

## A POTENTIAL SMOOTH COUNTEREXAMPLE IN DIMENSION 4 TO THE POINCARÉ CONJECTURE, THE SCHOENFLIES CONJECTURE, AND THE ANDREWS-CURTIS CONJECTURE

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WE DESCRIBE a homotopy 4-sphere  $\Sigma^4$ , built with the usual zero and 4-handle and two 1-handles and two 2-handles (Figure 28). Of course  $\Sigma^4$  is homeomorphic to  $S^4$  [Freedman] but considerable effort has not led to a proof that  $\Sigma^4$  is diffeomorphic to  $S^4$ .  $\Sigma^4$  has the following virtues:

(A) Although it is easy to construct smooth homotopy 4-spheres (e.g. the Gluck construction on knotted 2-spheres or via non-trivial presentations of the trivial group), this is the only (except  $S^4$ ) example we know without 3-handles and with so few handles altogether.

(B) The presentation of the trivial group arising from  $\Sigma^4$  (see §2) is  $\{x, y | xyx = yxy, x^5 = y^4\}$ ; it is easy to show that this group is trivial, but it seems difficult to do so using Andrews-Curtis moves ([1] or [10] 5.1).

(C) Let  $\Sigma_0$  be  $\Sigma$  without the 4-handle; then we can add two 2-handles and two 3-handles and a 4-handle to get (smoothly)  $S^4$  (see §2). Applying the topological Schoenflies theorem to  $\partial\Sigma_0$  in  $S^4$ , we see directly that  $\Sigma_0$  is homeomorphic to  $B^4$ .

(D)  $\partial\Sigma_0$  is an interesting smooth  $S^3$  in  $S^4$ . The smooth Schoenflies conjecture is unsettled in dimension 4 and  $\partial\Sigma_0$  is a good test case. So in §4, we give a critical level imbedding of  $\partial\Sigma_0$  in  $S^4$  (Fig. S1-S11). Scharlemann [12] has used critical level imbeddings to prove the conjecture for genus 2 imbeddings; this one is genus 51.

(E)  $\Sigma_0$  is the result of the Gluck construction on a knot  $K$  in  $S^4$  (Fig. 16).  $K$  is constructed from two distinct ribbons for the  $8_9$  knot (see [11]).

$\Sigma$  was first defined as the double cover of a certain exotic  $RP^4$  of Cappell and Shaneson [7]. It was built by decomposing  $RP^4$  into a 2-disk bundle over  $RP^2$  and the non-trivial 3-disk bundle over  $S^1$ , and then replacing the latter by a punctured 3-torus bundle over  $S^1$  with

monodromy  $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$ . We thought we proved (in [4]) that the double cover  $\Sigma$  of this

exotic  $RP^4$  was diffeomorphic to  $S^4$ . However Iain Aitchison and J. H. Rubenstein ([2], [3]) discovered an error (the last sentence on page 77 of [4] states that a framing is zero when it should be odd). We actually proved that  $\Sigma^4$  is the Gluck construction on a knotted 2-sphere in  $S^4$  (this was discussed in Remarks 2 and 3 of [4]). As mentioned above, we are still unable to prove that  $\Sigma^4$  is diffeomorphic to  $S^4$ . In the meantime however, Fintushel and Stern [8] have constructed, by different methods, an exotic  $RP^4$  whose double cover is  $S^4$ . Note that both these exotic  $RP^4$ 's are homotopy  $RP^4$ 's which are  $s$ -cobordant to  $RP^4$ , and then homeomorphic to  $RP^4$  by Freedman's recent proof of the topological  $s$ -cobordism theorem for many fundamental groups including  $Z/2$ .

After some definitions in §1, we begin in §2 with a handlebody description of  $\Sigma^4$  from [4, Fig. 5]. We simplify this handlebody presentation by sliding handles over other handles and by handle cancellations and births to get the properties of  $\Sigma^4$  mentioned above in (A), (B), (C) and (E). It is worth remarking that it is usually hard to see how to add a cancelling pair of handles (a birth) in any useful way; this is done with a (2-3)-pair in Fig. 19 and later in Fig. 35.

Some problems are suggested by this work:

- (1) Does every homotopy 4-ball with boundary  $S^3$  smoothly imbed in  $S^4$ ?
- (2) Do the results of this paper hold for the other fake  $RP^4$ 's of [3]?

- (3) Is  $S^4$  the double of  $\Sigma_0$ ?
- (4) Classify the different ribbons for a ribbon knot.

We would like to thank Andrew Casson for helping us in simplifying the fundamental group calculations.

§1

First, here are some definitions and notation. 1-handles were described in [4] by drawing their attaching maps, i.e. by drawing two 3-balls in  $S^3 = \partial B^4$ . They may also be described by an unknotted circle with a dot on it to distinguish it from the attaching map of a 2-handle (see [5], p. 260). This dotted circle means: delete from  $B^4$  the thickened, unknotted 2-ball which the circle bounds, obtaining  $B^4 \cup$  1-handle. Any arc going through the dotted circle goes over the 1-handle. This notation has the virtue that replacing the dot by a zero is the same as replacing the 1-handle by a 2-handle, i.e. surgering  $S^1 \times B^3$  to  $S^2 \times B^2$ . This notation can be extended to the case where the dotted circle is a slice knot and we are meant to delete the thickened slice from  $B^4$ . Since a slice knot has more than one slice (just connect sum any knotted  $S^2$ ) it is necessary to somehow specify the slice to be used; when the dotted knot is ribbon, this may be done by carrying along dotted arcs indicating which ribbon moves on the knot give the ribbon disk.

§2

Figure 1 is our starting point; it is a picture of the zero, one and two-handles of  $\Sigma^4$ , the double cover of  $Q$  (the fake  $RP^4$ ). It comes from Fig. 5 of [4] by changing the notation for the one-handle  $a_*$  (which is attached to the balls of the origin and at  $\infty$ ) to the "dotted circle", and by adding the last 2-handle  $\gamma$  with framing-1 (not 0 as erroneously claimed). The framing of the  $\alpha_i$  and  $\beta_i$ ,  $i = 1, 2, 3$ , are given by the normal vector field lying in the plane of the paper. There are three 3-handles and a 4-handle which are not drawn.

Before  $\gamma$  is added, but after the 3-handles are added, the boundary is  $S^1 \times S^2$  and what is missing is  $B^2 \times S^2$ , i.e.  $\gamma$  and the 4-handle. The core,  $0 \times S^2$ , is the "knotted" 2-sphere  $K$  in  $\Sigma^4$ . If we perform the Gluck construction on  $K$  we get  $S^4$ , since removing  $K \times B^2$  means removing  $\gamma$  and the 4-handle, and replacing  $K \times B^2$  with a twist means adding  $\gamma$  with 0-framing and the 4-handle; but this was what was proven in [4] to be  $S^4$ . Conversely then,  $\Sigma^4$  is the Gluck construction on  $K$  in  $S^4$ .

