

SMOOTHING LOCALLY FLAT IMBEDDINGS¹

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The fundamental imbedding problem for manifolds is to classify the imbeddings of an n -manifold into a q -manifold under ambient isotopy. We announce here that the differentiable and topological cases of this problem for differentiable manifolds are the same if $2q > 3(n+1)$ and $q \geq 8$.

This follows from Theorem 2 below which states that a locally flat imbedding of a compact differentiable manifold M^n into a differentiable manifold Q^q is ambient isotopic to a differentiable imbedding if $2q > 3(n+1)$ and $q \geq 8$. Since this ambient isotopy may be chosen arbitrarily close to the identity map, the set of differentiable imbeddings is dense in the set of locally flat imbeddings of M^n in Q^q .

It will then follow that two locally flat imbeddings of M^n into Q^q are ambient isotopic if they are homotopic; hence the classification problem reduces to a problem in homotopy theory.

THEOREM 1. *Let $f: B^n \rightarrow \text{int } Q^q$ be a locally flat imbedding of the unit n -ball into Q^q . Such an f always extends to $f: R^n \rightarrow \text{int } Q^q$. Let C^{n-1} be a compact differentiable submanifold of $\partial B^n = S^{n-1}$, and suppose that f is differentiable on a neighborhood of C^{n-1} in B^n . Let $q \geq 7$, $2q > 3(n+1)$ and $\epsilon > 0$. Then there exists an ambient ϵ -isotopy $F_t: Q^q \rightarrow Q^q$, $t \in [0, 1]$, satisfying*

- (1) $F_0 = \text{identity}$,
- (2) $F_t f$ is differentiable on $\text{int } B^n$ and on a neighborhood of C^{n-1} in B^n ,
- (3) $F_t = \text{identity}$ on $Q - N_\epsilon(f(B^n))$ and on $f(R^n - \text{int } B^n)$ for all $t \in [0, 1]$,
- (4) $|F_t(x) - x| < \epsilon$ for all $x \in Q^q$ and $t \in [0, 1]$. ($N_\epsilon(X)$ is the set of points within ϵ of X .)

THEOREM 2. *Let $f: M^n \rightarrow Q^q$ be a locally flat imbedding such that either $f(M^n) \subset \text{int } Q^q$ and $q \geq 7$ or $f^{-1}(\partial Q^q) = \partial M^n$ and $q \geq 8$. Let $2q > 3(n+1)$ and $\epsilon > 0$. Then there exists an ambient ϵ -isotopy $F_t: Q^q \rightarrow Q^q$, $t \in [0, 1]$, satisfying*

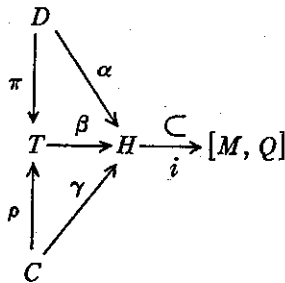
- (1) $F_0 = \text{identity}$,
- (2) $F_t f$ is a differentiable imbedding,

¹ This is an announcement of a portion of the author's dissertation at the University of Chicago written under Professor Eldon Dyer.

- (3) $F_t = \text{identity on } Q - N_\epsilon(f(M^n))$ for all $t \in [0, 1]$,
 (4) $|F_t(x) - x| < \epsilon$ for all $x \in Q^q$ and $t \in [0, 1]$.

The proof follows from Theorem 1 by considering the handlebody decomposition of M^n , and smoothing the imbedding of one handle at a time.

Only imbeddings of M^n into Q^q satisfying $f(M^n) \subset \text{int } Q^q$ or $f^{-1}(\partial Q^q) = \partial M^n$ will be considered. Let T be the set of equivalence classes of locally flat imbeddings of M^n into Q^q under equivalence by ambient isotopy. Similarly, let $D(C)$ be the set of equivalence classes of differentiable (combinatorial) imbeddings of M^n into Q^q under equivalence by ambient diffeotopy (ambient combinatorial isotopy). Let H be the homotopy classes of locally flat imbeddings of M^n into Q^q . H is a subset of $[M^n, Q^q]$, the homotopy classes of maps of M^n into Q^q . Then we have the following commutative diagram where the maps are the natural projections.



β is clearly onto for all n and q . Gluck has shown [1] that ρ and γ , and hence β and βi are isomorphisms for $q \geq 2n+2$. Haefliger has shown [2] that π is a monomorphism and that α is an isomorphism if $2q > 3(n+1)$.

It follows from Theorem 2 that π is also epimorphic if $2q > 3(n+1)$ and either $q \geq 7$ when $f(M^n) \subset \text{int } Q^q$ or $q \geq 8$ when $f^{-1}(\partial Q^q) = \partial M^n$. Then π and β are isomorphisms in this range of dimensions.

REFERENCES

1. H. Gluck, *Embeddings in the trivial range*, Ann. of Math. **81** (1965), 195-210.
2. A. Haefliger, *Plongements différentiables de variétés dans variétés*, Comment. Math. Helv. **36** (1961), 47-82.

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