & V(x) \end{pmatrix} z.$$

Do one more diagonal transformation

$$z = \begin{pmatrix} \exp\left(-\frac{i}{2k}\int_0^x V(t)\,dt\right) & 0 \\ 0 & \exp\left(\frac{i}{2k}\int_0^x V(t)\,dt\right) \end{pmatrix} w.$$

Then w satisfies

$$w' = \frac{i}{2k}\begin{pmatrix} 0 & -V(x)e^{-2ikx+\frac{i}{k}\int_0^x V(t)\,dt} \\ V(x)e^{2ikx-\frac{i}{k}\int_0^x V(t)\,dt} & 0 \end{pmatrix} w. \tag{2.2}$$

Let us denote

$$(Tf)(k) = \int_0^\infty e^{2ikx-\frac{i}{k}\int_0^x V(t)\,dt} f(x)\,dx, \tag{2.3}$$

an operator defined initially on integrable functions of compact support. We also introduce multilinear operators

$$\begin{aligned} (2.4)\quad & T_n(f_1,\dots,f_n)(x,k) \\ &= \left(\frac{i}{2k}\right)^n \int_x^\infty \int_{t_1}^\infty \cdots \int_{t_{n-1}}^\infty \prod_{j=1}^n \left[e^{(-1)^{n-j}(2ikt_j-\frac{i}{k}\int_0^{t_j} V(t)\,dt)} f_j(t_j)dt_j\right]. \end{aligned}$$

Iterating system (2.2) starting from the vector $(1,0)$ we obtain the following formal series expansion for one of the solutions of equation (1.1):

$$\begin{aligned} u_+(x,k) &= e^{ikx-\frac{i}{2k}\int_0^x V(t)\,dt}\sum_{n=0}^\infty (-1)^n T_{2n}(V,\dots,V)(x,k) \\ &+\ e^{-ikx+\frac{i}{2k}\int_0^x V(t)\,dt}\sum_{n=1}^\infty (-1)^n T_{2n-1}(V,\dots,V)(x,k) \end{aligned} \tag{2.5}$$

(we stipulate $T_0(V)(x,k) \equiv 1$ in the above formula). A similar representation can be derived for $u_-(x,k)$.

The existence of the individual terms of this expansion, and convergence of the infinite series, under the assumptions of Thereom 1.1, will be proven in later sections. We remark that a similar series can be derived, relating generalized eigenfunctions of $-D^2+q+V$ to those of $-D^2+q$, for various classes of background potentials q (for instance, periodic ones). We leave the consideration of that situation to a later paper.

Now we derive a different series representation suitable for the proof of Theorem 1.2. Let $\xi(x,k) = \sqrt{k^2-V_2}$. We can always assume that $k^2-V_2>0$ by replacing V_2 with its restriction to the complement of a suitably large interval, subsuming the remainder into an L^1 potential V_1. In a system (2.1) we do a different variation of parameters-type transformation

$$y = \begin{pmatrix} e^{i\int_0^x \xi\,dt} & e^{-i\int_0^x \xi\,dt} \\ i\xi e^{i\int_0^x \xi\,dt} & -i\xi e^{-i\int_0^x \xi\,dt} \end{pmatrix} z,$$

to get

$$z' = \frac{\xi'}{2\xi} \begin{pmatrix} -1 & e^{-2i\int_0^x \xi\, dt} \\ e^{2i\int_0^x \xi\, dt} & -1 \end{pmatrix} z.$$

Do one more transformation $z = \xi^{-1/2} w$. Then w satisfies

$$w' = \begin{pmatrix} 0 & -\frac{V_2'(x)}{4(k^2 - V_2(x))} e^{-2i\int_0^x \xi(t,k)\, dt} \\ -\frac{V_2'(x)}{4(k^2 - V_2(x))} e^{2i\int_0^x \xi(t,k)\, dt} & 0 \end{pmatrix} w. \tag{2.6}$$

Define

$$(Bf)(k) = \int_0^\infty \tfrac{1}{4}(k^2 - V_2(x))^{-1} e^{2i\int_0^x \xi(t,k)\, dt} f(x)\, dx\ . \tag{2.7}$$

In a similar manner, we introduce multilinear operators

$$\begin{aligned} &B_n(f_1, \dots, f_n)(x,k) \\ &\quad = \int_x^\infty \int_{t_1}^\infty \cdots \int_{t_n}^\infty \prod_{j=1}^n \left[\tfrac{1}{4}[k^2 - V(t_j)]^{-1}\, e^{2(-1)^{n-j} i \int_0^{t_j} \xi(t,k)\, dt} f_j(t_j) dt_j \right]\ . \end{aligned} \tag{2.8}$$

