CODIMENSION-TWO LOCALLY FLAT EMBEDDINGS HAVE NORMAL BUNDLES

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Let P^p and Q^q be topological manifolds of dimensions p and q, respectively, and let $i : P \to Q$ be a locally flat imbedding. If $p + 1 = q$ and $i(P)$ separates Q, then Brown has shown that $i(P)$ is flat, i.e., has a trivial normal bundle [1]. We have a similar theorem in codimension 2.

Theorem 1. *If $p + 2 = q$, $\exists$ a neighborhood E of $i(P)$ and a map $\pi : E \to i(P)$ which is a bundle ν with fiber R^2 and structural group $\mathscr{H}_0(R^2)$, the space (with the* CO*-topology) of homeomorphisms of R^2 which fix the origin. ν is unique up to ambient isotopy.*

If $\partial P \neq \varnothing$ and $\tilde{P} = P \cup (\partial P \times [0, 1))$, then i extends to a locally flat embedding $\tilde{\imath} : \tilde{P} \to Q$, which then has a normal bundle.

Since $\mathscr{H}_0(R^2)$ deforms to $O(2)$ (see [9]), $i(P)$ has a normal disk bundle.

Since $\mathscr{H}_0(R^2) \equiv \mathrm{TOP}_2 \simeq O(2) \simeq S^0 \times S^1$, and since there is a universal bundle $\mathrm{TOP}_2 \to E_{\mathrm{TOP}_2} \to B_{\mathrm{TOP}_2}$ with contractible total space E_{TOP_2}, we see that

$$\pi_i(B_{\mathrm{TOP}_2}) = \begin{cases} 0, & i \neq 1, 2, \\ Z_2, & i = 1, \\ Z, & i = 2. \end{cases}$$

The topological two-plane bundles over P are classified by maps $P \to B_{\mathrm{TOP}_2}$. Thus the oriented bundles over P are classified by $H^2(P; \pi_2(\mathrm{TOP}_2)) = H^2(P; Z)$.

If $q - p \geq 3$, it is known that there exist locally flat embeddings (in fact, PL embeddings) which have no normal disk bundles [3]. But if $q - p$ is large enough with respect to p, then normal bundles do exist [10]. Since normal bundles do not always exist, it would be nice to have normal block bundles and a good topological block bundle theory a la Rourke–Sanderson [11]. However, topological block bundles will have to differ somewhat from PL

block bundles because there are topological manifolds without handlebody structures in dimension four or five [7].

Section 1 contains notations and definitions, Section 2 has the main lemma, Theorem 1 is proved in Section 3, and Section 4 has an application on straightening handles.

1

Let R^n be euclidean n-space, $R^n_+ = \{x \in R^n | x_n \geq 0\}$, rB^n the ball of radius r in R^n, rS^{n-1} its boundary, and int B^n its interior.

$i : P \to Q$ is said to be locally flat if for each $i(p) \in Q$, there exists a neighborhood N such that $(N, N \cap i(P))$ is pairwise homeomorphic to (R^q, R^p), where p and q are the dimensions of P and Q. $i(P)$ is flat if i extends to an embedding $i : P \times R^{q-p} \to Q$; i.e., $i(P)$ has a trivial normal bundle. If $\partial P \neq 0$ and $i(P) \subset \operatorname{int} Q$, i is locally flat (flat) if i extends to a locally flat (flat) embedding of $P \cup$ (open collar on ∂P). This condition is equivalent to $(N, N \cap i(P))$ being pairwise homeomorphic to (R^q, R^p_+) for $p \in \partial P$. If i is proper ($i^{-1}(\partial Q) = \partial P$), then it is locally flat if $(N, N \cap i(P))$ is homeomorphic to (R^q_+, R^p_+) for $p \in \partial P$.

Let $\mathscr{H}_Y(X)$ be the space (with the compact open topology) of homeomorphisms of X which fix Y pointwise. A basis for the neighborhoods of the identity consists of sets of the form $N(C, \varepsilon) = \{h \in \mathscr{H}_Y(X) | d(h(x), x) < \varepsilon$ for all $x \in C\}$ for all compact sets C and $\varepsilon > 0$.

The following statements can be found in [2]. If L is locally flat in M and both L and M are compact or interiors of manifolds with boundaries, then $\mathscr{H}_L(M)$ is locally contractible. Let J and K be compact subsets with $J \subset \operatorname{int} K \subset M$. Given $\varepsilon > 0$, there exists $\delta > 0$ such that if $h_0 \in N(K, \delta)$, then $\exists$ a canonical isotopy $h_t : M \to M$, $t \in [0, 1]$ with $h_1 | J =$ identity, $h_t \in N(K, \varepsilon)$, and $h_t = h_0$ outside K.

