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The topology of 4-manifolds, by Robion C. Kirby. Springer-Verlag, Berlin, New York, 1989, 106 pp., $13.50. ISBN 3-540-51148-2

In the years between roughly 1975 and 1985, the modern theory of four-dimensional manifolds was born as workers came to recognize that there was a fundamental difference between the topological theory of these manifolds and the corresponding smooth theory. One of the most striking aspects of this difference is that there are smooth manifolds homeomorphic to 4-dimensional euclidean space which are not diffeomorphic to it, a phenomenon which happens in no other dimension.

During the 1950s and 60s, great progress was made on basic existence and classification questions for manifolds in dimensions greater than 4. Thom's theory of transversality and Smale's theory of handlebodies were used to reduce many outstanding problems to a mixture of algebraic K-theory and homotopy theory. Throughout much of this period the results applied only to smooth or PL manifolds, but in 1969, Kirby and Siebenmann were able to prove that these transversality and handlebody techniques were also valid for topological manifolds.

During this period, the 4-dimensional case had not been neglected. The most famous result was Rohlin's theorem, which says that a closed smooth 4-manifold with a spin structure on its tangent bundle must have signature divisible by 16. By the middle of the 60s, the homotopy theory of simply connected 4-manifolds was well understood [4, 9], and high-dimensional techniques such as the h-cobordism theorem and surgery theory were in place if one was willing to stabilize by taking connected sums with copies of $S^2 \times S^2$ [7, 8].

One of the early results from the 70s was the development, by Kirby, of an effective method of exhibiting a handle decomposition for 4-manifolds and for working out the effect of the standard handle moves on these pictures. He developed a similar description for 3-manifolds and, in this case, proved that two 3-manifold pictures yielded diffeomorphic 3-manifolds iff one could pass from one picture to the other by a sequence of elementary handle moves. This material forms the first chapter of the book under review. The treatment lacks the detail of the orginal papers, but gives a clear statement of the results and techniques which have proved useful in applications. Some applications, as well as several famous open problems, are discussed.

The second chapter set the tone for a good deal of the rest of the book. The intersection form for an oriented 4-manifold is developed from the point of view of intersection of submanifolds. The classification of simply connected 4-manifolds is derived (in a more geometric manner than is usual), and the algebraic theory of symmetric bilinear forms is recalled for the reader. The first two Stiefel-Whitney classes are introduced and there is a discussion of the signature.

In Chapter 3, Rohlin's theorem is recalled, Freedman's classification of simply connected 4-manifolds up to homeomorphism is given and some discussion of the nonsimply connected case ensues. Donaldson's nonexistence results in the smooth case are also discussed.

The next six chapters give proofs of well-known results, but using only immersion theory and the handlebody technique developed so far. Included are geometric proofs of Hirzebruch's signature theorem for 4-manifolds; Thom's theorem that oriented 4-dimensional bordism is isomorphic to the integers via the signature; and the result that the 4- dimensional spin bordism group is an infinite subgroup of the oriented bordism group.

Chapter 10 gives proofs of Wall's results on diffeomorphisms of simply connected 4-manifolds and on the h-cobordism theorem. The handle calculus developed in Chapter 1 is used to produce certain diffeomorphisms of $N \# S^2 \times S^2$, N simply connected. These diffeomorphisms generate the isometry group of the intersection form.

Chapter 11 revolves around Rohlin's theorem. The first section gives a very elementary proof of this fundamental result. In §2, the result is recast in terms of characteristic bordism. The major improvement over the earlier work of Freedman and Kirby [3] is to use spin structures on characteristic surfaces to define a homomorphisms $\phi: \Omega_4^{\text{char}} \to Z/2$. The third section uses the previous material to define the Arf invariant of a knot and to study imbedded surfaces with singular points. Some examples are worked out. This section seems to be the flaw in an other wise well-proofread book. Figure 4 (page 70) and Figure 5 (page 71) are switched and the formula in Corollary 7 on page 69 is missing the contribution from the quadratic form on the imbedded surface.

