

Dedekind sums, μ -invariants and the signature cocycle

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0 Introduction

The Dedekind eta function, defined by

$$\eta(z) = e^{\pi iz/12} \prod_{n=1}^{\infty} (1 - e^{2\pi in z})$$

for z in the upper half plane \mathbf{H} , plays a central role in number theory. It is a modular form of fractional weight whose 24th power is proportional to the fundamental discriminant cusp form of weight 12. In particular $\eta(z)^{24} dz^6$, is invariant under the modular group $\text{PSL}(2, \mathbf{Z})$, and so there is a function $\phi: \text{PSL}(2, \mathbf{Z}) \rightarrow \mathbf{Z}$ given by

$$(0.1) \quad \phi(A) = \frac{12}{\pi i} (\log \eta(Az) - \log \eta(z) - \mu(A)) \quad \text{for } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $\mu(A) = \frac{1}{2} \log \left(\frac{cz + d}{i \text{sign } c} \right)$ if $c \neq 0$, and $\mu(A) = 0$ if $c = 0$. Dedekind [D] gave a formula

$$(0.2) \quad \phi(A) = \begin{cases} \frac{b}{d} & \text{if } c = 0 \\ \frac{a+d}{c} - 12 \text{sign}(c) s(a, c) & \text{if } c \neq 0 \end{cases}$$

in terms of certain arithmetic sums $s(a, c)$ defined for coprime integers a and c by

$$(0.3) \quad s(a, c) = \sum_{k=1}^{|c|-1} \left(\left(\frac{k}{c} \right) \right) \left(\left(\frac{ka}{c} \right) \right)$$

where $((x)) = x - [x] - 1/2$. These sums, now called **Dedekind sums**, arise in many contexts and have been intensively studied during the past hundred years. Many of their fundamental properties were discovered by Rademacher, and the function ϕ is

often called the **Rademacher ϕ function**. References for this classical subject include the beautiful monograph [RadG] as well as basic texts in analytic number theory and modular forms such as [Ap] and [Kn].

Dedekind sums arise naturally in many topological settings. Hirzebruch was the first to notice this through computations with the equivariant signature theorem [Hir2]. In particular, $4cs(a, c)$ can be identified with the *signature defect* of the 3-dimensional lens space $L(c, a)$ (appropriately normalized) by means of Rademacher's cotangent formula

$$(0.4) \quad s(a, c) = \frac{1}{4|c|} \sum_{k=1}^{|c|-1} \cot\left(\frac{k\pi}{c}\right) \cot\left(\frac{ka\pi}{c}\right)$$

[Rad2].

It follows that Dedekind sums are present in other signature related invariants of lens spaces, such as their α -invariants [HirZ] and μ -invariants [NR]. They also appear in formulas for Meyer's *signature cocycle* $\sigma: \mathrm{SL}(2, \mathbf{Z}) \times \mathrm{SL}(2, \mathbf{Z}) \rightarrow \mathbf{Z}$ which arises in the study of signatures of torus bundles over surfaces [Mey], and in other topological contexts as well. An extensive discussion of many of these topological aspects of Dedekind sums appears in [At1] from the point of view of geometry (index theory of elliptic operators) and physics (gauge theory).

More recently, Dedekind sums have appeared in the *generalized Casson invariant* [Wa] and the new *SU(2)-quantum invariants* [Wi, ResT] of rational homology 3-spheres. The connection with quantum invariants was anticipated by our discovery of a relationship between the Casson invariant and the quantum invariants of integral homology spheres, and was first observed in our calculations for lens spaces in August 1990 [KM2] (see also [Gar, J]). We were aided in recognizing the Dedekind sum by a continued fraction formula in [Hic], brought to our attention by [Wa]. In fact our efforts to resolve a discrepancy between Hickerson's formula and ours led us to a new elementary geometric definition of the Rademacher ϕ function which was the starting point for the present paper.

The first two sections of the paper describe this geometric formulation of the ϕ function and Dedekind sums. Many basic properties follow easily, in particular the fundamental *Dedekind reciprocity law* $12(s(a, c) + s(c, a)) = a/c + c/a + 1/ac - 3 \operatorname{sign}(ac)$, which essentially characterizes the Dedekind sum.

The remaining sections of the paper show how this formulation illuminates the appearance of Dedekind sums in topology. Connections with signature defects and μ -invariants of lens spaces are discussed in Sect. 3 and Sect. 4, primarily by means of associated reciprocity laws. One interesting outcome of our viewpoint is the existence of certain integral lifts of the μ -invariants of lens spaces $L(c, a)$ which can be thought of as part of the associated Dedekind sums $s(a, c)$. These are also related to the *Brown invariants* which arise in the study of quantum invariants (see the end of Sect. 4). In Sect. 5 we give formulas for the μ -invariants of torus bundles over the circle, which are used in Sect. 6 to give a simple formula for the signature cocycle. The paper concludes with an appendix which gives surgery descriptions for lens spaces and torus bundles, and identifies the characteristic classes associated with various spin structures.

The rest of this introduction provides a more detailed survey of our results. The reader may find it convenient to use this both as an introduction and as a conclusion to the paper.

Our geometric definition of the ϕ function arises from the action of the modular group on the upper half plane \mathbf{H} . It can be expressed in terms of continued fractions as follows: For A as in (0.1), choose a list $\alpha = (a_1, \dots, a_n)$ of integers such that $a/c = \langle a_1, \dots, a_n \rangle$ and $b/d = \langle a_1, \dots, a_{n-1} \rangle$, where

$$(0.5) \quad \langle a_1, \dots, a_n \rangle \stackrel{\text{def}}{=} \frac{1}{a_1 - \frac{1}{a_2 - \frac{1}{\ddots - \frac{1}{a_n}}}}$$

(We shall always use this non-standard notation for continued fractions, as there is no uniformly accepted notation in the literature.) Geometrically α corresponds to an edge path in a certain ideal triangulation of \mathbf{H} (with vertices at the rational points on the circle at infinity) where the a_i specify the turns to be made at each vertex of the path (see Sect. 1). Let σ_α and τ_α denote the signature and trace of the associated matrix

$$(0.6) \quad M_\alpha = \begin{pmatrix} a_1 & 1 & & & \circ \\ 1 & a_2 & 1 & & \\ & & 1 & \ddots & \\ & & & \ddots & 1 \\ \circ & & & & 1 & a_n \end{pmatrix}$$

whose ij^{th} entry is a_i if $i = j$, 1 if $|i - j| = 1$ and 0 otherwise. Now define

$$(0.7) \quad \varphi(A) = 3\sigma_\alpha - \tau_\alpha = 3\sigma_\alpha - \sum_{i=1}^n a_i.$$

The geometric content of this definition is explained in Sect. 1, where it is shown to yield a well defined function $\varphi: \text{PSL}(2, \mathbf{Z}) \rightarrow \mathbf{Z}$ equal to the negative of the Rademacher ϕ function. Our normalization is convenient for many purposes, and we shall also call it the **Rademacher φ function**.

The **Dedekind symbol** of any (extended) rational number a/c can now be defined by

$$(0.8) \quad S\left(\frac{a}{c}\right) = \varphi\left(\begin{matrix} a & b \\ c & d \end{matrix}\right) + \frac{a+d}{c}$$

where b and d are chosen so that $ad - bc = 1$. Thus S is viewed as a function from $\mathbf{Q} \cup \infty$ to itself, which (using (0.2) or Dedekind reciprocity) is related to the classical Dedekind sums by the formula $S(a/c) = 12 \text{sign}(c)s(a, c)$. This definition is discussed in Sect. 2, where we show how the basic properties of Dedekind sums are easily proved.

Note that the expression $3\sigma - \tau$ first appeared in Hirzebruch's study of Hilbert modular surfaces [Hir3], and is there related to Dedekind sums through the work of Meyer. This expression also turned up in [FG] in connection with the study of quantum invariants. In particular, for a 3-manifold M obtained by surgery on a framed link L (see [K1]), $3\sigma - \tau$ measures the difference between the 2-framing arising from L and the canonical 2-framing of [At2]. (Here σ and τ are the signature and trace of

the linking matrix of L , and a 2-framing of M is a trivialization of twice its tangent bundle – the choice of a 2-framing is needed in Witten’s formulation of quantum invariants.)

The Dedekind symbol S and the Rademacher φ function are at the center of a number of topics arranged in the schematic diagram in Fig. 1.

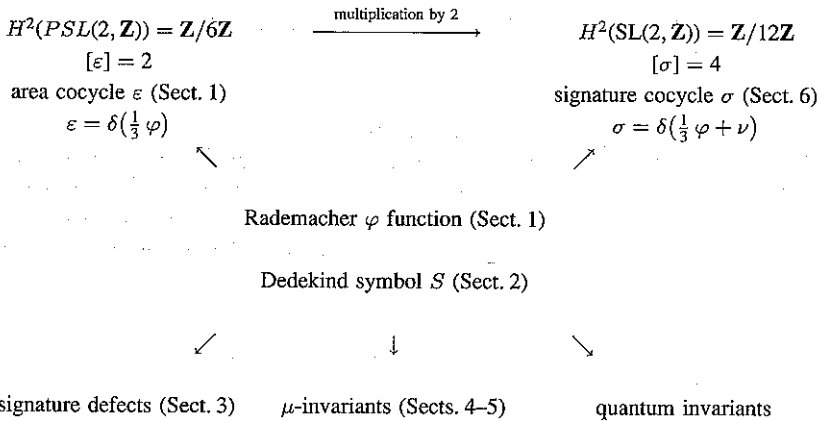


Fig. 1

The φ function can be viewed as the key to understanding the second cohomology of the special linear group $SL(2, \mathbf{Z})$ (the group of 2×2 unimodular integer matrices) and its central quotient $PSL(2, \mathbf{Z}) = SL(2, \mathbf{Z})/(\pm I)$ (the modular group), and thus to the understanding of their central extensions (see [Bro2] as a general reference for the cohomology of groups).

To fix notation, recall that $SL(2, \mathbf{Z})$ has a presentation $(S, T: S^2 = (ST)^3, S^4 = I)$ with respect to the generators

$$(0.9) \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

(see e.g. [Mag, Sect. III.1]). Thus $PSL(2, \mathbf{Z}) = (S, T: S^2 = (ST)^3 = I)$, where elements of $PSL(2, \mathbf{Z})$ are specified by *either* representative in $SL(2, \mathbf{Z})$ (determined up to sign).

It follows that $PSL(2, \mathbf{Z})$ is isomorphic to a free product $C_2 * C_3$ of cyclic groups of order 2 and 3, and so $H^* PSL(2, \mathbf{Z}) = H^* C_2 \oplus H^* C_3$. In particular

$$(0.10) \quad H^1(PSL(2, \mathbf{Z}), \mathbf{Z}) = 0 \quad \text{and} \quad H^2(PSL(2, \mathbf{Z}), \mathbf{Z}) = \mathbf{Z}/6\mathbf{Z}$$

since the integral cohomology of C_n vanishes in odd dimensions and is isomorphic to $\mathbf{Z}/n\mathbf{Z}$ in positive even dimensions. Thus

$$(0.11) \quad H^1(SL(2, \mathbf{Z}), \mathbf{Z}) = 0 \quad \text{and} \quad H^2(SL(2, \mathbf{Z}), \mathbf{Z}) = \mathbf{Z}/12\mathbf{Z}$$

by the universal coefficient theorem, or by a spectral sequence argument.

Now there are two important 2-cocycles that arise in many contexts, the *area cocycle* ϵ on $PSL(2, \mathbf{Z})$ and the *signature cocycle* σ on $SL(2, \mathbf{Z})$, mentioned above. (We thank R. Bott for bringing the former to our attention.) Both have simple geometric

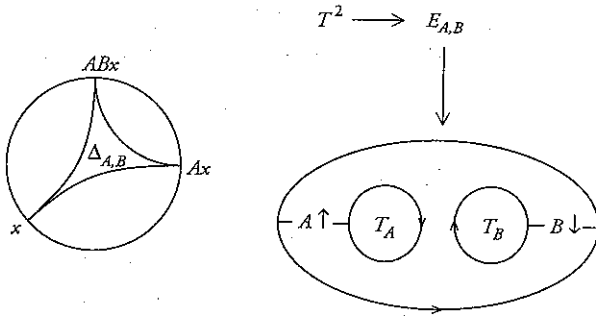


Fig. 2 $\varepsilon(A,B) = \text{area}(\Delta_{A,B}) / \pi$ $\sigma(A,B) = \text{signature}(E_{A,B})$

definitions: For unimodular matrices A and B (representing elements of $\text{PSL}(2, \mathbf{Z})$ or $\text{SL}(2, \mathbf{Z})$), let $\Delta_{A,B}$ denote the oriented ideal triangle in the hyperbolic plane with vertices at x, Ax, ABx (for some arbitrarily chosen point x on the circle at ∞), and let $E_{A,B}$ denote the total space of the torus bundle over the twice punctured disk with monodromies A and B . Then $\varepsilon(A, B)$ is defined to be $1/\pi$ times the area of $\Delta_{A,B}$ ($= -1, 0$ or 1 according to the orientation of $\Delta_{A,B}$), and $\sigma(A, B)$ is defined to be the signature of the 4-manifold $E_{A,B}$. (See Fig. 2.) Elementary geometric arguments (using Novikov additivity for the latter) show that both are cocycles.

It is easily shown that the cohomology classes represented by the area and signature cocycles are nontrivial. However, since the rational cohomology of $\text{PSL}(2, \mathbf{Z})$ and $\text{SL}(2, \mathbf{Z})$ vanish in dimensions 1 and 2, these cocycles must be coboundaries of unique rational 1-cochains. Indeed, it is shown in Theorems 1.7 and 6.1 that

$$(0.12) \quad \varepsilon = \delta\left(\frac{1}{3}\varphi\right) \quad \text{and} \quad \sigma = \delta\left(\frac{1}{3}\varphi + \nu\right)$$

where, in the second formula, φ is viewed as a cochain on $\text{SL}(2, \mathbf{Z})$ by composing with the natural projection $\text{SL}(2, \mathbf{Z}) \rightarrow \text{PSL}(2, \mathbf{Z})$, and $\nu: \text{SL}(2, \mathbf{Z}) \rightarrow \mathbf{Z}$ is defined by

$$\nu(A) = \begin{cases} \text{sign}(b) & \text{if } A \text{ is a power of } T \\ \text{sign}(c(a + d - 2)) & \text{otherwise} \end{cases}$$

for A as in (0.1).

It follows that $\sigma = \varepsilon + \delta\nu$ (where ε is viewed, as above, as a cocycle on $\text{SL}(2, \mathbf{Z})$). Now $\varepsilon(A, B)$ has a simple expression in terms of the entries of A and B :

$$\varepsilon(A, B) = -\text{sign}(c_A c_B c_{AB})$$

where c_X denotes the lower left entry of X . Combining this with (0.12) we obtain a simple formula for the signature cocycle:

$$\sigma(A, B) = \nu(A) + \nu(B) - \nu(AB) - \text{sign}(c_A c_B c_{AB}).$$

The first formula in (0.12) is proved by an elementary geometric argument, and was in part the motivation for our new definition of φ . It can also be deduced by analytic methods using properties of the η function [RadG] or by purely arithmetic methods [Rad1], although the latter appear quite lengthy.

The second formula in (0.12) (which is a simplification of the formula in [Mey]) is proved by elementary topological arguments. The key ingredient is the computation in Sect. 5 of the μ -invariant (mod 8) of the torus bundle Z_A over the circle (with

monodromy A) for any *Lie* spin structure (see Sect. 6). The term $\nu(A)$ is the signature of a natural bordism Y_A between T_A and the lens space L_A formed by gluing two copies of the solid torus together using A (see Lemma 5.2).

Note that the formula $\sigma = \delta(\frac{1}{3}\varphi + \nu)$ together with the continued fraction formula (0.7) for φ yields a formula for the signature cocycle in terms of the signature of matrices related to the monodromies. A similar formula was found previously by Szczec (see Remark 6.5).

Topological aspects of the Dedekind sum are discussed in Sects. 3–4, which are independent of the last two sections of the paper. The classical connection between Dedekind sums and signature defects of lens spaces is established in Sect. 3. Along the way a new formula (3.9) is found for the ac -signature of the (a, c) -torus knot.