Let $g_t : L \to M$ be a locally flat isotopy; i.e., $G = (g_t, \mathrm{id}) : L \times I \to M \times I$ is locally flat in a level-preserving way. Then g_t extends to an ambient isotopy of M. Furthermore, if g_t is small, then so is the extension and it is supported on a neighborhood of $G(L \times I)$.

We will say that A is a weak deformation retract of X if $\exists$ a homotopy $H_t : X \to X$, $t \in [0, 1]$, with $H_0 =$ identity, $H_1(X) \subset A$, and $H_t(A) \subset A$.

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The main new idea in Theorem 1 is contained in the next lemma. Then by applying it, using standard techniques, we get Theorem 1.

Lemma. *Let $h : M \times R^2 \to M \times R^2$ be a homeomorphism with $h | M \times 0 =$* id, *where M is a compact manifold. Then h is isotopic to a fiber* *preserving homeomorphism. Specifically,* $\exists\, h_t : M \times R^2 \to M \times R^2, t \in [0, 1]$ *with* $h = h_0$, $h_t|M \times 0 = \mathrm{id}$ *for all t and* $h_1(z \times R^2) = z \times R^2$ *for all* $z \in M$.

Proof. We will isotope h so that it becomes close enough to a rotation so as to apply local contractibility to move h to the rotation (= homeomorphism $\rho : M \times R^2 \to M \times R^2$ with $\rho(z, y) = (z, \rho_z(y))$).

Using polar coordinates for R^2, we describe points of $M \times R^2$ by triples (z, θ, t), $z \in M$, $(\theta, t) \in R^2$; h can be written $h(z, \theta, t) = (h_1(z, \theta, t), h_2(z, \theta, t), h_3(z, \theta, t))$. We will isotope h_1 and h_3 so that they are small enough (on $M \times KB^2$ for some $K > 0$) and h_2 so that it is close enough to a rotation.

Step 1. There is a well-known argument for making h_3 small. Let $C_t = M \times tS^1$ and $D_t = M \times tB^2$. We can assume by squeezing that $D_{1-\varepsilon} \subset h(D_1) \subset D_{1+\varepsilon}$ and $h(D_2) \subset D_{2+\varepsilon}$ (see Figure 1). We need to move $h(C_2)$ out between $C_{2-\varepsilon}$ and $C_{2+\varepsilon}$, without moving $h(C_1)$. For a small enough $r > 0$, $h(C_r) \subset D_{1-\varepsilon}$. We use the radial structure given by h to obtain a homeomorphism $f : M \times R^2 \to M \times R^2$ which slides $h(C_r)$ to $h(C_1)$ and fixes $h(C_2)$ so that $h(D_1) \subset f(D_{1-\varepsilon}) \subset h(D_2) \subset D_{2+\varepsilon}$. Then we use the radial structure given by f to slide $h(C_2)$ close enough to $C_{2+\varepsilon}$ so that it is between $C_{2-\varepsilon}$ and $C_{2+\varepsilon}$. Now $D_{1-\varepsilon} \subset h(D_1) \subset D_{1+\varepsilon} \subset D_{2+\varepsilon} \subset h(D_2) \subset D_{2+\varepsilon}$.

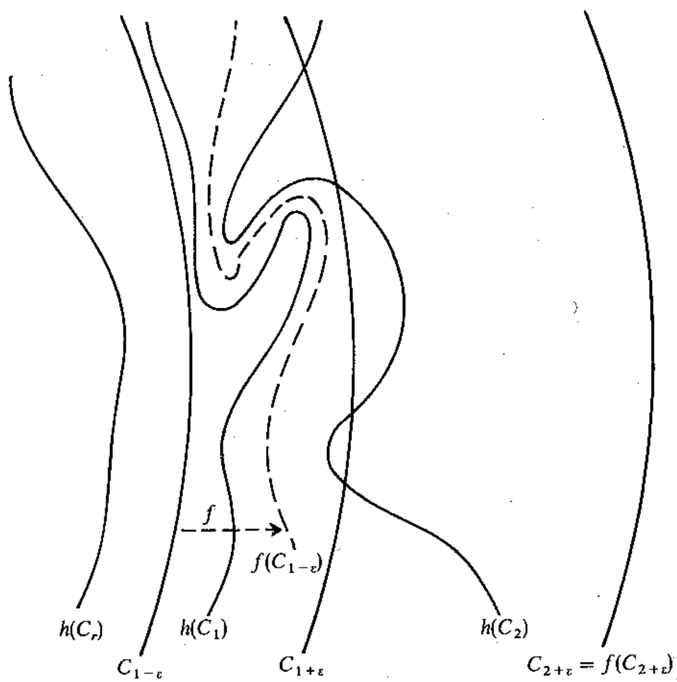

Figure 1

We may iterate this process countably many times so that $D_{t_i - \varepsilon_i} \subset h(D_{t_i}) \subset D_{t_i + \varepsilon_i}$, where the sequence $K = t_1 > t_2 > \cdots > 0$ forms an arbitrarily fine subdivision of $[0, K]$ and the ε_i are as small as desired. h_3 is now small on $M \times KB^2$. Call this new homeomorphism h'.

