CONVOLUTION, CURVATURE, AND COMBINATORICS
A CASE STUDY

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1. Introduction

Consider the convolution operator $Tf = f * \mu$ acting on $f \in L^p(\mathbb{R}^n)$, where μ is the measure defined by $\int_{\mathbb{R}^n} g\, d\mu = \int_{-1}^{1} g(h(t))dt$, and $h(t) = (t, t^2, t^3, \dots, t^n)$. We seek to determine for which exponents T maps $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.

Even though μ is a singular measure, its Fourier transform satisfies a decay estimate $\hat\mu(\xi) = O(1+|\xi|)^{-1/n}$, and the Sobolev embedding theorem thus implies that T maps L^2 to L^q where $q^{-1} = 2^{-1} - n^{-2}$. While this implies some gain in integrability, it does not yield the optimal gain for $p = 2$, except when $n = 2$; even then it does not capture the optimal gain for general p.

Define $\mathcal{R}_n \subset [0,1] \times [0,1]$ to be the convex hull of the four points (0,0), (1,1), Q_n, $\tilde Q_n$ where $Q_n = (p_n^{-1}, q_n^{-1})$, $\tilde Q_n = (1 - q_n^{-1}, 1 - p_n^{-1})$, and $p_n = \frac{n+1}{2}$, $q_n = \frac{n+1}{2} \cdot \frac{n}{n-1}$. Littman [7] proved that for $n = 2$, T maps L^p to L^q if and only if $(p^{-1}, q^{-1}) \in \mathcal{R}_2$. Oberlin [9] showed that $(p^{-1}, q^{-1}) \in \mathcal{R}_n$ is sufficient for boundedness in dimension $n = 3$. Carbery and Christ observed shortly thereafter that T is unbounded outside of $\mathcal{R}_n$, and asked whether $\mathcal{R}_n$ was the region of boundedness for all n. More recently Oberlin [10],[11] treated the four-dimensional case. There also exists a large body of related work on other classes of degenerate Fourier integral operators and Radon-like transforms.

In this paper we establish the optimal result in all dimensions, except at $Q_n, \tilde Q_n$.

Theorem. *For each $n \ge 2$, convolution with μ maps $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for all $(p^{-1}, q^{-1}) \in \mathcal{R}_n \backslash \{Q_n, \tilde Q_n\}$. Moreover this operator is of restricted weak type (p, q) if and only if $(p^{-1}, q^{-1}) \in \mathcal{R}_n$.*

One formulation of the second conclusion is simply that

$$\langle \chi_F, T(\chi_E)\rangle \le C|E|^{1/p}|F|^{1-1/q}$$

for arbitrary measurable sets $E, F \subset \mathbb{R}^n$, where χ_E denotes the characteristic function of E and $\langle f, g\rangle = \int_{\mathbb{R}^n} f(x)g(x)dx$. For $n = 2$ or 3, T is known to be bounded at Q_n and at $\tilde Q_n$. It remains an open question whether our argument might be sharpened to fully handle those endpoints.

Previous work on this and related questions has relied on such tools as the Fourier transform or more general oscillatory integral estimates, coupled with analytic families of operators or decompositions of phase space. We take a different approach, which is combinatorial in the sense that it consists in enumerating and counting

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points, though with respect to measure rather than cardinality. The impetus comes from work of Bourgain [1],[2], Wolff [16],[17], and Schlag [14] on the circular and Kakeya–Nikodym maximal operators. There is also a connection with the iteration device used in [4],[5],[6].

The question considered here is representative of a large class of problems, as is explained in §5. Although parts of our argument are rather particular, underlying the analysis is a method which may be of further use.

2. Initiation and a preliminary estimate

The restricted weak type estimate for Q_n implies the other conclusions. Indeed, restricted weak type at $\tilde{Q}_n$ follows by duality, while T is bounded for the vertices (0,0) and (1,1) simply because μ is a finite measure. Interpolating via the Lorentz space formulation of the Marcinkiewicz interpolation theorem then yields the full conclusion of the theorem.

In this section we set up the structure of the proof of the inequality for Q_n, and establish a weaker estimate.

2.1. Let $p = p_n,\ q = q_n$. We aim to show that for every $\alpha \in \mathbb{R}^+$ and every Borel set $E \subset \mathbb{R}^n$, the set $F = \{x : 2\alpha > T\chi_E(x) > \alpha\}$ satisfies

$$|F| \le C\alpha^{-q}|E|^{q/p} \ . \tag{2.1}$$

Here $q = \frac{n+1}{2}\,\frac{n}{n-1}$ and $q/p = n/(n-1)$. This is the restricted weak type estimate at Q_n; the theorem follows by interpolation, as was explained above.

Assume that E, F have positive measures and introduce

$$\beta = |E|^{-1}\langle \chi_F, T\chi_E\rangle \ .$$

Our central estimate is a lower bound for the measure of E:

$$|E| \ge c\alpha^{n(n+1)/2}\,(\beta/\alpha)^{n-1} \ . \tag{2.2}$$

$\beta \ge \alpha|F|/|E|$ by its definition; substituting that into (2.2) gives (2.1).

2.2. Define $\Phi_k : [-1,1]^k \to \mathbb{R}^n$ by

$$\Phi_k(t) = h(t_1) - h(t_2) + h(t_3) - \cdots + (-1)^{k+1}h(t_k) \ ,$$

where $t = (t_1, \dots, t_k)$. The next lemma is nearly a tautology.