Iterating the system (2.6) starting from the vector $(1,0)$, we obtain a series representation for one of the solutions of equation (1.1) with exactly the same structure as in the previous case:

$$\begin{aligned} u_+(x,k) &= (k^2 - V(x))^{-1/4} e^{i\int_0^x \xi(t,k)\, dt} \sum_{n=0}^\infty B_{2n}(V', \dots, V')(x,k) \\ &\quad + (k^2 - V(x))^{-1/4} e^{-i\int_0^x \xi(t,k)\, dt} \sum_{n=1}^\infty B_{2n-1}(V', \dots, V')(x,k). \end{aligned} \tag{2.9}$$

We remark that each individual term in the series (2.5) (respectively, (2.9)) is well-defined for a.e. k for any V (respectively, V') in $\ell^p(L^1)(\mathbb{R})$ according to the results of [3], even though the region of integration is infinite. We will provide another proof of this fact in Section 4. Therefore the proof of our main results reduces to the summation of these infinite series. This we were unable to accomplish in earlier work. The estimates for the multilinear transforms which we discuss in the next section are the essential ingredient of the proof.

3. The Multiplicative Estimate

Here we outline the general results regarding the estimation of a certain type of multilinear transforms. Some of the proofs appear in a companion paper [4].

Definition. A martingale structure on a subinterval $I \subset \mathbb{R}$ is a collection of subintervals $\{E_j^m : m \ge 0, 1 \le j \le 2^m\}$ of I that satisfy the following conditions, modulo endpoints.

- $I = \cup_j E_j^m$ for every m.
- $E_j^m \cap E_{j'}^m = \emptyset$ for every $j < j'$.
- If $j < j'$, $x \in E_j^m$, and $x' \in E_{j'}^m$, then $x < x'$.

- For every m, j, $E_j^m = E_{(2j-1)}^{m+1} \cup E_{2j}^{m+1}$.

We denote by χ_E the characteristic function of the set E, and introduce the special notation χ_j^m for the characteristic function of the interval E_j^m.

Definition. A martingale structure $\{E_j^m : m \geq 0, 1 \leq j \leq 2^m\}$ is said to be adapted in L^p to a function f if for all m, j

$$\|f \cdot \chi_j^m\|_{L^p}^p = 2^{-m}\|f\|_{L^p}^p.$$

It is said to be adapted in $\ell^p(L^1)$ to a function f if for all m, j

$$\|f \cdot \chi_j^m\|_{\ell^p(L^1)}^p \leq 2^{-m}\|f\|_{\ell^p(L^1)}^p.$$

Given any $f \in L^p(\mathbb{R})$, it is easy to construct a martingale structure adapted to f (which need not be unique) [3], [4]. The reason why we need to modify the notion of "adapted" in the $\ell^p(L^1)$ case is that for any disjoint measurable sets S_1, S_2,

$$\|f \cdot \chi_{S_1}\|_{\ell^p(L^1)}^p + \|f \cdot \chi_{S_2}\|_{\ell^p(L^1)}^p \leq \|f \cdot \chi_{S_1 \cup S_2}\|_{\ell^p(L^1)}^p,$$

as opposed to the equality in the L^p case.

Let us indicate the procedure by which, given $f \in \ell^p(L^1)(\mathbb{R})$, E_1^1, E_2^1 are to be chosen; further sets are obtained by applying the same procedure recursively. The set E_1^1 is chosen to be $(-\infty, x_1^1]$, where x_1^1 is chosen so that

$$\|f \cdot \chi_{(-\infty,x_1^1]}\|_{\ell^p(L^1)}^p = \|f \cdot \chi_{[x_1^1,\infty)}\|_{\ell^p(L^1)}^p.$$

Since $\|f \cdot \chi_{(-\infty,x]}\|_{\ell^p(L^1)}^p$ is continuous monotone increasing in x, such x_1^1 exists. Then $E_2^1 = [x_1^1, \infty)$. If x_1^1 is an integer, clearly

$$\|f \cdot \chi_{E_1^1}\|_{\ell^p(L^1)}^p + \|f \cdot \chi_{E_2^1}\|_{\ell^p(L^1)}^p = \|f\|_{\ell^p(L^1)}^p,$$

but if $n + 1 > x_1^1 > n$ for some integer n, we have

$$\begin{aligned} &\|f\|_{\ell^p(L^1)}^p - \|f \cdot \chi_{E_1^1}\|_{\ell^p(L^1)}^p - \|f \cdot \chi_{E_2^1}\|_{\ell^p(L^1)}^p \\ &\quad = \left(\int_n^{n+1} |f(x)|\, dx\right)^p - \left(\int_n^{n+1} |f(x)|\chi_{E_1^1}\, dx\right)^p - \left(\int_n^{n+1} |f(x)|\chi_{E_2^1}\, dx\right)^p \geq 0, \end{aligned}$$

since $(a+b)^p \geq a^p + b^p$ for any positive numbers a, b if $p \geq 1$. Hence generally

$$\|f \cdot \chi_{(-\infty,x_1^1]}\|_{\ell^p(L^1)}^p = \|f \cdot \chi_{[x_1^1,\infty)}\|_{\ell^p(L^1)}^p \leq 2^{-1}\|f\|_{\ell^p(L^1)}^p.$$

