HYPOELLIPTICITY: GEOMETRIZATION AND SPECULATION

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ABSTRACT. To any finite collection of smooth real vector fields X_j in $\mathbb{R}^n$ we associate a metric in the phase space $T^*\mathbb{R}^n$. The relation between the asymptotic behavior of this metric and hypoellipticity of $\sum X_j^2$, in the smooth, real analytic, and Gevrey categories, is explored.

To Professor P. Lelong, on the occasion of his 85th birthday.

1. Introduction

Let $\{X_j\}$ be a collection of real vector fields with C^∞ or C^ω coefficients, defined in a neighborhood of a point $x_0 \in \mathbb{R}^d$. Consider a second order differential operator $L = \sum_j X_j^2$. Under what conditions is L hypoelliptic, in C^∞ or C^ω? As is well–known, certain of these sums of squares operators provide a good model for aspects of the $\bar\partial$–Neumann problem in $\mathbb{C}^2$, and a grossly oversimplified but still illuminating window into the higher-dimensional case. Indeed, application of the method of boundary reduction to the $\bar\partial$–Neumann problem leads to a pseudodifferential equation on the boundary; microlocalizing and composing with an elliptic factor then leads in $\mathbb{C}^2$ to an operator closely related to $X^2 + Y^2$ for certain vector fields. We believe that the study of more general collections of vector fields than those directly relevant to the $\bar\partial$–Neumann has shed, and will continue to shed, light on the special cases of interest in complex analysis.

Sufficient conditions for C^∞ hypoellipticity formulated in [18],[22],[9], which subsume a wide variety of results and examples, involve the existence of auxiliary operators, having certain favorable commutation relations with L. The same holds for analytic hypoellipticity [26],[16], as well as for global C^∞ regularity [2].

Except in very special cases, there remains a large gap between the hypotheses of the sufficient conditions for hypoellipticity in various function spaces, and counterexamples suggesting what hypotheses are necessary. Many works on various types of hypoellipticity and regularity have relied on commutation methods to achieve results in the positive direction, whereas negative results have been attained through different methods.

Is commutation merely a useful device for proving hypoellipticity that is applicable in special circumstances, or is it tied to the problem in a more essential way? The aim of this note is to begin to more directly investigate this question, by:

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(1) Defining a metric ρ_L, adapted to L, on the cotangent bundle, and showing how a necessary condition for the existence of auxiliary operators having desirable commutation properties with L may be formulated in terms of ρ_L.
(2) Speculating on the connection between properties of ρ_L and hypoellipticity of L in C^ω, in Gevrey classes, and to a lesser extent in C^∞.
(3) Analyzing the behavior of ρ_L in fundamental examples.

A well-known construction associates to the collection $\{X_j\}$ of vector fields a Carnot-Caratheodory, or control, metric on the base space $\mathbb{R}^d$, rather than on the cotangent bundle. This metric, and its connection with the fundamental solution of $\sum X_j^2$, has been intensively studied by Fefferman and Phong [15], by Nagel, Stein and Wainger [23], and by Sánchez-Calle [25]. A cotangent bundle metric has been defined by Fefferman and Parmeggiani [14],[24]. Both of those metrics differ from ρ_L in fundamental respects, as will be explained in §4.

2. Definitions

Let $\mu : T^*\mathbb{R}^d \mapsto (0, \infty)$ be any positive nonvanishing real symbol belonging to the class $S^1_{1,0}$, and let $\{V_j\}$ be any finite collection of vector fields, defined on a conic open subset of the cotangent bundle $T^*\mathbb{R}^d$, with real C^∞ coefficients. Assume that each V_j takes the form $\sum_k a_{j,k}\partial/\partial x_j + \sum_k b_{j,k}\partial/\partial\xi_j$ where $a_{j,k} \in S^0_{1,0}$ and $b_{j,k} \in S^1_{1,0}$. Throughout the paper, ρ_0 will denote the standard metric $d\rho_0^2 = \langle\xi\rangle^2 dx^2 + d\xi^2$ on $T^*\mathbb{R}^d$.

Our first definition serves merely to facilitate the main definitions below.

Definition 1. *A C^1 function ψ defined on an open set in $T^*\mathbb{R}^d$ is said to be 1-Lipschitz relative to $(\mu, \{V_j\})$ if at every point $p = (x, \xi)$ of $T^*\mathbb{R}^d$,*

$$|V_j\psi(p)| \le \mu(p) \qquad \text{for all } j \le n \tag{1}$$

and

$$\langle\xi\rangle^{-2}|\nabla_x\psi(p)|^2 + |\nabla_\xi\psi(p)|^2 \le 1. \tag{2}$$

The main condition here is (1), which imposes a stringent upper bound on the variation of ψ in the directions V_j, at places where $\mu(x,\xi)$ is smaller than $\langle\xi\rangle$ and some vector field V_j is relatively large.

Next, let finitely many C^∞ real vector fields X_j be given on an open subset of the base space $\mathbb{R}^d$. Recall that to any smooth function $f : T^*\mathbb{R}^d \mapsto \mathbb{R}$ is associated its Hamiltonian vector field

$$H_f = \sum_{n=1}^{d} \left(\frac{\partial f}{\partial x_n}\frac{\partial}{\partial \xi_n} - \frac{\partial f}{\partial \xi_n}\frac{\partial}{\partial x_n} \right) .$$

Define $\sigma_j(x,\xi)$ to be the principal symbol of iX_j, which is real valued. Denote by H_{σ_j} the Hamilton vector field associated to σ_j.

In most of the discussion we assume the bracket hypothesis to be satisfied, to some order m. Then consider the set S consisting of all multi-indices $I = (I_1, \dots I_k)$ of degree $|I| = k \in \{1, 2, \dots m\}$ such that each $I_s \in \{1, 2, \dots n\}$. For each $I \in S$ define

$\sigma_I(x,\xi)$ to be the iterated Poisson bracket $\{\sigma_{I_k}, \sigma_{I'}\}$ where $I' = (I_1, \dots I_{k-1})$. Define the effective symbol[1] $\tilde{\sigma}$ of L to be

$$\tilde{\sigma}(x,\xi)^2 = \sum_{I \in S} |\sigma_I(x,\xi)|^{2/|I|}. \tag{3}$$

The bracket hypothesis is equivalent to the existence of $C, \varepsilon \in \mathbb{R}^+$ and m such that $\tilde{\sigma}(x,\xi) \geq C\langle\xi\rangle^{\varepsilon}$ for all (x,ξ).

The symplectic structure of the cotangent bundle is reflected in the next definition.

Definition 2. *A C^1 function ψ defined on $T^*\mathbb{R}^d$ is said to be microlocally 1-Lipschitz relative to $\{X_j\}$ if it is 1-Lipschitz relative to $(\mu, \{V_j\}) = (\tilde{\sigma}, \{H_{\sigma_j}\})$ where $\tilde{\sigma}$ is the effective symbol defined in (3), and H_{σ_j} is the Hamiltonian vector field associated to the principal symbol σ_j of iX_j.*

With this definition, even when $L = \sum X_j^2$ is elliptic, (1) by itself does not force ψ even to be continuous; when L has constant coefficients, for instance, each $V_j = H_{\sigma_j}$ is everywhere in the span of $\{\partial/\partial x_l\}$. The auxiliary condition (2), which does not depend on the vector fields X_j at all, is one of the inequalities defining the symbol class $S^1_{1,0}$; it ensures that ψ is locally 1-Lipschitz with respect to the standard metric ρ_0. (2) implies (1) when L is elliptic, but not in general.