The relationship between the μ -invariant(s) of a lens space $L(c, a)$ and its Dedekind sum $S(a/c)$ is studied in Sect. 4, and (as mentioned above) integer lifts of the μ -invariants are found which illuminate this relationship and provide connections with other 3-manifold invariants. These *integer μ -invariants* arise as follows:

Choose any edge path $\alpha = (a_1, \dots, a_n)$ with $a/c = \langle a_1, \dots, a_n \rangle$ (as above). We say that a vertex p/q is of **type 0** if $p/q \equiv 0/1 \pmod{2}$ and of **type 1** if $p/q \equiv 1/1 \pmod{2}$. Then for $k = 0$ and 1 , the integers

$$\mu_k(a/c) = \sigma_\alpha - \sum a_i \text{ for vertices of type } k$$

are invariants of $L(c, a)$ provided either c is even or $a - k$ is odd (cf. formula (0.7) for φ). Moreover, they reduce (mod 16) to the μ -invariant(s) of $L(c, a)$.

1 The area cocycle and the Rademacher phi function

The modular group $PSL(2, \mathbf{Z})$ acts by fractional linear transformations on the upper half plane \mathbf{H} . This action can be encoded in the well known diagram of Fig. 3a (or its image in the Poincaré disc under the transformation carrying $0, 1, \infty$ to $-i, 1, i$, shown in Fig. 3b). It consists of the triangulation K of \mathbf{H} by ideal triangles obtained by successive reflections from the triangle with vertices at $0, 1$ and ∞ . The vertices of K occur at the points of $\mathbf{Q}P^1 = \mathbf{Q} \cup \infty$, and p/q and r/s are joined by an edge if and only if the determinant $ps - qr$ is ± 1 . Thus the oriented edges of K can be identified with the elements of $PSL(2, \mathbf{Z})$; for example, the edge from $\infty = 1/0$ to $0 = 0/1$ corresponds to I , and its reverse corresponds to $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The action of $PSL(2, \mathbf{Z})$ on \mathbf{H} induces a simplicial action on K , which corresponds to left multiplication on the edges of K .

(1.1) *Remark.* The direction of an edge $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in K is determined by the sign (± 1 or 0) of $A^{-1}\infty = -d/c$ where by convention $\text{sign}(0) = \text{sign}(\infty) = 0$. Indeed, if $\text{sign}(A^{-1}\infty) = +1$, then $A^{-1}\infty$ lies to the left of I , and so ∞ lies to the left of A , i.e. A points to the right. Similarly $\text{sign}(A^{-1}\infty) = -1$ implies that A points to the left, and $\text{sign}(A^{-1}\infty) = 0$ implies that A is vertical.

The area cocycle ε

Define the **area 2-cocycle**

$$\varepsilon: PSL(2, \mathbf{Z}) \times PSL(2, \mathbf{Z}) \rightarrow \mathbf{Z},$$

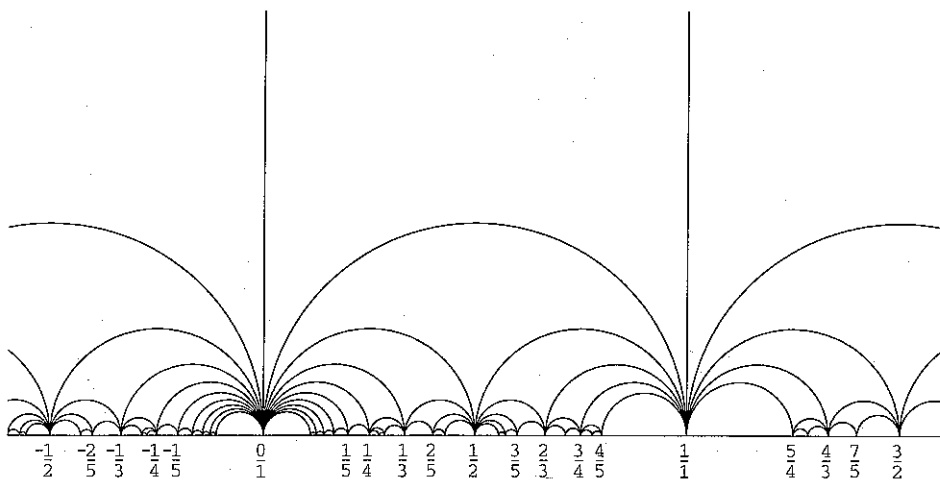


Fig. 3a

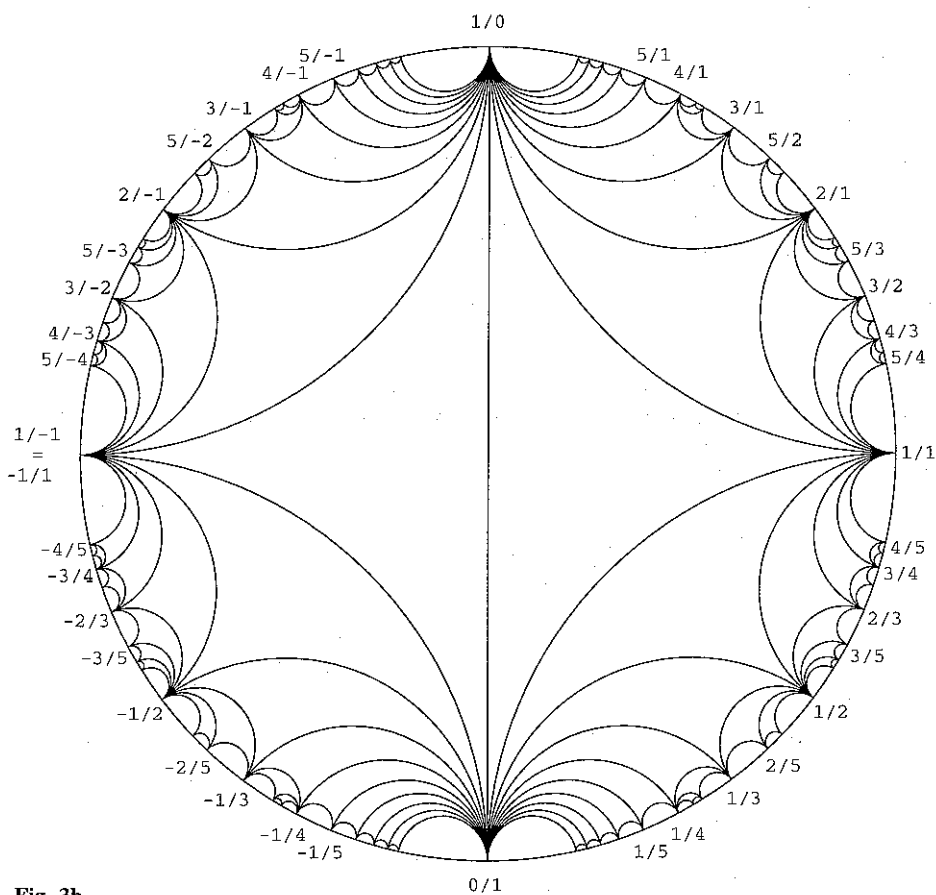


Fig. 3b

by $\varepsilon(A, B) = \frac{1}{\pi} \text{area}(\Delta_{A,B})$, where $\Delta_{A,B}$ is the oriented ideal triangle in the upper half plane \mathbf{H} with vertices at $\infty, A\infty$ and $AB\infty$. (Thus $\varepsilon(A, B)$ is 0 or ± 1 depending on the orientation of $\Delta_{A,B}$.) Equivalently $\varepsilon(A, B)$ is the sign of $AB\infty - A\infty$. A straightforward computation gives $AB\infty - A\infty = -c_B/(c_A c_{AB})$, where c_X is the lower left entry of X , and so

$$(1.2) \quad \varepsilon(A, B) = -\text{sign}(c_A c_B c_{AB}).$$

This is well defined for A and B in $\text{PSL}(2, \mathbf{Z})$ since changing the sign of either changes the sign of AB . To see that ε is a cocycle, note that its coboundary $\delta\varepsilon(A, B, C) = \varepsilon(B, C) - \varepsilon(AB, C) + \varepsilon(A, BC) - \varepsilon(A, B)$ is the area of the boundary of a collapsed tetrahedron in \mathbf{H} (with vertices at $\infty, A\infty, AB\infty$ and $ABC\infty$) which is clearly zero.

(1.3) **Lemma.** *There exists a unique rational 1-cochain $\text{PSL}(2, \mathbf{Z}) \rightarrow \mathbf{Q}$ whose coboundary is the area cocycle ε .*

Proof. Existence is immediate from the fact that $H^2(\text{PSL}(2, \mathbf{Z}), \mathbf{Q}) = 0$, and uniqueness follows from $H^1(\text{PSL}(2, \mathbf{Z}), \mathbf{Q}) = 0$ (see (0.10)) since the coboundary map on rational 0-cochains is zero. \square

(1.4) *Remark.* There is a formula for this cochain in terms of the Rademacher ϕ function (see Sect. 0) and thus in terms of Dedekind sums. Indeed, Rademacher has proved that $\delta\phi = -3\varepsilon$ [Rad1, Sect. 4] [RadG, Eq. (62)], and so $-\phi/3$ is the desired rational cochain.

It follows that ε represents an element of order 3 in $H^2(\text{PSL}(2, \mathbf{Z}), \mathbf{Z}) = \mathbf{Z}/6\mathbf{Z}$. Note that the other element of order 3 is represented by $-\varepsilon$; since $H^2(\text{PSL}(2, \mathbf{Z}), \mathbf{Z})$ classifies central extensions of $\text{PSL}(2, \mathbf{Z})$ by \mathbf{Z} , the two corresponding extensions $0 \rightarrow \mathbf{Z} \rightarrow \text{PSL}(2, \mathbf{Z})_{\pm\varepsilon} \rightarrow \text{PSL}(2, \mathbf{Z}) \rightarrow 1$ are not isomorphic, although the corresponding groups $\text{PSL}(2, \mathbf{Z})_{\pm\varepsilon}$ are [Mel1, Sect. 4].

The cochain φ

Here we give an elementary geometric construction of a cochain $\varphi: \text{PSL}(2, \mathbf{Z}) \rightarrow \mathbf{Z}$ with $\delta\varphi = 3\varepsilon$, which has its algebraic origins in the work of Rademacher (see [RadG, Sect. 4.C]). By the previous remark and the uniqueness statement in Lemma 1.3, φ is the negative of the Rademacher ϕ function, and will also be called the **Rademacher φ function**.

(1.5) **Definition.** For any element $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in the modular group $\text{PSL}(2, \mathbf{Z})$, let γ_A denote the oriented geodesic in \mathbf{H} from i to Ai (a Euclidean straight line segment in the Poincaré disc). Thus γ_A joins the edge I to the edge A in K . The intersection of γ_A with the interior of any ideal triangle Δ in K is either empty or a geodesic segment joining two sides of Δ . Set $\varphi_\Delta(A) = 0$ in the former case, and $\varphi_\Delta(A) = +1$ or -1 in the latter case according to whether the remaining side of Δ lies to the left or right of γ_A . Now define

$$\varphi(A) = \left(\sum \varphi_\Delta(A) \right) - \vartheta(A)$$

where the sum is over all triangles Δ in K (there are only finitely many nonzero terms), and $\vartheta(A)$ is an ‘‘orientation’’ term, defined as follows: $\vartheta(A) = 3 \text{sign}(ac)$ if

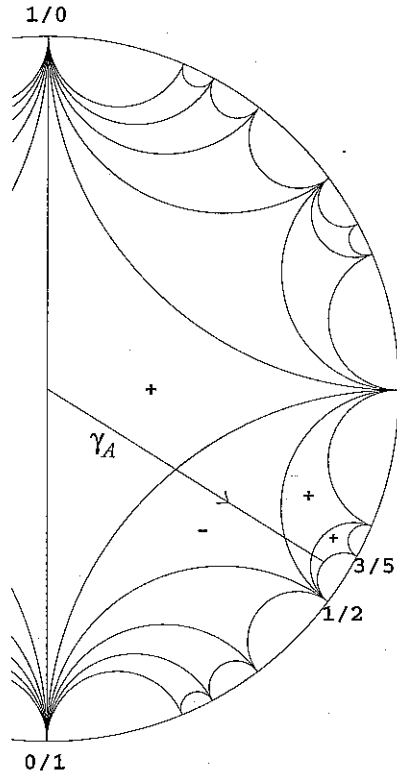


Fig. 4

A (viewed as an oriented edge in K) points toward 0 (i.e. if $|a/c| > |b/d|$) and $\vartheta(A) = 0$ otherwise. (See Fig. 4 for the example $A = \begin{pmatrix} 1 & -3 \\ 2 & -5 \end{pmatrix}$, where $\beta(A) = 0$ and so $\varphi(A) = 3 - 1 + 0 = 2$.)

(1.6) *Remark.* The definition of φ is related to the factorization of elements of $\text{PSL}(2, \mathbf{Z})$ in terms of the generators

$$L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad U = T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Indeed any element A of $\text{PSL}(2, \mathbf{Z})$ can be written uniquely in the form $S^\varepsilon W S^\delta$ (where ε and δ are 0 or 1, and W is a word in L and U), and $\sum \varphi_\Delta(A)$ is just the difference between the total exponents of L and U in W (cf. [RadG, Eq. (70)]). This factorization can be seen geometrically in terms of the geodesic γ_A : The initial direction of γ_A is determined by the factor S^ε , the passage of γ_A through a triangle Δ with one edge to the left (respectively right) of γ_A contributes a factor of L (respectively U), and the orientation on the edge corresponding to A is taken care of by the final factor S^δ . For the example shown in Fig. 4 above, $A = LUL^2S$.

(1.7) **Theorem.** $\delta\varphi = 3\varepsilon$, that is (by (1.2))

$$\varphi(AB) = \varphi(A) + \varphi(B) + 3 \text{sign}(c_A c_B c_{AB})$$

for all A and B in $\text{PSL}(2, \mathbf{Z})$.

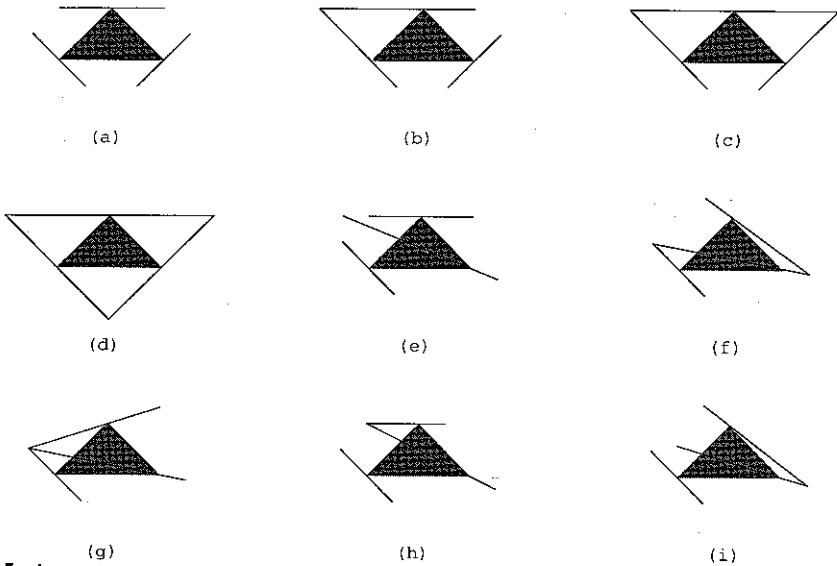


Fig. 5a-i

Proof. Fix A and B in $\text{PSL}(2, \mathbf{Z})$ and consider the oriented geodesic triangles E (with vertices i, Ai, ABi) and F (with vertices $\infty, A\infty$, and $AB\infty$). Write $\text{sign}(E)$ for the sign (± 1 or 0) of the oriented area of E , and similarly for F . Thus $\varepsilon(A, B) = \text{sign}(F)$, and we must show

$$\sum \delta\varphi_{\Delta}(A, B) - \delta\vartheta(A, B) = 3 \text{sign}(F).$$

There are two cases to consider.

Case 1. Some triangle Δ_0 in K intersects the interiors of all three sides of E . Then $\delta\varphi_{\Delta_0}(A, B) = 3 \text{sign}(E)$. Furthermore, any other triangle Δ in K is either disjoint from $\text{int}(E)$ or intersects exactly two sides of E , and so $\delta\varphi_{\Delta}(A, B) = 0$. Thus it suffices to show $\delta\vartheta(A, B) = 3(\text{sign}(E) - \text{sign}(F))$. The cases to be checked are indicated in Fig. 5(a-d), which gives all possible patterns (ignoring orientations) for E (shaded) and the edges I, A, AB . This is now a straightforward verification from the definition of ϑ and will be left to the reader. (It is useful to note that reversing the orientation on one edge frequently does not change $\delta\vartheta$. This is always the case, for example, in Fig. 5(a).)

