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Slice Knots and Property R

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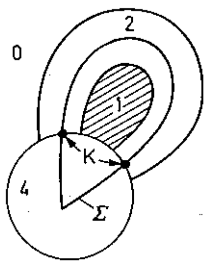
Theorem. Let M be the 4-manifold obtained by adding a 2-handle to B^4 along a knot K in S^3 with the zero framing. If ∂M is diffeomorphic to $S^2 \times S^1$, then

(1) K is slice (see Addendum)
(2) M is homeomorphic to $S^2 \times B^2$.

It is of considerable interest to know which 3-manifolds can be obtained by surgery on which knots. If surgery on a knot cannot give a homotopy 3-sphere, then the knot is said to satisfy Property P. If surgery cannot give $S^2 \times S^1$, the knot satisfies Property R.

The Theorem states that non-slice knots satisfy property R. It is known that knots with non-trivial Alexander polynomial [4], composite knots [4], doubled knots [4], and certain genus one knots [3] all satisfy Property R. There is a reference list for knots with Property P in [2]. We thank Larry Taylor for smoothing the proof of (1).

Proof of Theorem. First note that the union of the cone on K in B^4 and the core of the 2-handle is an embedded 2-sphere Σ which is smooth except at the cone point. We can add a 3-handle to M along $S^2 \times * \subset S^2 \times S^1$. The new boundary is S^3 to which we add a 4-handle, obtaining a closed smooth 4-manifold W. If we turn W upside down, we see that it is built with a 0, 1, 2, and 4-handle.



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W is a homotopy 4-sphere so $W_0=(W^4$ minus the 4-handle) is contractible. Thus $W_0\times I$ is a contractible 5-manifold built with a 0, 1, and 2-handle, and so the 1 and 2-handles must cancel algebraically. It follows from the Whitney trick that they must in fact cancel geometrically, and so $W_0\times I=B^5$. But $\partial B^5=S^4$ is the double $W\#-W$ of W. If the connected sum is taken away from Σ, then $\Sigma\subset S^4$. Removing a small 4-ball centered at the cone point, we observe a slice D of K in the complementary 4-ball. Thus (1) is proved.

The Topological Schoenflies Theorem applied to $W\#-W$ gives a homeomorphism between W and S^4. M is the complement in W of an open tubular neighborhood N (the 0 and 1-handles) of a smoothly embedded circle. Since W is simply connected, N lies in a coordinate chart in W, and so M is diffeomorphic to $S^2\times B^2\# W$. Thus M is homeomorphic to $S^2\times B^2$ and (2) is proved. □

Addendum. Larry Taylor and Mike Freedman showed us that K is in fact the intersection in S^4 of the standard S^3 and an unknotted S^2 which is invariant under reflection through S^3. We shall call such a knot a *symmetric slice of the unknot*. We give their proof here.

Doubling (B^4, D), where D is the slice disc for K constructed above, we obtain a pair (S^4, S) with K a symmetric slice of S.

To show that S is unknotted it suffices to show that the 4-manifold V gotten by surgery on S in S^4 is $S^1\times S^3$. Recall that the complement C of an open tubular neighborhood of D in B^4 is $S^1\times B^3\# W$, where $W\#-W=S^4$. Now V may be identified with the boundary of $C\times I$. Thus

$$\begin{aligned} V&=\partial((S^1\times B^3\# W)\times I)\\ &=S^1\times S^3\# W\#-W=S^1\times S^3. \quad \square \end{aligned}$$

Remark 1. One may alternatively deduce that S is unknotted from the following observation. If (B^5, B^3) denotes an unknotted ball pair, then there is a pairwise diffeomorphism

$$(B^4, D)\times I=(B^5, B^3). \tag{$*$}$$

This follows easily from the proof of the Theorem since

$$(B^4, D)=(W_0\cup h^1\cup h^2, \text{cocore } h^2)$$

where h^1 and h^2 are an algebraically cancelling pair of 1 and 2-handles attached to ∂W_0. As above $W_0\times I=B^5$, so we have

$$(B^4, D)\times I=(B^5\cup H^1\cup H^2, \text{ cocore } H^2)$$

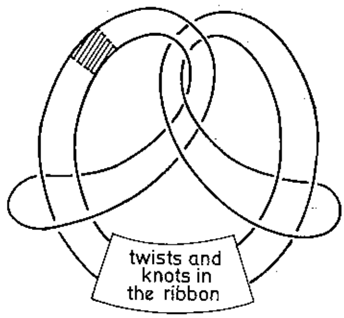
where H^1 and H^2 are a cancelling pair of 1 and 2-handles attached to $S^4=\partial B^5$. There are up to isotopy only two ways of attaching a cancelling pair, depending on the framing of the 2-handle, and it is easy to see that either way yields an unknotted ball pair (B^5, B^3).

In fact, it can be shown[1] that any symmetric slice of the unknot bounds a disc D in B^4 satisfying $(*)$. The converse is trivial.

[1] The proof we know uses duality on the infinite cyclic cover of C (defined in the Addendum) to show that C is a homotopy S^1, and a result of Shaneson and Wall to show that $C\times I=S^1\times B^4$

Gordon and Sumners [1] have studied such knots. They remark that the infinite cyclic cover of the knot complement must be acyclic, and so all the associated invariants vanish. In particular the Alexander polynomial must be equal to 1. They also produce an infinite collection of such knots, some of which are doubled knots and so satisfy Property R [4].

There are many other examples of such knots, however. For example any knot K of the form



bounds a disc D (obtained by slicing K as indicated by the shaded band) for which $(B^4, D) \times I = (B^5, B^3)$. In fact the complement C of an open tubular neighborhood of D in B^4 can be built as a handle-body with two 1-handles and one 2-handle which algebraically cancels each 1-handle. Thus $C \times I = S^1 \times B^4$ and the result follows.

Remark 2. Suppose we replace S^3 by a homology 3-sphere H. Then 0-surgery on a knot K in H yields $S^2 \times S^1$ if and only if H is the boundary of a contractible 4-manifold W built with a 0, 1 and 2-handle. In this case, as in the Addendum, K is the intersection in $S^4 = W \bigcup_{\partial} - W$ of $H = \partial W$ and an unknotted 2-sphere invariant under the obvious involution on S^4. If 0-surgery on K and K' in H both yield $S^2 \times S^1$, then is (H, K) pairwise diffeomorphic to (H, K')?

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