Lemma 1. *Let E, F, α, β be as above. Then there exist $c_0 \in \mathbb{R}^+$ depending only on n, a point $x_0 \in E$, and sets $\Omega_k \subset [-1,1]^k$ for $1 \le k \le 2n$, satisfying*

$$(t_1, \dots, t_{k+1}) \in \Omega_{k+1} \Rightarrow (t_1, \dots, t_k) \in \Omega_k \ ,$$
$$|\Omega_1| \ge c_0\beta$$

such that for each odd k and $t \in \Omega_k$,

$$|\{s \in [-1,1] : (t,s) \in \Omega_{k+1}\}| \ge c_0\alpha \ ,$$
$$|s - t_j| \ge c_0\alpha \quad \text{for all} \quad j \le k \quad \text{if} \quad (t,s) \in \Omega_{k+1}$$
$$x_0 + \Phi_{k+1}(\Omega_{k+1}) \subset E \ ,$$

and such that the preceding three conclusions hold for each even k if α, E are replaced respectively by β, F.

The sets Ω_k depend of course on E, F.

Proof. Denote by T^* the transpose of T. Define $E_0 = E,\ \ F_0 = F,$ and

$$\begin{aligned} E_{k+1} &= \{x \in E_k : T^*\chi_{F_k}(x) \ge \delta_k\beta\} \\ F_{k+1} &= \{y \in F_k : T\chi_{E_{k+1}}(y) \ge \varepsilon_k\alpha\} \end{aligned} \tag{2.3}$$

where ε_k, δ_k are small constants depending only on the dimension n and on k. If they are chosen to be sufficiently small then

$$\begin{aligned} \langle T^*\chi_{F_k}, \chi_{E_{k+1}}\rangle &\ge \eta_{k-1}\beta|E| \\ \langle \chi_{F_{k+1}}, T\chi_{E_{k+1}}\rangle &\ge \eta_k\alpha|F| \end{aligned} \tag{2.4}$$

for certain positive constants η_k depending only on n. Indeed, assuming this to hold for $1, 2, \dots, k$, we have

$$\langle T^*\chi_{F_k}, \chi_{E_{k+1}}\rangle = \langle \chi_{F_k}, T\chi_{E_k}\rangle - \langle T^*\chi_{F_k}, \chi_{E_k\setminus E_{k+1}}\rangle \ge \eta_{k-1}\alpha|F| - \delta_k\beta|E_k|\ ,$$

since $T^*\chi_{F_k} \le \delta_k\beta$ on $E_k\setminus E_{k+1}$. Thus it suffices to set both δ_k and η_k equal to $\eta_{k-1}/4$. The situation is symmetric, so the second conclusion of (2.4) follows in the same way.

(2.3) may be restated:

$$\begin{aligned} x \in E_{k+1} \quad &\Rightarrow \quad |\{s \in [-1,1] : x + h(s) \in F_k\}| \ge \delta_k\beta \\ y \in F_{k+1} \quad &\Rightarrow \quad |\{s \in [-1,1] : y - h(s) \in E_{k+1}\}| \ge \varepsilon_k\alpha\ . \end{aligned}$$

E_n is nonempty by (2.4); fix any $x_0 \in E_n$.

Choose a set Ω_1 of measure exactly $\varepsilon\beta$ such that $x_0 + h(t_1) \in F_{n-1}$ for all $t_1 \in \Omega_1$, where $\varepsilon = \frac{1}{2}\min(\varepsilon_1, \delta_1, \dots, \varepsilon_n, \delta_n)$. Next choose Ω_2 to be a measurable subset of the set of all (t_1, t_2) such that $t_1 \in \Omega_1,\ x_0 + h(t_1) - h(t_2) \in E_{n-1}$, and $|t_2 - t_1| \ge \varepsilon'\alpha$, where ε' is another small constant depending only on the dimension. Whenever necessary, omit some values of t_2 which satisfy all these criteria, so that for each $t_1 \in \Omega_1$, $|\{t_2 : (t_1, t_2) \in \Omega_2\}| = \varepsilon\alpha$. This is possible provided that we choose $\varepsilon' < \varepsilon/4$. Iterating this process yields the required sets Ω_k. □

2.3. Assume always that $n \ge 2$. Set $r_n = 1 + 3 + \cdots + (n-1)$ if n is even, and $r_n = 2 + 4 + \cdots + (n-1)$ if n is odd. Then $r_n \ge n-1$, and $r_n = n-1$ if and only if $n \in \{2, 3\}$. Thus the next lemma implies our theorem for $n = 2$ or 3, and it implies the main inequality (2.2) in the special case when β/α is bounded below by any constant, for all dimensions n.

Lemma 2. *For any $n \ge 2$, with the notation established above,*

$$|E| \gtrsim \alpha^{n(n+1)/2}(\beta/\alpha)^{r_n}. \tag{2.5}$$

Proof. Assume first that n is even. Let x_0, Ω_n be as in Lemma 1 and set $\Psi_n(t) = x_0 + \Phi_n(t)$. Then $E \supset \Psi_n(\Omega_n)$, so it suffices to establish the stated lower bound for $|\Psi_n(\Omega_n)|$.

The Jacobian determinant $J(t) = \det(\partial\Psi_n/\partial t)$ takes the form

$$J(t) = c_n \prod_{1\le i<j\le n} (t_i - t_j)$$

where c_n is a nonzero constant. Bezout's theorem implies that the set of all points $y \in \mathbb{R}^n$ having more than $n!$ preimages under Ψ_n is a proper algebraic subvariety, and in particular has Lebesgue measure zero.