We can see  $K$  by seeing  $0 \times S^2$  in  $S^1 \times S^2$ , the boundary before  $\gamma$  is added. Since the  $S^1 \times S^2$  is just  $S^1 \times \partial T_0^3$  where  $T_0^3$  is the punctured 3-torus, we can take  $0 \times S^2$  to be  $\partial[-1, 1]^3$

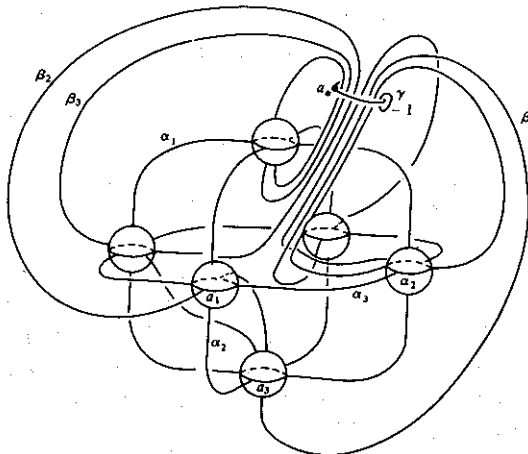


Fig. 1.

followed by surgery on  $a_1, a_2, a_3$  and  $\alpha_1, \alpha_2, \alpha_3$ , the three 1-handles and 2-handles of  $T_0^3 \times [-1, 1]$ . Thus an equator for  $0 \times S^2 = K$  is a circle close to and parallel to the dotted circle representing  $a_*$ ; call this equator  $A_*$ . One of the hemispheres it bounds in  $K$  is just the obvious flat 2-ball bounded by  $a_*$ , call it  $H_-$ ; the other,  $H_+$ , is harder to see because it goes over the handles.  $H_-$  looks unknotted, but it isn't. If we cancel  $a_1, a_2$ , and  $a_3$  with  $\beta_1, \beta_2 - \beta_1$ , and  $\beta_3$ , then  $H_-$  turns into a ribbon disk for the  $8_9$  knot in  $S^3$ , Fig. 16, as we shall see later. Similarly, if we turned our handlebody upside down and cancelled  $a_*$  with  $\gamma$  (with framing 0 so as to see  $K$  in  $S^4$ ) and  $\alpha_1, \alpha_2$ , and  $\alpha_3$  with the 3-handles, then  $H_+$  would become a different ribbon disk for  $8_9$ , (Figs. 15, 16).

We want to begin simplifying the description in Fig. 1 of  $\Sigma^4$ . It turns out that  $\alpha_2$  and  $\alpha_3$  are cancelled by 3-handles, and if we show this first then in later versions of Fig. 1 we will not have to draw  $\alpha_2$  and  $\alpha_3$ . To see that  $\alpha_2$  and  $\alpha_3$  are cancelled, go through the steps in [4], but carrying along  $a_*$  and  $\gamma$ . In going from Fig. 7 to Fig. 8 of [4],  $a_*$  (as a dotted circle) will be pulled over 1-handles, and then as  $a_1, a_2$  and  $a_3$  are cancelled,  $a_*$  becomes a knot ( $8_9$  in fact). Remember from the introduction that a knotted dotted circle must have a preferred ribbon disk which is to be removed (the same as adding a 1-handle if the dotted circle is unknotted). The three circles,  $\alpha_1, \alpha_2$  and  $\alpha_3$ , in Fig. 13 of [4] become the unlink tangled with  $a_*$ ; we get Fig. 2.

If we slide  $\alpha_2$  over  $\alpha_1$  it also becomes parallel to  $\alpha_1$ . Then slide both  $\alpha_2$  and  $\alpha_3$  over  $\alpha_1$  so that they become unlinked (with zero framings). Then  $\alpha_2$  and  $\alpha_3$  contribute two copies of  $S^1 \times S^2$  to the boundary which is  $S^1 \times S^2 \# S^1 \times S^2 \# S^1 \times S^2$ . Thus  $\alpha_2$  and  $\alpha_3$  must be cancelled by 3-handles. To put it another way, in Fig. 1,  $\alpha_2$  and  $\alpha_3$  may be slid over the other handles (excepting  $a_*$  and  $\gamma$ ) until they become an unlink separated from the rest.

Now we change the notation for the 1-handles in Fig. 1, switching to dotted circles. This is shown in Fig. 3, where  $\alpha_1$  is not drawn even though it is there. The attaching circles in Fig. 3 are drawn with care for their framings are determined by the push off which lies parallel in the figure. Thus the push off for  $\beta_1$  has one left crossing which, when straightened out, gives one full left twist with its pushoff, hence framing  $-1$ . Similarly  $\beta_2$  has framing  $+2$  and  $\beta_3$  gets  $-1$ .

To go from Fig. 3 to Fig. 4, we do some simple isotopies and slide  $\beta_1$  over  $a_1$  and  $\beta_3$  over  $a_3$ , thereby changing their framings. Next, slide  $\beta_1$  over  $\beta_2$  as indicated in Fig. 5, cancel  $a_1$  and  $\beta_2$  by erasing them, and isotop to Fig. 6.

We continue in Fig. 7 by sliding  $\alpha_1$  and  $\beta_1$  twice (algebraically zero) over  $\beta_3$  and then cancelling  $\beta_3$  and  $a_3$  by erasing them. Further isotopies give Figs. 8 and 9.

Now we want to describe  $K$  as the union of two ribbons for the  $8_9$  knot. Remember that Fig. 9 plus a 3 and 4-handle is  $\Sigma^4$ , and that if we remove the 4-handle and  $\gamma$  then we have the knot complement  $S^4 - (K \times \text{int } B^2) = \Sigma^4 - (K \times \text{int } B^2)$ . Also recall that a copy of  $K$  consists of a hemisphere  $H_-$  equal to the obvious disk bounded by the equator  $A_* = a_*$  of  $K$ , and a hemisphere  $H_+$  which is harder to see. Shrink  $\beta_1$ , turning  $a_*$  into a ribbon knot, as in Fig. 10;

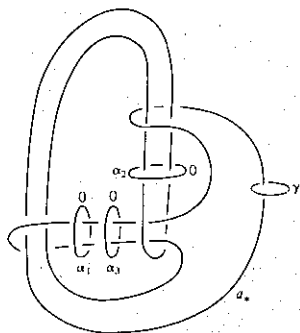


Fig. 2.

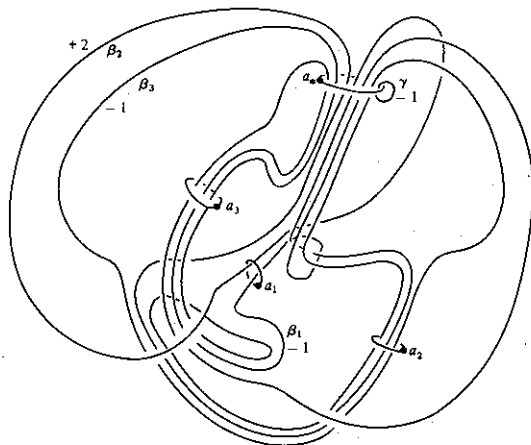


Fig. 3.