Our main definition is as follows. Associated to $\{X_j\}$, via $\tilde{\sigma}$ and the collection of Hamiltonian vector fields $\{H_{\sigma_j}\}$, is a metric ρ on $T^*\mathbb{R}^d$.

Definition 3. *Let X_j be real C^∞ vector fields in an open subset of $\mathbb{R}^d$. Then*

$$\rho_{\{X_j\}}(p,q) = \sup_{\psi} |\psi(p) - \psi(q)|, \tag{4}$$

where the supremum is taken over all functions ψ that are microlocally 1-Lipschitz relative to $\{X_j\}$. To $L = \sum X_j^2$ is associated the metric

$$\rho_L = \rho_{\{X_j\}}. \tag{5}$$

The motivation for this definition stems from proofs of hypoellipticity in various cases and function spaces [21],[18],[22],[26],[16],[9].

Although ρ_L is defined as a metric in the sense of point set topology, its definition could be modified to give a Riemannian metric with properties equivalent for our purpose. (2) implies that for any collection of vector fields X_j, there exists $C < \infty$ such that $\rho_{\{X_j\}} \leq C\rho_0$. Another simple property of ρ_L is that L is elliptic if and only if ρ_L is comparable on the infinitesimal level to ρ_0.

Remark. The construction of ρ_L is conformally invariant in the sense that if $L_2 = r^2 \cdot L_1$ for some constant $r \neq 0$, and if L_1 is a subelliptic sum of squares operator, then $\rho_1 \equiv \rho_2$.

Let $\{X_j\}$ be a collection of smooth real vector fields on a connected open set $U \subset \mathbb{R}^d$. The associated Carnot-Caratheodory metric on U is specified by saying

[1] In the absence of the bracket hypothesis, there is no satisfactory general description of the effective symbol. *Ad hoc* definitions can often be made in particular cases.

that ψ is 1-Lipschitz if $\sum_j |X_j\psi|^2 \le 1$ at every point. We next introduce a one-parameter family of metrics $\{\varrho_R : R \to \infty\}$ on the base space U, which combines features of ρ_L and the Carnot-Caratheodory metric. In Example 3 below it will help illustrate what is detected by the former but not by the latter.

Define the auxiliary quantity $\nu : U \times \mathbb{R}^+ \mapsto \mathbb{R}^+$ by

$$\nu(x,R) = \min_{|\xi|=R} \tilde{\sigma}(x,\xi). \tag{6}$$

For any $R, R' \ge 1$ satisfying $R \sim R'$, one has $\nu(x,R) \sim \nu(x,R')$, uniformly in x, R, R'. If the bracket condition holds to order m at x, then $\nu(x,R) \ge cR^{1/m}$ as $R \to \infty$.

Definition 4. *$\varrho_R(p,q)$ is the supremum of $|\psi(p)-\psi(q)|$, taken over all functions ψ satisfying both*

$$|X_j(\psi)(x)| \le \nu(x,R) \qquad \text{for all } j \tag{7}$$

$$R^{-1}|\nabla_x\psi| \le 1. \tag{8}$$

An equivalent formulation is that ϱ_R is the Carnot-Caratheodory metric associated to $\{\nu(x,R)^{-1}X_j\} \cup \{R^{-1}\partial/\partial x_k : 1 \le k \le d\}$. For each $R > 0$, this family of vector fields spans the tangent space at each point of U, but degenerates as $R \to 0$. If the bracket condition holds to order m at every point of U, then $\varrho_R(\cdot,\cdot) \ge cR^{1/m}$ on any compact subset of $U \times U$ minus the diagonal, as $R \to \infty$.

This construction takes a simpler form in the important special case where the set of vector fields X_j has cardinality $d-1$ and is everywhere linearly independent over an open set $U \subset \mathbb{R}^d$; this arises when U has a CR structure and the vector fields X_j represent the real and imaginary parts of $\bar{\partial}_b$. Then the characteristic variety Σ of $\sum X_j^2$ is a line bundle over U. In this case a quantity equivalent to $\nu(x,R)$ is obtained by deleting the supremum in (6) and instead taking ξ to be an element of the fiber of Σ lying over x, of magnitude R.

3. The Question

The following question may be fundamental. Let $s \in [1,\infty)$. We say that two sets $\Gamma, \Gamma' \subset T^*\mathbb{R}^d$ are c–separated if any two points $p = (x,\xi) \in \Gamma,\ p' = (x',\xi') \in \Gamma'$ satisfy $|x-x'| + (1+|\xi|+|\xi'|)^{-1}|\xi-\xi'| > c$. Let $\{X_j\}$ be a finite collection of C^ω real vector fields.

Main Question. *Consider the condition: For every $c > 0$, there exists $c' > 0$ such that for any two c–separated open sets $\Gamma, \Gamma' \subset T^*\mathbb{R}^d$,*

$$\rho_L(p,p') \ge c'\rho_0(p,p')^{1/s} \qquad \text{for all } p \in \Gamma, p' \in \Gamma'. \tag{9}$$

Is (9) or some closely related condition nearly equivalent to microlocal hypoellipticity of $\sum X_j^2$ in the Gevrey class G^s?

It is essential in (9) that Γ, Γ' be separated; thus no infinitesimal comparison of metrics is made.[2] Restricting attention to pairs (p,p') for which $\rho_0(p,p') \sim 1$, rather than merely $\gtrsim 1$, results in an equivalent question.

[2]When $s = 1$, the effective symbol may be replaced by $(1+\sum_j \sigma_j^2)^{1/2}$ in the definition of ρ_L without affecting the validity of (9).

In the remainder of this paper, the metric ρ_L will be analyzed for a number of examples, for all but one of which the optimal C^∞, analytic, and Gevrey class hypoellipticity properties of which are already known. In each known case, the result will be seen to be consistent with an affirmative answer to this question, with the word "nearly" omitted.

Nonetheless we have deliberately refrained from posing the question more precisely or labeling it a conjecture. Some variant of (9) may be more directly related to hypoellipticity; some such variants impose stronger restrictions on the functions ψ considered in the definition of ρ_L, involving upper bounds on their partial derivatives of orders greater than one, requiring them to belong to an appropriate symbol class; thus when $s = 1$, ψ should perhaps be an analytic symbol[3] as defined for instance in [28]. ρ_L is merely a leading order approximation to such a condition. Such variants of (9) are genuinely inequivalent, as shown in Example 9 below. To the extent of this author's knowledge, such a stronger variant is indeed satisfied with $s = 1$ for every operator that has so far been proved to be analytic hypoelliptic.

As will be seen in the subsequent discussion, (9) for $s = 1$ is related to a criterion for analytic hypoellipticity conjectured by Treves [29]; to further analyze this connection would be worthwhile. Our final Example 9 is a test case which might distinguish between the two points of view.