Given any locally integrable functions $h_1, \dots, h_n$, consider a multilinear expression

$$M_n(h_1, \dots, h_n)(x, x') = \int \cdots \int_{x \leq x_1 \leq x_2 \cdots \leq x_n \leq x'} \prod_{i=1}^n h_i(x_i) dx_i. \tag{3.1}$$

In the special case where each h_i is equal either to h or to its complex conjugate $\overline{h}$, we write simply $M_n(h)$. (In the latter case, all estimates stated below are independent of the choices of $h, \overline{h}$.) We set

$$M_n^*(h_1, \dots, h_n) = \sup_{x < x'} |M_n(h_1, \dots, h_n)(x, x')| \ .$$

and

$$M_n^*(h) = \sup_{x<x'} |M_n(h)(x,x')| \ .$$

Define

$$\tilde{G}_h = \sum_{m=1}^{\infty} m \left(\sum_{j=1}^{2^m} | \int_{E_j^m} h(t)\,dt|^2 \right)^{1/2} . \tag{3.2}$$

Here $\tilde{G}_h$ is either a nonnegative real number, or $+\infty$. Although this is not indicated explicitly in the notation, $\tilde{G}_h$ depends also on the underlying martingale structure $\{E_j^m\}$, which must be specified.

We have the following quite general multiplicative bounds.

Proposition 3.1. [4] *There exists $C < \infty$ such that for every martingale structure $\{E_j^m\}$ on $\mathbb{R}$, every $n \geq 1$ and all locally integrable functions $h, h_1, \dots, h_n$,*

$$M_n^*(h_1, \dots, h_n) \leq C^n \prod_{j=1}^{n} \tilde{G}_{h_j} \qquad \textit{and} \qquad M_n^*(h) \leq C^n \tilde{G}_h^n / \sqrt{n!} \ .$$

Here C is a universal constant. The numerical factor $1/\sqrt{n!}$ in the second bound plays an important role in our analysis. No relation is assumed here between the martingale structure involved in definition of $\tilde{G}$ and the functions h, h_j; in particular, the martingale structure need not be adapted to h. However, in the first bound, the same martingale structure is used to define each factor $\tilde{G}$. The proof of this proposition is combinatorial, exploiting the geometry of the region of integration in (3.1).

Let us show how the multilinear expressions M_n are relevant to our plan to sum the series (2.5), (2.9) for solutions. Let S be an integral operator,

$$(Sf)(k) = \int_{\mathbb{R}} S(k,t) f(t)\,dt.$$

Consider $M_n(S(k,\cdot)f(\cdot))(x,x')$. Then as $x' \to \infty$, we obtain expressions of the same type as in the series (2.5), (2.9) (with the operators T and B playing the role of S). Hence our goal is to justify the passage to the limit and to derive estimates on $M_n(S(k,\cdot)f(\cdot))(x,x')$, under assumptions valid for T and B.

Let us use the notation

$$G_{S(f)}(k) = \sum_{m=1}^{\infty} m \left(\sum_{j=1}^{2^m} |S(f \cdot \chi_j^m)(k)|^2 \right)^{1/2} . \tag{3.3}$$

Thus $G_{S(f)}(k) = \tilde{G}_{S(k,\cdot)f(\cdot)}$.

Proposition 3.1 has the following direct consequence.

Corollary 3.2. *For any k, f such that $S(x,k)f(x)$ is locally integrable,*

$$M_n^*(S(\cdot,k)f(\cdot)) \leq C^n G_{S(f)}(k)^n / \sqrt{n!} \ . \tag{3.4}$$

Now we derive one simple condition on S under which the bound (3.4) becomes useful.

Proposition 3.3. *Assume that S is a bounded linear or sublinear operator from $\ell^p(L^1)(\mathbb{R})$ to $L^q(J)$, where $p < 2 < q$ and J is an interval on the real axis. Then for any $f \in \ell^p(L^1)(\mathbb{R})$ and any martingale structure E_j^m adapted to f, we have*

$$\|G_{S(f)}(k)\|_{L^q(J)} \leq C\|f\|_{\ell^p(L^1)(\mathbb{R})},$$

where $C < \infty$ depends only on the operator norm of S.