Indeed, define $z \in \mathbb{C}^n$ to be generic if z is not the image of some point $t \in \mathbb{C}^n$ at which $J(t)$ vanishes. The set of nongeneric points is the image under an algebraic mapping of a proper algebraic variety, hence is an open set of full Lebesgue measure. Let $z = (z_1, \dots, z_n)$. For each $1 \le j \le n$ consider the variety $V_{j,z}$ of all t for which the jth coordinate of $\Psi_n(t)$ equals z_j. Temporarily regarding Ψ_n as an algebraic mapping of complex projective space $\mathbb{CP}^n$ to itself, we obtain n hypersurfaces in $\mathbb{CP}^n$, also denoted $V_{j,z}$; these are defined by polynomials of degrees j, and are in general position for generic z. Therefore Bezout's theorem, as stated on p. 198 of [15], asserts that for generic z, $\cap_j V_{j,z}$ consists of $n!$ points, counting multiplicity. Ignoring the hyperplane at infinity, returning to $\mathbb{C}^n$, and specializing to real points, we arrive at an upper bound of $n!$ on the number of preimages of any generic z.

Therefore

$$|\Psi_n(\Omega_n)| = |\Psi_n\big(\{t \in \Omega_n : J(t) \ne 0\}\big)| \ge (n!)^{-1} \int_{\Omega_n} |J(t)|\, dt \gtrsim \int_{\Omega_n} \prod_{i<j} |t_i - t_j|\, dt\ .$$

If $t \in \Omega_n$ and $i < j$ then by Lemma 1, $|t_i - t_j| \gtrsim \alpha$ if j is even, and $\gtrsim \beta$ if j is odd. Hence recalling that n is even,

$$t \in \Omega_n \Rightarrow \prod_{i<j} |t_i - t_j| \gtrsim \alpha^{n(n-1)/2} (\beta/\alpha)^\rho$$

where $\rho = 2 + 4 + \cdots + (n-2) = r_n - \frac{n}{2}$. Furthermore, $|\Omega_n| \gtrsim \alpha^{n/2}\beta^{n/2} = \alpha^n(\beta/\alpha)^{n/2}$ for even n, by Lemma 1. Multiplying this by the lower bound for $|J|$ gives the desired lower bound for $|E|$.

For odd dimensions n this argument goes awry because $x_0 + \Phi_n(\Omega_n)$ is not necessarily contained in E, but rather in F. To skirt this difficulty fix any $t_1 \in \Omega_1$, set $\tau = (t_2, \dots, t_{n+1})$, let $\Omega = \{\tau : (t_1, \tau) \in \Omega_{n+1}\}$, and set $\Psi(\tau) = x_0 + \Phi_{n+1}(t_1, \tau)$, so that $\Psi(\Omega) \subset E$. Then $|\Omega| \gtrsim \alpha^{(n+1)/2}\beta^{(n-1)/2}$, while a lower bound for the Jacobian is now $c\alpha^{n(n-1)/2}(\beta/\alpha)^{1+3+\cdots+(n-2)}$. Arguing as above leads to the stated lower bound, since $1 + 3 + \cdots + (n-2) + \frac{n-1}{2} = r_n$ for n odd. □

2.4 The remainder of the argument is merely an elaboration of the method already described. To motivate one aspect of what will follow, consider the situation where n is even, and where for each odd integer $3 \le j \le n-1$, one has $|t_j - t_{j-1}| \lesssim \beta$ for all points $t \in \Omega_n$. If β/α is much less than one, this implies that for all odd j, $|t_j - t_i| \gtrsim \alpha$ for all $i \le j-2$, because by construction of Ω_n, $|t_k - t_\ell| \gtrsim \alpha$ whenever k is even and $\ell < k$. Thus the Jacobian lower bound in Lemma 2 can be substantially improved in this special situation, and one obtains $|E| \gtrsim \alpha^{n(n+1)/2}(\beta/\alpha)^{n-1}$, which is the main inequality in its full strength. This situation corresponds to the third extremal case described in §5.

For $n = 4$, our theorem can be deduced by combining Lemma 2 with a small auxiliary analysis in this same spirit. But for larger n we have been unable to avoid further complication.

As additional motivation consider the situation where n is even, $\alpha \sim 1$, β is very small, and $|t_j - t_1| \lesssim \beta$ for all odd $3 \le j < n$. Then the lower bound $c\beta^{r_n}$ for $|\Psi_n(\Omega_n)|$ obtained in Lemma 2 cannot be improved. Yet one actually has $|E| \sim 1$; indeed, $E \supset \Psi_{2n}(\Omega_{2n})$, and $|\Psi_{2n}(\Omega_{2n})| \sim 1$. A proof of this lower bound is implicit in the arguments below, but this special case could be treated quite simply.

3. Proof of main estimate

In this section we demonstrate how the inclusions $E \supset \Psi_j(\Omega_j)$, for even $j > n$, can be exploited to obtain an improved lower bound for $|E|$. Throughout the rest of the paper we assume β/α to be majorized by a small constant depending only on n.

3.1 In §4 we will choose an integer $k \in \{n, n+1, \ldots, 2n-2\}$ and construct a set $\omega \subset \Omega_k$, together with a partition of $\{1, 2, \ldots, k\}$ into nonempty disjoint subsets, called bands. This construction depends on the sets Ω_i. No band will contain more than one even index, and no even index will belong to the same band as the index 1.