For C^∞ hypoellipticity the analogous condition would be that for every $A < \infty$ there should exist $C_A < \infty$ such that

$$\rho_L(p, p') \geq A \log \rho_0(p, p') - C_A \tag{10}$$

for all $p \in \Gamma, p' \in \Gamma'$, where ρ_L would denote the metric associated to an appropriately defined effective symbol of $\sum X_j^2$, if a meaningful effective symbol were to exist. The relevance of the logarithm is suggested by [21].

4. Examples

The notation $A \sim B$ will mean that the quantity A/B is bounded above and below by positive finite constants, uniformly in all relevant parameters; usually these parameters will be points in the cotangent bundle or certain subregions thereof, and a large parameter λ. Throughout the discussion, the zero section is always understood to be excluded from any cotangent bundle.

Example 1. Let $L = \sum X_k^2$ be an arbitrary sum of squares operator satisfying the bracket hypothesis to order m at every point of an open set $U \subset \mathbb{R}^d$. Then $\tilde\sigma(x, \xi) \geq c(1 + |\xi|)^{1/m}$. Therefore in the region where $|\xi| \sim \lambda$, the functions $\lambda^{1/m} x_j$ and $\lambda^{-(m-1)/m} \xi_j$ are microlocally 1-Lipschitz, up to a constant factor, uniformly for large λ.

From this it follows that

$$\rho_L(p, q) \geq c \rho_0(p, q)^{1/m} \tag{11}$$

[3]One reason for wariness is that such a conjecture takes little account of the behavior of the symbols σ_{X_j} in the complexified phase space.

whenever $\rho_0(p,q) \geq 1$. This is consistent with an affirmative answer to our main question, a theorem of Derridj and Zuily asserting that such operators are Gevrey class hypoelliptic of every order $s \geq m$, and the existence of such operators which are not Gevrey hypoelliptic of any smaller order, such as (13) below.

On the other hand, for any operator we have $\rho_L \leq C\rho_0$. In particular, when L is elliptic, ρ_L is pointwise comparable to ρ_0.

Remark. This conclusion for elliptic operators underscores the contrast between ρ_L and a metric $\tilde\rho_L$ (which will not be defined here) introduced by Fefferman [14] and Parmeggiani [24]. Firstly, the correspondence between L and $\tilde\rho_L$ has a fundamental monotonicity property: For any two operators L_1 and L_2 whose principal symbols are nonnegative and satisfy $\sigma_1(p) \leq \sigma_2(p)$ for every point $p \in T^*\mathbb{R}^n$, one has $\tilde\rho_{L_1}(p,p') \geq \tilde\rho_{L_2}(p,p')$ for all p,p'. The Carnot-Caratheodory metric construction on the base space $\mathbb{R}^d$ likewise shares this monotonicity. The mapping from L to ρ_L has by design no such monotonicity; nor should it, if it is to govern hypoellipticity, for the hypoellipticity of L_1 (in various function spaces) fails in general to imply hypoellipticity of L_2 when $\sigma_1 \leq \sigma_2$, as demonstrated for instance by the fundamental non-analytic hypoelliptic example $\partial_x^2 + (x^2+y^2)\partial_y^2$ of Métivier [20].

Secondly, up to order of magnitude, ρ_L is largest when L is elliptic, whereas $\tilde\rho_L$ is then smallest.

Example 2. The simplest type of obstruction to (9) in the case $s = 1$ is the possible existence of a nonvanishing vector field V in the span of $\{H_{\sigma_j}\}$, such that some integral curve γ of V is contained in the characteristic variety $\Sigma = \{p : \sigma_j(p) = 0 \text{ for all } j\}$. Treves [27] has conjectured that this obstruction should preclude analytic hypoellipticity of L.

To see that (9) must fail to hold with $s = 1$ in this situation, let $p = (x,\xi)$ and $p' = (x',\xi')$ be any two distinct points of γ, and for large $\lambda \in \mathbb{R}^+$ set $\lambda p = (x, \lambda\xi)$ and $\lambda p' = (x', \lambda\xi')$. Consider the integral curve γ_λ joining λp to $\lambda p'$, parametrized by $s \mapsto \exp(sV)(\lambda p)$; we have $\lambda p' = \exp(c_0 V)(\lambda p)$, where c_0 is a positive constant independent of λ, because the symbols σ_j of the underlying vector fields are homogeneous functions of degree one on $T^*\mathbb{R}^d$. Moreover γ_λ is contained in Σ by hypothesis. As is well-known, this implies a loss of at least one half of a derivative: along γ_λ,

$$\tilde\sigma(x,\xi) \leq C(1+|\xi|)^{1/2} \sim \lambda^{1/2}.$$

Any microlocally 1-Lipschitz function ψ must satisfy $|V(\psi)| \leq \tilde\sigma \leq C\lambda^{1/2}$ along γ_λ. Consequently $|\psi(\lambda p) - \psi(\lambda q)| \leq C\lambda^{1/2}$. Taking the supremum over all such ψ yields

$$\rho_{\{X_j\}}(\lambda p, \lambda p') \leq C\rho_0(\lambda p, \lambda p')^{1/2}$$

as $\lambda \to +\infty$. Consequently (9) fails to be satisfied for $s = 1$.

More precisely, if $\tilde\sigma(\lambda q) \leq C\lambda^{1/m}$ for every q belong to γ_λ, then

$$\rho_L(\lambda p, \lambda p') \leq C\rho_0(\lambda p, \lambda p')^{1/m}. \tag{12}$$

A concrete example is the Baouendi-Goulaouic type operator [1] in $\mathbb{R}^3$:

$$\partial_x^2 + \partial_y^2 + (x^{m-1}\partial_t)^2 \tag{13}$$

Take $p = (0,0,0;0,0,1)$ and $p' = (0,1,0;0,0,1)$ and $V = \partial_y$.

Remark. Assuming for simplicity that the collection of vector fields $\nabla_{x,\xi}\sigma_j$ has constant rank, so that Σ is a smooth manifold, the tangent vector $\dot\gamma$ of a curve $\gamma \subset \Sigma$ belongs to the span V of $\{H_{\sigma_j}\}$ at a point $p \in \gamma$, if and only if $\dot\gamma$ is orthogonal, with respect to the canonical symplectic form ω, to the tangent space $T_p\Sigma$. To see this, denote by $W^\perp$ the orthocomplement, with respect to ω, of any subspace W of $T_p\Sigma$, and recall that $(W^\perp)^\perp = W$. $T_p\Sigma$ is the set of all $v \in T_p(T^*\mathbb{R}^d)$ satisfying $v(\sigma_j)(p) = 0$ for all j. Since $v(\sigma_j) = \omega(v, H_{\sigma_j})$ for any $v \in T_pT^*\mathbb{R}^d$, we have $T_p\Sigma = V^\perp$, so equivalently $(T_p\Sigma)^\perp = V$.

Remark. A more general conjecture, also due to Treves [29], is formulated in terms of a stratification $\Sigma = \Sigma_0 \supset \Sigma_1 \supset \cdots$, based upon the vanishing not only of the symbols σ_j, but also of their iterated Poisson brackets, as well as on quantities such as the dimension of the span of their gradients and the rank of the bilinear form defined by the restriction to the tangent space of a stratum of the symplectic form. Conjecturally, L is not analytic hypoelliptic if and only if there exist k and a curve $\gamma \supset \Sigma_k \backslash \Sigma_{k+1}$ whose tangent is everywhere orthogonal, with respect to ω, to $T\Sigma_k$. I believe that the discussion of Example 2 may be generalized to conclude that the existence of such a curve γ implies a certain upper bound on ρ_L, depending on k, but have not verified the details.

