Convolution, Curvature and a bit of Combinatorics

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Main Theorem

Definitions.

- Measure μ in $\mathbb{R}^n$, $n \geq 2$:

$$\int f\,d\mu = \int_0^1 f(t, t^2, t^3, \ldots t^n)\,dt.$$

- $Q = (p_0^{-1}, q_0^{-1})$ where

$$p_0 = \frac{n+1}{2}, \quad q_0 = \frac{n+1}{2} \cdot \frac{n}{n-1}$$

- $Q^* =$ dual point.

Theorem. *Convolution with μ maps $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ $\iff$ (p^{-1}, q^{-1}) belongs to closed convex hull of*

$$\{Q, Q^*, (0,0), (1,1)\},$$

except perhaps at Q, Q^*.

Operator is of restricted weak type $\iff$ (p^{-1}, q^{-1}) belongs to closed convex hull.

$$\widehat{\mu}(\xi) = O(|\xi|^{-1/n}),$$

so by Sobolev embedding L^2 is mapped into L^q with $q^{-1} = 2^{-1} - n^{-2}$. This is not optimal, even for $p = 2$, in any dimension.

History

Littman 1973: Dimension 2.
Oberlin 1987: Dimension 3.
Carbery-Christ: Showed Oberlin's result optimal for $n = 3$; proved necessity for all n. Asked whether sufficient.
Oberlin 1997: Dimension 4 (except for some boundary points).

Method used: Analytic family of operators.

- $L^2 \mapsto L^2$ for worse operators at one endpoint.
- $L^r \mapsto L^s$ for better operators at other end.

Relied on numerology: special exponents such as $2, 4$ arise in low dimensions.

Asymptotically, for large n, this is half of the true gain for $p = 2$; for p near 1 or ∞ it leads to a very inferior result.

Work of many authors on related problems is relevant, including Greenleaf, Phong, Seeger, Stein ...

The better operators in Oberlin's 4D proof involve measures living on 2D manifolds. Half of ambient dimension ...

Cancellation/orthogonality played major role in analysis.

Features of Problem

- No good numerology for n large.
- Symmetry: all points on curve are "isomorphic". Operator is translation-invariant.
- Operators are positive. No cancellation.
- Higher-order curvature comes into play.

Examples showing necessity:

1) $E =$ ball of radius δ. $T(\chi_E) \sim \delta$ on a curved δ–tube.

2) $E =$ rectangle of dimensions $(\delta, \delta^2, \ldots, \delta^n)$. $T(\chi_E) \sim \delta$ on inner half of E.

3) Dual to 1): $E =$ curved δ–tube; $T(\chi_E) \sim 1$ on ball of radius δ.

The Plan

It suffices to prove that if

$$T(\chi_E) \sim \alpha$$

on a set F, then

$$|F| \leq C\alpha^{-q_0}|E|^{q_0/p_0}.$$

Define β by

$$\boxed{\beta|E| = \alpha|F|.}$$

Then equivalently we want

$$|E| \geq c\alpha^{n(n+1)/2}\,(\beta/\alpha)^{n-1} \; .$$

Strategy:

- Enumerate points of E.
- Count points of E.
- Account for duplications.

Proof of main Theorem is quite different. Essentially combinatorial; no cancellation. Inspired by Bourgain, Wolff, Schlag; but different.

It's a counting problem; of course points are "counted" according to Lebesgue measure.

Notation (Here $1 \leq k < \infty$):

$$h(s) = (s, s^2, \dots, s^n)$$
$$t = (t_1, \dots t_k)$$
$$\Phi_k(t) = h(t_1) - h(t_2) - \cdots + (-1)^{k+1} h(t_k)$$

Definition. A parameter space tower is a finite sequence of measurable sets $\Omega_j \subset [0,1]^j$ such that

$$(t_1, \dots, t_k) \in \Omega_k \Rightarrow (t_1, \dots t_{k-1}) \in \Omega_{k-1}$$
$$|\Omega_1| \geq \beta,$$

and such that there exists $x_0 \in E$ such that for all even k

$$t \in \Omega_k \Rightarrow |\{s \in [-1,1] : (t,s) \in \Omega_{k+1}\}| \geq c_0 \beta$$
$$x_0 + \Phi_k(\Omega_k) \subset E$$

and the same holds for odd k with α, E replaced respectively by β, F.

Near-Tautology

Lemma. *There always exists a parameter space tower* $\Omega_1, \ldots \Omega_{2n}$.

Proof:

• For Ω_1 this is Bourgain's bush construction.

• Successive Ω_j are obtained by descending induction on j, using a pruning operation.