Given a band structure, each index $j \in \{1, 2, \ldots, k\}$ will be designated as either free, quasi-free, or bound, according to the following rules. If a band contains an even index or 1, then that index is designated as free. For bands containing neither an even index nor 1, the smallest index in the band is designated as free. Thus each band contains exactly one free index.

Each remaining index is either quasi-free or bound. If a band $\mathcal{B}$ contains exactly two elements then the second one is quasi-free. Otherwise all remaining indices are bound.

Each bound index is bound to the unique free index belonging to its band. Each quasi-free index is quasi-bound to the unique free index in its band, but is not said to be bound. Free indices are neither bound nor quasi-bound. The number of indices bound to any free index is thus either zero, or at least two.

Associated to ω will be a pair (δ, δ') of small positive real parameters; the set of all such pairs which arise will be finite and will depend only on n. All of the following will hold, for every $t \in \omega$.

(i) The number of free indices plus the number of quasi-free indices equals n.

(ii)
$$|\omega| \gtrsim \begin{cases} \alpha^{k/2}\beta^{k/2} & \text{if } k \text{ is even} \\ \alpha^{(k-1)/2}\beta^{(k+1)/2} & \text{if } k \text{ is odd.} \end{cases}$$

(iii) $|t_i - t_j| \ge \delta\alpha$, unless i, j belong to the same band.

(iv) If i is bound to j then $|t_i - t_j| \le \delta'\alpha$.

(v) If i is quasi-bound to j then $c\beta \le |t_i - t_j| \le \delta\alpha$.

(vi) δ' is sufficiently small, relative to δ, to satisfy the hypothesis of Lemma 3 below.

A consequence is that the number M of quasi-free indices satisfies

$$\begin{cases} M+(k/2) & \le n-1 \qquad \text{if } k \text{ is even} \\ M+(k-1)/2 & \le n-1 \qquad \text{if } k \text{ is odd.} \end{cases} \tag{3.1}$$

Indeed if k is even then the number N of free indices is at least $1+(k/2)$, since all even indices are free and 1 is free. For odd k the same reasoning gives $N \ge (k+1)/2$. Since $M = n - N$, the claim follows.

In the remainder of §3 we show how these conditions lead to a good lower bound for $|E|$.

3.2. Let $\Lambda = \{m_1, \dots, m_n\}$ be the set of all indices that are free or quasi-free, and define $\tau = (t_{m_1}, \dots, t_{m_n}) \in \mathbb{R}^n$. For each bound index $i \le k$ let $B(i)$ be the unique free index to which i is bound, and set

$$s = (t_i - t_{B(i)})_{i \notin \Lambda} \in \mathbb{R}^{k-n} \ .$$

The map $t \mapsto (\tau, s)$ is a diffeomorphism; we regard t as a function of (τ, s) and introduce

$$G_s(\tau) = \Psi_k(t(\tau, s)) \qquad \text{and} \qquad J(\tau, s) = \det(\partial G_s / \partial \tau) \ .$$

The following lemma will be established in §3.3.

Lemma 3. *For each $\delta \in \mathbb{R}^+$, there exist $\delta', c \in \mathbb{R}^+$ such that for all indices k and parameter sets ω possessing the structure described in* §3.1*, for each point (τ, s) such that $t(\tau, s) \in \omega$,*

$$|J(\tau, s)| \ge c\alpha^{n(n-1)/2} (\beta/\alpha)^M$$

where M is the number of quasi-free indices associated to ω. Here c depends only on n, δ, not on α, β.

With the aid of this lemma, the proof of the main inequality (2.2) may be concluded as follows, under the additional assumption that k is even. By Bezout's theorem, as in §2, for each $s \in \mathbb{R}^{k-n}$ any generic point in $\mathbb{R}^n$ has at most $n!$ preimages under G_s. For any fixed s, since k is assumed to be even, $E \supset \{G_s(\tau) : t(\tau, s) \in \omega\}$ and hence

$$\begin{aligned} |E| &\ge |\{G_s(\tau) : t(\tau, s) \in \omega\}| \\ &\ge (n!)^{-1} \int_{\{\tau : t(\tau, s) \in \omega\}} |J(\tau, s)| d\tau \\ &\gtrsim \alpha^{n(n-1)/2} (\beta/\alpha)^M |\{\tau : t(\tau, s) \in \omega\}| \ . \end{aligned}$$

Now $|s| \le \delta' \alpha$ by condition (iv) of §3.1, whenever $t(\tau, s) \in \omega$. Therefore integrating over the ball $B = \{s \in \mathbb{R}^{k-n} : |s| \le \delta' \alpha\}$ on both sides of the preceding inequality yields

$$|B| \cdot |E| \gtrsim \alpha^{n(n-1)/2} (\beta/\alpha)^M |\omega| \ ,$$

whence by the lower bound (ii) on $|\omega|$,

$$|E| \gtrsim \alpha^{n(n+1)/2} (\beta/\alpha)^{M + \frac{k}{2}} \ .$$

Since $\beta \le \alpha$ and $M + (k/2) \le n-1$, this gives (2.2).

The case of odd k remains, and will be treated in §3.4.