Example 3. Suppose throughout this example that $X_1, \dots, X_{d-1}$ are linearly independent at each point of $U \subset \mathbb{R}^d$, and satisfy the bracket hypothesis to order two at every point of U. Then $\rho_L(p,q) \ge c\rho_0(p,q)^{1/2}$ for separated points. According to Example 2, this lower bound cannot in general be improved.

If however the characteristic variety $\Sigma = \{(x,\xi) : \sigma_j(x,\xi) = 0 \text{ for all } j\}$ is a symplectic manifold and the symbols have linearly independent gradients, then each Hamiltonian vector field H_{σ_j} is everywhere transverse to Σ. As one might expect, it can be shown that in this circumstance $\rho_L(p,q) \ge c\rho_0(p,q)^1$ for all separated points, consistent with the theorem of Tartakoff and of Treves asserting analytic hypoellipticity in the symplectic situation.

Now consider the one-parameter family of metrics ϱ_R of Definition 4. Σ is a line bundle over $U \subset \mathbb{R}^d$, and $\nu(x,R) \sim R^{1/2}$ for all x, R. From this and the order two bracket hypothesis it is easily deduced that $\varrho_R(x,y)$ is comparable to $R^{1/2}$ times the associated Carnot-Caratheodory metric at distances greater than $R^{-1/2}$ with respect to the latter metric. In particular, $\varrho_R(x,y) \sim R^{1/2}$ as $R \to \infty$, for fixed $x \ne y$, whether or not Σ is symplectic. Thus the family ϱ_R fails to detect a property of the vector fields which is fundamental for analytic hypoellipticity and is detected by ρ_L.

Example 4. Let $W \subset \mathbb{R}^d$ be any connected open set. Let Γ be a conic open subset of T^*W. Suppose that L is elliptic in Γ. Then for separated points p, q lying in distinct topological components of $T^*W \backslash \Gamma$, $\rho_L(p,q) \ge c\rho_0(p,q)$. Indeed, any $\psi \in S^1_{1,0}$ whose gradient is supported in Γ is microlocally 1-Lipschitz, up to multiplication by a bounded constant factor, and there exist such functions satisfying $|\psi(p) - \psi(q)| \sim \rho_0(p,q)$.

More generally, if $\tilde\sigma(x,\xi) \ge C|\xi|^\delta$ for all $(x,\xi)\in\Gamma$, then by the same reasoning $\rho_L(p,q)\ge c\rho_0(p,q)^\delta$ for all p,q satisfying the same hypothesis.

The next example concerns C^∞ hypoellipticity. In the C^∞ case one seeks only to have $\rho(p,p') \gg \log\rho_0(p,p')$ for separated p,p'. Under the very weak hypothesis that L has an effective symbol $\tilde\sigma(x,\xi)$ which tends to infinity as $|\xi|\to\infty$, this bound is easily obtained whenever $|\xi|$ and $|\xi-\xi'|$ are large, because the function $\psi(x,\eta)=\log(|\eta-\xi|)$ is always locally 1-Lipschitz modulo a constant factor, and an arbitrarily large constant multiple of ψ is 1-Lipschitz relative to $\{X_j\}$ for sufficiently large $|\xi|$, provided that $\tilde\sigma\to\infty$.

Example 5. In $\mathbb{R}^2$ with coordinates (x,t) consider $L=X^2+Y^2$ where $X=\partial_x,\ Y=a(x)\partial_t$. Suppose that $a\in C^\infty$, and that $a(x)=0$ if and only if $x=0$. All such operators are C^∞ hypoelliptic [12], regardless of their degree of degeneracy. Let us see that this is consistent with our point of view.

The characteristic variety is $\Sigma=\{x=\xi=0\}$ (which is symplectic). Because of the remark made two paragraphs above, the condition $\rho_L(p,q)\gg\log\rho_0(p,q)$ is most likely to fail when $p,q\in\Sigma$ and both have the same τ coordinate. Thus let $p=(0,0,0,\tau)$ and $q=(0,0,1,\tau)$ with τ large and positive. Consider $\psi(x,t,\xi,\tau)=t\tau$. Certainly $|\psi(p)-\psi(q)|=\tau$ is $\gg\log\rho_0(p,q)\sim\log\tau$, and we claim that ψ is 1-Lipschitz. Indeed, the two Hamiltonian vector fields are ∂_x and $a(x)\partial_t-\tau a'(x)\partial_\xi$. The former annihilates ψ, while applying the latter to ψ gives $a(x)\tau$, which equals the symbol of Y and hence is acceptably bounded.

Example 6. Let (x,t) be coordinates in $\mathbb{R}^2$ and $(x,t;\xi,\tau)$ be coordinates in the cotangent bundle. Consider

$$L_1=\partial_x^2+(x\partial_t)^2 \tag{14}$$

$$L_2=\partial_x^2+(x\partial_t)^2+(t\partial_t)^2. \tag{15}$$

The associated effective symbols are up to order of magnitude equal to $\tilde\sigma_1\sim 1+|\xi|+|x\tau|+|\tau|^{1/2}$, and $\tilde\sigma_2\sim 1+|\xi|+|x\tau|+|\tau|^{1/2}+|t\tau|$. Thus $\tilde\sigma_1\le C\tilde\sigma_2$.

The Hamiltonian vector fields associated to L_1 are ∂_x and $x\partial_t-\tau\partial_\xi$. Associated to L_2 is the additional Hamiltonian field $t\partial_t-\tau\partial_\tau$. Thus there arise two competing effects in comparing ρ_{L_2} to ρ_{L_1}: $\tilde\sigma_2$ is larger, but the presence of an additional Hamiltonian vector field introduces additional constraints on microlocally Lipschitz functions ψ.

To compare the two metrics, let $\lambda\in\mathbb{R}^+$ be a large parameter and consider the two points $p=(0,0;0,\lambda)$ and $q=(0,0;0,2\lambda)$. The function $\psi(x,t;\xi,\tau)=\tau$ is microlocally Lipschitz with respect to the metric associated to L_1, because it belongs to $S^1_{1,0}$ and is annihilated by the two Hamiltonian vector fields. Therefore $\rho_{L_1}(p,q)\ge c\lambda$; it is comparable to the distance between these points in the metric associated to any elliptic operator, even though L_1 fails to be elliptic.

Consider instead L_2, and consider the restriction of any microlocally 1-Lipschitz function ψ to the line segment joining p to q. The Hamiltonian vector field $t\partial_t-\tau\partial_\tau$ is tangent to this line segment, and $\tilde\sigma_2$ has order of magnitude $\lambda^{1/2}$ on the whole segment, so the main condition (1) forces $|\psi(p)-\psi(q)|\le C\lambda^{1/2}$. Thus $\rho_{L_2}(p,q)\le$

$C\lambda^{1/2} \ll \rho_{L_1}(p,q)$. Indeed, Métivier [20] has proved that L_2 is hypoelliptic in G^s only for $s \geq 2$, whereas L_1 is G^s hypoelliptic for all $s \geq 1$ by a theorem of Grušin [17].