More generally, if $\{E_j^m\}$ is not necessarily adapted to f but satisfies

$$\|f\chi_j^m\|_{\ell^p(L^1)}^p \leq A^p 2^{-m}$$

for some numerical quantity A, then

$$\|G_{S(f)}(k)\|_{L^q(J)}^q \leq CA^{q-p}\|f\|_{\ell^p(L^1)(\mathbb{R})}^p.$$

Proof.

$$\int_J \left(\sum_{j=1}^{2^m} |S(f\cdot\chi_j^m)(y)|^2\right)^{q/2} dy \leq 2^{m(\frac{q}{2}-1)} \int_J \sum_{j=1}^{2^m} |S(f\cdot\chi_j^m)(y)|^q\, dy$$

$$\leq C2^{m(\frac{q}{2}-1)} \sum_{j=1}^{2^m} \|f\cdot\chi_j^m\|_{\ell^p(L^1)}^q \leq C\|f\|_{\ell^p(L^1)}^q 2^{-mq(\frac{1}{p}-\frac{1}{2})}.$$

In the first step we used the general inequality $(\sum_{n=1}^N a_n)^s \leq N^{s-1}\sum_{n=1}^N a_n^s$, which holds for any a_n and $s \geq 1$. Hence

$$\|G_{S(f)}\|_{L^q(J)} \leq C\|f\|_{\ell^p(L^1)} \sum_{m=1}^{\infty} m2^{-m(\frac{1}{p}-\frac{1}{2})} \leq C_1\|f\|_{\ell^p(L^1)}\ .$$

The second conclusion follows from the same reasoning. □

To any S we associate a maximal operator

$$S^*f(k) = \sup_{y\in\mathbb{R}} |\int_y^\infty S(k,t)f(t)\, dt|.$$

To S^* and some martingale structure $\{E_j^m\}$, we associate a function $G_{S^*(f)}(k)$, by replacing S by S^* in (3.3). A simple but fundamental observation (see the second remark in section 2 of [4]) is that $S^*f(k) \leq G_{Sf}(k)$, for any S, f, k.

Lemma 3.4. *The mapping $f \mapsto G_{S^*(f)}(k)$ also satisfies the conclusions of Proposition 3.3.*

Proof. By Proposition 3.3, boundedness of G_{S^*} would follow from boundedness of S^*. We claim that whenever S maps $\ell^p(L^1)(\mathbb{R})$ boundedly to L^q of some measure space, then so does S^*, provided that $q \geq 2 > p$. For any martingale structure, any operator S and any function f, S^*f is dominated pointwise by G_{Sf} [4]. By choosing a martingale structure adapted to f in $\ell^p(L^1)$ and applying Proposition 3.3 to G_f, we conclude that S^* is bounded. □

Actually boundedness of S implies boundedness of S^* whenever $q > p$, without further restriction on the exponents, as follows quite directly from the argument in [4]. We have proved only this more restricted statement here because the required assertion is not explicit in [4].

Finally, we observe that under our hypotheses on the potential V, the operators T and B do actually satisfy $\ell^p(L^1)$ to L^q norm bounds, which make the above results on multilinear transforms applicable.

Proposition 3.5. *Let the operators T and B be defined by*

$$(Tf)(k) = \int_0^\infty e^{2ikx - \frac{i}{k}\int_0^x V(t)\,dt} f(x)\,dx$$

and

$$(Bf)(k) = \int_0^\infty (4(k^2 - V_2(x)))^{-1}\, e^{2i\int_0^x \xi(t,k)\,dt} f(x)\,dx.$$

Let $p^{-1}+q^{-1}=1$, and $1 \le p \le 2$. Then T, under the conditions of Theorem 1.1, and B, under the conditions of Theorem 1.2, satisfy

$$\|Tf\|_{L^q(J)} \le C_J \|f\|_{\ell^p(L^1)(\mathbb{R})} \quad \textit{and} \quad \|Bf\|_{L^q(J)} \le C_J \|f\|_{\ell^p(L^1)(\mathbb{R})} \tag{3.5}$$

for any compact interval $J \subset \mathbb{R}$ disjoint from zero (or, in the case of the operator B, disjoint from $[-\sqrt{A}, \sqrt{A}]$ with $A = \limsup_{x\to\infty} V_2(x)$).

Proof of Proposition. The operators T and B clearly map $L^1(\mathbb{R})$ to $L^\infty(J)$, so it suffices to show the $\ell^2(L^1)(\mathbb{R}) \mapsto L^2(J)$ bounds. Then (3.5) will follow by complex interpolation.

The proof of the $\ell^2(L^1)(\mathbb{R}) \mapsto L^2(J)$ bound for T was given in [3]. Here we will demonstrate (in a similar way) the bound for B. For simplicity, we will prove the bound on any compact interval $J' \subset (0,\infty)$ for a related operator

$$(B'f)(E) = \int_0^\infty \tfrac{1}{4}(E - V_2(x))^{-1} e^{2i\int_0^x \xi(t,\sqrt{E})\,dt} f(x)\,dx;$$

since J is disjoint from 0, the bound for B will follow.

