A CURIOUS CATEGORY WHICH EQUALS TOP

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Given a locally defined collection (called CAT) of continuous functions from open sets in Euclidean space to Euclidean space, which is closed under composition whenever composition is defined (so that the invertible functions form a pseudo-group), then we can form a category, CAT, of manifolds and maps where the manifolds are covered by coordinate charts whose coordinate transformations belong to CAT. One wishes to find useful categories which contain all topological manifolds.

It is known that an m-manifold M^m admits a measure zero preserving structure when $m \neq 4 \neq \dim(\partial M)$; that is the coordinate transformations take sets of r-measure zero to the same, for all r. With the same dimensional restrictions M admits a D^1-structure, where a D^1 map is differentiable everywhere, but not necessarily with a continuous differential. It is a good guess that every manifold has a Lipschitz structure, because there are different PL structures which are Lipschitz equivalent [5].

We will describe here a category, CCG, which turns out to be precisely TOP. That is, every continuous map belongs to CCG. Consider the continuous functions from open sets in R^n to R^p which have smooth graphs, that is, whose graphs are C^r submanifolds of $R^n \times R^p$, $r \geq 1$. An illustrative example is $f(x) = x^{1/3}$. We call such a function C^rG for C^r graph, and write CG when the value of r is unimportant.

Two CG functions do not necessarily compose to give a CG function. If

$$f(x) = \begin{cases} x^3 \text{ for } x \geq 0 \\ 0 \text{ for } x \leq 0 \end{cases}$$

and $g(x) = x^{1/3}$, then

$$f(g(x)) = \begin{cases} x \text{ for } x \geq 0 \\ 0 \text{ for } x \leq 0 \end{cases}$$

is not smooth at the origin. The function f is only C^2, but a C^∞ example comes from using $\exp(-1/x^2)$.

In order to get a locally defined category we must arbitrarily enlarge the collection of CG functions by including all functions which are locally finite compositions of CG functions. If $f_i\colon U_i \longrightarrow V_i$, $i = 1,\ldots,k$ are CG functions from open sets $U_i \subset R^{n_i}$ to open sets $V_i \subset R^{p_i}$ and $V_i \subset U_{i+1}$, then we call $f_k f_{k-1} \cdots f_1\colon U_1 \longrightarrow V_k$ a CCG function (C stands for composition).

We define a CCG manifold as one whose coordinate transformations are CCG homeomorphisms.

Beware that a CCG homeomorphism is not locally a composition of CG homeomorphisms, just CG functions (but see MCCG below). This gives us the CCG category which we now show is the same as TOP.

We cite the following well-known result:

LEMMA: If A is any closed subset of R^n, then there is a C^∞ function $f: M \longrightarrow [0,\infty)$ such that $f^{-1}(0) = A$.

THEOREM: Let $h: R^n \longrightarrow R$ be continuous. Then we can write $h = Hi$ where $R^n \xrightarrow{i} R^n \times R \xrightarrow{H} R$, $i(x) = (x,0)$ and H has a C^∞ graph (H is $C^\infty G$). Thus h is CCG.