Case 2. Every triangle Δ in K is either disjoint from $\text{int}(E)$ or intersects exactly two sides of E . Then $\sum \delta\varphi_{\Delta}(A, B) = 0$ by the argument in case 1, and it suffices to show $\delta\vartheta(A, B) = -3 \text{sign}(F)$. Once again, this is a straightforward verification; the cases for E nondegenerate are shown in Fig. 5(e-i). \square

Continued fractions and signatures

For many purposes, it is useful to have a formula for φ in terms of continued fractions. Such formulas have been developed in special contexts by [Rad1, Sect. 4.59] and [Hic], and can be deduced in general from work of Sczech [MeyS, Theorem 3.2] (see

Remark 6.5). It is our purpose here to give a completely general treatment from our geometric perspective. It turns out that the general formula involves signatures, and therefore illuminates the appearance of Dedekind sums in many signature formulas in topology (see Sects. 3–6).

(1.8) **Definition.** A based path in K is an oriented edge path α with initial edge I (from ∞ to 0).

Note that α is uniquely specified by the list (a_1, \dots, a_n) of integers defined geometrically as follows: the first edge of α turns through a_1 triangles to reach the second (with $a_1 > 0$ if and only if the turn is counterclockwise), the second turns through a_2 triangles to reach the third, and so on. Thus we often write

$$\alpha = (a_1, \dots, a_n).$$

(See Fig. 6 for an example.) The final edge of α , oriented backwards, determines an element A_α of the modular group $\text{PSL}(2, \mathbf{Z})$. In particular

$$A_\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

if the final edge goes from b/d to a/c .

Algebraically, a based path corresponds to a continued fraction expansion of its final vertex. This is established in the next result, along with various other useful identities. (Recall (0.5) the notation $\langle a_1, \dots, a_n \rangle = -1/(a_1 - 1/(a_2 - \dots - 1/a_n) \dots)$ for continued fractions used here.)

(1.9) **Lemma.** Let $\alpha = (a_1, \dots, a_n)$ be a based path whose final edge goes from b/d to a/c . Then

- (a) $a/c = \langle a_1, \dots, a_n \rangle$ and $b/d = \langle a_1, \dots, a_{n-1} \rangle$,
- (b) $d/c = \langle a_n, \dots, a_1 \rangle$ and $b/a = \langle a_n, \dots, a_2 \rangle$,
- (c) $A_\alpha = S(T^{a_1}S) \dots (T^{a_n}S)$ (in $\text{PSL}(2, \mathbf{Z})$)

where S and T are the standard generators of the modular group.

Proof. First observe that there is a simple formula for a_1, \dots, a_n in terms of the vertices $p_0/q_0 = 1/0 = \infty$, $p_1/q_1 = 0/1 = 0$, $p_2/q_2, \dots, p_n/q_n = b/d$, $p_{n+1}/q_{n+1} = a/c$ of α . In particular

$$a_i = \det \begin{pmatrix} p_{i-1} & p_{i+1} \\ q_{i-1} & q_{i+1} \end{pmatrix}$$

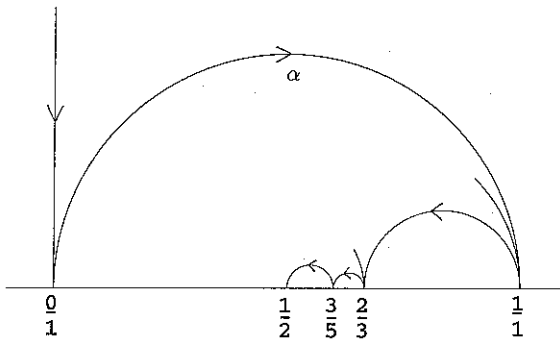


Fig. 6

$$\alpha = (-1, 2, 2, 1)$$

$$A_\alpha = \pm \begin{pmatrix} 1 & -3 \\ 2 & -5 \end{pmatrix}$$

provided the p_i and q_i are chosen so that each determinant

$$\det \begin{pmatrix} p_i & p_{i+1} \\ q_i & q_{i+1} \end{pmatrix} = +1$$

(cf. [Mel2, Sect. 2]).

Conversely, the vertices may be recovered from a_1, \dots, a_n by solving for p_{i+1} and q_{i+1} in the previous two equations. This gives a recursive formula

$$\begin{aligned} p_{i+1} &= a_i p_i - p_{i-1}, \\ q_{i+1} &= a_i q_i - q_{i-1}, \end{aligned}$$

which yields the continued fraction expression

$$p_{i+1}/q_{i+1} = \langle a_1, \dots, a_i \rangle$$

by a standard inductive argument (see e.g. [HW, Sect. 10.2]). This proves (a). The factorization (c) also follows by induction using the recursive formula.

For (b), consider the based paths $-\alpha = (-a_1, \dots, -a_n)$ and $\bar{\alpha} = (a_n, \dots, a_1)$. Then $A_{-\alpha}$ and $A_{\bar{\alpha}}$ are given by

$$(1.10) \quad A_{-\alpha} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}, \quad A_{\bar{\alpha}} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}.$$

The formula for $A_{-\alpha}$ is immediate from the fact that $-\alpha$ is the reflection of α in its initial edge (the choice of sign for $A_{-\alpha}$ is immaterial since we are in $\text{PSL}(2, \mathbf{Z})$). For $A_{\bar{\alpha}}$, first observe that the path $-\bar{\alpha}$ in K can be obtained from α by reversing its orientation and then moving the edge A_α (from a/c to b/d) to the edge I (from $1/0$ to $0/1$) by the isometry A_α^{-1} (cf. Remark 1.1). Thus $A_{-\bar{\alpha}} = A_\alpha^{-1}$, and the formula for $A_{\bar{\alpha}}$ follows. (Alternatively, this formula can be derived using (c).) Now apply (a) to the path $\bar{\alpha}$. \square

(1.11) *Remark.* In view of the sign conventions in the proof above, the factorization in (c) actually determines A_α as a matrix in $\text{SL}(2, \mathbf{Z})$, with $a = p_{n+1}$, $b = -p_n$, $c = q_{n+1}$ and $d = -q_n$. For example $A_{(-1,2,2,1)} = \begin{pmatrix} -1 & 3 \\ -2 & 5 \end{pmatrix}$, cf. Fig. 6.

Now we show how to compute the values of the function φ using based paths.

(1.12) **Theorem.** *Given A in $\text{PSL}(2, \mathbf{Z})$,*

$$\varphi(A) = 3\sigma_\alpha - \tau_\alpha$$

for any based path $\alpha = (a_1, \dots, a_n)$ in K with $A = A_\alpha$. Here σ_α and τ_α denote the signature and trace of the matrix

$$M_\alpha = \begin{pmatrix} a_1 & 1 & & & & \circ \\ 1 & a_2 & 1 & & & \\ & & 1 & \ddots & & \\ & & & \ddots & \ddots & \\ \circ & & & & \ddots & 1 \\ & & & & & 1 & a_n \end{pmatrix}$$

whose i_j^{th} entry is a_i if $i = j$, 1 if $|i - j| = 1$ and 0 otherwise.

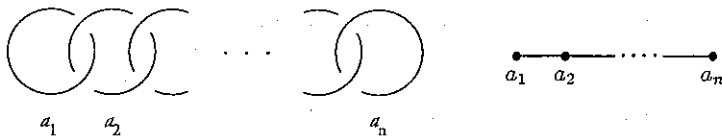


Fig. 7

L_α

G_α

(1.13) *Remarks.* (a) The condition $A = A_\alpha$ in $\text{PSL}(2, \mathbf{Z})$ means that $A = \pm A_\alpha$ as matrices (or equivalently $a/c = \langle a_1, \dots, a_n \rangle$ and $b/d = \langle a_1, \dots, a_{n-1} \rangle$ by (1.9a)). The path α may be modified if necessary to arrange $A = A_\alpha$ in $\text{SL}(2, \mathbf{Z})$ (since $A_{\alpha 00} = A_\alpha T^0 S T^0 S = A_\alpha S^2 = -A_\alpha$); this will be needed in Sects. 5–6.

(b) The matrix M_α is the linking matrix of the framed link L_α shown (along with the corresponding weighted graph G_α) in Fig. 7. There is an associated 4-manifold W_α formed by adding 2-handles to B^4 along L_α (or equivalently by plumbing disc bundles over S^2 according to G_α) [K1]. M_α represents the intersection form on $H_2(W_\alpha; \mathbf{Z})$, and so σ_α can be thought of as the signature of W_α .

(c) Both σ_α and τ_α have simple geometric interpretations in terms of the based path α . This is obvious for $\tau_\alpha = \sum a_i$, since the a_i are defined geometrically in terms of the turns in α . For σ_α , observe that M_α can be diagonalized formally (from the top down) to get a matrix D_α with diagonal entries

$$\alpha_i = a_i - \frac{1}{a_{i-1} - \frac{1}{\ddots - \frac{1}{a_1}}} = \frac{-1}{\langle a_i, \dots, a_1 \rangle}$$

for $i = 1, \dots, n$. If $\alpha_i \neq \infty$ for all i , then this gives the formula

$$\sigma_\alpha = \sum_{i=1}^n \text{sign}(\alpha_i)$$

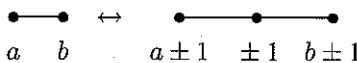
which in fact holds in general by an easy induction. But

$$\text{sign}(\alpha_i) = \text{sign}(\langle -a_i, \dots, -a_1 \rangle) = \text{sign}(A_{(a_1, \dots, a_i)}^{-1} \infty).$$

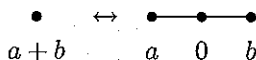
It follows from Remark 1.1 that σ_α is just the difference between the number of edges of α which point left and the number which point right (ignoring the vertical edges, which have ∞ as a vertex). For example $\varphi \begin{pmatrix} 1 & -3 \\ 2 & -5 \end{pmatrix} = 6 - 4 = 2$, cf. Fig. 6.

(d) The formula $3\sigma_\alpha - \tau_\alpha$ can be taken as the definition of $\varphi(A)$ once it is shown independent of the choice of α . But this is easy: First observe that one may pass from any path α with $A = A_\alpha$ to any other by a sequence of moves of the following two kinds (cf. [Mel2, Sect. 2]):

Move 1. Shift across a triangle



Move 2. Retrace an edge



(From the 4-manifold viewpoint, these correspond to forming the connected sum with $\pm CP^2$ in move 1 and with an S^2 -bundle over S^2 in move 2.) But σ_α and τ_α change by ± 1 and ± 3 under move 1 and remain unchanged under move 2, so $3\sigma_\alpha - \tau_\alpha$ is unchanged by either move. Theorem 1.7 can now be proved directly, starting with the signature computation in Remark 1.13(c). Its simple geometric content, however, is disguised by this approach.

Proof of Theorem 1.12. The proof is by induction on n , starting with $\varphi(S) = 0$ when $n = 0$. For $n > 0$, set $\beta = (a_1, \dots, a_{n-1})$. Then $A = A_\beta(T^{a_n}S)$, by the factorization in (c) of Lemma 1.9. By induction, $\varphi(A_\beta) = 3\sigma_\beta - \tau_\beta$, and one easily computes $\varphi(T^a S) = -a$. Thus

$$\varphi(A) = 3(\sigma_\beta + \text{sign}(dc)) - \tau_\alpha$$

by Theorem 1.7. But $\sigma_\beta + \text{sign}(dc) = \sigma_\alpha$ by the signature formula in Remark 1.13(c), since $\text{sign}(dc) = \text{sign}(d/c) = \text{sign}(\alpha_n)$, and the theorem is proved. \square

Certain arithmetic properties of the function φ will be discussed in the next section, and so we introduce the necessary prerequisites here. These notions will also be used in the discussion of μ -invariants in Sects. 4–5.

(1.14) **Definition.** For any prime p , the **p-type** of a vertex a/c of K (i.e. an element of $\mathbf{Q} \cup \infty$) is

$$(a/c)_p = \begin{cases} a\bar{c} \pmod{p} & \text{if } p \text{ is prime to } c, \text{ where } c\bar{c} \equiv 1 \pmod{p} \\ \infty & \text{if } p \text{ divides } c. \end{cases}$$

This is an element of $\mathbf{Z}/p\mathbf{Z} \cup \infty$.

We are primarily interested in the cases $p = 2$ and $p = 3$, where the p -types are $0, 1, \infty$ and $0, 1, -1, \infty$ respectively.

The **p-type** of an oriented edge of K (i.e. an element of $\text{PSL}(2, \mathbf{Z})$) is the ordered pair of p -types of its vertices. For example, the 2-type of T^{-1} is $(\infty, 1)$, while its 3-type is $(\infty, -1)$. Note that for $p = 2$ or 3 , the p -type of any element of $\text{PSL}(2, \mathbf{Z})$ uniquely determines it as an element of $\text{PSL}(2, \mathbf{Z}/p\mathbf{Z})$. (This fails for all larger p .) Thus in these cases, a diagram K_p for $\text{PSL}(2, \mathbf{Z}/p\mathbf{Z})$ can be obtained by identifying triangles of K whenever the p -types of their corresponding edges coincide. In particular, K_2 is a triangle (with vertices $0, 1, \infty$) and K_3 is a tetrahedron (with vertices $0, 1, -1, \infty$).

For $p = 2$, it is often useful to consider the full subcomplex of K containing all vertices of type 0 and 1. This is easily seen to be a tree, called the **01-tree** in K (or equivalently the **10-tree**). Similarly define the **0 ∞ -tree** and the **1 ∞ -tree** in K .

(1.15) **Lemma.** If $p = 3$ and A is any element in the modular group, then $\varphi(A) \pmod{p}$ depends only on the p -type of A . The same is true for $p = 2$ provided $c_A \neq 0$ (i.e. A is not a power of T).

Proof. Choose a based path α in K with $A_\alpha = A$. By Theorem 1.12, $\varphi(A) = 3\sigma_\alpha + \tau_\alpha$. Thus $\varphi(A) \equiv \tau_\alpha \pmod{3}$. But it is easily seen that τ induces a mod 3 cocycle on K_3 , and so $\tau_\alpha \pmod{3}$ depends only on the 3-type of A_α .

For $p = 2$, the signature term $3\sigma_\alpha$ cannot a priori be ignored. However, if B is any other element of $\text{PSL}(2, \mathbf{Z})$ (i.e. oriented edge in K) of the same 2-type (s, t) as A , then α can be extended by a path γ lying entirely in the st -tree to a based path $\beta = \alpha\gamma$ with $A_\beta = B$. Since the turn between any two adjacent edges in this tree is even, τ_α and τ_β have the same parity. Furthermore, as long as neither endpoint of γ is ∞ (i.e. neither A nor B is a power of T), the path γ must have an even number of non-vertical edges and so $3\sigma_\alpha$ and $3\sigma_\beta$ have equal parities as well (by Remark 1.13(c)). Thus $\varphi(A) \equiv \varphi(B) \pmod{2}$. \square

2 Dedekind sums

In this section, Dedekind sums are characterized axiomatically in terms of Dedekind reciprocity (cf. [Hir2]). Their existence and elementary arithmetic properties are established using the function φ defined in Sect. 1 and another function $\tau: \text{PSL}(2, \mathbf{Z}) \rightarrow \mathbf{Q} \cup \infty$ defined as follows:

(2.1) **Definition.** For any element $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in the modular group, define $\tau(A) = (a + d)/c$. Also for any nonzero rational number $x = a/c$ (with a and c coprime) set $\{x\} = 1/ac$.

(2.2) **Theorem.** There exists a unique function $S: \mathbf{Q} \cup \infty \rightarrow \mathbf{Q} \cup \infty$ (the Dedekind symbol) carrying ∞ to itself and satisfying the three axioms

(D1) $S(-x) = -S(x)$

(D2) $S(x + 1) = S(x)$

(D3) (Dedekind reciprocity) $S(x) + S(1/x) = x + 1/x + \{x\} - 3 \text{sign}(x)$ for all rational numbers x (with $x \neq 0$ in (D3)). In particular, S is given by

$$S(x) = \varphi(A) + \tau(A)$$

for any A in $\text{PSL}(2, \mathbf{Z})$ with $A\infty = x$. (See Remark 2.3(b) below for the continued fraction formula for S .)

Proof. Uniqueness follows easily from existence. Indeed, (D1–2) force S to vanish on any integer. The value on any other rational number can now be computed by repeatedly applying (D2–3) to reduce the denominator.