Let $0 \le \psi \le 1$ be a $C_0^\infty((0,\infty))$ function equal to 1 on J. Then

$$\begin{aligned}\|B'f\|^2_{L^2(J)} &\le \tfrac{1}{16}\int_{\mathbb{R}}\int_{\mathbb{R}} f(x)\overline{f(y)} \int \frac{\psi(E)}{(E-V_2(x))(E-V_2(y))} e^{2i\int_y^x \xi(t,\sqrt{E})\,dt}\,dE\,dxdy \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(x)\overline{f(y)} M(x,y)\,dxdy.\end{aligned}$$

It suffices to prove boundedness of B from $\ell^p(L^1)(R,\infty)$ for suitably large R, since B maps L^1 to L^∞, and $\ell^p(L^1)(0,R] \subset L^1$.

Under the conditions of Theorem 1.2, for all $t \ge R$ for sufficiently large R, we have $d\xi(t,\sqrt{E})/dE = \frac{1}{2}[E - V_2(t)]^{-1/2} > c > 0$. Hence there exists N such that if $|x-y| > N$, $|d\int_y^x \xi(t,E)\,dt/dE| > (c/2)|x-y|$. A double integration by parts in the integral defining $M(x,y)$, integrating

$$e^{2i\int_y^x \xi(t,\sqrt{E})\,dt} \int_y^x (E - V_2(t))^{-1/2}\,dt$$

and differentiating the other factor, gives

$$|M(x,y)| \le C_{N,c}(1+|x-y|)^{-2}).$$

Hence the required bound for B follows from the elementary bound

$$\int\int |f(x)f(y)|(1+|x-y|)^{-2}\,dxdy \le C\|f\|^2_{\ell^2(L^1)(\mathbb{R})}\ .$$

□

4. The Asymptotic Behavior of Solutions

Proposition 3.5 allows us to apply the maximal estimate that we derived in the previous section to the multilinear transforms T_n and B_n, and thus to prove an effective estimate on the terms in the series (2.5), (2.9). However, it must be shown that the definitions of T_n, B_n make sense for general functions in $\ell^p(L^1)$, not merely those having compact support.

Let us denote $V_y = V\cdot\chi_{(-\infty,y]}$.

Proposition 4.1. *Let $1\le p<2$. For any $V\in\ell^p(L^1)(\mathbb{R})$, the multilinear transforms $T_n(V,\dots,V)(x,k)$ defined by* (2.4) *are well-defined as the limits*

$$T_n(V,\dots,V)(x,k) = \lim_{y\to\infty} T_n(V_y,\dots,V_y)(x,k),$$

which exist for almost every k. Moreover,

$$|T_n(V,\dots,V)(x,k)| \le C^n G_{T(V-V_x)}(k)^n/\sqrt{n!} \tag{4.1}$$

for some constant $C<\infty$. Similarly, for any $V'\in\ell^p(L^1)(\mathbb{R})$, the multilinear transforms $B_n(V',\dots,V')(x,k)$ are well-defined in the same sense, and satisfy

$$|B_n(V',\dots,V')(x,k)| \le C^n G_{B(V'-V'_x)}(k)^n/\sqrt{n!}\ . \tag{4.2}$$

Proof. It suffices to restrict consideration to an arbitrary compact interval J disjoint from 0. The cases of T_n and B_n are completely parallel, so we discuss only the former. Recall the maximal operator $T^*f(k) = \sup_y|T(f\cdot\chi_{[y,\infty)})(k)|$, the associated quadratic expression $G_{T^*(\cdot)}$, and their mapping properties discussed in Lemma 3.4.

We wish to show that

$$\limsup_{y,z\to\infty}\sup_x |T_n(V_y,\dots,V_y)(x,k) - T_n(V_z,\dots,V_z)(x,k)| = 0$$

for almost every $k\in J$. The difference is majorized by $\sum_{r=1}^n |T_n(V_y,\dots,V_y,V_y - V_z,V_z,\dots,V_z)(x,k)|$, where V_y-V_z appears in the r-th position. Fix any $1\le r\le n$, and assume that $y,z\ge M$. Then by Proposition 3.1,

$$\begin{aligned}\sup_x |T_n(V_y,\dots,V_y,V_y-V_z,V_z,\dots,V_z)(x,k)| &\le C^n G^{r-1}_{T(V_y)}(k)G_{T(V_y-V_z)}(k)G^{n-r}_{T(V_z)}(k)\\ &\le C^n G^{n-1}_{T^*(V)}(k)G_{T^*(V\cdot\chi_{[M,\infty)})}(k)\ .\end{aligned}$$

Here the martingale structure used to define each quantity G is always a structure adapted to V, regardless of whether the subscript is $T^*(V), T(V_y), T(V_z), T(V_y-V_z)$, or $T^*(V\cdot\chi_{[M,\infty)})$. To obtain the last line, we have rewritten V_y-V_z as $-V\chi_{[y,\infty)} + V\chi_{[z,\infty)}$. The factor $G_{T^*(V\cdot\chi_{[M,\infty)})}(k)$ tends to zero in $L^q(J)$, as $M\to\infty$, by Lemma 3.4

and the second conclusion of Proposition 3.3, while $G_{T^*(V)} \in L^q$. By summing over r, we conclude that the lim sup vanishes in $L^{q/n}(J)$.