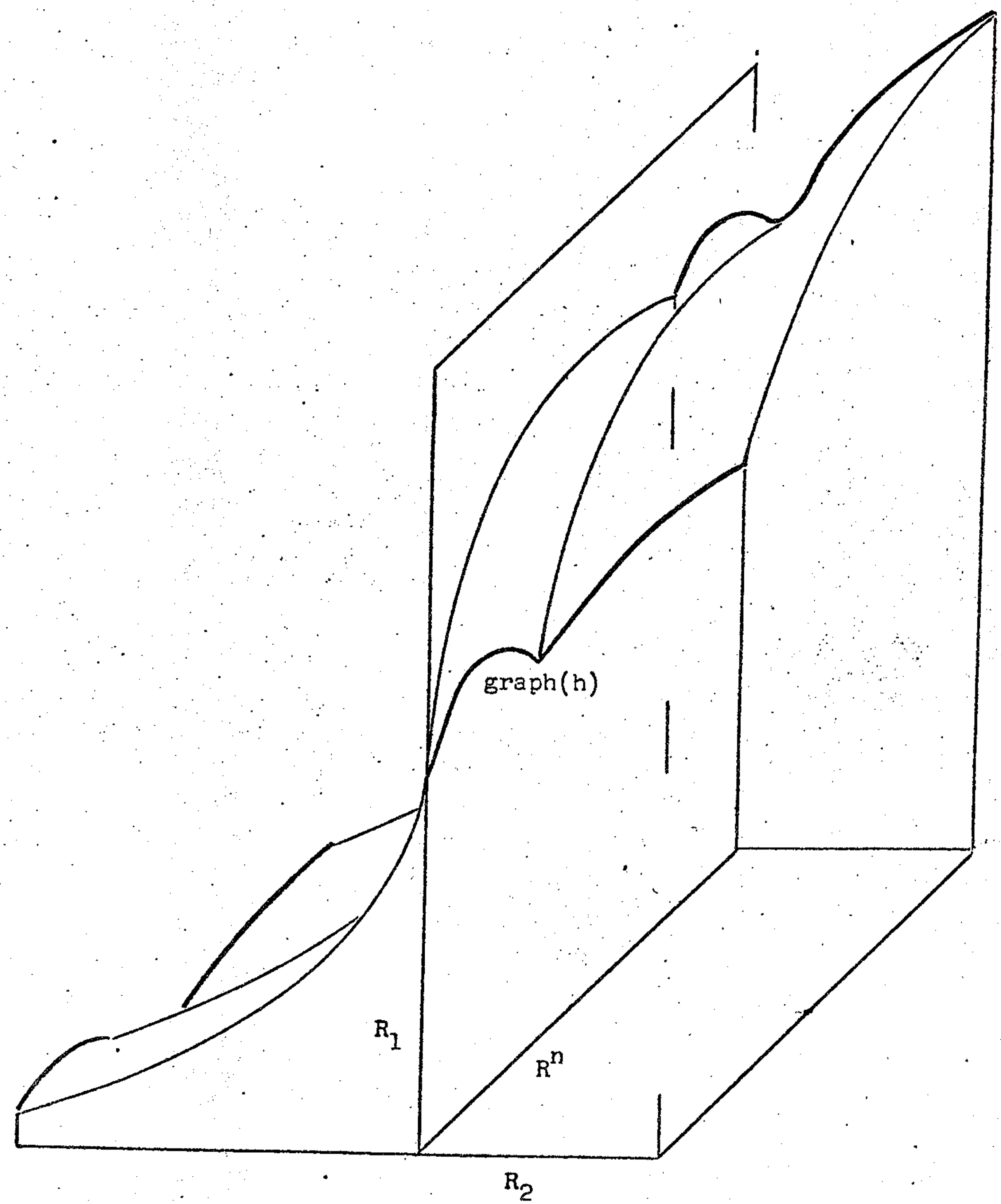
Proof: It will be convenient to specify two copies of the real numbers, R_1 and R_2, and to assume $h: R^n \longrightarrow R_1$. For simplicity we assume that $|h|$ is bounded by a constant K. By the Lemma, there is a C^∞ map $g_1: R^n \times R_1 \longrightarrow R_2$ for which $g_1^{-1}(0) = \{(x,t) \in R^n \times R_1 \mid t \leq h(x)\}$ and $g \geq 0$. Define $f_1: R^n \times R_1 \longrightarrow R_2$ by

$$f_1(x,t) = \int_{-K}^{t} g_1(x,s)\, ds.$$

f_1 is a C^∞ function and $\partial f_1(x,t)/\partial t = g_1(x,t) \begin{cases} > 0 \text{ if } t > h(x) \\ = 0 \text{ if } t \leq h(x) \end{cases}$

Similarly we define g_2 so that $g_2^{-1}(0) = \{(x,t) \in R^n \times R_1 \mid t \geq h(x)\}$ and $g_2 \geq 0$. Then $f_2: R^n \times R_1 \longrightarrow R_2$ is given by

$$f_2(x,t) = -\int_{t}^{K} g_2(x,s)\, ds,$$


graph(h)
R_1
R^n
R_2

so that f_2 is C^∞ and $\partial f_2(x,t)/\partial t = g_2(x,t) \begin{cases} >0 \text{ if } t < h(x) \\ = 0 \text{ if } t \geq h(x) \end{cases}$

Now let $f = f_1 + f_2 \colon R^n \times R_1 \longrightarrow R_2$. Then $f^{-1}(0) = \text{graph } (h)$

and $\partial f(x,t)/\partial t = (g_1 + g_2)(x,t) \begin{cases} \geq 0 \text{ if } t \neq h(x) \\ = 0 \text{ if } t = h(x) \end{cases}$.

If g_1 and g_2 are chosen appropriately, we can assume $f_x = f \big| \{x\} \times R_1 : \{x\} \times R_1 \longrightarrow R_2$ is onto for any x. Since f_x has positive derivative except at $(x,h(x))$, it is a homeomorphism. See the figure.