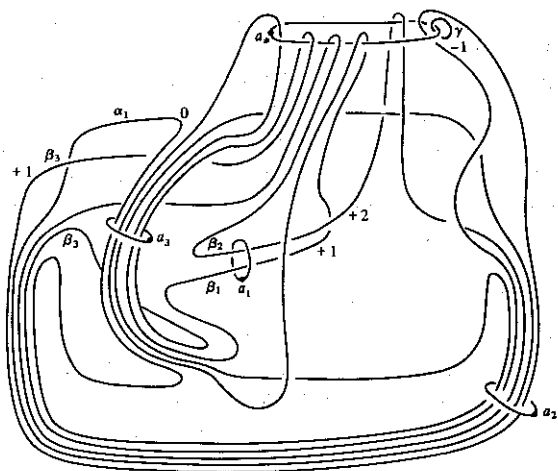


Fig. 4.

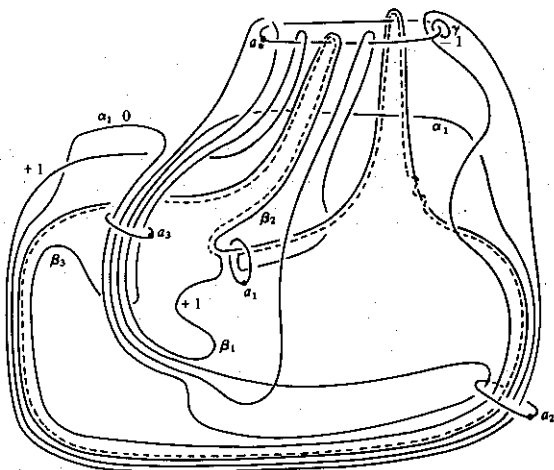


Fig. 5.

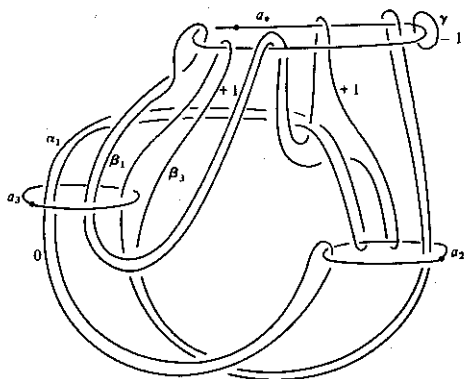


Fig. 6.

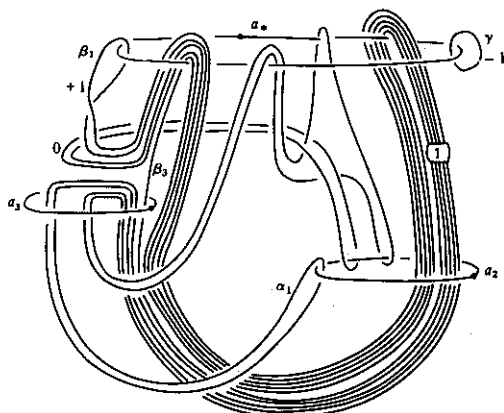


Fig. 7.

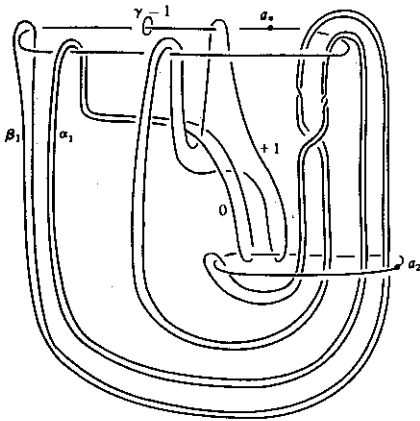


Fig. 8.

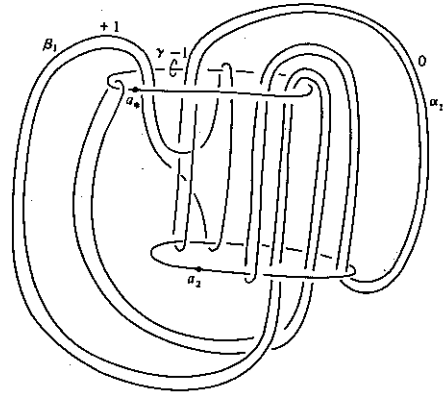


Fig. 9.

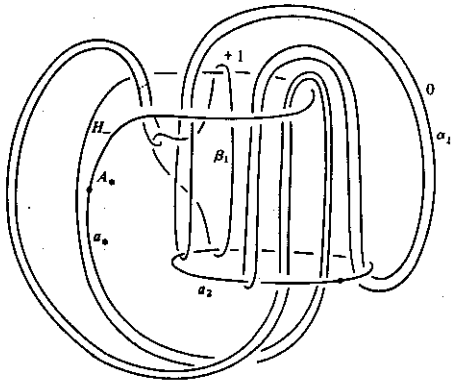


Fig. 10.

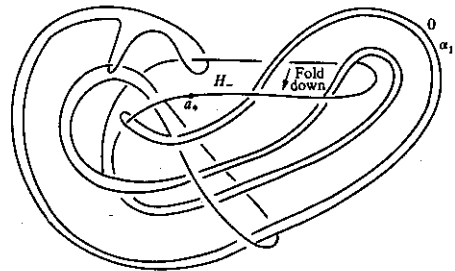


Fig. 11.

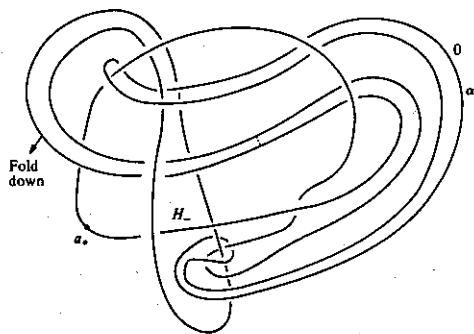


Fig. 12.

the ribbon move is indicated by a dotted arc which can be isotoped along the band to put it in a convenient position. Now cancel  $a_2$  with  $\beta_1$  by sliding over  $\beta_1$  eight times, obtaining Fig. 11.