The bounds (4.1) and (4.2) now follow immediately from the maximal estimate (3.4). $\square$

We are now ready to complete the proofs of our main results.

Proof of Theorems 1.1 and 1.2. It is sufficient to consider k from an arbitrary compact interval $J = [a, b] \subset (0, \infty)$. Let us recall the series representation (2.5):

$$\begin{aligned} u_+(x,k) &= e^{ikx - \frac{i}{2k}\int_0^x V(t)\,dt} + e^{ikx - \frac{i}{2k}\int_0^x V(t)\,dt} \sum_{n=1}^{\infty} (-1)^n T_{2n}(V, \ldots, V)(x,k) \\ &\quad + e^{-ikx + \frac{i}{2k}\int_0^x V(t)\,dt} \sum_{n=1}^{\infty} (-1)^n T_{2n-1}(V, \ldots, V)(x,k). \end{aligned}$$

Its terms are well-defined by Proposition 4.1, and it converges uniformly as a function of $x \in \mathbb{R}$. An application of Corollary 3.2 shows that this series converges uniformly in $\mathbb{R}$ as a function of x for almost every k, and leads to the estimate

$$\sup_x |u_+(x,k)| \le 1 + \sum_{n=1}^{\infty} \frac{B^n G^n_{T(V)}(k)}{\sqrt{n!}}. \tag{4.3}$$

Lemma 4.2. *$u_+(x,k)$ satisfies $(-D^2 + V - k^2)u_+ = 0$, for almost every $k \in \mathbb{R}$.*

Proof. Since the series representation (2.5) is obtained from a corresponding series representation for a solution of the first-order system (2.2), it suffices to prove that the latter series defines a solution of (2.2). Write (2.2) as $w' = \mathcal{V}(x,k)w$, where

$$\mathcal{V} = \frac{i}{2k} \begin{pmatrix} 0 & -V(x)e^{-2ikx + \frac{i}{k}\int_0^x V} \\ V(x)e^{2ikx - \frac{i}{k}\int_0^x V} & 0 \end{pmatrix}.$$

A formal solution, obtained from the Ansatz $w_0(x) \equiv$ transpose of (1, 0), is

$$w(x,k) = \sum_{n=0}^{\infty} (-1)^n \mathcal{T}^n w_0(x,k),$$

where $\mathcal{T}w(x,k) = \int_x^\infty \mathcal{V}(t,k)w(t,k)\,dt$ and the formal n-fold iterate $\mathcal{T}^n$ of $\mathcal{T}$ is defined rigorously not as an iterate, but rather by applying Proposition 4.1 to the n-fold multiple integral in the formal expression for the iterate. Again, the series converges in $L^\infty(\mathbb{R})$ for almost every k, by Proposition 4.1 and Corollary 3.2.

Set $w_N(x,k)$ equal to the partial sum, from $n = 0$ to $n = N$. Then for almost every k, $w_N' - \mathcal{V}w_N = \pm \mathcal{V}\mathcal{T}^N w_0$; this is obvious for all k for compactly supported potentials, and follows for general $V \in \ell^p(L^1)$ via approximation, justified by Proposition 4.1. For almost every k, the right-hand side of this last equation converges, as $N \to \infty$, to zero in $\ell^p(L^1)(\mathbb{R})$, while the left-hand side converges to $w'(x,k) - \mathcal{V}w(x,k)$ in the sense of distributions in x. Therefore $w(x,k)$ satisfies the equation in the sense of distributions, for almost every k. $\square$

Notice that

$$\frac{B^{2n+1}G^{2n+1}}{\sqrt{(2n+1)!}} \le \sqrt{\frac{B^{2n}G^{2n}}{\sqrt{(2n)!}}\frac{B^{2n+2}G^{2n+2}}{\sqrt{(2n+2)!}}} \le \frac{1}{2}\left(\frac{B^{2n}G^{2n}}{\sqrt{(2n)!}} + \frac{B^{2n+2}G^{2n+2}}{\sqrt{(2n+2)!}}\right), \tag{4.4}$$

and hence we can simplify the estimate (4.3) to

$$\sup_x |u_+(x,k)| \le 2\sum_{n=0}^{\infty} \frac{B^{2n}G^{2n}_{T(V)}(k)}{n!} = 2e^{B^2 G^2_{T(V)}(k)}.$$

A similar estimate holds for u_-. Therefore,

$$\int_a^b \log(1+\sup_x |u_\pm(x,k)|)\,dk \le C\left(1 + B^2\int_a^b G^2(k)\,dk\right) < \infty, \tag{4.5}$$

since

$$\|G\|_{L^2([a,b])} \le C\|G\|_{L^q([a,b])} \le C_1\|V\|_{\ell^p(L^1)(\mathbb{R})}.$$

The solution u_- satisfies the same bound by complex conjugation, and moreover, $\partial_x u_\pm$ also satisfy the same bounds, since our estimates apply to the solutions of the 2×2 system introduced above.