To prove existence, we first show that the proposed S is well defined, independent of the choice of A . If B is any other element of $\text{PSL}(2, \mathbf{Z})$ with $B\infty = x$, then A and B (viewed as edges in K) share the same initial vertex x . Thus A can be rotated about this vertex through some number of triangles, say k , to reach B . (As usual, the sign of k determines the direction of the rotation, with a positive sign corresponding to a counterclockwise turn.) In algebraic terms, $B = AT^k$. It is now evident that $\varphi(B) = \varphi(A) - k$ (either from the geometric Definition 1.5 or from Theorem 1.7 since $\varphi(T) = -1$), whereas $\tau(B) = \tau(A) + k$. (Also see Remark 2.3(b) below.) Thus $\varphi(B) + \tau(B) = \varphi(A) + \tau(A)$.

Finally, we verify axioms (D1–3). Observe that K is (setwise) invariant under the three hyperbolic isometries T_1, T_2 and T_3 defined as follows: T_1 and T_3 are the reflections through the imaginary axis and the unit circle, respectively, and T_2 is the translation carrying z to $z + 1$ (corresponding to the action of the element T of $\text{PSL}(2, \mathbf{Z})$). If A_i denotes the image of the edge A under T_i , then $A_1\infty = -x$, $A_2\infty = x + 1$ and $A_3\infty = 1/x$. It is geometrically evident that $\varphi(A) = -\varphi(A_1) = \varphi(A_2) - 1$, and a straightforward computation gives $\tau(A) = -\tau(A_1) = \tau(A_2) + 1$. Axioms (D1–2) follow. Also $\varphi(A) = -\varphi(A_3) - 3 \text{sign}(x)$ and $\tau(A_3) = (c - b)/a$. Reciprocity follows since $\tau(A) + \tau(A_3) = (a + d)/c + (c - b)/a = a/c + c/a + (ad - bc)/ac = a/c + c/a + 1/ac = x + 1/x + \{x\}$. \square

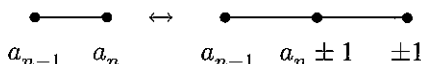
(2.3) **Remarks.** (a) The classical Dedekind sums $s(a, c)$ provide another construction for S by setting $S(a/c) = 12s(a, c)$ for positive a and c , and extending to negative rationals using (D1). Thus $S(a/c) = 12 \text{sign}(c)s(a, c)$ for general a and $c \neq 0$, since $s(-a, c) = -s(a, c) = -s(a, -c)$ [RadG, Eq. (33a–b)]. Axioms (D2–3) are well known properties of Dedekind sums.

(b) $S(x)$ can be computed using continued fractions as follows: Choose a based path $\alpha = (a_1, \dots, a_n)$ in K with final vertex x (i.e. $x = \langle a_1, \dots, a_n \rangle$). Note that $A_\alpha \infty = x$. Now

$$S(x) = 3\sigma_\alpha - \sum a_i + \langle a_1, \dots, a_n \rangle + \langle a_n, \dots, a_1 \rangle$$

where σ_α is the signature of the matrix M_α of Theorem 1.12. (The first two terms give $\varphi(A_\alpha)$ and the last two give $\tau(A_\alpha)$ by Lemma 1.9(a-b).) This formula may of course be taken as the definition of S . To show that it does not depend on the choice of α , observe that any two based paths with final vertex x can be connected by Moves 1-2 of Remark 1.13(d) (which do not change φ or τ) together with

Move 3. Shift last edge across a triangle



which changes φ by ± 1 and τ by ∓ 1 . The axioms (D1-3) are easily verified.

We now derive some well known elementary arithmetic properties of Dedekind sums [RadG, Sect. 3A].

(2.4) **Theorem.** (a) If $a\bar{a} \equiv 1 \pmod{c}$, then $S(a/c) = S(\bar{a}/c)$.

(b) $S(a/c)$ is an integer if and only if c divides $a^2 + 1$, in which case it is zero.

(c) $cS(a/c)$ is always an even integer, and in fact a multiple of 6 for $c \not\equiv 0 \pmod{3}$.

Proof. Choose A in $\text{PSL}(2, \mathbf{Z})$ so that $A\infty = a/c$, say $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. By definition $S(a, c) = \varphi(A) + \tau(A)$.

For (a), arrange further that $d = \bar{a}$. It follows that $A^{-1}\infty = -\bar{a}/c$, and so $S(\bar{a}/c) = -(\varphi(A^{-1}) + \tau(A^{-1}))$ by (D1). Evidently $\tau(A^{-1}) = -\tau(A)$, and $\varphi(A^{-1}) = -\varphi(A)$ (by Theorem 1.7 for example). Thus $S(\bar{a}/c) = S(a/c)$.

For (b), observe that $S(a/c)$ is an integer if and only if c divides $a + d$ (since $\varphi(A)$ is integral) which holds if and only if c divides $a^2 + 1$ (since $a^2 + 1 = a^2 + ad - bc = a(a + d) - bc$ and a and c are relatively prime). This proves the first statement. For the second, note that if c divides $a + d$, then a/c and $-d/c$ differ by an integer. Thus $S(a/c) = S(-d/c) = -S(d/c) = -S(a/c)$ by axioms (D1-2) and part (a), and so $S(a/c) = 0$.

For (c), the congruence $c\varphi(A) + (a + d) \equiv 0 \pmod{p}$ must be proved for $p = 2$, and for $p = 3$ provided c is prime to 3. By Lemma 1.15, it suffices to first reduce A modulo p . The proof is then a simple verification for the 6 elements of $\text{PSL}(2, \mathbf{Z}/2\mathbf{Z})$ and the 12 elements of $\text{PSL}(2, \mathbf{Z}/3\mathbf{Z})$, which we leave to the reader. \square

We conclude with a remark about the connection between Dedekind sums and 3-dimensional topology, which will be expanded upon in subsequent sections.

(2.5) *Remark.* Recall that the lens space $L(c, a)$ (for $c > 0$) is the quotient of the unit sphere S^3 in \mathbf{C}^2 by the action $z(u, v) = (z^a u, zv)$ of the cyclic group of c^{th} roots of 1. We shall give it the natural orientation induced from S^3 as the boundary of the unit ball, with the “outward first” convention for orienting boundaries. Equivalent descriptions are (1) the Seifert manifold $M(0; (a, c))$ (see for example [NR]), (2) $-c/a$ Dehn surgery on the unknot in S^3 , and (3) surgery on the framed link L_α shown in Fig. 7, where $\alpha = (a_1, \dots, a_n)$ corresponds to a continued fraction expansion $a/c = \langle a_1, \dots, a_n \rangle$, as first discussed in Hirzebruch’s 1950 thesis [Hir1].

It is well known that $L(c, a)$ and $L(c', a')$ are of the same oriented homeomorphism type if and only if $c = c'$ and either $a \equiv a'$ or $aa' \equiv 1 \pmod{c}$. It follows from axiom (D2) and Theorem 2.4(a) that $S(a/c)$ is an invariant of $L(c, a)$ (it's *Dedekind sum*), which negates under orientation reversal (by axiom (D1)). It turns out to be a very discerning invariant of oriented lens spaces, although incomplete; for example, $L(25, 4)$ and $L(25, 9)$ have equal Dedekind sums.

3 Signature defects

It is a well known consequence of the G -signature theorem that Dedekind sums are related to the signature defects of lens spaces [Hir2]. In this section, we establish this relationship by elementary means, proving a reciprocity formula (3.3) for defects. Our exposition is very much in the spirit of Hirzebruch's lovely treatment, but with a milder dependence on the signature theorem. As a byproduct we find a simple formula $-\frac{1}{3}(a^2 - 1)(c^2 - 1)$ for the ac -signature of the (a, c) -torus knot (3.9).

First recall the definition of the signature defect. Let M be a closed oriented 3-manifold with a finite regular c -fold cover $\pi: \tilde{M} \rightarrow M$. It can be shown that π extends to a cover $\tilde{W} \rightarrow W$ of compact 4-manifolds (where $\partial\tilde{W} = \tilde{M}$ and $\partial W = M$) branched along a closed connected surface Y in W (see for example [CG, Lemma 2.2]). Suppose that the branching degree is m (that is the normal circle to Y lifts to c/m circles, each of which covers the normal circle m times). Then the **signature defect** of M with this specific cover π is

$$(3.1) \quad \text{def}(M, \pi) = c\sigma(W) - \sigma(\tilde{W}) - \frac{m^2 - 1}{3m} Y \cdot Y.$$

That $\text{def}(M, \pi)$ is well defined follows from the G -signature theorem (Sect. 4 in [Hir2]). Observe that $Y \cdot Y$ is divisible by m , and so $3 \text{def}(M, \pi)$ is an integer.

Now set

$$(3.2) \quad \delta(a/c) = 3 \text{def}(L(c, a), \pi)$$

where $\pi: S^3 \rightarrow L(c, a)$ is the universal cover. With our orientation conventions for lens spaces (2.5) this corresponds to $-3 \text{def}(c; a, 1)$ in Sect. 4 of [Hir2].

$$(3.3) \textbf{ Theorem (Hirzebruch). } \delta(a/c) = cS(a/c) \text{ (for } c > 0 \text{)}.$$

We shall prove the theorem by establishing a reciprocity formula (3.4) for signature defects of lens spaces (cf. Sect. 6 in [Hir2]) – the theorem then follows from the uniqueness statement in Theorem 2.2. It can also be proved using Rademacher's cotangent formula for S by applying the G -signature theorem directly to the action of the cyclic group of order c on B^4 whose quotient is the cone on $L(c, a)$ (see Sect. 5 in [Hir2]).

$$(3.4) \textbf{ Theorem (reciprocity for defects). } a\delta(c/a) + c\delta(a/c) = a^2 + c^2 + 1 - 3ac \text{ (for positive } a \text{ and } c \text{). Thus is general, } |a|\delta(c/a) + |c|\delta(a/c) = \text{sign}(ac)(a^2 + c^2 + 1 - 3ac).$$

Proof. First build a 4-manifold W_0 by adding a 2-handle to B^4 along the (a, c) -torus knot K with framing ac . Then $-\partial W_0 = L(a, c) \# L(c, a)$. To see this, observe that K lies on the torus T of a genus 1 Heegaard splitting $H_1 \cup H_2$ of $S^3 = \partial B^4$, and the ac -framing corresponds to the pushoff of K in T . The effect on the boundary of adding the 2-handle is to surger S^3 along K . This turns T into a 2-sphere S which decomposes ∂W_0 . Indeed, the solid torus $B^2 \times S^1$ which is glued in by the

surgery splits into two pieces, $H'_1 = B^2 \times S^1_+$ and $H'_2 = B^2 \times S^1_-$, which may be viewed as 2-handles attached to H_1 and H_2 , respectively, along the (a, c) and (c, a) torus knots. Thus $-\partial W_0$ is the union of $-H_1 \cup -H'_1 = L(c, a) - B^3$ and $-H_2 \cup -H'_2 = L(a, c) - B^3$ along S . (This argument is given in greater generality in [MelK, Lemma 1] and originated in [Rou].)

Next form W by adding a 3-handle to W_0 along the decomposing 2-sphere in ∂W_0 , so that $-\partial W$ is the disjoint union of $L(a, c)$ and $L(c, a)$. Thus the signature defect of ∂W with its ac -fold cyclic cover π_{ac} satisfies

$$(3.5) \quad -3 \operatorname{def}(\partial W, \pi_{ac}) = a\delta(c/a) + c\delta(a/c).$$

Now π_{ac} extends to a branched cover $\tilde{W} \rightarrow W$, branched along the surface Y in W which is the union of the Seifert surface F for K (pushed into the interior of B^4) and the core of the 2-handle. Thus (3.1) yields

$$(3.6) \quad \begin{aligned} \operatorname{def}(\partial W, \pi_{ac}) &= ac\sigma(W) - \sigma(\tilde{W}) - \frac{a^2c^2 - 1}{3ac} Y \cdot Y \\ &= ac - \sigma(\tilde{W}) - \frac{1}{3}(a^2c^2 - 1) \end{aligned}$$

since $\sigma(W) = 1$ and $Y \cdot Y = ac$.

It remains to compute $\sigma(\tilde{W})$. Observe that \tilde{W} consists of the ac -fold branched cover V of B^4 along F , a 2-handle which is the ac -fold cover of the original 2-handle branched along its core, and some 3-handles which can be ignored in the calculation of $\sigma(\tilde{W})$. The 2-handle contributes exactly 1 to $\sigma(\tilde{W})$. Indeed, ∂V is the ac -fold branched cover of S^3 along K , so it contains a copy of K and its Seifert surface F . When the 2-handle is added to V along K (with framing 1), its core union F defines a class in $H_2(\tilde{W}; \mathbf{Z})$ with self intersection 1 which clearly does not intersect any class in $H_2(V; \mathbf{Z})$. Hence $\sigma(\tilde{W}) = \sigma(V) + 1$, and so substituting into (3.6) and combining with (3.5) gives

$$(3.7) \quad a\delta(c/a) + c\delta(a/c) = a^2c^2 + 2 + 3\sigma(V) - 3ac.$$

Brieskorn has given a formula for $\sigma(V)$ in terms of integer lattice points inside the rectangular solid $S = [0, a] \times [0, c] \times [0, ac]$ in \mathbf{R}^3 [Bri] (see also [L]). In particular, let T and T' denote the tetrahedra with vertices $(0, 0, 0)$, $(a, 0, 0)$, $(0, c, 0)$, $(0, 0, ac)$ and (a, c, ac) , $(0, c, ac)$, $(a, 0, ac)$, $(a, c, 0)$, respectively, and $M = S - (T \cup T')$. Write t, t' and m for the number of interior lattice points in T, T' and M . Then

$$\sigma(V) = t + t' - m = 2t - m$$

where the last equality holds since T and T' are *equivalent* (by an automorphism of \mathbf{R}^3 preserving lattice points).

Now it is easy to compute

$$2t + m = (a - 1)(c - 1)(ac - 2),$$

as this is the difference of the total number $(a - 1)(c - 1)(ac - 1)$ of lattice points in S and the number $(a - 1)(c - 1)$ on the two interior faces. It is also clear that m is twice the number of interior lattice points in the pyramid P with base $[0, a] \times [0, c] \times ac$ and vertex at the origin – indeed S is the union of two pyramids equivalent to P and two tetrahedra equivalent to T . An elementary argument counting lattice points in layers parallel to the base of P [Rad1, Sect. 2.4] gives

$$m = \frac{1}{6}(a - 1)(c - 1)(4ac + a + c - 5)$$

and so

$$(3.8) \quad \sigma(V) = -\frac{1}{3}(a^2 - 1)(c^2 - 1)$$

by a straightforward calculation. The theorem follows by substitution in (3.7). \square

(3.9) *Remark.* $\sigma(V)$ can be identified with the total ac -signature $\sigma_{ac}(K)$ of the (a, c) -torus knot $K [V, L]$ (that is the sum of the signatures of $(1 - \zeta)A + (1 - \bar{\zeta})A^\dagger$ for all ζ with $\zeta^{ac} = 1$, where A is a Seifert matrix for K). Thus (3.8) yields the formula

$$\sigma_{ac}(K) = -\frac{1}{3}(a^2 - 1)(c^2 - 1)$$

which has apparently not been noticed before.

An alternative proof of (3.8) is based on Mordell's beautiful formula for the number $N(p, q, r)$ of lattice points in the closed tetrahedron with vertices $(0, 0, 0)$, $(p, 0, 0)$, $(0, q, 0)$, $(0, 0, r)$, for p, q, r pairwise coprime:

$$\begin{aligned} 12N(p, q, r) &= 2pqr + 3(pq + qr + rp + p + q + r) \\ &\quad + pq/r + qr/p + rp/q + 1/pqr + 24 \\ &\quad - S(pq, r) - S(qr, p) - S(rp, q) \end{aligned}$$

[Mor] (see also [RadG, Theorem 5], and [Hir2] for a proof using the G -signature theorem). Applying this formula with $p = a$, $q = c$ and $r = ac + 1$, and adjusting for the lattice points on the boundary, gives $12t = (a - 1)(c - 1)(2ac - a - c - 7)$ and (3.8) follows. Of course, if one is willing to use the G -signature theorem to prove Theorem 3.3 then (3.8) follows readily from (3.7).

4 μ -invariants of lens spaces

Dedekind sums are intimately related to μ -invariants of lens spaces. In particular, it is well known that the Dedekind sum of an *odd* lens space (see (2.5) for the definition) determines its μ -invariant (4.12). For an *even* lens space, there are two μ -invariants and they are in general *not* determined by the Dedekind sum (4.14a). We show here (4.5) how to define integer lifts of the μ -invariants of lens spaces which can be thought of as part of the associated Dedekind sums, although neither is determined by the other (4.14b). Various reciprocity laws for these lifts are given in (4.8) and are applied to establish the connections between μ -invariants, Dedekind sums and other invariants of lens spaces (e.g. Brown invariants (4.16)).