Further analysis shows that $\rho_{L_2}(p,q) \geq c\rho_0(p,q)^{1/2}$ for any points p, q. L_2 is Gevrey hypoelliptic of order s if and only if $s \geq 2$, so this example is consistent with an affirmative answer to our main question.

So far we have derived upper bounds on ρ_L by examining the variation of microlocally Lipschitz functions ψ along integral curves $\gamma_\lambda \subset \Sigma$ of Hamiltonian vector fields. Our next two examples demonstrate subtler effects due to the influence of one-parameter families of integral curves γ_λ which do not quite lie in the characteristic variety Σ, but merely tend to Σ at certain rates as $\lambda \to \infty$.

Example 7. With coordinates $(x, y, t; \xi, \eta, \tau)$ for $T^*\mathbb{R}^3$, consider

$$\partial_x^2 + (x^{k-1}\partial_y)^2 + (x^{m-1}\partial_t)^2 \tag{16}$$

where $2 \leq k \leq m$. Here

$$\tilde{\sigma} \sim |\xi| + |x^{k-1}\eta| + |x^{m-1}\tau| + |\eta|^{1/k} + |\tau|^{1/m}.$$

Consider the associated Hamiltonian vector field $V = x^{k-1}\partial_y - (k-1)x^{k-2}\eta\partial_\xi$. Let $p = (\delta, 0, 0; 0, 0, \lambda)$ where $\delta = \lambda^{-1/m}$, and $q = \exp(TV)(p)$ where $T = \delta^{1-k}$. The integral curve γ joining p to q takes the simple form $\exp(sV)(p) = (\delta, s\delta^{k-1}, 0; 0, 0, \lambda)$. In particular, $q = (\delta, 1, 0; 0, 0, \lambda)$. Along γ, $\tilde{\sigma} \sim \lambda^{1/m}$ because δ was chosen so that $\delta^{m-1}\lambda \sim \lambda^{1/m}$. Thus any microlocally 1-Lipschitz function ψ must satisfy $|V\psi| \leq C\lambda^{1/m}$, whence $|\psi(p) - \psi(q)| \leq CT\lambda^{1/m} = C\lambda^{(k-1)/m}\lambda^{1/m} = C\lambda^{k/m}$.

On the other hand, the function $\psi(x, y, t; \xi, \eta, \tau) = \lambda^{k/m}y$ is microlocally Lipschitz in the region where $|(\xi, \eta, \tau)| \sim \lambda$, modulo a uniformly bounded factor, because setting $\theta = (k-1)/(m-1) \in (0, 1]$,

$$|V\psi| = |x|^{k-1}\lambda^{k/m} = (|x|^{m-1}\lambda)^\theta(\lambda^{1/m})^{1-\theta} \lesssim |x|^{m-1}\lambda + \lambda^{1/m}.$$

Since $|\psi(p) - \psi(q)|$ is comparable to $\lambda^{k/m}$. we conclude

$$\rho_L(p,q) \sim \lambda^{k/m} \sim \rho_0(p,q)^{k/m}.$$

This is consistent with both an affirmative answer to our main question, and the fact that L is hypoelliptic in the Gevrey class of order s if and only if $s \geq m/k$, assuming that $m \geq k$ [6].

Example 8. Fix any integers $m, r > 1$. In $\mathbb{R}^2$ consider

$$\partial_x^2 + (x^{m-1}\partial_t)^2 + (t^r\partial_t)^2; \tag{17}$$

the case $r = 1$ has been discussed above. One of the associated Hamiltonian vector fields is $V = t^r\partial_t - rt^{r-1}\tau\partial_\tau$. The effective symbol has order of magnitude $|\xi| + |x^{m-1}\tau| + |t^r\tau| + |\tau|^{1/m}$.

Given a large quantity λ, define $\delta = \lambda^{-(m-1)/mr}$, so that $\delta^r\lambda = \lambda^{1/m}$. Let $p = (0, \delta; 0, \lambda)$ and $q = \exp(-TV)(p)$ where $T = \delta^{1-r}$. Denote by γ the segment of integral curve $\exp(-sV)(p)$, $0 \leq s \leq T$. For every point of γ, uniformly in λ, $\exp(-sV)(p) = (0, t(s); 0, \tau(s))$ where $\delta \geq t(s) \geq c\delta$ and $\lambda \leq \tau(s) \leq C\lambda$. Moreover $q = (0, t(T); 0, \tau(T))$ where $\tau(T) - \lambda$ is positive and has order of magnitude λ. By the choice of δ and the fact that $|t(s)| \le \delta$ for every point of γ, $\tilde\sigma \le C\lambda^{1/m}$ at every point of γ.

Every microlocally 1-Lipschitz function ψ must therefore satisfy $|V\psi| \le C\lambda^{1/m}$ at every point of γ, and consequently $|\psi(p) - \psi(q)| \le CT\lambda^{1/m}$. Since $T = \delta^{1-r}$, we conclude that

$$\rho(p,q) \le C\lambda^{(mr-m+1)/mr}. \tag{18}$$

This upper bound on $\rho_L(p,q)$ is also a lower bound, up to a constant factor. To see this consider the function $\psi(x,t;\xi,\tau) = \lambda^a \cdot \tau$, where $a+1 = (mr-m+1)/mr < 1$. Then in the region where $|(\xi,\tau)| \sim \lambda$, ψ is a symbol in $S^1_{1,0}$ uniformly in λ, and $|V(\psi)| \le C|t|^{r-1}\lambda^{a+1}$. To show that ψ is microlocally Lipschitz in this region relative to the vector fields in question, we must verify that $|V\psi| \le C\tilde\sigma$. Indeed, this holds because $\tilde\sigma(x,t;\xi,\tau) \ge c\lambda^{1/m} + c|t|^r\lambda$ there, and $|t|^{r-1}\lambda^{a+1}$ equals the logarithmically convex average $(|t|^r\lambda)^\theta \cdot (\lambda^{1/m})^{1-\theta} \le C\tilde\sigma$ where θ equals $(r-1)/r$. Thus ψ is microlocally Lipschitz, and

$$|\psi(p) - \psi(q)| \sim \lambda^{(mr-m+1)/mr}, \tag{19}$$

whence $\rho_L(p,q) \ge c\lambda^{(mr-m+1)/mr}$. It has indeed been proved in [8], as well as in [4] and [19], that L is G^s hypoelliptic for all s satisfying $s^{-1} \le (mr-m+1)/mr$. In unpublished work this author has outlined a proof that the threshold $mr/(mr-m+1)$ is optimal, based on the method introduced in [5] and [7].

Remark. Both a conjectured characterization of analytic hypoellipticity by Treves [29], and a conjectured sufficient condition for Gevrey hypoellipticity by Bove and Tartakoff [4], are formulated in terms of properties of a certain stratification of the characteristic variety Σ of L. This stratification does not distinguish between different parameters r in Example 8. Nonetheless one of the two versions of the conjecture in [29] concerning the analytic case is consistent with all examples known to this author.

5. A Subtler Example

We next discuss an operator for which analytic hypoellipticity apparently remains an open question. It is the principal part of the Kohn Laplacian for a three-dimensional pseudoconvex CR manifold, and is of particular interest both because it is one of the simplest operators whose analytic hypoellipticity is undecided, and because it illustrates the distinction between two variants of our main question; functions ψ which are merely Lipschitz relative to ρ_L can vary much more rapidly than those which also belong to the symbol class $S^1_{1,0}$ and hence satisfy higher derivative bounds.