A sequence of isotopies (Figs. 12, 13 and 14) end with the  $8_9$  knot, a dotted line indicating the ribbon move, and a 2-handle. The reader can check that this is the same knot as in Fig. 2 with  $\alpha_1$  in the same position and the dotted arc giving the same ribbon move. The other ribbon is given by  $\alpha_1$ . Figure 15 has a movie of  $K$  in  $S^4$  and simultaneously a handlebody description of  $S^4 - K$ . We begin at the left with an unlink, and two 3-handles and a 4-handle (one 3-handle cancels the 4-handle) which are not drawn. The middle picture is a slice of  $K$  the

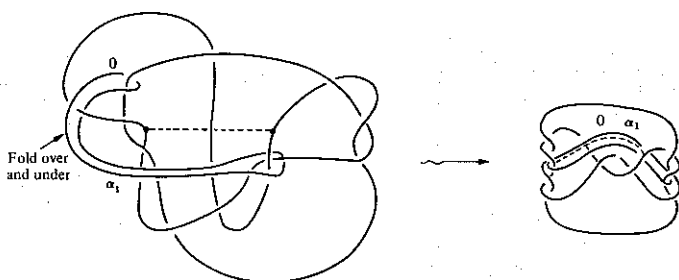


Fig. 13.

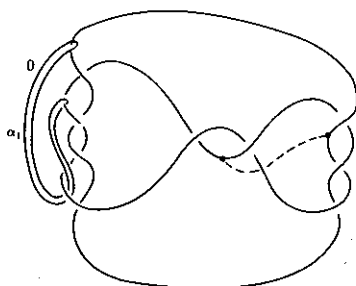


Fig. 14.

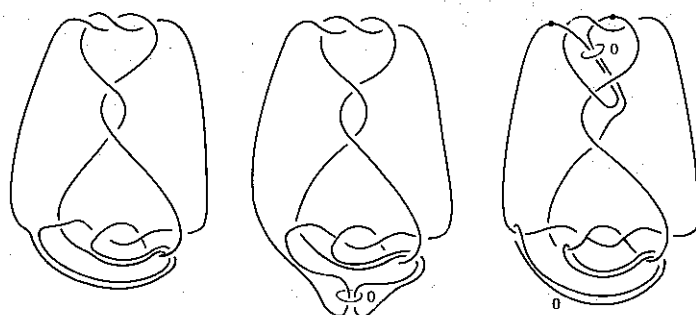


Fig. 15.

$8_9$  knot just after a ribbon move has occurred and a 2-handle has been added to  $S^4 - K$ . The left hand picture is the unlink again just after another ribbon move and another 2-handle has been added. Finally the two 1-handles of  $S^4 - K$  are given by dotting the unlink (see also [5]).

Figure 16 is a more artistic way of drawing the  $8_9$  knot with its ribbon moves; this picture makes obvious an orientation reversing involution of the  $8_9$  knot which switches the ribbon moves (just rotated by  $\pi$  and reflect).

We redraw the left hand picture in Fig. 15 and add  $\gamma$  to get Fig. 17; together with a 3 and 4-handle it gives  $\Sigma^4$ . Cancel  $b_2$  and  $\gamma$  to get Fig. 18. (This figure is actually  $-\Sigma^4$ ).

Next, we add a cancelling 2-3 pair, with the 2-handle added to a circle parallel to the twist (Fig. 19). This circle must be trivial on the boundary,  $S^1 \times S^2$ , of the link in Fig. 18. To check this, draw a circle  $\mu$  parallel to the "twist" circle, blowing up a  $-1$  circle and changing both  $+1$  framings to 0, slide  $\mu$  over the  $-1$  circle so that it links only the  $-1$  circle, surger the 1-handle to a 2-handle by replacing the dot by a zero, and shrink a 2-handle as in Fig. 20. Then change the shrunk 2-handle to a 1-handle, slide the  $-1$  2-handle over the 2-handle which cancels the 1-handle, and cancel. Isotopies and a few slides reduce to the simple link in Fig. 20

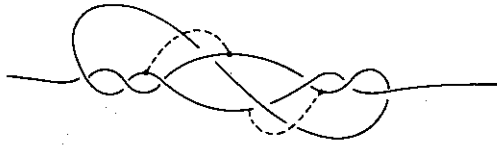


Fig. 16.

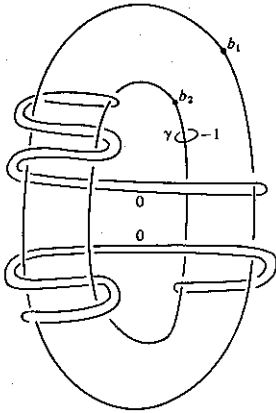


Fig. 17.

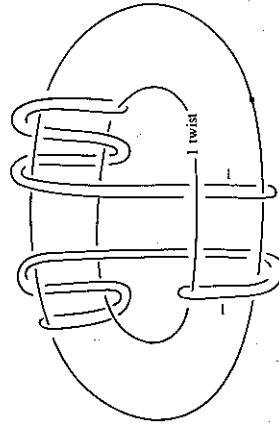


Fig. 18.

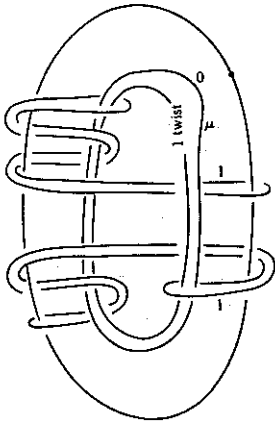


Fig. 19.

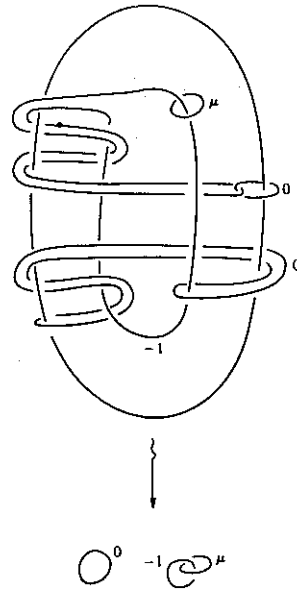


Fig. 20.

which shows that  $\mu$  is trivial in the boundary so that we may add a cancelling 3-handle along the 2-sphere that  $\mu$  defines.

Now slide a 2-handle over the 2-handle  $\mu$  so that it becomes unlinked from the twist, and note that its framing changes to  $-1$  (Fig. 21). The  $-1$  handle cancels the 1-handle which leaves two 2-handles and two 3-handles. We want to turn this handlebody over to get two 1-handles and two 2-handles; this is done by drawing the dual 2-handles,  $\sigma$  and  $\tau$ , changing the interior to the handlebody  $0^0 0^0$  via handle slides while carrying along  $\sigma$  and  $\tau$ , and then changing this unlink to two 1-handles (Figs. 21–28).

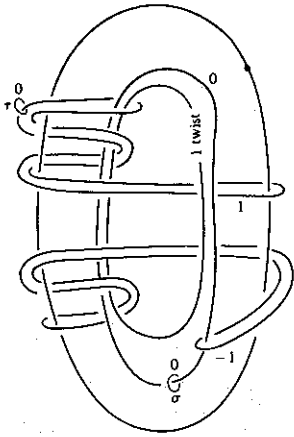


Fig. 21.