First recall that the μ -invariant $\mu(M, \theta)$ of a spin 3-manifold (M, θ) is defined to be the signature (mod 16) of any compact spin 4-manifold W bounded by (M, θ) . (This is well defined by Rohlin's theorem.) If W is not spin but merely oriented, then there is still a formula in terms of the signature $\sigma(W)$ and also the self intersection and Arf invariant of a characteristic surface for θ in W (see Sect. IV.3 in [K2] for a general discussion of this formula, which goes back to [Rob]). In particular, if $W = W_L$ is obtained by attaching 2-handles to B^4 along a framed link L , then

$$(4.1) \quad \mu(M, \theta) \equiv \sigma(W_L) - C \cdot C + 8 \text{Arf}(C) \pmod{16}$$

where C is the sublink of L consisting of all components K of L for which θ does not extend across the handle attached to K (see Appendix C in [KM1]). Note that C is *characteristic* (that is, $C \cdot K \equiv K \cdot C \pmod{2}$ for each component K of L). In general, there is a natural one-to-one correspondence between the spin structures

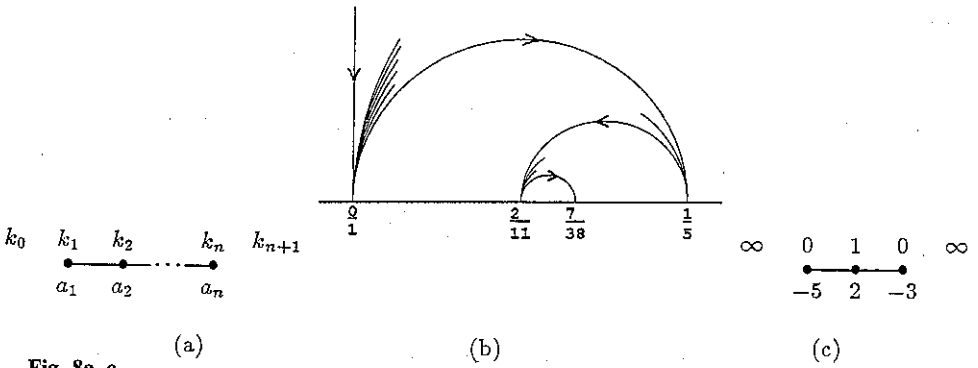


Fig. 8a-c

on $M = \partial W_L$ and the characteristic sublinks of L , equal in number to the order of $H_1(M; \mathbf{Z}/2\mathbf{Z})$ (see for example Sect. C.1 in [KM1]).

Now consider the lens space $M = L(c, a)$. It is called an **odd** or **even** lens space according to whether c is odd even. Note that M has a unique spin structure in the former case (since $H^1(M; \mathbf{Z}/2\mathbf{Z}) = 0$) and exactly two spin structures in the latter (since $H^1(M; \mathbf{Z}/2\mathbf{Z}) = \mathbf{Z}/2\mathbf{Z}$). To study the associated μ -invariant(s), it is convenient to partition the extended rationals as follows:

(4.2) **Definition.** Decompose $\mathbf{Q} \cup \infty$ as the disjoint union $\mathbf{Q}_0 \cup \mathbf{Q}_1 \cup \mathbf{Q}_\infty$, where \mathbf{Q}_k denotes the set of all extended rationals of 2-type k . (Recall (1.14) that a/c is of 2-type 0, 1 or ∞ according to whether $a/c \equiv 0/1, 1/1$ or $1/0 \pmod{2}$.)

To each element a/c of $\mathbf{Q} \cup \infty$, associate the lens space $L(c, a)$. (Thus \mathbf{Q}_∞ corresponds to the even lens spaces and $\mathbf{Q}_0 \cup \mathbf{Q}_1$ to the odd ones.) Define an equivalence relation \sim on $\mathbf{Q} \cup \infty$ by $a/c \sim a'/c'$ if and only if $L(c, a) = L(c', a')$. By the classification of lens spaces (see Remark 2.5) \sim is generated by $a/c \sim (a + c)/c$ and $a/c \sim d/c$ for $ad \equiv 1 \pmod{c}$. Of course each equivalence class of \sim lies entirely in \mathbf{Q}_∞ or entirely in $\mathbf{Q}_0 \cup \mathbf{Q}_1$.

We now propose to define functions μ_0 and μ_1 which yield integer lifts of the μ -invariants of lens spaces. First consider a based path $\alpha = (a_1, \dots, a_n)$ in the triangulation K of the hyperbolic plane (described in Sect. 1), with associated signature σ_α and trace $\tau_\alpha = \sum a_i$ (see Theorem 1.12). Each vertex p_i/q_i of α (for $i = 0, \dots, n + 1$) has a 2-type k_i equal to 0, 1 or ∞ . (See Fig. 8(a) for the associated graph G_α ; note that $k_0 = \infty$ and $k_1 = 0$ always since the first two vertices are $1/0$ and $0/1$). Set

$$\tau_{\alpha,k} = \sum_{k_i=k} a_i, \quad \mu_{\alpha,k} = \sigma_\alpha - \tau_{\alpha,k}$$

(where $a_0 = a_{n+1} = 0$ by convention). Thus $\tau_{\alpha,k}$ is just the sum of the “turns” at the (internal) vertices of 2-type k .

(4.3) **Example.** Consider the based path $(-5, 2, -3)$ shown in Fig. 8(b) (and schematically in Fig. 8(c)). The internal vertices are of types 0, 1, 0 (respectively) so $\tau_{\alpha,0} = -5 - 3 = -8$, $\tau_{\alpha,1} = 2$ and $\tau_{\alpha,\infty} = 0$. Thus $\mu_{\alpha,0} = 7$, $\mu_{\alpha,1} = -3$ and $\mu_{\alpha,\infty} = -1$ (since $\sigma_\alpha = -1$, by (1.13c) for example).

(4.4) **Lemma.** $\mu_{\alpha,k} = \mu_{\beta,k}$ provided either (1) α and β share the same final edge, or (2) α and β share the same final vertex, and this vertex is not of type k .

Proof. For (1) recall that β can be obtained from α by a sequence of Moves 1 and 2 (described in Remarks 1.13(d)) and these change both σ_α and $\tau_{\alpha,k}$ by ± 1 . For (2), it suffices to establish invariance under Move 3 (described in Remark 2.3(c)). Once again σ_α changes by ± 1 , as does $\tau_{\alpha,k}$ since one of the last two internal vertices must be of type k (the three vertices of any triangle in K are of distinct 2-types). \square

It follows that there are well defined functions (for $k = 0, 1$ and ∞)

$$\varphi_k: \text{PSL}(2, \mathbf{Z}) \rightarrow \mathbf{Z} \quad \text{and} \quad \mu_k: (\mathbf{Q} \cup \infty) - \mathbf{Q}_k \rightarrow \mathbf{Z}$$

given by $\varphi_k(A) = \mu_{\alpha,k}$ (for any α with $A = A_\alpha$) and $\mu_k(a/c) = \mu_{\alpha,k}$ (for any α with final vertex a/c). Evidently the functions φ_k are related to the function φ of Sect. 1 – indeed $\varphi = \varphi_0 + \varphi_1 + \varphi_\infty$ by Theorem 1.12. Similarly the functions μ_k are related to Dedekind sums, and (as we shall see below) to μ -invariants of lens spaces.

We are primarily interested in the restrictions

$$\mu_0, \mu_1: \mathbf{Q}_\infty \rightarrow \mathbf{Z}$$

of μ_0 and μ_1 to the intersection of their domains, and their union

$$\mu: \mathbf{Q}_0 \cup \mathbf{Q}_1 \rightarrow \mathbf{Z}$$

on the symmetric difference of their domains ($\mu = \mu_0$ on \mathbf{Q}_1 and $\mu = \mu_1$ on \mathbf{Q}_0).

(4.5) **Theorem.** *The function μ (on $\mathbf{Q}_0 \cup \mathbf{Q}_1$) and the (unordered) set of functions $\{\mu_0, \mu_1\}$ (on \mathbf{Q}_∞) provide integer lifts of the μ -invariants of lens spaces. In other words μ and $\{\mu_0, \mu_1\}$ respect the equivalence relation \sim of (4.2), and $L(c, a)$ has μ -invariant $\mu(a/c) \pmod{16}$ for c odd and μ -invariants $\mu_0(a/c) \pmod{16}$ and $\mu_1(a/c) \pmod{16}$ for c even.*

Proof. Given $x \sim y$, we must show $\mu(x) = \mu(y)$ for x in $\mathbf{Q}_0 \cup \mathbf{Q}_1$ and

$$\{\mu_0(x), \mu_1(x)\} = \{\mu_0(y), \mu_1(y)\}$$

for x in \mathbf{Q}_∞ . By (4.2) and the definition of μ , it suffices to prove the stronger

(4.6) **Assertion.** (i) $\mu_0(x) = \mu_1(x \pm 1)$ for all x in $\mathbf{Q}_1 \cup \mathbf{Q}_\infty$.

(ii) Let $ad \equiv 1 \pmod{c}$. If c is odd, then $\mu(a/c) = \mu(d/c)$. If c is even, then $\mu_0(a/c) = \mu_0(d/c)$ and $\mu_1(a/c) = \mu_1(d/c)$ when $ad \equiv 1 \pmod{2c}$, and $\mu_0(a/c) = \mu_1(d/c)$ otherwise.

For (i), choose a based path $\alpha = (a_1, \dots, a_n)$ with final vertex x . Then $\beta = (0, \pm 1, a_1, \dots, a_n)$ has final vertex $x \pm 1$, and so we must show $\mu_{\alpha,0} = \mu_{\beta,1}$. The first two internal vertices of β are of 2-type 0 and ∞ , and the remaining vertices of β are just unit translations of the corresponding vertices of α . Since this translation maps \mathbf{Q}_∞ to itself and swaps \mathbf{Q}_0 and \mathbf{Q}_1 , the vertices of type ∞ in α and β correspond, while those of type 0 in α correspond to those of type 1 in β (and vice versa). Thus $\tau_{\beta,\infty} = \tau_{\alpha,\infty} \pm 1$, $\tau_{\beta,0} = \tau_{\alpha,1}$ and $\tau_{\beta,1} = \tau_{\alpha,0}$. Evidently $\sigma_\alpha = \sigma_\beta$, and so $\mu_{\alpha,0} = \mu_{\beta,1}$ as desired.

For (ii), set $b = (ad - 1)/c$ (so that $ad - bc = 1$). Choose a path $\alpha = (a_1, \dots, a_n)$ with final edge from b/d to a/c . Then $\bar{\alpha} = (a_n, \dots, a_1)$ has final vertex d/c , so $\mu_k(a/c) = \mu_{\alpha,k}$ and $\mu_k(d/c) = \mu_{\bar{\alpha},k}$. As in the proof of (i), set up a correspondence between the vertices of α and $\bar{\alpha}$ by identifying the i^{th} internal vertex of α with the $(n - i)^{\text{th}}$ internal vertex of $\bar{\alpha}$.

Assume first that c is odd. Then a/c and d/c are each of type 0 or 1. If both are of type 0, then vertices of type 1 in α and $\bar{\alpha}$ correspond, and so $\tau_{a,1} = \tau_{\bar{a},1}$. Thus $\mu(a/c) = \mu_{\alpha,1} = \mu_{\bar{a},1} = \mu(d/c)$ (since evidently $\sigma_\alpha = \sigma_{\bar{\alpha}}$). If a/c is of type 0 and d/c is of type 1, then the vertices of type 1 in α correspond to those of type 0 in $\bar{\alpha}$. Thus $\tau_{\alpha,1} = \tau_{\bar{a},0}$ and so $\mu(a/c) = \mu_{\alpha,1} = \mu_{\bar{a},0} = \mu(d/c)$. The other cases are analogous.

Now assume that c is even. Then a/c and d/c are both of type ∞ , and b/d is either of type 0 or type 1 depending upon whether b is even or odd (equivalently whether $ad \equiv 1 \pmod{2c}$ or not). In the former case, the correspondence between vertices of α and $\bar{\alpha}$ preserves 2-type, and so $\tau_{\alpha,k} = \tau_{\bar{\alpha},k}$ for all k . Thus $\mu_k(a/c) = \mu_{\alpha,k} = \mu_{\bar{\alpha},k} = \mu_k(d/c)$ for $k = 0, 1$. In the latter case, vertices of type 0 in α correspond to those of type 1 in $\bar{\alpha}$, and vice versa, and so $\mu_0(a/c) = \mu_1(d/c)$ as above. This completes the proof of (4.6).

To establish the relationship with the μ invariant, recall (Remark 2.5) that the lens space $M = L(c, a)$ can be obtained by surgery on the framed link L_α shown in Fig. 7, where $\alpha = (a_1, \dots, a_n)$ is a based path with final vertex a/c . The components L_i of L_α correspond naturally to the internal vertices of α , and so have associated 2-types k_i (for $i = 1, \dots, n$). As above, set $k_0 = \infty$, the 2-type of the initial vertex of α , and write k_{n+1} for the 2-type of its final vertex a/c . (Thus $k_{n+1} = 0$ or 1 if c is odd, and $k_{n+1} = \infty$ if c is even.) Let $L_{\alpha,k}$ denote the sublink of all L_i with $k_i = k$.

Observe that since the three vertices of any triangle in K are of distinct 2-types, the k_i have the following properties:

- (1) $k_i \not\equiv k_{i+1}$ (for $0 \leq i \leq n$) – i.e. neighboring components have distinct 2-types
- (2) $k_{i-1} = k_{i+1}$ if and only if a_i is even (for $0 < i \leq n$).

It follows that $L_{\alpha,k}$ is characteristic (i.e. $L_{\alpha,k} \cdot L_i \equiv a_i \pmod{2}$) for $1 \leq i \leq n$) provided $k \not\equiv k_0$ and $k \not\equiv k_{n+1}$. To see this, consider any component L_i . If $k_i = k$, then (by (1)) L_i is an isolated component of $L_{\alpha,k}$ and so $L_{\alpha,k} \cdot L_i = a_i$. So assume that $k_i \not\equiv k$. If $1 < i < n$, then L_i has exactly two neighbors. If a_i is odd, then (by (2)) exactly one of these is of type k , and so $L_{\alpha,k} \cdot L_i = \pm 1 \equiv a_i \pmod{2}$. Similarly, if a_i is even, then either both are of type k or neither is. Thus $L_{\alpha,k} \cdot L_i = \pm 2$ or 0, which is congruent to $a_i \pmod{2}$. Finally suppose that $i = 1$ or n . Then L_i has only one neighbor which (by (2) and the hypothesis on k) will have type k if and only if a_i is odd. Therefore $L_{\alpha,k} \cdot L_i = \pm 1$ or 0, depending upon whether a_i is odd or even, as desired. (See below for an example.)

We can now compute the μ -invariant(s) of M using (4.1) and the 4-manifold $W = W_{L_\alpha}$. Observe that $\sigma(W) = \sigma_\alpha$ (Remark 1.13(b)). By the previous paragraph, exactly one of $L_{\alpha,0}$ or $L_{\alpha,1}$ is characteristic if c is odd (depending upon whether a/c is of type 1 or 0), and both are characteristic if c is even. Noting that the components of $L_{\alpha,k}$ are isolated, we have $L_{\alpha,k} \cdot L_{\alpha,k} = \tau_{\alpha,k}$ and $\text{Arf}(L_{\alpha,k}) = 0$. Thus $\mu_{\alpha,k}$ is the μ -invariant of M with the spin structure corresponding to $L_{\alpha,k}$ (provided it is characteristic), and the theorem follows. \square

(4.7) *Example.* Consider the lens space $M = L(38, 7)$ given by the based path $\alpha = (-5, 2, -3)$ with final vertex $7/38$ (discussed in Example 4.3). The associated link L_α is a chain of three components with the middle one of type 1 (comprising one characteristic sublink) and the outer two of type 0 (comprising the other). Thus (as in (4.3)) $\mu_0(7/38) = 7$ and $\mu_1(7/38) = -3$, and so the μ -invariants of M are 7 and 13 (mod 16).