This has nothing to do with curvature.

Crummy Lower Bound for $|E|$

Define

$$r_n = \begin{cases} 1+3+\ldots(n-1) & \text{if } n \text{ is even} \\ 2+4+\cdots+(n-1) & \text{if } n \text{ is odd.} \end{cases}$$

Lemma. $|E| \geq c\alpha^{n(n+1)/2} \cdot (\beta/\alpha)^{r_n}$

Proof for n even: Let $\Psi(t) = x_0 + \Phi(t)$.

- $E \supset \Psi_n(\Omega_n)$, and

$$|\Psi_n(\Omega_n)| \geq \frac{1}{n!} \int_{\Omega_n} |\partial\Psi_n/\partial t| \, dt.$$

- The Jacobian is

$$|\partial\Psi_n/\partial t| = c \prod_{i<j} |t_i - t_j|$$

and simple estimations exploiting the tower structure of Ω_n conclude the proof.

Factor of $n!$ comes from upper bound on multiplicity (except for set of parameters t of measure zero).

For n odd we use Ω_{n+1} but only those points t whose first coordinate has a fixed value.

For $n = 2, 3$ this proves the theorem (also for $\beta \geq c\alpha$). A small extra trick makes it work for $n = 4$, but for about $n = 6$ and larger it's less easy to fix.

For the particular configurations that arise in analyzing this construction for the extremal examples, the set $\Psi_n(\Omega_n)$ *does* satisfy the desired lower measure bound. The problematic cases turn out to be irrelevant ones where E is actually much larger than needed; the problem is to prove it.

(I don't know how close I come to proving that the known extremals are essentially the *only* worst cases.)

Modified Strategy

Device: Use also $\Psi_k(\Omega_k)$ for $k > n$.

Drawback: Ψ_k is not finite-to-one.

Solution: $E \supset \Psi_k($ typical n-dimensional slice $)$

What remains is merely an elaboration of the scheme I've described, but in terms of number of written words it's actually the bulk of the argument. Because it's a bit particular I'll describe the rest only in vague terms.

I need all k between n and $2n-2$.

Insert picture

We have choice of how to slice.

Good Parameter Sets, $k > n$
(Oversimplified)

Assume $\beta \ll \alpha$.

Definition: band structure on $\{1, 2, \ldots k\}$ is a partition into nonempty disjoint subsets, called bands.

In each band, one index is designated as free; others are bound to the free index, except that for bands having exactly 2 indices, the second index is quasi-free (and is quasi-bound to the associated free index).

For odd k there are very similar requirements.

Main Organizational Lemma

It is always possible to find
• an index $n \le k \le 2n-2$ and a set $\omega \subset \Omega_k$,
and to assign a
• band structure to set of all indices $1 \le j \le k$ such that (for even k):

(i) The number of free indices plus the number of quasi-free indices equals n.

(ii) $|\omega| \ge c\alpha^{k/2}\beta^{k/2}$

(iii) For all $i \ne j$,

$$\boxed{|t_i - t_j| \ge \delta\alpha}$$

unless one of i, j is bound or quasi-bound to the other, or both are bound to the same index.

(iv) If i is bound to j then $|t_i - t_j| \le \delta'\alpha$.

(v) If i is quasi-bound to j then $c\beta \le |t_i - t_j| \le \delta\alpha$.

(vi) δ' is sufficiently small relative to δ to satisfy the Jacobian lemma below.

Parameter Space Slicing

- $\Lambda = \{m_1, \ldots, m_n\}$ dendnotes set of all free or quasi-free indices.
- $\tau = (t_{m_1}, \ldots, t_{m_n}) \in \mathbb{R}^n$.
- For each bound index $i \leq k$, $B(i)$ = unique free index to which i is bound.

- Make change of variables

$$\boxed{s = (t_i - t_{B(i)})_{i \notin \Lambda} \in \mathbb{R}^{k-n}}$$

- Regard t as a function of (τ, s).

- Slices of ω: Hold s constant.

- Define

$$G_s(\tau) = \Psi_k(t(\tau, s)) \ .$$

Jacobian Lemma

Lemma. *For each* $\delta \in \mathbb{R}^+$*, there exist* $\delta', c \in \mathbb{R}^+$ *such that for all indices* k *and parameter sets* ω *possessing the structure described above, whenever* $t(\tau, s) \in \omega$

$$|\partial G_s / \partial \tau| \geq c\alpha^{n(n-1)/2}(\beta/\alpha)^M$$

where $M =$ *number of quasi-free indices.*

Applying this lemma to a typical slice leads to

$$|E| \geq c\alpha^{n(n+1)/2}(\beta/\alpha)^{M+\frac{k}{2}}$$

and the main organizational lemma guarantees that

$$M + \frac{k}{2} \leq n - 1.$$

Note that lemma gives a *uniform* lower bound for the Jacobian.