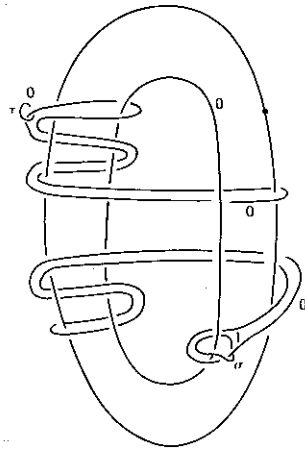


Fig. 22.

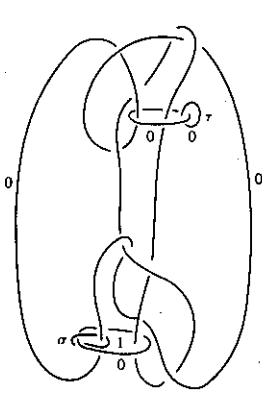


Fig. 23.

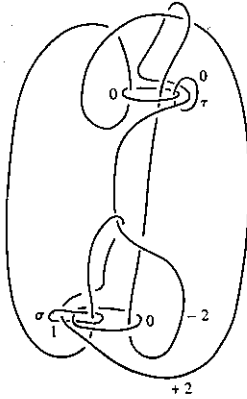


Fig. 24.

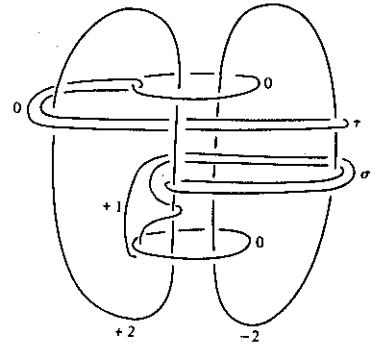


Fig. 25.

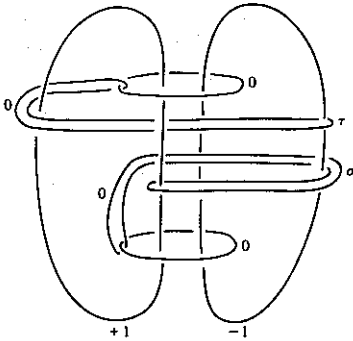


Fig. 26.

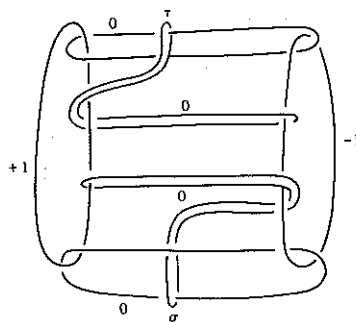


Fig. 27.

To get to Fig. 22, we blow up a  $-1$  circle parallel to the twist, eliminating the twist and changing framings to zero; then slide the new  $-1$  circle over the  $0$ -framed  $2$ -handle and blow it down changing framings as indicated. Surger the  $1$ -handle and isotop to get Fig. 23. Two handle slides give Fig. 24. Another isotopy leads to Fig. 25. To remove the full right hand twist between the  $+1$  and  $+2$  curves, we blow up a  $-1$  curve around them, slide it over the lower horizontal  $0$  curve and blow it down, obtaining Fig. 26 (note the framing changes). An isotopy gives Fig. 27 which is invariant under rotation by  $\pi$  followed by reflection.



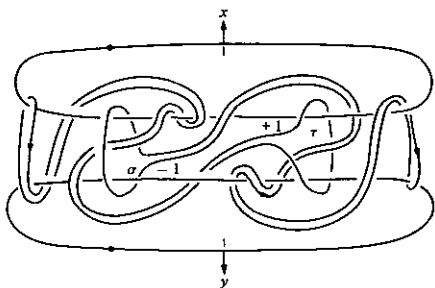


Fig. 28.

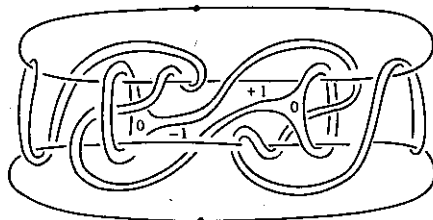


Fig. 29.

Now blow down both the +1 and -1 handles, isotop to Fig. 28 and surger the 0 curves to 1-handles. This figure is worth examination; it is a non-trivial picture of a homotopy 4-ball with boundary  $S^3$ , it has symmetry, and it gives an interesting presentation of the trivial group.

Labeling the 1-handles by  $x$  and  $y$  and starting at the arrows, we read off the relations

$$1 = yx^{-2}yxy^{-2}x(xy^{-1}x^{-1}) \quad \text{and} \quad (1)$$

$$1 = yx^{-2}yxy^{-2}x(y^{-1}x^{-1}y) \quad \text{so we deduce} \quad (2)$$

$$xy^{-1}x^{-1} = y^{-1}x^{-1}y \quad \text{or} \quad (3)$$

$$xyx = yxy. \quad (4)$$

Using (3) and (4) we see that (1) becomes

$$\begin{aligned} 1 &= yx^{-2}(yx)y^{-2}x^2(y^{-1}x^{-1}) \\ &= y(yx)y^{-4}(y^{-1}x^{-1})y^2 \\ &= x^{-5}(yx)x^{-1}y^3 \\ &= x^{-5}y^4 \end{aligned} \quad (5)$$

This presentation  $\{x, y \mid xyx = yxy, x^5 = y^4\}$  is seen to be the trivial group as follows:  $xyx = yxy$  implies that  $y = (yx)^{-1}x(yx)$  so  $y^5 = (yx)^{-1}x^5(yx) = (yx)^{-1}y^4(yx) = x^{-1}y^4x = x^{-1}x^5x = x^5 = y^4$  so  $y = 1$  and  $x = 1$ . (Note that this proof works for the group  $\{x, y \mid xyx = yxy, x^{n+1} = y^n\}$ .)

We do not know whether this presentation, let along the original one with relations (1) and (2), can be trivialized by Andrews-Curtis moves (see [1], [10] Prob. 5.2); if so, then  $\Sigma^4$  would be homeomorphic to  $S^4$  by a standard argument [1].

We shall show that  $\Sigma^4$  is homeomorphic to  $S^4$  by adding two 2-handles to the handlebody of Fig. 28 in such a way that the new manifold is  $S^2 \times B^2 \# S^2 \times B^2$ ; then two 3-handles and a 4-handle can be added to give  $S^4$ . Since the boundary of Fig. 28 is  $S^3$ , an application of the topological Schoenflies theorem gives that Fig. 28 is homeomorphic to  $B^4$ .

Add the 2-handles as in Fig. 29 and do the obvious two handle slides over the new 2-handles to get Fig. 30. Cancel the two 1-handles with the +1 and -1 handles to obtain  $S^2 \times B^2 \# S^2 \times B^2$  in Fig. 31. This finishes a rather long series of moves which shows that  $\Sigma^4$  is homeomorphic to  $S^4$ .