3.3. To prove Lemma 3, regard $G_s(\tau)$ as a row vector in $\mathbb{R}^n$, indexed by $p \in \{1, 2, \ldots, n\}$. Its pth entry is

$$\sum_{j=1}^{k}(-1)^{j+1}t_j^p = \sum_{j\in\Lambda}\left((-1)^{j+1}\tau_j^p + \sum_{i\Rightarrow j}(-1)^{i+1}(\tau_j + s_i)^p\right)$$

where the notation $i \Rightarrow j$ means that i is bound to j and the sum is taken over all such i. Expansion in Taylor series with respect to s, about the origin, yields

$$\sum_{j\in\Lambda}\left((-1)^{j+1}\tau_j^p + \sum_{i\Rightarrow j}(-1)^{i+1}\sum_{r=0}^{n}\frac{s_i^r}{r!}\,\frac{\partial^r}{\partial T_j^r}T_j^p\,|_{T_j=\tau_j}\right)\ . \tag{3.2}$$

Define

$$\theta_j = (-1)^{j+1} + \sum_{i\Rightarrow j}(-1)^{i+1}\ .$$

Observe that no coefficient θ_j can vanish. For if j is even then $(-1)^{j+1} = -1$. Every index i bound to j must be odd, so $(-1)^{i+1} = +1$. Since the number of indices bound to j is by construction either zero or at least two, θ_j is either -1 or is ≥ 1. If on the other hand j is odd, then $(-1)^{j+1} = +1$, while still $(-1)^{i+1} = +1$ for every index bound to j.

(3.2) equals

$$\sum_{j\in\Lambda}\left(\theta_j\tau_j^p + \sum_{i\Rightarrow j}(-1)^{i+1}\sum_{r=1}^{n}\frac{s_i^r}{r!}\,\frac{\partial^r}{\partial T_j^r}\ T_j^p\,|_{T_j=\tau_j}\right)\ ,$$

which we write more simply as

$$\sum_{j\in\Lambda}\left(\theta_j\tau_j^p + \sum_{i\Rightarrow j}(-1)^{i+1}\sum_{r=1}^{n}\frac{s_i^r}{r!}\,\frac{\partial^r}{\partial \tau_j^r}(\tau_j^p)\right)\ . \tag{3.3}$$

The matrix $(\partial G_s/\partial\tau)$ is indexed by $j \in \Lambda$ and by $1 \leq p \leq n$. Its pth column is the gradient with respect to τ of (3.3). For each $j \in \Lambda$ decompose the jth row as the principal term

$$\theta_j(\partial\tau_j^p/\partial\tau_j)_{1\leq p\leq n}$$

plus a sum of rows

$$\left(\frac{(-1)^{i+1}}{r!}\ s_i^r\ \frac{\partial^{r+1}}{\partial\tau_j^{r+1}}(\tau_j^p)\right)_{1\leq p\leq n}\ .$$

For each j there is one such secondary term for each possible pair (r, i) with $1 \leq r \leq n$ and i bound to j.

Write

$$V(\tau) = \prod_{i<j,\ i,j\in\Lambda}(\tau_j - \tau_i)\ .$$

By multilinearity of the determinant, $J(\tau,s)$ equals $(\prod_{j\in\Lambda}\theta_j)\cdot V(\tau)$ plus a finite linear combination, whose coefficients are bounded by a constant depending only on n, of determinants

$$\det\left(s_{i_j}^{r_j}\frac{\partial^{r_j+1}}{\partial\tau_j^{r_j+1}}\tau_j^p\right)_{j\in\Lambda,\ 1\le p\le n}$$

where $i_j \Rightarrow j$ or $i_j = j$, $0\le r_j\le n$, and at least one r_j is nonzero. Each such determinant takes the form

$$\left(\prod_{j\in\Lambda} s_{i_j}^{r_j}\frac{\partial^{r_j}}{\partial\tau_j^{r_j}}\right)V(\tau) \tag{3.4}$$

times a multiplicative constant. We aim to find a lower bound for the main term, when $t(\tau,s)\in\omega$, and to show that every other term is much smaller, provided that δ' is sufficiently small relative to δ. For the main term we have

$$|V(\tau)|\ge c\alpha^{n(n-1)/2}(\beta/\alpha)^M$$

where M is the number of quasi-free indices, by (iii) and (v) of §3.1.

We claim that each determinant (3.4) is majorized by $\varepsilon|V(\tau)|$ provided that $t(\tau,s)\in\omega$, where $\varepsilon\to 0$ as $\delta'\to 0$, uniformly in α,β for fixed δ,n. Indeed, the ratio of (3.4) to $V(\tau)$ equals a finite linear combination of terms

$$\prod_{j\in\Lambda}\left(s_{i_j}^{r_j}\Big/\prod_{m=1}^{r_j}(\tau_j-\tau_{\ell_m^j})\right)$$

where associated to each j are r_j indices $\ell_m^j\in\Lambda$, not equal to j. Moreover $r_j=0$ unless j is free and some index is bound to j. Thus it will suffice to have

$$|s_i|\le\varepsilon|\tau_j-\tau_\ell|$$

whenever j is free, i is bound to j, $\ell\in\Lambda$, and $\ell\ne j$. On one hand, $|s_i|\le\delta'\alpha$. On the other hand, $|\tau_j-\tau_\ell|\ge\delta\alpha$ because j is free, $\ell\in\Lambda$, $\ell\ne j$, and ℓ is not quasi-bound to j. Indeed, it is impossible for ℓ to be quasi-bound to j, because i is bound to j and quasi-bound indices are found only in bands having exactly two elements.