I use Bezout's lemma to control maximal number of preimages for generic point in target space $\mathbb{R}^d$. This ought to be avoided.

This is one of those things which takes a lot of words to say but is essentially trivial. (i) If there are too many free plus quasi-free indices, then one drops the last index. (ii) If many indices are either far apart or close together then all is OK. In bad case where they're in between we redefine what is meant by “far apart” by a large constant factor, getting more free plus quasi-free indices. Then we repeat.

A Cautionary Note

Suppose $u \in C^2$ satisfies

$$\frac{\partial^2 u}{\partial x \partial y} \geq 1 \text{ on } [0,1]^2.$$

Question. Is it true that for any α, β and any parameter space tower Ω_1, Ω_2,

$$\int_{\Omega_2} |u(x,y)| \, dx \, dy \geq c\alpha^2\beta^2 \text{ ?}$$

(c *is* permitted to depend on u but not on α, β, Ω_j.)

Proposition.

• *There exists $u \in C^\infty$ for which the answer is NO.*

• *There exist $u \in C^2$, sequences $\alpha, \beta \to 0$, and sets Ω_j such that*

$$\int_{\Omega_2} |u(x,y)| \, dx \, dy \leq c\alpha^2\beta^2 \ / \ \log(1/\alpha).$$

If one were to attempt to apply this general method to other similar operators, one would arrive at a family of questions of the following type. Here I state the simplest nontrivial one.

The proposition is part of ongoing work with A. Carbery and J. Wright.

This also helps to explain why I use explicit form of Jacobians (Vandermonde determinants) in proving theorem, rather than merely exploiting lower bounds for partial derivatives of Jacobians.

• If one asks for a uniform lower bound for all u satisfying $\partial^2 u/\partial x \partial y \geq 1$ then for $\beta \leq \alpha$, the integral is $\geq c\beta^2\alpha^3/\log(\beta^{-1})$. This is best possible modulo the log factor.

Deficiency

More ambitious goal: optimal smoothing estimates $L^p \mapsto L^q_s$.

Problem: For $s > 0$, cancellation plays essential role.

Expectation: This method won't settle the case $s > 0$, but perhaps it can be combined with endpoint results for $q = p$ plus interpolation and/or analytic family of operators to get full picture.

Core of method should be quite generally applicable; does not require convolution operators, nor curves.

Optimal $q = p$ smoothing estimates are presently known only for dimension 2.

Related Work of Carbery-C-Wright

Theorem. Suppose $\partial^2 u/\partial x \partial y \geq 1$ on $[0,1]^2$. Then

$$|\{(x,y) : |u(x,y)| < \delta\}| \leq C\delta^{1/2}\sqrt{\log(1/\delta)}.$$

• The constant C is universal; it is independent of any upper bound on higher derivatives of u.

• Example $-(x-y)^2/2$ shows exponent is best possible. We don't know whether the log is needed.

• This estimate is equivalent to having

$$\int_E |u| \geq c|E|^3/\log(|E|^{-1})$$

for all small sets E.

There are generalizations to higher-order derivatives and higher dimensions, and there are oscillatory integral estimates of van der Corput type in the same spirit, for all dimensions.

Related to Combinatorial Problems

Let E = set where $|u| < \delta$
and $E_x = \{y : (x,y) \in E\}$.

$\partial^2 u/\partial x \partial y \geq 1$
$\Rightarrow$
For any polygonal Jordan curve γ with sides parallel to axes and all corners in E, Area of region enclosed is $\leq \delta$ times number of corners
$\Rightarrow$
Any rectangle with sides parallel to axes and all corners in E has area $\leq \delta$
$\Rightarrow$
$|E_x \cap E_{x'}| \leq \delta/|x - x'|$ for all x, x'
$\Rightarrow$
$|E| \leq c\delta^{1/2}\sqrt{\log(1/\delta)}$.

Proof goes via a sequence of implications:

Thus the proof has an combinatorial aspect, as well. It seems to be an open problem whether the rectangle condition implies the measure estimate with no logarithmic factor.

Counterexample

The condition

$$|E_x \cap E_{x'}| \leq \delta / |x - x'|$$

does not imply

$$|E| \leq c\delta^{1/2}.$$

The construction is based on the Kakeya/Besicovitch set.