Step 2. Let $h''(x, \theta, t) = (h_1'(x, \theta, \delta t), h_2'(x, \theta, \delta t), (1/\delta)h_3'(x, \theta, \delta t))$. We claim that if δ is small enough, then h_1'' is arbitrarily small on $M \times KB^2$, and that h_3'' is still small enough. The first follows because h' is continuous and is the identity on $M \times 0$, and the second follows if $\{t_i\}$ is fine enough.

Step 3. We will isotope h'' so that on $M \times KB^2$ it is close enough to the rotation ρ defined by $\rho(z, \theta, t) = (z, \theta + h_2''(z, 0, t), t)$. Let $g = \rho^{-1}h''$. If h_1'' and h_3'' are small enough on $M \times KB^2$, then g will be small enough near $M \times 0 \times (0, K]$. In particular, given $\varepsilon > 0$, $\exists\, \delta > 0$ such that $g(z, \theta, t) \in M \times [-\varepsilon, \varepsilon] \times (0, \infty)$ for $\theta \in [-\delta, \delta]$ and $t \in (0, K]$ (see Figure 2).

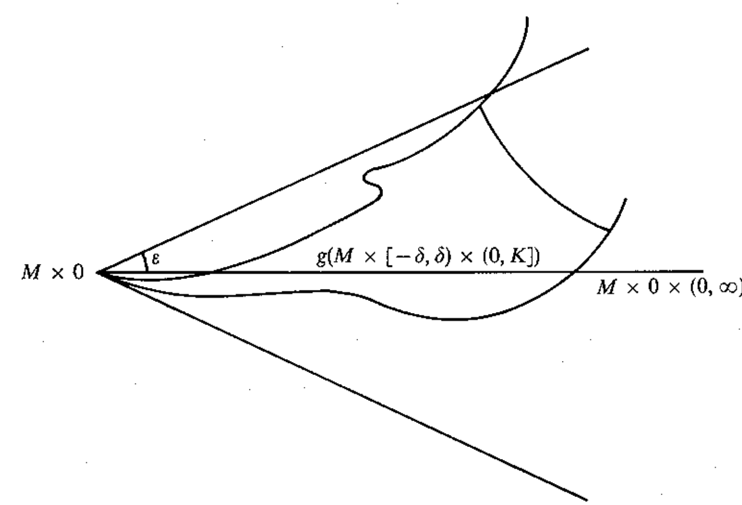

Figure 2

Let $\hat{g}$ be a certain finite covering of g:

$$\begin{array}{ccc} M \times R^2 & \xrightarrow{\hat{g}} & M \times R^2 \\ \downarrow \lambda & & \downarrow \lambda \\ M \times R^2 & \xrightarrow{g} & M \times R^2. \end{array}$$

Let $\lambda_1 : S^1 \to S^1$ be defined by $\lambda_1(\theta) = n\theta(2\pi)$, the n-fold covering map. Let λ_2 be an approximation to λ_1 with the properties that $\lambda_2 = \mathrm{id}$ on $[-\varepsilon, \varepsilon]$ and $\lambda_2 = \lambda_1$ outside $[-2\varepsilon, 2\varepsilon]$, where we need to have chosen ε small enough. Finally, let $\lambda = \mathrm{id}$ on $M \times 0$ and $\lambda(z, \theta, t) = (z, \lambda_2(\theta), t)$ for $t > 0$. Then it is easy to see that g lifts to a homeomorphism $\hat{g}$ with $g = \hat{g}$ on the wedge $M \times [-\delta, \delta] \times (0, K]$. Therefore, g is isotopic to $\hat{g}$, via an isotopy fixing $M \times 0$.

It is not hard to verify that we can make $\hat{g}$ arbitrarily small on $M \times KB^2$ by taking n large enough. By local contractibility, $\hat{g}$ is isotopic to the identity on $M \times KB^2$, and hence on $M \times R^2$.