The remainder of the book is a description of Freedman's work and a description of some exotic smooth structures on R^4. The starting point for Freedman's work was some work of Casson's from the early 1970s making some progress on the Whitney trick. Almost from the beginning, it was realized that the Whitney trick played a fundamental role in the high-dimesional theory. It was also realized early on that the full power of the Whitney trick could not be made to work in dimension 4. However, the full power of the Whitney trick is rarely used even in the high-dimensional theory, so it was not clear whether the high-dimensional theory could be made to work substantially unchanged in dimension 4 or not. Casson's fundamental contribution was to show that Casson handles could be used in place of Whitney disks in the simply connected theory. Casson handles are open manifolds with boundary where the boundary is $S^1 \times R^2$ and the manifold has the proper homotopy type of $D^2 \times R^2$ and are replacements for the Whitney disks used in the Whitney trick. Chapter 12 contains a very good discussion of Casson handles.

It is not clear from Casson's work how to use the imbedded Casson handles to remove double points in the way that Whitney disks are used in the high-dimensional theory. Freedman was the one who saw how to make further progress.

The first of Freedman's results were the reimbedding theorems. These orginated in his 1979 paper [1] and are covered in §3 of Chapter 13. After some improvements by Gompf, the end result of these reimbedding theorems is that the first five stages of any Casson handle contains the first six stages of some other Casson handle. This allowed Freedman in 1979 to construct a compactum, K, inside a Casson handle by successive intersections of reimbedded five-stage Casson handles so that K looked a lot like a flat 2-disk. Using this construction, Freedman was able to do a certain amount of surgery theory, and, in particular, construct a smooth manifold which had the proper homotopy type of $S^3 \times R$ but was not diffeomorphic to it. The proofs in this section of the book are often sketchy but the key points are covered, with the reader referred to the orginal sources for the full details. (One might have appreciated a bit more attention to precise references at this point, but it does force one to read more of the original sources than one might otherwise be inclined to do.) Siebenmann's first Bourbaki article [5] also makes excellent reading at this point.

Freedman's epic 1982 paper [2] takes things a step further and is the subject of the remainder of Chapter 13 of the book. Sections 1 and 2 of this chapter give an introduction to decomposition space theory, including a proof of the fact that R^3 with the Whitehead continuum smashed to a point is a manifold factor, explicitly $(R^3/Wh)\times R$ is homeomorphic to R^4. Section 4 gives an overview of the grand design of Freedman's proof, but again the reader is referred to the orginal articles for details. Siebenmann's second Bourbaki article [6] may be profitably consulted at this point.

In the last chapter, Kirby outlines two constructions of "exotic" smooth structures on R^4. All known examples of this phenomenon have two steps: one first uses Freedman's work to construct an open subset of some smooth manifold. One does the construction so that one sees that the resulting manifold is homeomorphic to R^4 using Freedman's characterization of R^4 up to homeomorphism. One then argues that if this manifold were diffeomorphic to R^4, one could then construct an example which we know can not exist. The first construction in Chapter 14 produces an "exotic" R^4 which is an open subset of $S^2 \times S^2$, but we know that it is exotic because it can not be an open subset of S^4. The second example is more subtle and constructs an exotic R^4 which is an open subset of the standard one. Both of these constructions rely on work of Donaldson to produce the contradiction. Taubes was able to extend Donaldson's work and produce an uncountable number of exotic R^4's, none of which imbed in the standard one. The reviewer has recently received a preprint from DeMichelis and Freedman proving that there are also uncountably many "exotic" R^4's imbedded in the standard R^4.

This book is a tour through the geometric theory of 4-manifolds, guided by the author, rather than an encyclopedic treatment of any part of the theory. The expressed goal of this book is to take a look at the major theorems in geometric 4-manifold theory from an "elmentary" viewpoint. The treatment is not always uniform: most readers would feel that immersion theory (which is used but not really treated) is intrinsically more difficult than the intersection form (which is discussed in some detail). Still, what the author has decided to do he does well, he is clear as to what he is skipping, and he does give references for the omitted results. This book should be useful to anyone needing a guide into this fascinating subject and it is delightful read for anyone with even a passing interest in the material.

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