Consider the domain $\Im(z_2) > b(z_1)$ in $\mathbb{C}^2$, where b is assumed to be subharmonic so that this domain will be pseudoconvex. Its boundary may be identified with $\mathbb{R}^3$, with coordinates (x,y,t), in such a way that a Cauchy-Riemann operator is $\bar\partial_b = X + iY$ where $X = \partial_x - b_y\partial_t$ and $Y = \partial_y + b_x\partial_t$. Here $b_x = \partial_x b$, $b_y = \partial_y b$, and $b(x,y) \equiv b(x+iy)$. The Kohn Laplacian is then $(X+iY)(X-iY)$, which equals

X^2+Y^2 modulo a lower order term.[4] Analytic or Gevrey hypoellipticity of such an operator depends only on Δb, where Δ denotes the Laplacian in $\mathbb{R}^2$, rather than on b itself, for any operator may be transformed into any other with the same invariant Δb via a change of variables $(x,y,t)\mapsto(x,y,t-\phi(x,y))$.

Example 9. In $\mathbb{R}^3$ consider $L=X^2+Y^2$ with

$$X=\partial_x-b_y\partial_t,\quad Y=\partial_y+b_x\partial_t$$

where b is a polynomial depending only on x,y, satisfying $b(0)=0$, $\nabla b(0,0)=0$, and

$$\Delta b(x,y)=x^k+y^k+x^2y^2,$$

where $k\ge 6$ is an even integer. Define $\lambda(x,y)=\Delta b=x^k+y^k+x^2y^2$.

For general $b(x,y)$ it has been shown by Grigis and Sjöstrand [16] that if Δb vanishes only at $x=y=0$ and is a polyhomogeneous function of (x,y), that is, $\Delta b(r^kx,r^ly)\equiv r^n\Delta b(x,y)$ for some positive integers k,l,n, then L is analytic hypoelliptic. In our example $\lambda=\Delta b$ is polyhomogeneous only for $k=4$.

Since $[X,Y]=\lambda(x,y)\partial_t$ and λ vanishes only on the line $\{(x,y,t):x=y=0\}$ in $\mathbb{R}^3$, the bracket hypothesis is satisfied to order 2 everywhere except on that line. Where $x=y=0$, it is satisfied to order precisely $m=6$, because of the presence of the term x^2y^2 in λ. The characteristic variety $\Sigma\subset T^*\mathbb{R}^3$ of L is a manifold of codimension two. It is symplectic, at every point at which the principal symbol of $[X,Y]$ is nonzero. Thus Σ is symplectic everywhere except on the submanifold $\Sigma'\subset\Sigma$ given by $\Sigma'=\{(x,y,t;\xi,\eta,\tau):x=y=\xi=\eta=0\}$. Hence by virtue of the theorem of Tartakoff and of Treves, L is microlocally analytic hypoelliptic everywhere except possibly on Σ'.

Any particular iterated commutator of X,Y has a principal symbol which either vanishes identically on Σ', or vanishes nowhere on Σ'. Therefore the stratification defined by Treves [29] takes the simple form $\Sigma\supset\Sigma'\supset\emptyset$. Because Σ' is itself a symplectic manifold, his conjecture then predicts that L should be analytic hypoelliptic.

For simplicity we set $k=6$ for the remainder of the discussion. All results generalize to $k\ge 6$, but the details of the analysis change slightly.

We will establish two contrasting conclusions concerning this example. Set $p_T=(0,0,0;0,0,T)$ and $q_T=(0,0,1;0,0,T)$, where $T\in\mathbb{R}^+$ is a large parameter.

Proposition 5.1. *There exists $c>0$ such that for each large T there exists a function φ_T which is microlocally 1-Lipschitz relative to $\{X,Y\}$, and satisfies $|\varphi_T(q_T)-\varphi_T(p_T)|\ge cT$.*

Proposition 5.2. *It is not the case that there exist $c>0$ and a one-parameter family of functions $\varphi_T\in C^9$ such that for each large $T\in\mathbb{R}^+$, φ_T is microlocally 1-Lipschitz relative to $\{X,Y\}$, $|\varphi_T(q_T)-\varphi_T(p_T)|\ge cT$, and*

$$|\partial^\alpha_{x,y,t}\partial^\beta_{\xi,\eta,\tau}\varphi_T|\le C_{\alpha,\beta}(1+|(\xi,\eta,\tau)|)^{1-|\beta|}\qquad \textit{for all } |\alpha|+|\beta|\le 9.$$

[4] This lower order term is not insignificant, but the questions of interest regarding $(X+iY)(X-iY)$ concern its microlocal hypoellipticity properties in a subset of the cotangent bundle, where its properties coincide with those of X^2+Y^2 in all known cases.

Thus whether our philosophy predicts analytic hypoellipticity or not, depends on precisely which variant of ρ_L we consider.[5]

We first prove Proposition 5.1, by constructing a single function φ which has the required properties simultaneously for all T. Consider a function

$$\varphi(x,y,t;\xi,\eta,\tau) = (t+h)\tau + f\cdot(\xi-\tau b_y) + g\cdot(\eta+\tau b_x)$$

where f,g,h are functions of x,y alone. Our aim is construct f,g,h so that φ is microlocally 1-Lipschitz relative to $\{X,Y\}$. Since $\varphi(q_T)-\varphi(p_T) = T-0 = T$, we can then conclude that $\rho_L(p_T,q_T) \ge cT$.

Because the characteristic variety of L is $\{x=y=\xi=\eta=0\}$ and because the function $(x,y,t;\xi,\eta,\tau)\mapsto\tau$ is annihilated by both H_X, H_Y, it is easy to conclude using the same function φ that more generally, for any separated points p,q, one has $\rho_L(p,q) \ge c\rho_0(p,q)$.

The Hamiltonian vector fields associated to X,Y are

$$\begin{aligned} H_X &= \partial_x - b_y\partial_t + \tau b_{yx}\partial_\xi + \tau b_{yy}\partial_\eta, \\ H_Y &= \partial_y + b_x\partial_t - \tau b_{xx}\partial_\xi - \tau b_{xy}\partial_\eta. \end{aligned}$$

Thus

$$\begin{aligned} H_X\varphi &= -\tau b_y - \tau f b_{xy} + \tau g b_{xx} + \tau f b_{xy} + \tau g b_{yy} + \tau h_x && \text{plus error terms} \\ &= \tau[-b_y + g\Delta b + h_x] && \text{plus error terms} \\ H_Y\varphi &= -\tau f b_{yy} + \tau g b_{xy} + \tau b_x - \tau b_{xx} f - \tau b_{xy} g + \tau h_y && \text{plus error terms} \\ &= \tau[b_x - f\Delta b + h_y] && \text{plus error terms.} \end{aligned}$$

Each of the aforementioned error terms is a product of one of f_x, f_y, g_x, g_y with either $(\xi-\tau b_y)$ or $(\eta+\tau b_x)$. The latter two quantities are the principal symbols of X,Y, hence are dominated by $\tilde\sigma$. If f,g are constructed so as to be Lipschitz continuous functions of (x,y), then each error term will be majorized by $\tilde\sigma$. Since our goal is to have $|H_X\varphi| + |H_Y\varphi| \le C\tilde\sigma$, it therefore suffices to construct Lipschitz functions f,g and a function h such that

$$b_y - g\Delta b = h_x \qquad \text{and} \qquad -b_x + f\Delta b = h_y. \tag{20}$$