It is tempting to think of the functions μ_0 and μ_1 individually as defining μ -invariants of lens spaces. Unfortunately they do not respect \sim , and so it does not

make sense to talk about μ_0 (or μ_1) of a lens space. For example, $\mu_0(7/38) = 7$ and $\mu_1(7/38) = -3$ (as we have just shown) whereas $\mu_0(45/38) = -3$ and $\mu_1(45/38) = 7$ (by Assertion 4.6(i)). Since $M = L(38, 7) = L(38, 45)$, the invariant $\mu_0(M)$ is ambiguous.

Next we derive reciprocity laws for the functions μ_0 and μ_1 , and thus for the μ -invariants of lens spaces (by the previous theorem). From these we deduce relationships between Dedekind sums, μ -invariants of lens spaces, and the Brown invariant (see [Bro1] and Sect. 6 in [KM1]).

(4.8) **Theorem** (reciprocity for μ_0 and μ_1). (a) $\mu_1(a/c) + \mu_1(c/a) = -\text{sign}(ac)$ for a/c in $\mathbf{Q}_0 \cup \mathbf{Q}_\infty$.

(b) $\mu_0(a/c) + \mu_0(c/a) \equiv ac - \text{sign}(ac) \pmod{16}$ for a/c in \mathbf{Q}_1 . (Note that $\mu = \mu_0$ on \mathbf{Q}_1).

Proof. The proof of (a) is easy. Choose a based path $\alpha = (a_1, \dots, a_n)$ ending at a/c . Then the path $\gamma = (0, -a_1, \dots, -a_n)$ ends at c/a , and we must show $\mu_{\alpha,1} + \mu_{\gamma,1} = -\text{sign}(ac)$. As in the proof of (4.6.i), the vertices of type 1 in α and γ correspond (after omitting the first vertex of γ), and so $\tau_{\alpha,1} + \tau_{\gamma,1} = 0$. Since $\sigma_\alpha + \sigma_\gamma = -\text{sign}(ac)$ (by Remark 1.13(c) for example), it follows that $\mu_{\alpha,1} + \mu_{\gamma,1} = -\text{sign}(ac)$.

The proof of (b) is similar, except that we are now interested in the vertices of type 0 in α and γ . But the vertices of type 0 in γ correspond (as above) to the vertices of type ∞ in α , and so it suffices to prove the first of the following three useful congruences:

(4.9) **Assertion.** (i) $\mu_0(a/c) - \mu_\infty(a/c) \equiv ac \pmod{16}$ for a/c in \mathbf{Q}_1 .

(ii) $\mu_0(a/c) - \mu_1(a/c) \equiv ac \pmod{8}$ for a/c in \mathbf{Q}_∞ .

(iii) $\mu_1(a/c) - \mu_\infty(a/c) \equiv ac \pmod{8}$ for a/c in \mathbf{Q}_0 .

To prove this, choose as above a based path $\alpha = (a_1, \dots, a_n)$ ending at a/c . Assume by induction that the assertion has been verified for the endpoints of all shorter paths. (It is trivially verified if $n = 0$ or 1 , so assume $n > 1$.) Write p/q , b/d and (of course) a/c for the final three vertices of α , and t for the ‘‘turn’’ a_n at b/d . Adopting the usual sign conventions we have $pd - bq = bc - ad = 1$ and $t = pc - aq$. Thus $a = tb - p$ and $c = td - q$ (cf. the proof of Lemma 1.9), and so $ac = bdt^2 - (pd + bq)t + pq$. Rewriting the coefficient of t as $-1 + 2pd$ or $1 + 2bq$ (using $pd - bq = 1$) and completing the square yields

$$(4.10) \quad ac = pq + \varepsilon t + bdt(t - 2) + 2ts_\varepsilon$$

for $\varepsilon = \pm 1$, where $s_{+1} = d(b - p)$ and $s_{-1} = b(d - q)$.

Now in (i) we are given a/c of type 1. Suppose first that t is even, or equivalently that p/q is also of type 1. Then $\mu_0(a/c) - \mu_\infty(a/c) = \mu_0(p/q) - \mu_\infty(p/q) + \varepsilon t \equiv pq + \varepsilon t \pmod{16}$ (by induction), where $\varepsilon = +1$ or -1 depending upon whether b/d is of type ∞ or type 0. By (4.10), it suffices to show $bdt(t - 2) + 2ts_\varepsilon \equiv 0 \pmod{16}$. But this is immediate from parity considerations: t is even, exactly one of b or d is even (the latter iff $\varepsilon = +1$), and p and q are odd. This proves (i) when t is even. If it is odd, then consider the path $(a_1, \dots, a_{n-1} + 1, 1, a_n + 1)$ which has a new vertex r/s of type 1 between p/q and b/d . The assertion holds for r/s by induction, and so also for a/c by the preceding argument.

The arguments for (ii) and (iii) are analogous except that the parity considerations only yield congruence mod 8. This proves the assertion and thus the theorem as well. \square

(4.11) *Remark.* Theorem 4.8(a) provides a recursive computation for the values of μ_1 , and coupled with Assertion 4.6(i), for the values of μ_0 as well. This yields a quick recursive method for computing the μ -invariant(s) of lens spaces (cf. the algorithm in [NR, p. 181] for odd lens spaces). For example, let $M = L(38, 7)$. One computes $\mu_0(7/38) = \mu_1(-31/38) = \mu_1(38/31) + 1 = \mu_1(-24/31) + 1 = \mu_1(31/24) + 2 = \dots = \mu_1(10/3) + 5 = \mu_1(-2/3) + 5 = \mu_1(3/2) + 6 = \mu_1(-1/2) + 6 = \mu_1(2) + 7 = 7$ and $\mu_1(7/38) = \mu_1(4/7) - 1 = \mu_1(1/4) - 2 = \mu_1(-4) - 3 = -3$. Thus the μ -invariants of M are $\{\mu_0(7/38), \mu_1(7, 38)\} \pmod{16} = \{7, 13\}$ (cf. Example 4.7).

The well known fact that the μ -invariant of an odd lens space is determined by its Dedekind sum is easily deduced from Theorem 4.8(b):

(4.12) **Corollary.** *If c is odd, then $3\mu(L(c, a)) \equiv c^2S(a/c) \pmod{16}$.*

Proof. The left hand side of the congruence is $3\mu(a/c)$ by Theorem 4.5. Noting that the restriction of μ to \mathbf{Q}_1 is characterized by the reciprocity law (4.8.b) together with the properties $\mu(-x) = -\mu(x)$ and $\mu(x+2) = \mu(x)$, it suffices to establish the corresponding reciprocity law for $c^2S(a/c)$ (with a and c odd):

$$(4.13) \quad c^2S(a/c) + a^2S(c/a) \equiv 3ac - 3 \operatorname{sign}(ac) \pmod{16}.$$

(The additional properties $c^2S(-a/c) = -c^2S(a/c)$ and $c^2S((a/c)+2) = c^2S(a/c)$ are immediate from axioms (D1-2) for the Dedekind sum (2.2).) Using $\mu(-x) = -\mu(x)$ we need only consider the case when a and c are positive. Then the reciprocity formula for Dedekind sums (D3) (multiplied by a^2c^2) yields $a^2(c^2S(a/c)) + c^2(a^2S(c/a)) = ac(a^2+c^2+1) - 3a^2c^2$. The left hand side is congruent to $c^2S(a/c) + a^2S(c/a) \pmod{16}$ (since $c^2S(a/c) \equiv 0 \equiv a^2S(c/a) \pmod{2}$ by (2.4.c) and $a^2 \equiv 1 \equiv c^2 \pmod{8}$). Thus it remains to show $ac(a^2+c^2+1) - 3a^2c^2 \equiv 3ac - 3 \pmod{16}$, which is readily verified (again using $a^2 \equiv 1 \equiv c^2 \pmod{8}$). \square

(4.14) *Remark.* (a) For even lens spaces, the μ -invariants are not determined by the Dedekind sum. For example $L(64, 9)$ and $L(64, 25)$ have the same Dedekind sum $-63/32$ but different μ -invariants: $\{\mu_0(9/64), \mu_1(9/64)\} = \{7, -9\}$ (both congruent to $7 \pmod{16}$) and $\{\mu_0(25/64), \mu_1(25/64)\} = \{-1, -1\}$ (both congruent to $-1 \pmod{16}$).

(b) In general (i.e. for both even and odd lens spaces), the integer lifts of the μ -invariants given in Theorem 4.5 are independent of the Dedekind sums. For example $L(85, 7)$ and $L(85, 22)$ have the same Dedekind sum $84/17$ but different integer μ -invariants: 12 and -4 respectively. Similarly for $L(64, 9)$ and $L(64, 25)$, as remarked in (a). Perhaps more striking are $L(100, 9)$ and $L(100, 29)$, which have the same Dedekind sum $99/50$, the same μ -invariants $\{7, 11\}$, but different integer lifts of their μ -invariants: $\{11, -9\}$ and $\{7, -5\}$ respectively. Conversely, there are lens spaces with the same integer μ -invariants but different Dedekind sums. For example, $L(11, 4)$ and $L(11, 5)$ both have μ -invariant 2 while their Dedekind sums $18/11$ and $-30/11$ differ. Similarly $L(28, 5)$ and $L(28, 13)$ have the same set $\{3, -1\}$ of μ -invariants but distinct Dedekind sums, $39/14$ and $-57/14$.

Finally, we establish a relationship between the (integer) μ -invariants of a lens space and its Brown invariant, which is useful in the study of quantum invariants of lens spaces [KM2]. Recall [KM1] that the **Brown invariant** $\beta(M)$ of a 3-manifold M is defined if and only if the μ -invariants of M are all congruent mod 4 (which for a lens space $L(c, a)$ means that $a \not\equiv 2 \pmod{4}$) by (4.9.ii). In that case, describe M by surgery on a framed link L in the 3-sphere with linking matrix A . Then $\beta(M) = \sigma_A - \lambda_A$, where σ_A is the signature of A and λ_A is the $\mathbf{Z}/8\mathbf{Z}$ Arf invariant

of A (viewed as a $\mathbf{Z}/4\mathbf{Z}$ -valued quadratic form on the $\mathbf{Z}/2\mathbf{Z}$ -inner product space given by A) [Bro1]. This is easily shown independent of the choice of L by the calculus of framed links [K1] and is a $\mathbf{Z}/8\mathbf{Z}$ -valued homotopy invariant.

It is shown in Sect. 6 of [KM1] that the Brown invariant of any $\mathbf{Z}/2\mathbf{Z}$ -homology sphere is determined by its μ -invariant and its first betti number. In particular

$$(4.15) \quad \beta(L(c, a)) = \mu(L(c, a)) + (c^2 - 1)/2 \pmod{8}$$

for c odd. A similar formula holds for c divisible by 4:

$$(4.16) \textbf{ Theorem. } \textit{If } c \equiv 0 \pmod{4}, \textit{ then } \beta(L(c, a)) = \mu_1(a/c) + (a^2 - 1)/2 \pmod{8}.$$

Proof. Set $\beta(a/c) = \beta(L(c, a))$. Observe that $\beta(a/c)$ satisfies the following reciprocity law:

$$(4.17) \quad \beta(a/c) + \beta(c/a) = -\text{sign}(ac).$$

To see this, choose a based path $\alpha = (a_1, \dots, a_n)$ ending at a/c (as in Sect. 1). Thus $L(c, a)$ is surgery on the framed link L_α described in Fig. 7, and similarly $L(a, c)$ is surgery on L_γ for $\gamma = (0, -a_1, \dots, -a_n)$ (which is a based path to c/a). By definition $\beta(a/c) = \sigma_\alpha - \lambda_\alpha$, where σ_α and λ_α are the signature and $\mathbf{Z}/8\mathbf{Z}$ Arf invariant of the linking matrix M_α of L_α (see (1.12)), and similarly $\beta(c/a) = \sigma_\gamma - \lambda_\gamma$. Now it is easy to show that $\sigma_\alpha + \sigma_\gamma = -\text{sign}(ac)$ using the geometric interpretation of the signatures in Remark 1.13(c) (as in the proof of Theorem 4.8(a)). Thus it remains to show $\lambda_\alpha = \lambda_\gamma$. But this can be seen from the fact that the Arf invariant λ_α can be computed by diagonalizing M_α as a $\mathbf{Z}/4\mathbf{Z}$ -valued quadratic form, first stabilizing if necessary [KM1]. In particular, diagonalizing M_α (from the bottom up) yields $0 \oplus D$ for some $\pmod{2}$ diagonal matrix D (since $c \equiv 0 \pmod{4}$), while M_γ reduces to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus D$ by the same process. Thus $\lambda_\alpha = \lambda_\gamma$ and (4.17) is proved.

Now by (4.15) (with a and c reversed) we have $\beta(c/a) = \mu_1(c/a) - (a^2 - 1)/2 \pmod{8}$. (Note that the last term is 0 or 4 since a is odd, and so its sign is irrelevant.) Combining this with (4.17) gives

$$\begin{aligned} \beta(a/c) &\equiv -\mu_1(c/a) - \text{sign}(ac) + (a^2 - 1)/2 \pmod{8} \\ &\equiv \mu_1(a/c) + (a^2 - 1)/2 \pmod{8} \end{aligned}$$

by the reciprocity law (4.9.a) for μ_1 .

(4.18) *Remark.* The Brown invariant of an even lens space $L(c, a)$ is not determined by its μ -invariants $\pmod{8}$ and c . For example $L(12, 1)$ and $L(12, 5)$ have equal μ -invariants $\{1, 5\}$ but distinct Brown invariants: 1 and 5 respectively.

5 μ -invariants of torus bundles

The μ -invariants of torus bundles over the circle are related to Dedekind sums, as in the case of lens spaces, but now the situation is subtler. For one thing, there are (in general) several spin structures, and so one must find a natural way to distinguish between them (this is a problem with even lens spaces as well). In addition, the dependence on the Dedekind sum is indirect and is more naturally expressed in terms of the φ -function of Sect. 1. Furthermore, there is a sign subtlety due to the fact that the monodormy matrix lies in $\text{SL}(2, \mathbf{Z})$ rather than $\text{PSL}(2, \mathbf{Z})$.

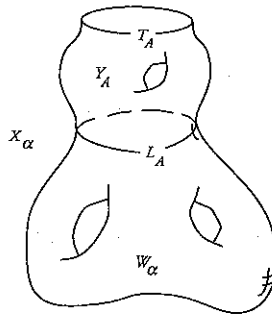


Fig. 9

Here we give a formula (5.4) for the μ -invariants associated with the *Lie spin structures* on torus bundles over the circle (i.e. structures which restrict to each fiber as the Lie group framing plus a normal vector to the fiber) in terms of the entries of the monodromy matrix A (in $SL(2, \mathbf{Z})$), the image of A under a suitable abelianization map $SL(2, \mathbf{Z}) \rightarrow H_1 SL(2, \mathbf{Z}) = \mathbf{Z}/12\mathbf{Z}$, and $\varphi(A)$. (This formula will be applied in the next section to compute the signature cocycle which defines the universal central extension of $SL(2, \mathbf{Z})$.) The formula for non-Lie spin structures involves the functions φ_k in the decomposition $\varphi = \varphi_0 + \varphi_1 + \varphi_\infty$ developed in Sect. 4, and is given in Table 1.

Fix a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in $SL(2, \mathbf{Z})$, and let

$$T_A = T^2 \times I / (x, 1) \sim (Ax, 0)$$

be the torus bundle over the circle with monodromy A . Write L_A for the associated lens space $L(c, a)$, obtained from the union $V \cup -V$ of two solid tori $V = B^2 \times S^1$ by identifying x in $\partial V = T^2$ with Ax in $\partial(-V)$. There is an oriented bordism Y_A between L_A and T_A , obtained from $I \times L_A$ by attaching a round 1-handle to $1 \times L_A$ along the two oriented cores of V and $-V$. In particular, $\partial V_A = T_A \cup -L_A$. (See the Appendix for a more complete discussion of orientation conventions.)

Recall from (1.13b) (also see Remark 1.11) that any based path $\alpha = (a_1, \dots, a_n)$ with $A = ST^{a_1}S \dots T^{a_n}S$ yields an oriented 4-manifold W_α (obtained by attaching 2-handles to the 4-ball along the simple chain L_α shown in Fig. 7) with $\partial W_\alpha = L_A$. Let X_α denote the union of W_α and Y_A along L_A , so $\partial X_\alpha = T_A$ (see Fig. 9).

A framed link picture for X_α is given in the Appendix, obtained from L_α by adding two 0-framed components J and K , where J closes the chain and K (the "1-handle") encircles it (Fig. 19).