Since $\rho^{-1}h'' = g$ is isotopic to the identity, it follows that h'', and therefore h, is isotopic to the rotation ρ (fixing $M \times 0$ throughout), finishing the proof of the lemma.

REMARK 1. Suppose h was fiber preserving on a neighborhood of a subset L of M. Then we can find an isotopy h_t with the additional property of being fiber preserving on a smaller neighborhood of L in M. The isotopies constructed in steps 1, 2, and 3 are all clearly fiber preserving if h is, except possibly the isotopy constructed using local contractibility. $\hat{g}$ is small on each fiber over the neighborhood of L, so we isotope $\hat{g}$ to the identity on each fiber separately. This is done in a continuous way (using local contractibility) so we get a fiber-preserving isotopy $\hat{g}_t : M \times R^2 \to M \times R^2$, with $\hat{g}_0 = \hat{g}$ and $\hat{g}_1 =$ identity on a neighborhood of L (see Section 1 on local contractibility). $\hat{g}_1$ can still be small enough to be isotopic to the identity elsewhere, using the relative form of local contractibility.

REMARK 2. If M is not compact, then we may find h_t with h_1 fiber preserving on an arbitrarily large compact submanifold. All the steps in the lemma rely on compactness. For example, in step 3, g may wind $M \times R^2$ around $M \times 0$ more and more as one approaches the open ends of M. But on a compact subset, this winding is bounded so some finite cover $\hat{g}$ is small enough.

REMARK 3. We can require that h_t have compact support if we only require h_1 to be fiber preserving near $M \times 0$. Specifically, we get $h_t =$ id outside a compact set and $h_1(z \times KB^2) \subset z \times KB^2$ for z in a compact subset of M. One just checks that all constructions can be done in a neighborhood of $C \times 0$, C compact in M.

REMARK 4. The lemma still holds when h is only an embedding (with $h|M \times 0 =$ id). This follows because we have noted (in Remark 3) that all constructions are done in a neighborhood of $M \times 0$. (If $\partial M \neq \emptyset$, we need to assume that h is a proper embedding.)

3

Proof of Theorem 1. Let P_0 be an open submanifold of P with a normal bundle ν_0 over $i(P_0)$ in Q. Let R^p be a coordinate patch in P with $i(R^p)$ flat in Q, and let $M = P_0 \cap R^p$. Suppose at first that $\nu_0|M$ is trivial so that there is an embedding $\alpha : M \times R^2 \to Q$ with $\alpha(M \times R^2) = E(\nu_0|M)$. Since $i(R^p)$ is flat, let $\beta : M \times R^2 \to Q$ be the flat structure on $i(M)$. We can assume that $\beta(M \times R^2) \subset \alpha(M \times R^2)$, so as to consider $\alpha^{-1}\beta : M \times R^2 \to M \times R^2$. By the lemma (including the remarks, particularly 4), $\alpha^{-1}\beta = h$ is isotopic (with compact support) to an embedding h_1 with $h_1(z \times KB^2) \subset z \times KB^2$ for all z in some large compact subset of M. Then

$$\beta_t = \begin{cases} \alpha h_t & \text{on } M \times R^2, \\ \beta & \text{on } (R^p - M) \times R^2 \end{cases}$$

is an isotopy fixing $R^p \times 0$ with $\beta_1(z \times KB^2) \subset \alpha(z \times KB^2)$ for z in the above compact subset of M. By restricting β_1 to $R^p \times \operatorname{int} KB^2$ and then applying the microbundles-are-bundles argument in [8], we get that $\beta_1(z \times R^2) = \alpha(z \times R^2)$ for the above z. Thus, taking appropriate refinements of P_0 and R^p, say $\tilde{P}_0$ and $\tilde{R}^p$, we have extended ν_0 to a bundle over $i(\tilde{P}_0 \cup \tilde{R}^p)$.

Now suppose $\nu_0|M$ is not trivial. Then we cover a large enough compact subset of M with open sets $M_1, \ldots, M_k$ on which ν_0 is trivial. We proceed as above with M_1. Then for $M_2, \ldots, M_k$, we use the relative form of the lemma in Remark 1, to make $\alpha^{-1}\beta$ fiber preserving over large compact subsets of $\bigcup_{i=1}^{j} M_i$, $j = 2, 3, \ldots, k$. So as before ν_0 extends over $i(\tilde{P}_0 \cup \tilde{R}^p)$.

If P is compact (with $\partial P \neq \emptyset$), we construct ν coordinate patch by coordinate patch, as above, using appropriate refinements; when P is open, paracompactness is sufficient for the same construction to work. This sort of argument is well known and we omit further details.