3.4. For odd k the argument is modified as follows. Set

$$\tilde\omega=\{\tilde t=(t_2,\dots,t_{k+1}):(t_1,\dots,t_k)\in\omega \text{ and } (t_1,\dots,t_{k+1})\in\Omega_{k+1}\}\ .$$

There exists $t_1\in\Omega_1$ such that

$$|\tilde\omega|\gtrsim\frac{\alpha}{\beta}|\omega|\ ,$$

as is seen by averaging $|\tilde\omega|$ over all $t_1\in\Omega_1$ and recalling that for each $t\in\Omega_k$, the set of all t_{k+1} for which $(t,t_{k+1})\in\Omega_{k+1}$ has measure $\sim\alpha$, while $|\Omega_1|\sim\beta$.

Since $k+1$ is even, $|t_{k+1}-t_j|\ge c\alpha$ for all $1\le j\le k$, for all $\tilde t\in\tilde\omega$. Therefore we designate the index $k+1$ to be free relative to $\tilde\omega$. All indices $1<j\le k$ retain the same designations relative to $\tilde\omega$ that they had relative to ω, except those in the band which contained $j=1$. If that band contained only $j=1$ then no indices remain to be designated, and the new set of free and quasi-free indices is $\tilde\Lambda=(\Lambda\cup\{k+1\})\backslash\{1\}$.

If that band contained 1 and one other index i, then 1 was formerly free and i was quasi-free; now i is designated as free, and again $\tilde{\Lambda} = (\Lambda \cup \{k+1\}) \setminus \{1\}$. Finally, if that band contained three or more indices, then all of those indices are now designated as bound to $j = 1$, and $\tilde{\Lambda} = (\Lambda \cup \{k+1\}) \setminus \{1\}$. In every case, $\tilde{\Lambda}$ has the same cardinality n as Λ. A new feature is that some indices may now be bound to the omitted index $j = 1$.

Introduce $(\tau, s) \in \mathbb{R}^n \times \mathbb{R}^{k-n}$ as before; τ has one component for each index in $\tilde{\Lambda}$, while s has one component for each $i \in \{2, \ldots, k+1\} \setminus \tilde{\Lambda}$, with $s_i = t_i - t_j$ where i is bound to j; here j could equal one. Retracing our steps, the indices bound to $j = 1$ do not essentially alter the argument, and we obtain

$$|\det(\partial G_s / \partial \tau)| \gtrsim \alpha^{n(n-1)/2} (\beta/\alpha)^M \ ,$$

and then

$$|E| \gtrsim \alpha^{n(n-1)/2} (\beta/\alpha)^M \ \alpha^{n-k} |\tilde{\omega}| \gtrsim \alpha^{n(n-1)/2} (\beta/\alpha)^{M + \frac{k-1}{2}}$$

by (ii), since $|\tilde{\omega}| \gtrsim (\alpha/\beta)|\omega|$. Now (3.1) guarantees that $M + \frac{k-1}{2} \le n - 1$, so this is the sought-after lower bound for $|E|$.

4. Parameter space and band construction

In this section we construct an index k, a set ω, and a band structure satisfying conditions (i) through (vi) of §3.1. It may help the reader to understand in advance why there is no essential difficulty in doing so. Oversimplifying matters momentarily for the sake of exposition, the favorable situation, according to Lemma 3, is when for each pair $i \ne j$, either $|t_i - t_j| \ge \delta\alpha$, or $\le \delta'\alpha$, where $\delta' \ll \delta$. If the set of all unfavorable parameters t has relatively small measure then that set may be disregarded. If it has large measure, then for some fixed bound pair $i \ne j$, $|t_i - t_j| \ge \delta'\alpha$ for a set of t of large measure; because of the nature of the zero locus of the Vandermonde determinant, this is a more favorable situation because fewer coordinates are close together. We replace δ by δ', and δ' by a sufficiently smaller δ'', and begin anew. Provided that a favorable situation is eventually reached after only a fixed finite number of such replacements, the desired lower bound for $|E|$ is achieved.

4.1. Choose a permutation σ of $\{1, 2, \ldots, 2n-2\}$ such that

$$|\{t \in \Omega_{2n-2} : t_{\sigma(1)} < t_{\sigma(2)} < \cdots < t_{\sigma(2n-2)}\}| \ge |\Omega_{2n-2}|/(2n-2)! \ . \tag{4.1}$$

This permutation will remain fixed throughout the remainder of the argument. For each $1 \le r \le 2n-2$, replace Ω_r by the set $\tilde{\Omega}_r$ of all $t \in \Omega_r$ satisfying $t_{\sigma(i)} < t_{\sigma(j)}$ for all $i < j$ such that $1 \le \sigma(i), \sigma(j) \le r$.

We still have

$$|\tilde{\Omega}_r| \sim \begin{cases} \alpha^{r/2}\beta^{r/2} & \text{if } r \text{ is even} \\ \alpha^{(r-1)/2}\beta^{(r+1)/2} & \text{if } r \text{ is odd.} \end{cases} \tag{4.2}$$

Indeed, recall that for each $t \in \Omega_r$,

$$|\{s : (t, s) \in \Omega_{r+1}\}| \sim \begin{cases} \alpha & \text{if } r \text{ is odd} \\ \beta & \text{if } r \text{ is even.} \end{cases} \tag{4.3}$$

Thus for even r, $|\tilde{\Omega}_{2n-2}| \lesssim (\alpha\beta)^{(2n-2-r)/2}|\tilde{\Omega}_r|$, and (4.2) follows by the choice of σ. The same reasoning applies for odd r.