§3

It is interesting to have an example of a framed link  $L$  of two components, with framings and linking number all zero, whose 4-manifold  $W_L$  has boundary  $S^1 \times S^2 \# S^1 \times S^2$  but is not obviously  $S^2 \times B^2 \# S^2 \times B^2$ . It is easy to construct nontrivial links with  $\partial W_L = S^1 \times S^2 \# S^1 \times S^2$  by adding an unknot to a ribbon knot, e.g. Fig. 32; but in these cases  $W^4$  is obviously  $S^2 \times B^2 \# S^2 \times B^2$ . If  $L$  has one component, then  $\partial W^4 = S^1 \times S^2$  implies that  $L$  is slice [9] and may imply that  $L$  is the unknot.

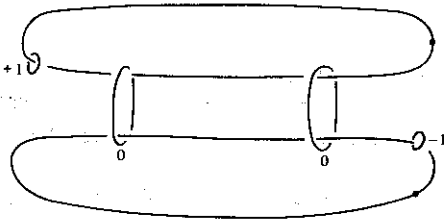


Fig. 30.

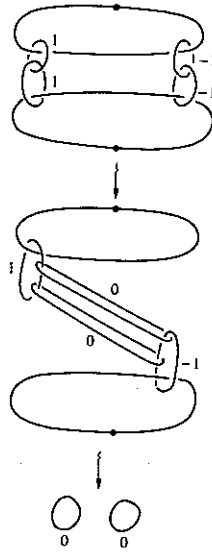


Fig. 31.

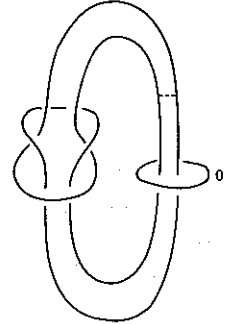


Fig. 32.

Our example begins with Fig. 9 and we cancel  $a_*$  and  $\gamma$  to get Fig. 33. We want to add a cancelling 2–3-handle pair with the 2-handle  $\delta$  being attached to an unknotted circle parallel to the twist in Fig. 33. We need to know that  $\delta$  represents a trivial circle with framing zero in the boundary of Fig. 33. To see this we construct a diffeomorphism of the boundary to  $S^1 \times S^2 = \partial(0^4)$  and check that  $\delta$  is trivial. The first step is to shrink  $\alpha_1$ , change it to a 1-handle and change  $a_2$  to a 2-handle (see Fig. 34). Cancel  $\alpha_1$  with  $\beta_1$  by first sliding twice over  $\beta_1$  and then erasing  $\alpha_1$  and  $\beta_1$ . Surprisingly,  $a_2$  and  $\delta$  become the unlink, as can be seen after a long isotopy.

So add the 2-handle  $\delta$  and slide  $\beta_1$  over  $\delta$  so that the end of  $\beta_1$  no longer goes through the twist, Fig. 35. Shrink  $\beta_1$  (Fig. 36) and cancel  $a_2$  and  $\beta_1$  by sliding over  $\beta_1$  eight times and then erasing  $a_2$  and  $\beta_1$ . A further isotopy gives our example in Fig. 37. One can independently check that the boundary in Fig. 37 is  $S^1 \times S^2 \# S^1 \times S^2$ . First, blow up a  $-1$  circle parallel to the twist, thus removing the twist; then slide the  $-1$  circle over  $\alpha_1$  using a band connected sum along the dotted arc and isotop the  $-1$  circle to the other end of  $\alpha_1$ ; finally blow down the  $-1$  circle and check via isotopies and a few handle slides that one obtains the unlink.

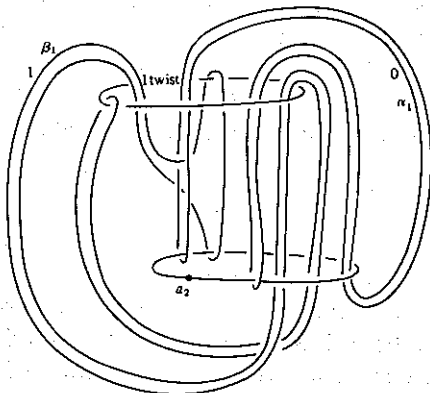


Fig. 33.

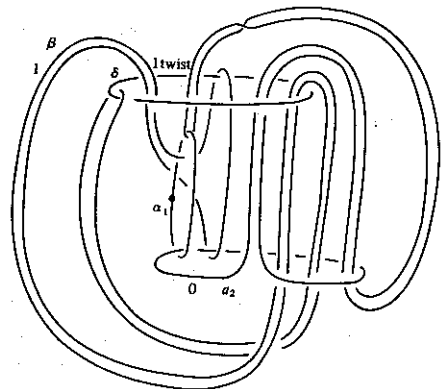


Fig. 34.

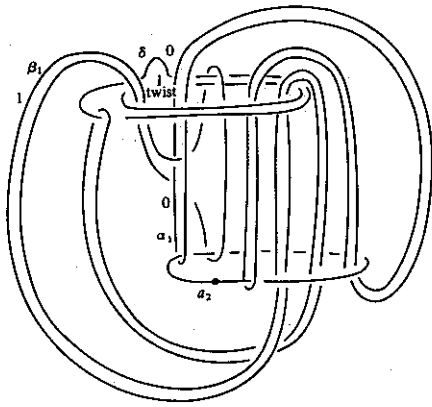


Fig. 35.

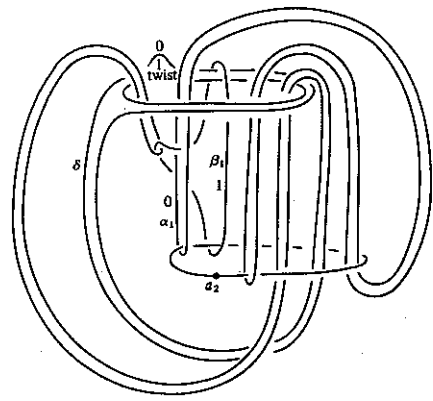


Fig. 36.

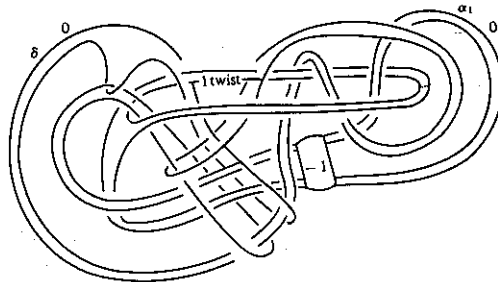


Fig. 37.

§4

$\Sigma^4$  would be diffeomorphic to  $S^4$  if the smooth 4-dimensional Schoenflies conjecture could be proved. It may help in analyzing this conjecture to have a non-trivial example of an imbedded  $S^3$  in  $S^4$ . To this end we draw (in Figs. S1–S11) a “movie” of  $\partial(\Sigma^4\text{-4-handle}) = S^3$  in  $S^4$ . For simplicity we refer to this  $S^3$  as  $\partial\Sigma_0$ .