Recalling that $\Delta b = \lambda$, a necessary condition for (20) is that $b_{yy} - (g\lambda)_y = -b_{xx} + (f\lambda)_x$, which may be rewritten as

$$\lambda = (f\lambda)_x + (g\lambda)_y. \tag{21}$$

Conversely, if f,g are bounded solutions of (21) then there exists a solution h of (20) in the sense of distributions. But the left hand sides of both equations in (20) are bounded since b is a polynomial, so h is Lipschitz after possible redefinition on a set of measure zero; in fact $h\in C^{1,1}$ when f,g are Lipschitz. Thus in order to show that φ is microlocally Lipschitz relative to $\{X,Y\}$, it suffices to prove the existence of Lipschitz continuous (in the ordinary sense) solutions f,g of (21).

[5]We explicitly refrain from venturing any conjecture as to whether this particular operator is analytic hypoelliptic.

Lemma 5.3. *There exist Lipschitz continuous functions f, g, defined in a neighborhood of the origin in $\mathbb{R}^2$, satisfying $\lambda \equiv (f\lambda)_x + (g\lambda)_y$ almost everywhere.*

If λ were homogeneous or polyhomogeneous, then by Euler's identity there would exist a solution of the form $f = k_1 x$, $g = k_2 y$ for certain constants k_j. To construct Lipschitz solutions of $\lambda = f \cdot \lambda_x + g \cdot \lambda_y$ in the present inhomogeneous case is relatively easy, but is not what is required. In the course of the proof of Proposition 5.2 it will be shown that there exist no C^6 solutions f, g; perhaps this indicates that the complications of the following argument are not altogether avoidable.

Proof. We write $A \lesssim B$ to mean that the ratio A/B of nonnegative functions is bounded above by a finite constant, and $A \sim B$ to mean that $A \lesssim B$ and $B \lesssim A$. Different constructions will be used in different regions of the plane. Consider first the region Γ_1 where $|x| \le 2|y|$. We set $g \equiv 0$ and solve $(f\lambda)_x = \lambda$ by defining

$$f(x, y) = \lambda(x, y)^{-1} \int_0^x \lambda(s, y)\, ds.$$

In this region $\lambda \sim y^6 + x^2y^2$, so $|\int_0^x \lambda(x, y)\, ds| \lesssim |x|\lambda(x, y)$. Hence $|f(x, y)| \lesssim |x|$.

In analyzing ∇f it will be useful to note that

$$|\lambda_x| \lesssim \lambda/|x| \qquad \text{and} \qquad |\lambda_y| \lesssim \lambda/|y|.$$

To analyze ∇f consider first

$$f_x = 1 - \lambda^{-2}\lambda_x \cdot \int_0^x \lambda(s, y)\, ds.$$

Since $|\lambda_x| \lesssim \lambda/|x|$ and the integral is $\lesssim |x|\lambda$, the second term is uniformly bounded. Similarly

$$f_y = -\lambda^{-2}\lambda_y \int_0^x \lambda + \lambda^{-1} \int_0^x \lambda_y(s, y)\, ds.$$

In absolute value the first term is $\lesssim \lambda^{-2} \cdot (\lambda/|y|) \cdot (|x|\lambda)$, which is uniformly bounded in Γ_1. The absolute value of the second term is $\lesssim \lambda^{-1} \int_0^x \lambda/|y|\, ds \lesssim |x|/|y|$, so likewise is bounded.

Fix an auxiliary function ϕ which is homogeneous of degree zero in $\mathbb{R}^2$, is C^∞ except at the origin, is identically equal to one where $|x| \le |y|/2$, and is supported where $|x| \le |y|$. Set $\tilde{f} = f \cdot \phi$. This formula makes sense only where $|x| \le |y|$, but we extend the definition by setting $\tilde{f} \equiv 0$ where $|x| > |y|$. Then $\tilde{f}$ is Lipschitz, because $|\nabla\phi(z)| \lesssim |z|^{-1}$ and $|f(z)| \lesssim |z|$. We claim that where $|y|/2 \le |x| \le |y|$ and $|(x, y)|$ is sufficiently small,

$$|\nabla^\alpha \tilde{f}(z)| \le C_\alpha |z|^{1-|\alpha|}$$

for every multi-index α. Because ϕ is homogeneous and smooth except at the origin, it suffices to verify this for f. One has explicitly

$$f(x, y) = \frac{xy^6 + x^3y^2/3 + x^7/7}{y^6 + x^2y^2 + x^6}.$$

In the conic region in question, $\lambda(z) \sim |z|^4$, the terms x^6, y^6 in the denominator being comparatively negligible. Hence $|\nabla^\alpha f(z)| \lesssim |z|^{1-|\alpha|}$ where $|x| \sim |y|$. The claim follows.

In the (overlapping) region where $|y| \le 2|x|$, we solve instead $(g\lambda)_y = \lambda$, taking $f \equiv 0$. Because λ is a symmetric function of (x,y), conclusions parallel to those above may be obtained. Define $\tilde{g}(x,y) = g(x,y)\phi(y,x)$, and again define $\tilde{g} \equiv 0$ where $|y| > |x|$. We have then $\lambda \equiv (\tilde{f}\lambda)_x + (\tilde{g}\lambda)_y$ where $|x| \le |y|/2$ and also where $|y| \le |x|/2$.

Define

$$\tilde{\lambda} = \lambda - (\tilde{f}\lambda)_x - (\tilde{g}\lambda)_y.$$

$\tilde{\lambda}$ is supported in the conic region Γ_3 where $|y|/2 \le |x| \le 2|y|$. It satisfies $|\nabla^\alpha \tilde{\lambda}(z)| \lesssim |z|^{4-|\alpha|}$ for all $z \in \Gamma_3$ and all α.

In order to complete the proof of the lemma, it suffices to construct Lipschitz functions F, G satisfying

$$\tilde{\lambda} = (F\lambda)_x + (G\lambda)_y.$$

To accomplish this we set $F = xh$ and $G = yh$ and solve $\tilde{\lambda} = (xh\lambda)_x + (yh\lambda)_y$ for the single unknown h. In polar coordinates (r,θ) the equation becomes

$$\tilde{\lambda} = \lambda r h_r + h \cdot (2\lambda + r\lambda_r).$$

Define

$$\beta(x,y) = (2 + r\lambda_r/\lambda)$$

and note that $2 \le \beta$ and moreover that in Γ_3, $|\nabla^\alpha \beta(z)| \lesssim |z|^{-|\alpha|}$. The equation to be solved is

$$h_r + h\beta/r = r^{-1}\tilde{\lambda}/\lambda. \tag{22}$$

It suffices to produce a bounded solution whose gradient is $\lesssim r^{-1}$.

Note for future use that

$$|\beta_\theta(r,\theta)| \lesssim r$$

in Γ_3, not merely $\lesssim 1$. The fact that λ is homogeneous of degree 4 modulo higher order terms and nonvanishing in Γ_3 leads by Euler's identity to the conclusions $\beta(r,\theta) = 6 + O(r)$ and $\nabla\beta = O(1)$ in that cone. This strengthened bound is essential to our construction.