If $\partial P \neq \emptyset$, and $i(P) \subset \operatorname{int} Q$, we add an open collar to ∂P, extend i, and proceed as above. If i is proper $[i^{-1}(\partial Q) = \partial P]$, then we construct ν on ∂P in ∂Q, extend to collars, and continue as above.

4

Proposition. *$\mathscr{H}_{R^k}(R^n)$ is a weak deformation retract of $\mathscr{H}_{R^k - B^k}(R^n)$.*

Proof. $\mathscr{H}_{R^k - B^k \cup 0}(R^n)$ is clearly a strong deformation retract of $\mathscr{H}_{R^k - B^k}(R^n)$. But by adding a point at infinity, compactifying each homeomorphism, and removing the origin, we see that there is a homeomorphism $\Omega : \mathscr{H}_{R^k - B^k \cup 0}(R^n) \to \mathscr{H}_{B^k}(R^n)$. Now $\mathscr{H}_{B^k}(R^n)$ deforms to $\mathscr{H}_{R^k}(R^n)$ by Theorem 1 of [8]. Applying Ω^{-1} to this deformation shows that $\mathscr{H}_{R^k}(R^n)$ is a weak deformation retract of $\mathscr{H}_{R^k - B^k \cup 0}(R^n)$.

Furthermore, if $h_0 \in \mathscr{H}_{R^k - B^k}(R^n)$ and h_t is the deformation taking h into $\mathscr{H}_{R^k}(R^n)$, then h_t has compact support. This follows directly from Kister's proof.

Theorem 2. *Let $h : B^k \times R^2 \to B^k \times R^2$ be a homeomorphism with $k \neq 1$ and $h =$ identity on $S^{k-1} \times R^2$. Then h is isotopic to the identity, fixing $h|\partial$.*

Proof. We can assume that h = identity on $(B^k - \frac{1}{2}B^k) \times R^2$. Then the interior of $B^k \times R^2$ is R^{k+2}, and from the proposition above, it follows that h is isotopic (rel ∂) to a homeomorphism $\bar{h}$ which fixes $B^k \times 0$. It is now easy to see that the method of proof of the lemma works here to give the desired isotopy of $\bar{h}$ to the identity (rel ∂). [Since $\bar{h}|\partial$ = identity, it is not necessary to alter it on the ∂ in steps 1 or 2; since $k \neq 1$ and $\pi_k(\mathscr{H}_0(R^2)) = 0$, no alteration is necessary in step 3 either.]

Now consider the problem of straightening 3-handles (see [4], [5], and [6]). Let $h : B^3 \times R^n \to V$ be a homeomorphism, PL on the boundary, onto a PL manifold V. We wish to straighten h, i.e., find an isotopy h_t, $t \in [0, 1]$, with h_1 PL and $h_t|\partial$ PL. If $n \geq 3$, Sullivan has shown that $h|\partial$ extends to a PL homeomorphism $\bar{h} : B^3 \times R^n \to V$. However, $\exists$ nonstraightenable 3-handles for $n \geq 2$, so $\exists$ homeomorphisms $g = \bar{h}^{-1}h : B^3 \times R^n \to B^3 \times R^n$, identity on ∂, which are not isotopic to PL homeomorphisms rel ∂, when $n \geq 3$.

On the other hand, when $n = 2$, we have just seen in Theorem 2 that any homeomorphism $g : B^3 \times R^2 \to B^3 \times R^2$, g = identity on ∂, is isotopic to the identity, rel ∂. Thus, since $\exists$ nonstraightenable 3-handles $h : B^3 \times R^2 \to V$, we see that V cannot be PL homeomorphic to $B^3 \times R^2$ rel ∂. Therefore, $B^3 \times R^2$ has more than one PL structure rel ∂ (in fact, two).

So nonstraightenable 3-handles $h : B^3 \times R^n \to V$, $n \geq 2$, arise in two ways: if $n = 2$, V is not PL homeomorphic to $B^3 \times R^n$ rel ∂ so of course h cannot be straightened; if $n \geq 3$, V is PL homeomorphic to $B^3 \times R^n$ rel ∂, but the homeomorphism h is bad.

We say that two PL structures on a manifold are equivalent up to isotopy (homotopy) if the identity is isotopic (homotopic) to a PL homeomorphism.

From [4], [5], [6], and the above, the PL structures (rel ∂) up to isotopy on $B^3 \times R^n$, $n \geq 2$, correspond to $H^3(B^3 \times R^n, \partial; Z_2) = Z_2$ and the PL structures (rel ∂) up to homotopy correspond to Z_2 if $n = 2$, 0 if $n \geq 3$.

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