Since f is C^∞, graph(f) is a C^∞ submanifold of $R^n \times R_1 \times R_2$ for which $\text{graph}(f) \cap R^n \times R_1 \times \{0\} = \text{graph}(h)$. The projection $p_2 \colon \text{graph}(f) \longrightarrow R^n \times R_2$ given by $p_2(x,t,f(x,t)) = (x,f(x,t))$ is a homeomorphism. For f_x is onto and if $p_2(x,t,f(x,t)) = p_2(x',t',f(x',t'))$ then $x = x'$ and $t = t'$ because f_x is one-to-one.

Now let H be the compositon $R^n \times R_2 \xrightarrow{p_2^{-1}} \text{graph}(f) \xrightarrow{p_1} R_1$ and note that graph(H) = graph(f) so H is $C^\infty G$. Clearly the compositon $R^n \xrightarrow{i} R^n \times R_2 \xrightarrow{H} R_1$ is h.

COROLLARY 1: If $h \colon R^n \longrightarrow R^p$ is continuous, then there is a $C^\infty G$ map H so that the composition $R^n \xrightarrow{i} R^n \times R^p \xrightarrow{H} R^p$ is equal to h. So h is CCG.

Proof: Apply the theorem to each component of $h = (h_1,\ldots,h_p)$, obtaining $H_1,\ldots,H_p$. Let $H(x,t_1,\ldots,t_p) = (H_1(x,t_1),\ldots,H_p(x,t_p))$.

COROLLARY 2: CCG = TOP

Proof: ∴ Any coordinate transformation in a topological manifold is CCG (take n=p in Corollary 1) and any continuous function between CCG manifolds is locally CCG, hence CCG.

We had hoped to use CCG to prove transversality in the TOP category, particularly in the cases in which it is not known [2],[3]. There was some reason to believe transversality could be proven, because a CG map is locally a projection except where the differential exists and has less than maximal rank. Once we knew that CCG = TOP, it seemed that a "stronger" category was needed to solve delicate problems like transversality. Since a difficulty with CCG is that a continuous map is factored through a space of higher dimensions, we offer the following definition.

Let MCCG be monotone compositions of CG functions, that is, compositions of CG functions $f_i: R^{n_i} \longrightarrow R^{n_{i+1}}$, $i = 1,\ldots,k$, where either $n_i \geq n_{i+1}$ for all i or $n_i \leq n_{i+1}$ for all i.

First of all, MCCG $\neq$ TOP. A function $f: R \longrightarrow R$ which is nowhere differentiable cannot be the composition of finitely many CG functions $f_i: R \longrightarrow R$. Locally, each f_i is C^r, or where it has vertical slope it has a C^r inverse. To get a point t of non-differentiability for $f_{i+1}f_i$, we must have $f_i'(t) = 0$ and $f_{i+1}'(f_i(t)) = \infty$, or vice versa, as in the first example in the paper. No f_i can be locally constant, so the critical points of each f_i form a subset of a Cantor set. Near points of vertical slope, f_i is a homeomorphism and therefore takes Cantor sets to Cantor sets. It follows that f can only be non-differentiable on a Cantor set. See [1] for some related examples.

However it seems likely that MCCG contains PL. To show that a PL map $f: R^n \longrightarrow R$ is MCCG, one should find a sequence of homeomorphisms $g_i: R^n \longrightarrow R^n$ such that $fg_{n-1}\cdots g_i$ and $g_{n-1}\cdots g_i$ are C^∞ on the complement of the (i-1)-skeleton of a triangulation of R^n; then $f = fg_{n-1}\cdots g_0(g_{n-1}\cdots g_0)^{-1}$ is MCCG. The g_i's might be constructed by squeezing an open tubular neighborhood of an open i-simplex in towards the i-simplex so that g_i is a diffeomorphism off the i-simplex, but has zero partial derivative in directions normal to the i-simplex.

The basic problem to solve in constructing the g_i's is this: given a function $h: R^n \longrightarrow R$ which is C^∞ differentiable off of R^k, $0 \leq k < n$, and given a squeeze $g: R^n \longrightarrow R^n$ such as

$$g(x,y) = (\exp(-1/|x|^2)\cdot x/|x|, y)$$

where $(x,y) \in R^{n-k} \times R^k = R^n$, then what conditions must h satisfy so that hg will be C^∞ differentiable? For example, when $n = 1$, h must have bounded variation. Also it is not clear how to handle the higher order derivatives which mix partial derivatives in the R^{n-k} and R^k directions.

We do not know whether every TOP manifold has an MCCG structure, and what the answer "should" be is not clear. The C^∞ Hauptvermutung [4], which was originally motivated by just this problem, precludes a composite of globally defined CG functions of smooth manifolds from changing PL structures, but a CCG function need only be the composite of CG functions locally.

References

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