Our starting point is Fig. 29 which is a framed link picture of  $S^4$ . In it we can see  $\partial\Sigma_0$  as  $\partial(0\text{-handle})$  with surgery on the 1-handles and the 2-handles with framing  $\pm 1$ . We want to cancel the 1, 2 and 3-handles so that  $S^4$  is constructed with only a 0 and 4-handle; as we cancel handles we keep track of  $\partial\Sigma_0$  and also isotop it to a critical level imbedding. This means that if we think of  $S^4$  as  $(S^3 \times \mathbb{R}) \cup \{-\infty\} \cup \{+\infty\}$ , then projection to  $\mathbb{R}$  is a Morse function,  $f: \partial\Sigma_0 \rightarrow \mathbb{R}$ , when restricted to  $\partial\Sigma_0$ . Figures S1–S11 show  $f^{-1}(t_i)$ , for increasing  $i$ .

Assume that  $S^3 \times (-\infty, 0] \cup \{-\infty\}$  is the 0-handle of  $S^4$ . Then except for the surgeries,  $\partial\Sigma_0 \cup S^3 \times 0$ . We begin with Fig. S4 which shows the boundary of  $\{S^3 \text{ with two } S^0 \times B^3\text{'s and two } S^1 \times B^2\text{'s removed}\}$ . This 2-manifold can be thought of as four  $S^2$ 's (the boundaries of the two  $S^0 \times B^3$ 's) and twenty two tubes connecting them; the tubes correspond to the boundaries of the attaching maps of the two handles. The interior of  $\{S^3 \text{ with two } S^0 \times B^3\text{'s and two } S^1 \times B^2\text{'s removed}\}$  has been pushed down into  $S^3 \times (-\infty, 0)$  and is drawn in Figs. S1–S3. First, the surface in Fig. S4 is “unknotted” by an isotopy as in Fig. S3; next the obvious holes are filled in as we pass sixteen critical points of index 1 to reach Fig. S2; then we unknot again by an isotopy, fill in three holes to get a 2-sphere; (Fig. S1) and finally cap off *outside*.

To proceed upwards from Fig. S4, we must see what happens to  $\partial\Sigma_0$  as the 2-handles slide as in Figs. 29, 30. The reader should work through the lower dimensional case in which  $F$  is the boundary of  $B^3 \cup 2\text{-handle}$ , a second 2-handle is added to  $B^3$ , and the first 2-handle is slid over the second. To begin with, push  $F$  into  $B^3$  except for the  $S^1 \times S^0$  which remains in  $\partial B^3$  (see Fig. 38). Then, using a collar  $S^2 \times [0, 1]$  on  $\partial B^3$ , we see the  $S^1 \times S^0$  get pushed around the

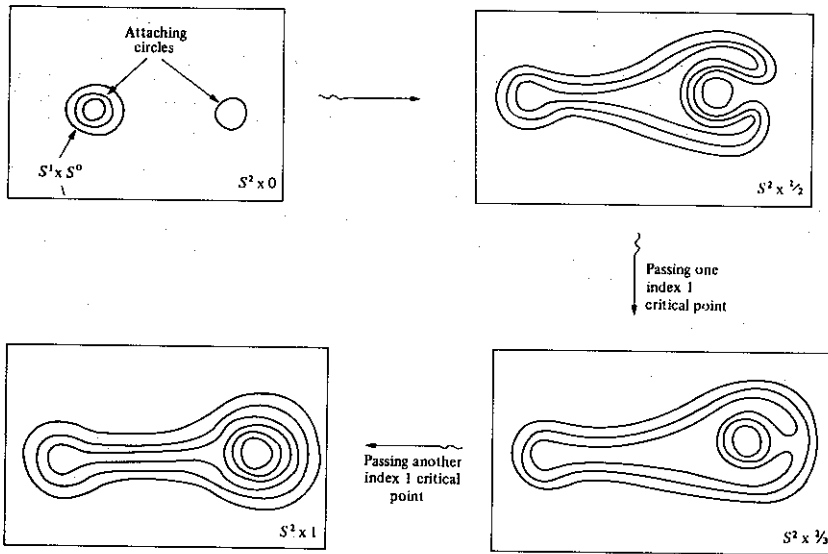


Fig. 38.

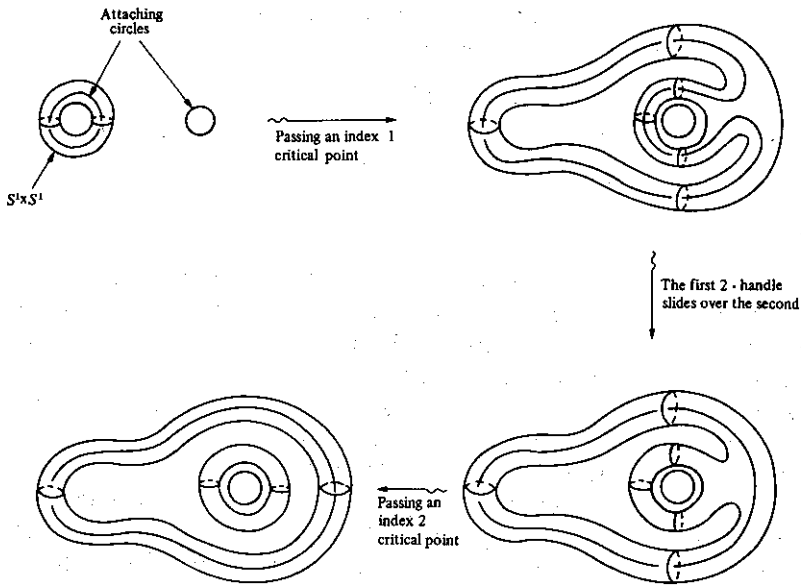
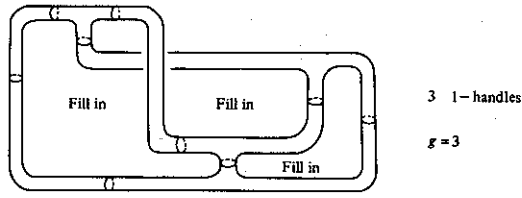


Fig. 39.

attaching circle of the second 2-handle as we simultaneously push the first 2-handle and  $F$  over the second 2-handle (Fig. 38). In Fig. 39 we draw the same case but in dimension 4. Notice that we freely add collars to  $B^3$  (or  $B^4$ ) when pushing  $F$  (or  $\partial\Sigma_0$ ) around, and similarly we fatten the 2-handles when convenient.

To get from Fig. S4 to Fig. S6 we must push  $\partial\Sigma_0$  as we slide two 2-handles; the picture is more complicated than Fig. 39 because there are 1-handles present and the 2-handles go over them. Figure S5 shows part of the movie between Figs. S4 and S6. We see what happens as the  $-1$  2-handle (not drawn since it lies inside the obvious tube) slides over the drawn 2-handle and then begins to isotop off the 1-handle.