4.2. Choose $\delta > 0$ so that for all $t \in \Omega_{2n-2}$,

$$|t_i - t_j| \geq 2n\delta\alpha \quad \text{if } j \text{ is even and } i < j. \tag{4.4}$$

Such a δ exists, and may be taken to depend only on the dimension n, by the construction of the sets Ω_ℓ.

Begin with $r = 2n-2$. Decompose $\tilde{\Omega}_r$ as a disjoint union of sets ω_γ^r, as follows. Enumerate all possible choices of integers $1 \leq R \leq r$ and $1 = L_1 < L_2 < \cdots < L_R \leq r$; associate to each such choice an element γ of an unnamed index set. Define ω_γ^r to be the set of all $t \in \tilde{\Omega}_r$ such that for every $2 \leq j \leq r$,

$$t_{\sigma(j)} \geq t_{\sigma(j-1)} + \delta\alpha \quad \text{if and only if} \quad \sigma(j) \in \{L_2, L_3, \ldots, L_R\}. \tag{4.5}$$

Set $\omega^{2n-2} = \omega_\gamma^{2n-2}$, where γ is chosen to satisfy

$$|\omega_\gamma^{2n-2}| \geq |\tilde{\Omega}_{2n-2}|/2^{2n-3}\ .$$

4.3. Divide the indices $1, 2, \ldots, 2n-2$ into bands. The first band consists of $\sigma(L_1), \ldots, \sigma(L_2 - 1)$; the second band is $\{\sigma(L_2), \sigma(L_2+1), \ldots, \sigma(L_3 - 1)\}$, and so on. Note that whenever i, j belong to different bands we have $|t_i - t_j| \geq \delta\alpha$, while $|t_i - t_j| \leq n\delta\alpha$ if they belong to the same band.

(4.4) ensures that no band can contain more than one even index, and that no even index can belong to the same band as 1. Consequently for $r = 2n-2$, the total number R of bands must be at least n. Designate indices to be free, quasi-free, and bound according to the rules of §3.1. Choose δ' sufficiently small to satisfy the hypothesis of Lemma 3.

Temporarily set $k = 2n-2$ and $\tilde{\omega} = \omega^{2n-2}$. Thus this construction yields all the conditions (i) through (vi), except possibly (i) and (iv).

4.4. Let N, M denote respectively the numbers of free and of quasi-free indices; we have $N + M \geq n$. If it equals n, go on to §4.5. Otherwise replace $\tilde{\omega} \subset \tilde{\Omega}_k$ by $\{(t_1, \ldots, t_{k-1})$: there exists t_k such that $(t_1, \ldots, t_k) \in \tilde{\omega}\} \subset \tilde{\Omega}_{k-1}$. The indices $1, \ldots, k-1$ inherit the prevailing band structure; omission of the index k causes one band to have one fewer element. Each index retains its status as free, quasi-free or bound, except for those in the band $\mathcal{B}$ that had contained k.

If $\mathcal{B}' = \mathcal{B} \backslash \{k\}$ is empty then nothing more must be done. Otherwise its indices are redesignated as free, quasi-free or bound following the rules in §3.1, according to the cardinality of $\mathcal{B}'$. As a result, $N + M$ either decreases by one, remains unchanged, or increases by one (if $\mathcal{B}$ has cardinality three and $\mathcal{B}'$ has cardinality two).

Since $|\tilde{\omega}^k| \lesssim \alpha|\tilde{\omega}^{k-1}|$ if k is even and $\lesssim \beta|\tilde{\omega}^{k-1}|$ if k is odd, $|\tilde{\omega}^{k-1}|$ inherits the lower bound (ii). Thus the pair $(k-1, \tilde{\omega}^{k-1})$ satisfies all the conditions (i) through (vi), except possibly (i) and (iv); we still have $N + M \geq n$ for the revised counts N, M of free and quasi-free indices, because $N + M$ cannot decrease by more than one.

Continue this process, decreasing k with each step, until $N + M = n$. Equality must eventually occur, for if k is ever reduced to n, then $N + M$ cannot exceed the total number n of indices available.

4.5. We now have $(k, \tilde{\omega})$ satisfying all the conditions listed in §3.1, except (iv). Denoting by $B(j)$ the index to which j is bound, if any, consider

$$\omega = \{t \in \tilde{\omega} : |t_j - t_{B(j)}| \leq \delta' \alpha \text{ for all bound indices } j \in \{1, 2, \ldots, k\}\}. \tag{4.6}$$

If $|\omega| \geq |\tilde{\omega}|/2$ then ω satisfies all of conditions (i) through (vi), which was our goal.

4.6. Suppose that $|\omega| < |\tilde{\omega}|/2$, instead. We permanently discard ω, and work with $\tilde{\omega} \backslash \omega$. Introduce a refined band structure, subdividing existing bands of cardinality ≥ 3 by beginning a new band at $\sigma(j)$ whenever

$$t_{\sigma(j)} \geq t_{\sigma(j-1)} + \delta' \alpha \ . \tag{4.7}$$

Bands of cardinality less than 3 are not subdivided. $\tilde{\omega} \backslash \omega$ is decomposed as a disjoint union of at most 2^{k-1} subsets ω_γ^k, each of which is compatible with a fixed band structure in the sense that the indices L_i dictated by (4.7) are independent of $t \in \omega_\gamma^k$.