Define next

$$b(r,\theta) = -\int_r^1 \beta(\rho,\theta)\, d\rho/\rho.$$

Then $b(r,\theta) < 0$ for $r < 1$, and $|b(r,\theta)| \sim \log r^{-1}$. Because $\beta/\rho \ge 2/\rho$, $e^b(\rho,\theta) \lesssim \rho^2$ as $\rho \to 0$.

Define a solution h of (22) by

$$h(r,\theta) = e^{-b} \int_0^r e^b(\rho,\theta) \frac{\tilde{\lambda}}{\lambda}(\rho,\theta)\, \frac{d\rho}{\rho};$$

the integral converges since $\tilde{\lambda}/\lambda$ is bounded and $e^b \lesssim \rho^2$. Formally h satisfies the required equation. Note that h is supported in Γ_3, since the integrand vanishes identically outside it.

To complete the proof of the lemma it suffices to show that $|h| \lesssim 1$, and that $|\nabla h(r,\theta)| \lesssim r^{-1}$. Because we have seen above that $\tilde\lambda/\lambda$ is bounded, an upper bound for $|h|$ is $e^{-b}\int_0^r e^b\,d\rho/\rho$. Since b_r is between $2/r$ and C/r, the factor e^b is monotone increasing, and $e^b(r/2,\theta) \le \frac{1}{4}e^b(r,\theta)$. Consequently this last integral has the same order of magnitude as e^b, and hence h is uniformly bounded.

From the boundedness of h and the differential equation (22) it follows immediately that the partial derivative h_r is $\lesssim r^{-1}$ in absolute value. It remains to show that $|h_\theta| \lesssim 1$. Differentiation in the definition of h yields several terms; the simplest is $-b_\theta$ times h. We have

$$|b_\theta| \le \int_r^1 |\beta_\theta|\,d\rho/\rho,$$

and $|\beta_\theta(\rho,\theta)| \le \rho|\nabla\beta| \lesssim \rho$ in Γ_3, as observed above. Thus b_θ is uniformly bounded. Hence this simplest term is $\lesssim h$, hence uniformly bounded.

A second term arises when on differentiating h with respect to θ, the derivative falls upon the factor of e^b inside the integral. An additional factor of b_θ results, causing no harm since it is uniformly bounded; the analysis indicated above for h itself applies also to this term. The third and last term arises when the derivative falls on the factor of $\tilde\lambda/\lambda$ inside the integral. Now

$$|\partial_\theta(\tilde\lambda/\lambda)(r,\theta)| \le Cr|\nabla(\tilde\lambda/\lambda)| \lesssim 1$$

in Γ_3, since $|\nabla\tilde\lambda| \lesssim |z|^3$ in Γ_3, the same holds for λ, and $|z|^4 \lesssim \lambda(z)$ there. Thus the same analysis applies once more. Hence $|\nabla h| \lesssim r^{-1}$, and the proofs of both the lemma and Proposition 5.1 are complete. □

Proof of Proposition 5.2. Suppose to the contrary that functions φ_T possessing the indicated properties were to exist. Note that H_X, H_Y are homogeneous of degree zero with respect to the dilations $(x,y,t;\xi,\eta,\tau) \mapsto (x,y,t;r\xi,r\eta,r\tau)$ for $r \in \mathbb{R}^+$. By applying Arzela-Ascoli and a rescaling argument based on these dilations and homogeneity to the collection $\{\varphi_T\}$, we could then extract a single function φ which in any fixed relatively compact subset of $T^*\mathbb{R}^3$ minus the zero section belonged to C^8 and satisfied $\varphi(p) \ne \varphi(q)$, where $p = p_1, q = q_1$, and

$$|H_X\varphi| + |H_Y\varphi| \le |\xi - \tau b_y| + |\eta + \tau b_x|. \tag{23}$$

Other terms from the right hand side of (1) scale differently, and contribute zero in the rescaled limit.

The vector fields H_X, H_Y are tangent to each level set of τ, so the restriction of φ to $\{\tau = 1\}$ must satisfy the preceding inequality. Henceforth we restrict attention to that level set, freezing $\tau = 1$. Since $\varphi(p) \ne \varphi(q)$, there must exist t such that $\partial_t\varphi(0,0,t;0,0,1) \ne 0$. Both H_X, H_Y are translation invariant with respect to t, so we may translate the coordinates so that $\partial_t\varphi(0,0,0;0,0,1) \ne 0$.

Next, expanding in Taylor series with respect to t,ξ,η about $(t, b_y(x,y), -b_x(x,y))$ and rewriting the result slightly gives

$$\varphi(x,y,t;\xi,\eta,1) = h + \gamma t + f\cdot[\xi - b_y] + g\cdot[\eta + b_x] + E$$

where $E(x,y,t;\xi,\eta) = O(t,(\xi - b_y),(\eta + b_x))^2$ and the coefficients γ, f, g, h are C^7 functions of x, y alone. Moreover $\gamma(0,0) \neq 0$ because $\partial_t\varphi \neq 0$; by dividing through by a constant we may assume that $\gamma(0,0) = 1$.

Applying H_X, H_Y, evaluating at $t = \xi - b_y = \eta + b_x = 0$, and invoking (23) yields now the equations (20) that we arrived at in the proof of Proposition 5.1. Consequently $\lambda \equiv (f\lambda)_x + (g\lambda)_y$.

Consider the Taylor expansions of both sides of this last equation about the origin, and compare terms of equal degrees. By examining terms homogeneous of degree three, one finds immediately that $f(0) = g(0) = 0$. Equality for degree 4 forces $f = c_1 x + O^2(x,y)$ and $g = c_2 y + O^2(x,y)$, where $3c_1 + 3c_2 = 1$, in order to reproduce the term x^2y^2 on the left without introducing other monomials of degree 4. The contribution to $(f\lambda)_x + (g\lambda_y)$ of the sum of all terms homogeneous of degree 2 in the Taylor expansions of f, g is a homogeneous polynomial of degree 5 plus a remainder which is $O^7(x,y)$. The part of degree 5 must vanish identically, since no such terms are present in the expansion of λ. Consider finally the coefficient of x^6 on the right. It equals $7c_1 + c_2$, for the monomials x^2y^2 and y^6 in λ cannot possibly lead to any x^6 term on the right. Similarly the coefficient of y^6 equals $c_1 + 7c_2$. Therefore $7c_1 + c_2 = 1 = c_1 + 7c_2$. The unique solution is $c_1 = c_2 = 1/8$, but this is incompatible with $3c_1 + 3c_2 = 1$. This completes the proof of nonexistence of φ. □

Final Remarks. 1. In view of a wide variety of examples, it may be tempting to conjecture a precise link between the growth of ρ_L and hypoellipticity in various function spaces. However Example 9 suggests that such a conjecture may be overly simplistic. The metric ρ_L may represent merely an approximation to the correct condition. To decide whether Example 9 is analytic hypoelliptic would be illuminating.

2. There are indications that ρ_L may also be related to the question of *global* regularity in C^∞ for the $\bar\partial$–Neumann problem, in both the positive [2],[3] and negative [10] directions.

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