Expand out to  $\infty$

$g = 0$

Fig. S1

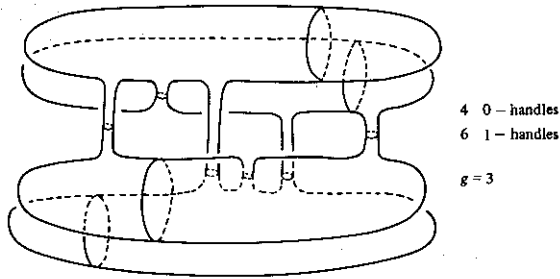


Fig. S2.

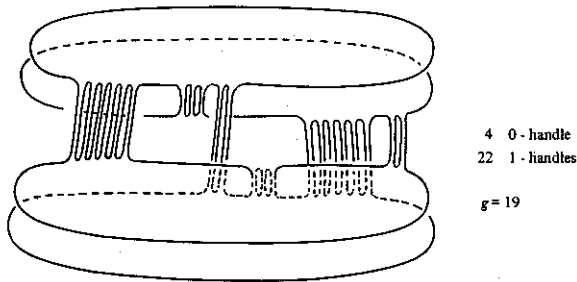


Fig. S3.

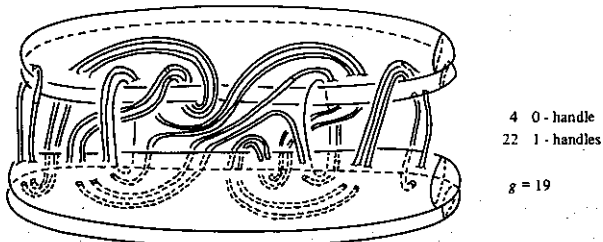


Fig. S4.

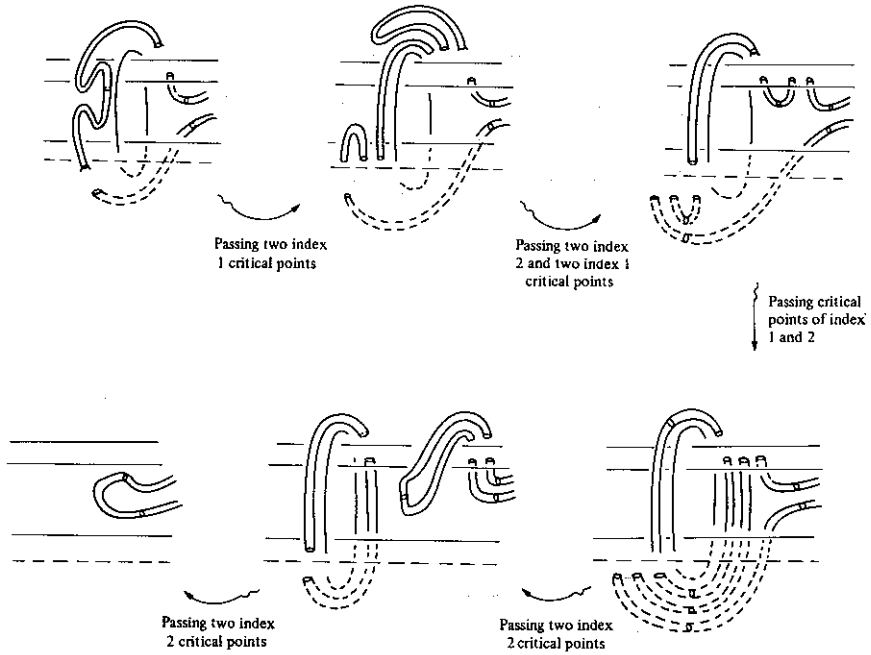


Fig. S5.

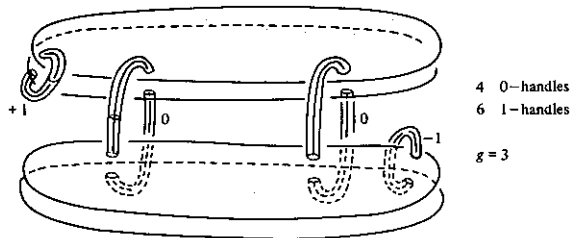


Fig. S6.

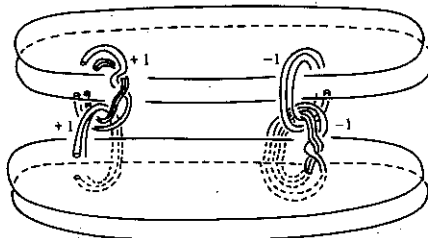


Fig. S7.

The same technique is used in the handle slides going from Fig. S6 to S8. In Fig. S8, the tori parallel to the 0-framed circles are isotoped away from the rest. To see what happens when the 1 and 2-handles are cancelled, it is enough to examine the case in dimension 3; this is left to the reader. From Fig. S9 to Fig. S10, we pass four critical points of index 2. Finally in

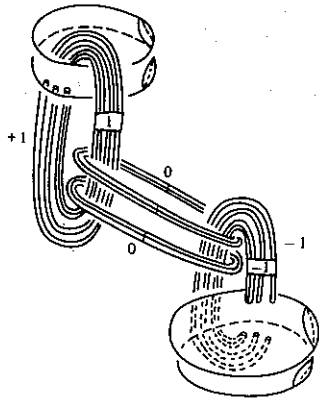


Fig. S8.

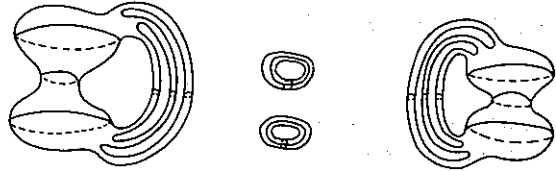


Fig. S9.

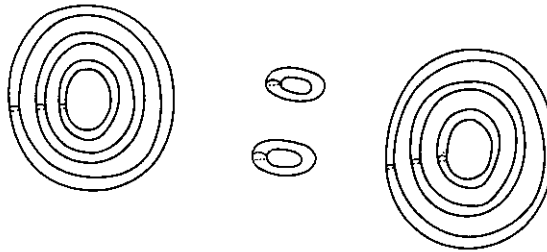
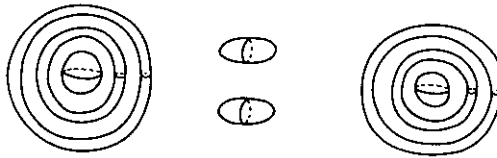


Fig. S10.



And continue to add 2 - handles to longitudes and 3 - handles to 2- spheres

Fig. S11.

Fig. S10 we cap off each torus by passing a critical point of index 2 and then index 3, as indicated in Fig. S11.

This description of  $\partial\Sigma_0$  has one critical point of index 0, 51 of index 1, 58 of index 2 and 8 of index 3. Many of these can be cancelled, but we have not tried to simplify the movie; it is simply the translation of the handlebody picture. We humbly offer  $\partial\Sigma_0$  to the devotees of the Schoenflies conjecture.

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