To compute the μ -invariants of T_A (using (4.1) with $L = L_\alpha \cup J \cup K$) we must first compute the signature $\sigma(X_\alpha)$, which equals $\sigma(W_\alpha) + \sigma(Y_A)$ by Novikov additivity. The first term $\sigma(W_\alpha) = \sigma_\alpha$ was computed in Sect. 1 and is related to the φ -function, while the second $\sigma(Y_A)$ is easily determined by a direct topological argument. First define $\nu: SL(2, \mathbf{Z}) \rightarrow \{-1, 0, 1\}$ by

$$(5.1) \quad \nu \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \text{sign}(b) & \text{if } c = 0 \text{ and } a + d = 2 \\ \text{sign}(c(a + d - 2)) & \text{otherwise.} \end{cases}$$

(Thus the first case applies only to the powers $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ of T .)

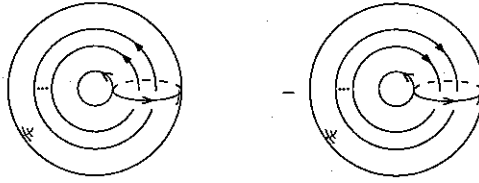


Fig. 10

(5.2) **Lemma.** $\sigma(Y_A) = \nu(A)$.

Proof. In this proof, we drop the subscript A , so $L_A = L$ and $Y_A = Y$. Note that Y/L has the homotopy type of $S^2 \vee S^1$, and so (working with rational coefficients) we have $H_2(Y, L) = \mathbf{Q}$, generated by the core $Z = B^1 \times S^1$ of the round handle, and $H_3(Y, L) = 0$. This yields the exact sequence

$$0 \rightarrow H_2(L) \rightarrow H_2(Y) \rightarrow \mathbf{Q} \rightarrow H_1(L)$$

extracted from the sequence of the pair (Y, L) .

First assume $c \neq 0$. Then $H_1(L) = H_2(L) = 0$ and so $H_2(Y) = \mathbf{Q}$. A generating surface F can be made from c copies of the core Z of the round handle capped off appropriately in $I \times L$. In particular, $\partial(cZ)$ consists of c core circles in each of V and $-V$, the solid tori which make up L , as drawn in Fig. 10.

The matrix A takes the c circles in V (after moving them to ∂V) to $cbm + cd\ell$ in $\partial(-V)$, so the total in $\partial(-V)$ is $cbm + c(d - 1)\ell = (ad - 1)m + c(d - 1)\ell = (a - 1)m + (d - 1)(am + c\ell)$. Now we can cap off $\partial(cZ)$, by letting the $a - 1$ copies of the meridian m bound disks in $-V$, and the $d - 1$ copies of $am + c\ell$, which is a meridian in ∂V , bound disks in V .

We compute the self-intersection $F \cdot F$ as follows: Take a parallel copy cZ' of cZ . We capped off cZ in $1 \times L$ to get F , so to get the push off F' of F we push $\partial(cZ')$ through $I \times L$ and cap off in $0 \times L$. Then the intersections of F with F' occur between the meridional disks ($a - 1$ in $-V$ and $d - 1$ in V) in F and the boundary components of cZ' . Thus we get $c(d - 1)$ positive intersections in V and $c(a - 1)$ positive intersections in $-V$ (note that $-V$ has the left hand orientation.) Altogether, $F \cdot F' = c(a + d - 2)$ and so $\sigma(Y) = \text{sign}(c(a + d - 2))$.

Now suppose that $c = 0$ and so $A = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ (when $\text{tr}(A) = 2$) or $\begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix}$ (when $\text{tr}(A) = -2$). In either case $L = S^1 \times S^2$ and so $H_1(L) = H_2(L) = \mathbf{Q}$. This gives the exact sequence

$$0 \rightarrow \mathbf{Q} \rightarrow H_2(Y) \rightarrow \mathbf{Q} \rightarrow \mathbf{Q}$$

where the last map λ is the boundary map from $H_2(Y, A)$ to $H_1(L)$. If $\text{tr}(A) = -2$ then λ is multiplication by 2. Thus $H_2(Y) = \mathbf{Q}$ with generator S^2 in $L = S^1 \times S^2$ of self-intersection zero, and so $\sigma(Y) = 0 = \text{sign}(c(a + d - 2))$ as desired. In contrast, if $\text{tr}(A) = 2$ then λ is the zero homomorphism and so $H_2(Y) = \mathbf{Q}^2$. It is still true that S^2 intersects everything zero times, but the other generator ρ may have non-zero self-intersection. We construct a surface F representing ρ from the cylinder Z which meets ∂V in ℓ and $\partial(-V)$ in $-\ell$. The matrix A takes ℓ to $bm + \ell$, so $\partial Z = bm + \ell - \ell = bm$ in $\partial(-V)$, which bounds m disks in either V or $-V$. Both cases give b positive intersection points, so $\sigma(Y) = \text{sign}(b)$. \square

Now it is a straightforward task to compute all the μ -invariants of the torus bundle T_A : Identify all the characteristic sublinks C of $L = L_\alpha \cup J \cup K$; the associated μ -invariants are $\sigma_\alpha + \nu(A) - C \cdot C + 8 \text{Arf}(C)$ (see Sect. 4). (See [Wo] for formulas for the μ -invariants of related homology spheres.)

The result is most interesting for the two Lie spin structures on T_A (see the Appendix (A.6)) and is expressed in terms of the abelianization homomorphism $h: \text{SL}(2, \mathbf{Z}) \rightarrow \mathbf{Z}/12\mathbf{Z}$ given by

$$(5.3) \quad h(S) = 3, \quad h(T) = -1$$

on the generators S and T (see Sect. 0).

(5.4) **Theorem.** *Let Θ and Θ' be the two Lie spin structures on the torus bundle T_A over the circle. Then*

$$\begin{aligned} \mu(T_A, \Theta) &\equiv \mu(T_A, \Theta') + 8 \pmod{16} \\ &\equiv \frac{1}{3} \varphi(A) + \frac{2}{3} h(A) + \nu(A) \pmod{8}. \end{aligned}$$

(See Sect. 1, (5.1) and (5.3) for the definitions of φ , ν and h).

Proof. As above, view T_A as the boundary of the 4-manifold $X_\alpha = W_\alpha \cup Y_A$ obtained by surgery on the link $L = L_\alpha \cup J \cup K$. In the Appendix (A.6) it is shown that the two Lie spin structures correspond to the characteristic sublinks $L - K$ (the closed chain) and L , both with self intersection $\sum a_i - 2(n + 1)$ but with distinct Arf invariants. The first congruence follows immediately. For the second, use Lemma 5.2 and the formula $\varphi(A) = 3\sigma_\alpha - \sum a_i$ Theorem 1.12) to write

$$\begin{aligned} \mu(T_A, \Theta) &\equiv \sigma_\alpha + \nu(a) - \sum a_i + 2(n + 1) \pmod{8} \\ &\equiv \frac{1}{3} \varphi(A) + \frac{2}{3} \left(3(n + 1) - \sum a_i \right) + \nu(A) \pmod{8}. \end{aligned}$$

Since h is a homomorphism, $h(A) = h(ST^{a_1}S \dots T^{a_n}S) = 3(n+1) - \sum a_i \pmod{12}$, which completes the proof. \square

For the non-Lie spin structures on T_A , we must first identify the associated characteristic sublinks of the framed link $L = L_\alpha \cup J \cup K$. Recall that the number of spin structures is the order of $H_1(T_A, \mathbf{Z}/2\mathbf{Z}) = \mathbf{Z}/2\mathbf{Z} \oplus \text{coker}(A - I)$ (where the first factor is generated by a section and the second comes from the fiber). This is either 2, 4 or 8 (according to whether $\text{rk}_{\mathbf{Z}/2\mathbf{Z}}(A - I) = 2, 1$ or 0), and so the number of non-Lie spin structures is either 0, 2 or 6. They naturally come in pairs, just as for the Lie spin structures, where each pair has characteristic sublinks S and $S \cup K$ for a suitable sublink S of $L - K$. Using the method of proof of Theorem 4.5, it is easy to see that S must be of the form $L_{\alpha,k}$ (for $k = 0, 1$ or ∞), consisting of all the components of L_α of type k , with J thrown in when $k = \infty$. In particular S and $S \cup K$ are unlinks and so the Arf invariant is always zero. The resulting μ -invariants are then naturally expressed in terms of the μ -invariant of the associated lens space L_A (when it is odd, i.e. c is odd), or in terms of the invariants $\varphi_k(A)$ defined in Sect. 4 (which are also of course related to the μ -invariants of L_A).

The results are given in Table 1 below. The first column gives the matrix $A \pmod{2}$, the second gives the number of pairs of non-Lie spin structures on T_A , and the third gives the μ -invariants for one spin structure in each pair.

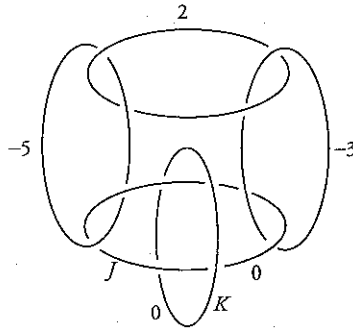


Fig. 11

Table 1

Matrices	#	μ -invariants
$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	0	
$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	2	$\mu(L_A) + \nu(A)$
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	6	$\varphi_0(A) + \nu(A), \varphi_1(A) + \nu(A), \varphi_\infty(A) + \nu(A)$
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	2	$\varphi_\infty(A) + \nu(A)$

(5.5) *Example.* For $A = \begin{pmatrix} 7 & 2 \\ 38 & 11 \end{pmatrix} = ST^{-5}ST^2ST^{-3}S \equiv I \pmod{2}$, the torus bundle T_A has 6 non-Lie spin structures, corresponding to the characteristic sublinks $L_{\alpha,0}, L_{\alpha,1}, J$ (with their partners $L_{\alpha,0} \cup K, L_{\alpha,1} \cup K, J \cup K$) of the framed link for T_A shown in Fig. 11. Here $L_{\alpha,0}$ is the union of the components with framings -5 and -3 and $L_{\alpha,1}$ is the 2-framed component. (Note that $L_{\alpha,\infty}$ is empty.) Since $\nu(A) = 1, \varphi_0(A) = 7, \varphi_1(A) = -3$ and $\varphi_\infty(A) = -1$ (see Example 4.3), the associated μ -invariants are 8, 14 and 0 (mod 16).

Finally observe that $\varphi(A) = 3$ and $h(A) = 6$, and so the Lie spin structures on T_A have μ -invariants $\frac{1}{3}3 + \frac{2}{3}6 + 1 = 6$ and 14 by Theorem 5.4.

6 The signature cocycle

Central extensions of $SL(2, \mathbf{Z})$ by \mathbf{Z} are classified by $H^2(SL(2, \mathbf{Z}); \mathbf{Z}) = \mathbf{Z}/12\mathbf{Z}$ (see (0.11)). Of particular importance in string theory and Jones-Witten theory is the central extension

$$0 \rightarrow \mathbf{Z} \rightarrow \widetilde{SL}(2, \mathbf{Z}) \rightarrow SL(2, \mathbf{Z}) \rightarrow 1$$

[At2] which corresponds to the *signature 2-cocycle*

$$\sigma: SL(2, \mathbf{Z}) \times SL(2, \mathbf{Z}) \rightarrow \mathbf{Z}.$$

(Thus $\widetilde{SL}(2, \mathbf{Z})$ is the set $\mathbf{Z} \times SL(2, \mathbf{Z})$ with multiplication $(m, A)(n, B) = (m + n + \sigma(A, B), AB)$.) In particular, $\sigma(A, B)$ is defined to be the signature $\sigma(E_{A,B})$

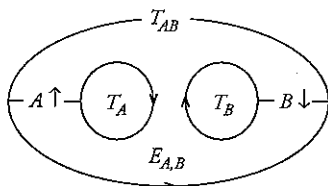


Fig. 12

of the T^2 -bundle $E_{A,B}$ over the thrice punctured 2-sphere P (a pair of pants) with monodromies A, B and AB on the boundary components, oriented as shown in Fig. 12 so that $\partial E_{A,B} = T_A \cup T_B \cup -T_{AB}$ (see Sect. 5).

A formula for the signature cocycle was first worked out in [Mey] from an algebraic point of view, and explained geometrically in [At1]. We derive a simple formula using our calculations in Sect. 5 of the μ -invariants of torus bundles. It is expressed in terms of the lift to $SL(2, \mathbf{Z})$ of the Rademacher φ function defined in Sect. 1, which we also call φ , and the function $\nu: SL(2, \mathbf{Z}) \rightarrow \mathbf{Z}$ defined in (5.1).

(6.1) **Theorem.** *The signature cocycle is the coboundary of a unique rational 1-cochain, namely $\frac{1}{3}\varphi + \nu$. In particular $\sigma = \varepsilon + \delta\nu$, where ε is the area cocycle (by Theorem 1.7). Equivalently*

$$\sigma(A, B) = \nu(A) + \nu(B) - \nu(AB) - \text{sign}(c_A c_B c_{AB})$$

for all A and B in $SL(2, \mathbf{Z})$.

Proof. The existence and uniqueness of this 1-cochain follows at once from the fact that the 1 and 2-dimensional rational cohomology groups of $SL(2, \mathbf{Z})$ vanish (cf. the proof of Lemma 1.3).

Now fix A and B in $SL(2, \mathbf{Z})$ and assign Lie spin structures Θ_A, Θ_B and Θ_{AB} to the torus bundles T_A, T_B and T_{AB} . Together these give a spin structure Θ on $\partial E = T_A \cup T_B \cup -T_{AB}$ (writing E for $E_{A,B}$), which evidently extends across $E - F$ for any T^2 -fiber F . Since $F \cdot F = 0$ it follows from the definition of the μ -invariant (4.1) that

$$\begin{aligned} \sigma(E) &\equiv \mu(\partial E, \Theta) \pmod{8} \\ &\equiv \mu(T_A, \Theta_A) + \mu(T_B, \Theta_B) - \mu(T_{AB}, \Theta_{AB}) \pmod{8}. \end{aligned}$$

Viewing μ as a $\mathbf{Z}/8\mathbf{Z}$ -valued 1-cochain on $SL(2, \mathbf{Z})$, defined by $\mu(A) = \mu(T_A, \Theta_A) \pmod{8}$ for any Lie spin structure Θ_A on T_A (this is independent of the choice of Θ_A by Theorem 5.4), this simply says $\sigma \equiv \delta\mu \pmod{8}$. Using the formula for $\mu(A)$ in (5.4), we get $\sigma \equiv \delta(\frac{1}{3}\varphi + \frac{2}{3}h + \nu) \pmod{8}$. But $\delta h = 0$ since h is a homomorphism, and so

$$(6.2) \quad \sigma \equiv \delta(\frac{1}{3}\varphi + \nu) \pmod{8}.$$

To remove the mod 8 restriction, first recall from Theorem 1.7 that $\delta(\frac{1}{3}\varphi) = \varepsilon$, where ε is the area cocycle given by $\varepsilon(A, B) = -\text{sign}(c_A c_B c_{AB})$ (1.2). Thus

$$(6.3) \quad \sigma(A, B) \equiv \nu(A) + \nu(B) - \nu(AB) - \text{sign}(c_A c_B c_{AB}) \pmod{8}.$$

Since $|\nu| \leq 1$ (it is a sign), the right hand side is clearly bounded in absolute value by 4. The left hand side, by the lemma below, is bounded by 3 and so the congruences (6.2) and (6.3) are indeed equalities. \square

(6.4) **Lemma.** $|\sigma(E_{A,B})| \leq 3$.