We have proved a variant of (1.3), with $\tilde{u}_\pm$ replaced by $u_\pm$. To estimate $\tilde{u}_+$, we now exploit the asymptotic WKB behavior (1.2) of $u_\pm$, which will be established in the next paragraph. Define coefficients $c_\pm(k)$, independent of x, by

$$\begin{pmatrix} \tilde{u}_+(x,k) \\ \partial_x \tilde{u}_+(x,k)\end{pmatrix} = \begin{pmatrix} u_+(x,k) & u_-(x,k) \\ \partial_x u_+(x,k) & \partial_x u_-(x,k)\end{pmatrix}\cdot\begin{pmatrix} c_+ \\ c_-\end{pmatrix}.$$

Now $\tilde{u}_+(0,k) = 1$, $\partial_x\tilde{u}_+(0,k) = ik$, and we may solve for $c_\pm$ by evaluating the matrix equation at $x = 0$. Inverting the 2×2 matrix on the right, we obtain a cofactor matrix whose entries satisfy the desired bound, times the reciprocal of the Wronskian $W(x,k) \equiv W(k)$ of the solutions $u_\pm$. $W(k)$ may be evaluated by using the WKB asymptotics (1.2) in the following form:

$$\begin{pmatrix} u_+(x,k) \\ \partial_x u_+(x,k)\end{pmatrix} = \begin{pmatrix} e^{ikx - \frac{i}{2k}\int_0^x V} \\ ike^{ikx-\frac{i}{2k}\int_0^x V}\end{pmatrix} + \varepsilon(x,k) \tag{4.6}$$

where the vector $\varepsilon(x,k) \to 0$ as $x \to +\infty$, for almost every k. Granting these asymptotics, it follows that for a.e. k, $W(x,k) \to 2ik$ as $x \to +\infty$; but W is independent of x, hence is identically equal to $2ik$. Thus $\tilde{u}_+$ is a finite linear combination, with coefficients locally bounded in k, of products of two factors, each of which satisfies the desired bound. The logarithm in (4.5) converts these products to sums, so $\tilde{u}_+$ also satisfies (4.5).

To analyze asymptotic behavior, we notice that by (2.5), (4.4), and Proposition 4.1,

$$\begin{aligned}\sup_{x\le y}|u_+(y,k) - e^{iky - \frac{i}{2k}\int_0^y V(t)\,dt}| &\le \left(BG_{T(V-V_x)}(k) + 2\sum_{n=1}^{\infty}\frac{B^{2n}G^{2n}_{T(V-V_x)}(k)}{n!}\right) \\ &\le C\left(e^{B^2G^2_{T(V-V_x)}(k) + BG_{T(V-V_x)}(k)} - 1\right).\end{aligned}$$

From this estimate we infer that for a.e. $k \in \mathbb{R}$

$$u_+(x,k) - e^{ikx - \frac{i}{2k}\int_0^x V(t)\,dt} \to 0$$

as $x \to +\infty$. The solution $u_-(x,k)$ with WKB asymptotic behavior is simply the complex conjugate of u_+.

To obtain the vector asymptotics (4.6), it suffices to observe that with the notation of Section 2,

$$\begin{pmatrix} u \\ \partial_x u \end{pmatrix} = y = \begin{pmatrix} e^{ikx} & e^{-ikx} \\ ike^{ikx} & -ike^{-ikx} \end{pmatrix} \begin{pmatrix} \exp(\frac{-i}{2k}\int_0^x V) & 0 \\ 0 & \exp(\frac{i}{2k}\int_0^x V) \end{pmatrix} w,$$

where the above analysis proves that $w(x,k)$ equals the transpose of $(1,0)$ plus a vector-valued function that tends to zero as $x \to +\infty$, for almost every k.

The proof of Theorem 1.2 follows along the same lines, when $V_1 \equiv 0$. The addition of the L^1 potential V_1 constitutes a further perturbation, which again preserves the WKB asymptotic behavior, by a classical theorem of Levinson. This can also be shown by elementary estimates employing a variation of parameters transformation and integral equation for the solution. $\square$

5. A more general ODE application

It remains to prove Theorem 1.5. Observe first that the operator

$$Tf(k) = \int_0^\infty e^{i(2kx - k^{-1}\int_0^x W(t,k)\,dt)} f(x)\,dx$$

maps $L^p(\mathbb{R})$ boundedly to $L^q(J)$, where $q = p/(p-1)$, for all $1 \le p \le 2$. Indeed, the proof of Proposition 3.5 still applies; to obtain the bound $|M(x,y)| \le C|x-y|^{-2}$ we require control of three derivatives of W with respect to k.