Fix γ such that $|\omega_\gamma^k| \gtrsim |\tilde{\omega}|$. Designate each index $1 \leq j \leq k$ as free, quasi-free, or bound relative to the refined band structure on ω_γ^k, according to the same rules used in §3.1.

Subdivision of bands can never cause $N + M$ to decrease; and because at least one band is subdivided because of the failure of (iv), and because that band had cardinality ≥ 3 and hence had one free element and no quasi-free elements, $N + M$ has increased by at least one. Hence the number of bound indices has decreased by at least one.

Redefine δ to be the quantity formerly denoted by δ', and introduce a new, smaller δ', as dictated by Lemma 3. With that modification, $\omega = \omega_\gamma^k$ satisfies all conditions of §3.1, except possibly (i) and (iv).

4.7 Return now to the iterative procedure in §4.4, whose goal was to reduce $N + M$ to n. That procedure never increased the number of bound indices, and each iteration decreases the number of coordinates t_i by one. Thus eventually we achieve $N + M = n$, and return to §4.5, with a smaller value of δ than before, and with fewer bound indices. If the set ω defined in (4.6) has sufficiently large measure then the proof is complete. If not, we revisit §4.6, refine the band structure, replace δ by δ', and continue.

No infinite loop can arise, for the number of bound variables never increases, and decreases every time §4.6 is reached. If the number of bound indices ever reaches zero then condition (iv), which concerns proximity of bound coordinates to free coordinates, cannot possibly be violated. Then the set defined in (4.6) will equal $\tilde{\omega}$, hence will have sufficiently large measure to satisfy (ii), completing the construction.

5. Necessity and remarks

For completeness we indicate why the condition $(p^{-1}, q^{-1}) \in \mathcal{R}_n$ is necessary for T to map L^p to L^q, and indeed even to be of restricted weak type (p, q). Let $\delta > 0$ be a small parameter and let E_δ be the ball of radius δ centered at 0. Then $(\chi_{E_\delta} * \mu)(x) \geq c\delta$ for all x such that $|x - h(t)| \leq \delta/2$ for some $t \in [-\frac{1}{2}, \frac{1}{2}]$. The set of all such points x

has measure $\gtrsim \delta^{n-1}$. The restricted weak type estimate would imply that

$$\delta \cdot \delta^{(n-1)/q} \le C\delta^{n/p}$$

for all small δ, with C independent of δ. Therefore a necessary condition is

$$1 + \frac{n-1}{q} \ge \frac{n}{p}\ . \tag{5.1}$$

Boundedness of T from $L^{p,1}$ to $L^{q,\infty}$ implies boundedness from the dual of $L^{q,\infty}$ to the dual of $L^{p,1}$, so a bit of algebra gives the second necessary condition

$$\frac{n}{q} \ge \frac{n-1}{p}\ . \tag{5.2}$$

A third necessary condition is found by taking E_δ to be the characteristic function of a rectangular parallelipiped centered at the origin, with sides parallel to the coordinate axes, having side lengths $(\delta, \delta^2, \dots, \delta^n)$. Then $|E_\delta| = \delta^{n(n+1)/2}$. If $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $|x_j| \le \delta^j/4$ for all $j \le n$, and if $|t| < \delta/4$, then $x - h(t) \in E_\delta$. Thus $T(\chi_{E_\delta}) \ge \delta$ on a set of measure $\ge 4^{-n}|E_\delta|$. This leads as above to the necessary condition

$$1 + \frac{n(n+1)}{2q} \ge \frac{n(n+1)}{2p}\ . \tag{5.3}$$

This condition is self dual.

Equality holds in (5.2) and in (5.3) at Q_n, where $p = (n+1)/2$ and $q = pn/(n-1)$. By duality, equality holds in (5.1) and (5.3) at the dual point $\tilde{Q}_n$. Equality holds in (5.3) when $p = q = \infty$, and in (5.1) for $p = q = 1$.

A fourth necessary condition is that $q \ge p$.

We conclude with some remarks. First, a natural generalization is to replace $\{(t, t^2, \dots)\}$ by any C^n curve whose first n derivatives are linearly independent at every point. Generalizing to that setting would require an analysis of the multiplicity of preimages which we have sidestepped by exploiting the algebraic character of $\{(t, t^2, \dots)\}$. We have also benefitted from explicit formulas for Vandermonde determinants.

Second, a more general problem is to determine the mapping properties, in Lebesgue and Sobolev spaces, of operators

$$Tf(x) = \int_{M_x} f(y)\, d\sigma_x(y)\ , \tag{5.4}$$

where $\{M_x : x \in \mathbb{R}^n\}$ is a smoothly varying family of submanifolds of $\mathbb{R}^m$, and σ_x is an appropriate measure on M_x. A necessary and sufficient geometric criterion has been established [6] (in the equidimensional case $m = n$) for T to map L^p to L^q for some $q > p$. But to determine the precise relation between the exponents is a deeper problem which has been resolved only in special cases.

The approach taken here is not restricted to convolution operators, nor to one-dimensional manifolds M_x. It furnishes for instance an alternative proof of optimal L^p–L^q mapping properties (though only restricted weak type at an endpoint) for families of hypersurfaces under the diffeomorphism-invariant condition called rotational curvature by Phong and Stein [13]. In outline it is applicable to operators (5.4) in great generality, but the implementation must vary widely from one instance to another, and substantial difficulties appear to arise even in relatively simple situations.

Third, another recent work involving some combinatorial analysis is [3].

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