Proof. The Euler characteristic of $E = E_{A,B}$ is zero since E is the union of $T_A \times I$ and $T_B \times I$ along $T^2 \times I$, all spaces of Euler characteristic zero. Writing b_k for the rank of $H_k(E)$ (with integer coefficients), we have $b_3 = 2$ (by a Mayer-Vietoris argument using the same decomposition of E) and $b_1 \leq 4$ (since $H_1(E)$ is generated by the image of $H_1(F)$ and $H_1(K)$, each of rank 2, where F is a fiber in E and K is a section over a figure eight retraction of the base P). Thus $b_2 = b_1 + 1 \leq 5$. Furthermore, $b_1 = 4$ if and only if $A = B = I$ in which case $E = P \times T^2$ has zero signature. Thus we may assume $b_2 \leq 4$. Since there is always a class in $H_2(E)$ of zero self-intersection (namely the fiber F), it follows that $|\sigma(E)| \leq 3$. \square

(6.5) *Remark.* Substituting the continued fraction expression for φ in Theorem 1.12 into the formula in the theorem yields a formula for the signature cocycle in terms of signatures of matrices related to the monodromies. In particular, $\sigma = \delta(\sigma(M) - \frac{1}{3} \text{tr}(M) + \nu)$ where $M(A) = M_\alpha$ for $\alpha = (a_1, \dots, a_n)$ satisfying $A = ST^{a_1}S \dots T^{a_n}S$. A similar formula was obtained previously by Sczech using different methods, $\sigma = \delta(\frac{1}{3} \text{tr}(N) - \sigma(N))$ where $N(A) = N_\alpha$ for $\alpha = (a_0, \dots, a_n)$ satisfying $A = T^{a_0}ST^{a_1}S \dots T^{a_n}S$ [MeyS, Sc]. Here $N_\alpha = \bar{M}_{-\alpha}$ with 1's placed in the upper right and lower left corners. The equivalence of these two formulas for σ can be deduced from Lemma 3 in Sczech's Diplomarbeit [Sc]. (Note that N_α is the intersection matrix of a "circular plumbing" with boundary T_A , as described in the Appendix.) This can be used to give an alternative continued fraction formula for φ , namely $\varphi(A) = \text{tr}(N_\alpha) - 3\sigma(N_\alpha) - 3\nu(A)$.

Appendix. Framed links

The purpose of this appendix is to give framed link descriptions for 3-dimensional lens spaces and torus bundles over the circle. For the case of torus bundles we also identify the characteristic sublinks for Lie spin structures. Link pictures for lens spaces are of course well known; nevertheless we give a careful treatment, taking orientations into account, which makes the pictures for torus bundles more transparent. In particular we develop the notion of surgery on oriented links with framings in $SL(2, \mathbf{Z})$.

Throughout, we let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

denote a fixed matrix in $SL(2, \mathbf{Z})$.

Lens spaces

The oriented lens space $L_A = L(c, a)$ is constructed by taking two copies of the solid torus $V = B^2 \times S^1$, choosing oriented meridians $m = \partial B^2 \times 1$, longitudes $\ell = 1 \times S^1$ and orientations $((n, m, \ell)$ for V and $(-n, m, \ell)$ for $-V$) as in Fig. 13, and forming $V \cup -V/x \sim Ax$ (for x in ∂V). Thus the meridian and longitude of ∂V are glued to $\partial(-V)$ by $m \mapsto am + c\ell$ and $\ell \mapsto bm + d\ell$.

Equivalently, this may be thought of as a Dehn surgery on the unknot K in S^3 , identifying V with a tubular neighborhood of K (so that K and ℓ have zero linking number) and $-V$ with the complementary solid torus V' . If we choose the latter so

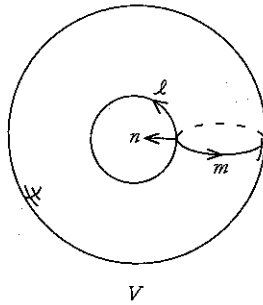


Fig. 13

that $\partial(-V)$ is glued to $\partial V'$ by the matrix $-S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ with respect to the basis m, ℓ (note that $\partial V'$ is marked with the curves m and ℓ from its identification with ∂V before surgery), then the surgery is achieved by removing V and regluing using the matrix $-SA = \begin{pmatrix} c & d \\ -a & -b \end{pmatrix}$. We shall call $-SA$ the $SL(2, \mathbf{Z})$ -framing for the surgery (see the definition below); the more traditional rational framing is $-c/a$ (Fig. 14(a)).

It is well known (see e.g. [Rol] or [O, Sect. 2]) that L_A can also be described by the integer framed link L_α in S^3 shown in Fig. 14(b), for any $\alpha = (a_1, \dots, a_n)$ for which $-c/a = a_1 - 1/(a_2 - \dots - 1/a_n \dots)$ (or equivalently $A = \pm ST^{a_1}S \dots T^{a_n}S$). (Note that α also specifies a 4-manifold W_α with boundary L_A , obtained by adding 2-handles to the 4-ball along L_α .) We recall how to show this, taking extra care with orientations since we will also want to know the location in Fig. 14(b) of the oriented core circles of V and $-V$. It is convenient to introduce a notion of surgery on an oriented $SL(2, \mathbf{Z})$ -framed link.

(A.1) Definition. Let L be an oriented link in S^3 with components labelled by elements of $SL(2, \mathbf{Z})$ (the framings). Denote by M_L (surgery on L) the 3-manifold obtained by removing an oriented tubular neighborhood $V = B^2 \times S^1$ of each component, and regluing using the associated framing matrix (with respect to the usual oriented basis $m = \partial B^2 \times 1$ and $\ell = 1 \times S^1$).

For example, the lens space $L_{SA} = V \cup -V$ is described by the oriented A -framed unknot, as shown in Fig. 15(a) and explicitly described in Fig. 15(b); the arrows marked A indicate that the solid torus V_A (the removed tubular neighborhood of A) should be glued by A to its complement in S^3 .

The oriented core of the surgery on a component K with framing A is the image of $0 \times S^1 \subset V$ after surgery, which can be identified with the (b, d) -cable about K

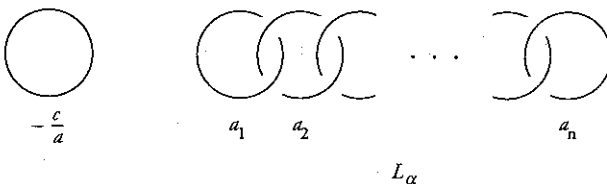


Fig. 14

(a)

(b)

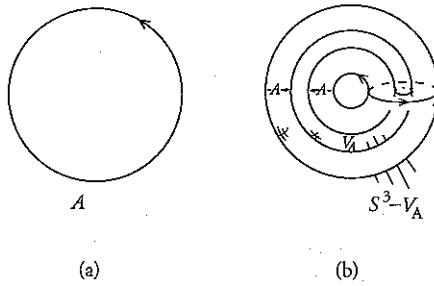


Fig. 15a, b

(since the core of V is isotopic to \emptyset). If $A = T^a S = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}$ for some integer a , then the framing will be called **integral** (corresponding to the traditional integer surgery with framing a). Note that for integral surgeries on K , the oriented core of the surgery will be a negatively oriented meridian of K (i.e. have linking number -1 with K).

Observe that $M_L = M_{L'}$ if L' is obtained from L by reversing the orientation of a component and negating its framing. A more interesting move is described by:

(A.2) **Lemma.** *Let L be an oriented $SL(2, \mathbf{Z})$ -framed link in S^3 . Suppose that L has two components K_A (with integral framing A) and K_B (with arbitrary framing B), and that K_B is a negatively oriented meridian for K_A . Let L' denote the framed link obtained from L deleting K_B and replacing K_A by K_{AB} (the same oriented curve but with framing AB). Then there is a homeomorphism between M_L and $M_{L'}$ which identifies the oriented cores of K_B and K_{AB} . (See Fig. 16(a).)*

Proof. Choose appropriately oriented solid tori V_A and V_B about K_A and K_B . Since A is integral, we can shrink V_B down to ∂V_A and then inside V_A where it appears (after surgery) as the core of V_A (preserving orientations) and we see Fig. 16(b). As usual, the B -labeled circle means cut out V_B and glue back by B , as in Fig. 16(c). Thus V_B is glued in by the composition AB . \square

Repeated application of this lemma shows that the lens space $L_A = V \cup -V$ can be obtained by surgery on the framed link L_α (Fig. 14(b)) for any $\alpha = (a_1, \dots, a_n)$ with $A = \pm ST^{a_1} S \dots T^{a_n} S$. (Note that $-SA = (T^{a_1} S) \dots (T^{a_n} S)$ and each $T^{a_i} S$ is integral.) The last statement in the lemma shows that the oriented core of V is a meridian for the last component of L_α (i.e. the one with framing a_n) with orientation determined by the sign in the expression for A . In particular:

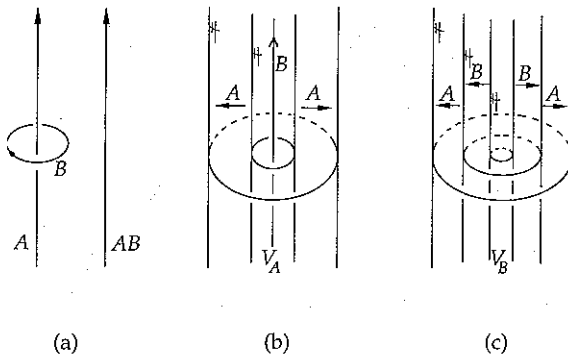


Fig. 16a-c

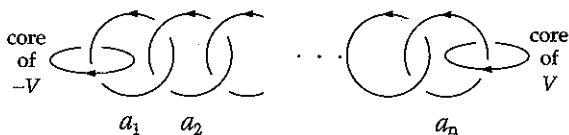


Fig. 17

(A.3) **Corollary.** *If $A = ST^{a_1}S \dots T^{a_n}S$, then the oriented lens space $L_A = V \cup -V$ can be obtained by surgery on the framed link L_α shown in Fig. 14(b) (where $\alpha = (a_1, \dots, a_n)$). Moreover, the oriented core of $-V$ is a positively oriented meridian of the first component of L_α (the one with framing a_1), and the oriented core of V is a negatively oriented meridian of the last component, as shown in Fig. 17.*

Torus bundles

The oriented torus bundle T_A is formed from L_A by removing the *solid cores* $(\frac{1}{2}B^2 \times S^1)$ of V and $-V$ and gluing the two resulting boundary components by the identity. (Of course, this is equivalent to identifying the ends of $T^2 \times I$ by $(x, 1) \mapsto (Ax, 0)$, viewing T^2 as $\partial B^2 \times S^1$.)

Observe that there is a 4-dimensional bordism Y_A from L_A to T_A (with boundary $-L_A \cup T_A$) obtained by adding a round 1-handle $I \times B^2 \times S^1$ along the cores of V and $-V$ in $I \times L_\alpha$ (i.e. attach $0 \times B^2 \times S^1$ to the solid core of V and $1 \times B^2 \times S^1$ to the solid core of $-V$).

The union X_α of W_α (see above) and Y_A along L_A is a 4-manifold with boundary T_A whose link picture is now easy to describe. W_α is given by the chain link L_α of Fig. 14(b). Adding the round 1-handle is equivalent to adding a 1-handle and a 2-handle as indicated in Fig. 18.

The 1-handle can be shown by an unknotted circle K with a dot on it (see [K2]), which can be surgered and replaced by the same circle with a 0-framing (indicating a 2-handle). The 2-handle is attached by a circle J which passes twice over the 1-handle (in opposite directions) and goes over both oriented cores with opposite orientations. Thus J closes the chain while K encircles it, giving the framed link $L = L_\alpha \cup J \cup K$ for T_A (Fig. 19). Note that L has $\mathbf{Z}/(n+1)\mathbf{Z}$ -fold symmetry (ignoring framings), and all linking numbers are -1 (excepting the “1-handle” K). In summary we have shown:

(A.4) **Theorem.** *If $A = ST^{a_1}S \dots T^{a_n}S$, then the oriented torus bundle T_A can be obtained by surgery on the framed link $L_\alpha \cup J \cup K$ shown in Fig. 19 (where $\alpha = (a_1, \dots, a_n)$), corresponding to the 4-manifold X_α bounded by T_A .*

(A.5) **Examples.** If $A = I$, then clearly $T_A = T^3$. Since $A = ST^0ST^0ST^0S$ in $SL(2, \mathbf{Z})$, the resulting link picture is shown in Fig. 20(a), which simplifies to the well

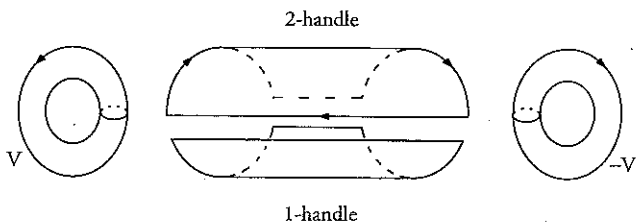


Fig. 18

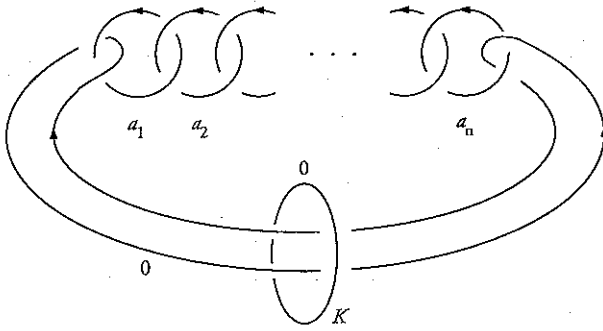


Fig. 19

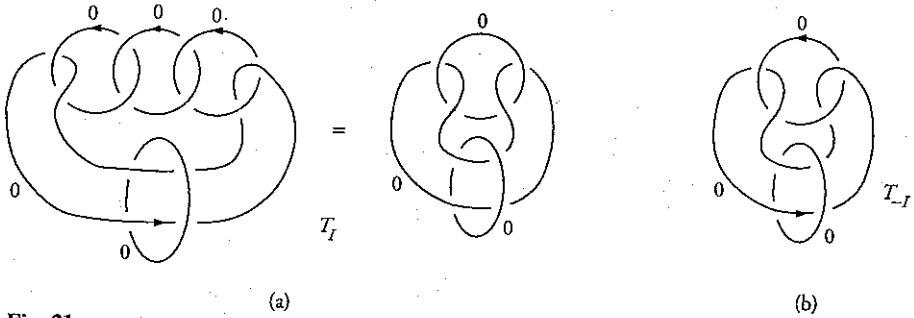


Fig. 21

known description of T^3 as 0-surgery on the Borromean rings. Compare this with the picture for T_A where $A = -I = ST^0S$, shown in Fig. 20(b).

It is easy to see the torus fibers in the link picture of T_A in Fig. 19. One fiber is evidently constructed from the twice punctured disk in S^3 spanned by K (the “1-handle”), as shown in Fig. 21.

We can move to the right across the clasp because the enclosed region is just $I \times T^2$ (for the same reason that the complement of the Hopf link in S^3 is $\mathbf{R} \times T^2$), and the meridian and longitude are switched (as by the action of the matrix S). Moving still further takes the left annulus (on the a_1 -framed circle) to the right annulus. This happens across the sewn in solid torus and amounts to a_1 Dehn twists along the meridian (as by the action of T^{a_1}). Of course if one moves all the way around, one gets the monodromy $-SA = T^{a_1}S \dots T^{a_n}S$.

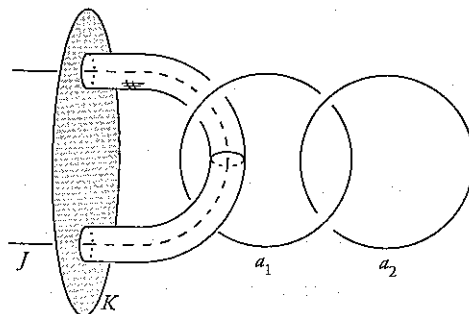


Fig. 20a, b

This picture of the fibration is important for understanding the *Lie spin structures* on T_A (i.e. structures which restrict to each fiber as the Lie group framing plus a normal vector to the fiber). There are exactly two Lie spin structures on T_A since $\pi_0 \text{SL}(2, \mathbf{R}) = 0$ and $\pi_1 \text{SL}(3, \mathbf{R}) = \mathbf{Z}/2\mathbf{Z}$ (they differ on a cross section of the bundle by the generator of $\pi_1 \text{SL}(3, \mathbf{R})$).

(A.6) **Lemma.** *The two Lie spin structures on T_A do not extend across the 4-manifold X_α . In terms of the framed link picture for X_α given in Figure 19, the two associated characteristic sublinks (representing integral homology classes dual to the second Stiefel-Whitney class) are L and $L-K$. (Recall that K corresponds to the "1-handle".) The self-intersection of either characteristic class is $\sum a_i - 2(n+1)$, while their Arf-invariants (in $\mathbf{Z}/2\mathbf{Z}$) are distinct.*

Proof. Consider any component λ of $L-K$, and let μ denote a meridian of λ (which lies in a fiber). The Lie spin structure obviously frames the tangent bundle of T_A restricted to μ with the tangent to μ , the normal to μ in the fiber, and the normal to the fiber. This framing is the one which does not extend across the normal disk that μ bounds (after surgery on λ). Hence λ belongs to the characteristic sublink. The formula for the self-intersection of the characteristic class follows by inspecting the linking matrix of L . Finally, it is easily verified that the Arf-invariant of $L-K$ is 0 or 1 according to whether n is even or odd, while the reverse is true for L .

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