Let

$$\Omega = \sum_{r=0}^{1} \iint_{\mathbb{R}\times J} |\partial_\rho^r W(x,\rho)|^p\,dx\,d\rho\ .$$

Choose a martingale structure $\{E_j^m\}$ so that for all m, j,

$$\iint_{E_j^m \times J} \sum_{r=0}^{1} |\partial_\rho^r W(x,\rho)|^p\,dx\,d\rho = 2^{-m}\Omega. \tag{5.1}$$

The pointwise in k estimate (3.4) still applies, and gives

$$\sup_y |T_n(W(\cdot,k),\dots,W(\cdot,k))(y,k)| \le C^n G_{T(W(\cdot,k))}(k)^n/\sqrt{n!}\ ,$$

where

$$G_{T(W(\cdot,k))}(k) = \sum_{m=1}^{\infty} m \left(\sum_{j=1}^{2^m} |T(W(x,k)\cdot\chi_j^m)(k)|^2 \right)^{1/2}. \tag{5.2}$$

Here $T(W(x,k)\cdot\chi_j^m)(\rho)$ denotes the function obtained by letting T act on the function $x \mapsto W(x,k)\chi_j^m$, restricted to $\rho = k$. All proofs from the previous sections would extend to this case if we could apply Proposition 3.3 to estimate G in terms of

W. However, the bound (3.5) is not applicable because of the k–dependence of W. Instead, we will bound $G_{T(W(\cdot,k))}(k)$ directly. The proof of Theorem 1.5 reduces to the following result.

Proposition 5.1. *Let $1 \leq p < 2$. Assume that $W(x,k)$ satisfies the hypotheses of Theorem 1.5, and let E_j^m be a martingale structure satisfying (5.1). Then $G_{T(W(\cdot,k))}(k)$, defined by (5.2), satisfies*

$$\|G_{T(W(\cdot,k))}(k)\|_{L^q(J)} \leq C\Omega^{1/p}, \tag{5.3}$$

where $q^{-1}+p^{-1}=1$.

Proof. Clearly,

$$G_{T(W(\cdot,k))}(k) \leq \sup_{\rho\in J} G_{T(W(\cdot,\rho))}(k),$$

so it suffices to bound the latter expression,

$$\sup_{\rho\in J} G_{T(W(\cdot,\rho))}(k) = \sup_{\rho\in J}\sum_{m=1}^{\infty} m\left(\sum_{j=1}^{2^m}|T(W(x,\rho)\cdot\chi_j^m)(k)|^2\right)^{1/2}.$$

Notice that

$$\begin{aligned}
&\int_J \sup_{\rho\in J}\left(\sum_{j=1}^{2^m}|T(W(x,\rho)\cdot\chi_j^m)(k)|^2\right)^{q/2} dk\\
&\leq C2^{m(\frac{q}{2}-1)}\sum_j\int_J\sup_{\rho\in J}|T(W(x,\rho)\cdot\chi_j^m(x))(k)|^q\,dk\\
&\leq C2^{m(\frac{q}{2}-1)}\sum_j\int_J(\int_J\sum_{r=0}^{1}|\partial_\rho^r T(W(x,\rho)\chi_j^m(x))(k)|^p\,d\rho)^{q/p}\,dk,
\end{aligned}$$

by the Sobolev embedding theorem applied in $L^p(J,d\rho)$. Since ∂_ρ commutes with T, we may continue by writing

$$\begin{aligned}
&= C2^{m(\frac{q}{2}-1)}\sum_j\int_J(\int_J\sum_{r=0}^{1}|T(\partial_\rho^r W(x,\rho)\chi_j^m(x))(k)|^p\,d\rho)^{q/p}\,dk\\
&\leq C2^{m(\frac{q}{2}-1)}\sum_{r=0}^{1}\sum_j(\int_J(\int_J|T(\partial_\rho^r W(x,\rho)\chi_j^m(x))(k)|^q\,dk)^{p/q}\,d\rho)^{q/p}\\
&\leq C2^{m(\frac{q}{2}-1)}\sum_{r=0}^{1}\sum_j(\int_J\int_{\mathbb{R}}|\partial_\rho^r W(x,\rho)\chi_j^m(x)|^p\,dx\,d\rho)^{q/p}\\
&\leq C2^{m(\frac{q}{2}-1)}\sum_{r=0}^{1}\sum_j 2^{-m(\frac{q-p}{p})}\Omega^{\frac{q-p}{p}}\iint_{J\times\mathbb{R}}|\partial_\rho^r W(x,\rho)\chi_j^m(x)|^p\,dx\,d\rho\\
&\leq C2^{mq(\frac{1}{2}-\frac{1}{p})}\Omega^{q/p}.
\end{aligned}$$

We have invoked first Minkowski's integral inequality, then the $L^p \mapsto L^q$ bound for T. Finally, the triangle inequality implies that

$$\|G_{T(W(\cdot,k))}(k)\|_{L^q(J)} \leq C\Omega^{1/p} \sum_{m=1}^{\infty} m 2^{(1/2-1/p)m} \leq C\Omega^{1/p}.$$

$\square$

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