Notes on prismatic cohomology

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Abstract

These are notes from a graduate topics course[1] given at UC San Diego during the spring 2021 quarter; recordings of the lectures can be found at the same web site. The notes were typeset using PreTeXt[2] so as to produce matching HTML[3] and PDF[4] versions.

The course was closely modeled on [18], which in turn presented material from [25]. No claim of originality is made except for errors.

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1 Introduction and overview

Reference. [18], Lecture I.

We begin by describing some of the (global) context for the study of prisms and prismatic cohomology. We then take a more local view to explain what we are trying to do (with no proofs at this point). Keep in mind that it is *not* necessary to know about all of the topics I describe here in order to understand the rest of the course!

1.1 (Co)homology of complex varieties

The dual notions of **homology** and **cohomology** first appeared in topology as ways to "linearize nonlinear geometry"; that is, to attach "linear" invariants (abelian groups, modules over commutative rings) to complicated geometric objects. This project proved to be quite successful, to the point that nowadays there are even significant *real-life* applications of these constructions; see for example [56].

In the theory of manifolds, there are two traditional approaches to homology and cohomology.

1. One is the combinatorial approach, in which one views a global space as being glued together from a small number of simple pieces (e.g., a

[1] kskedlaya.org/math206a-spr21/
[2] pretextbook.org
[3] kskedlaya.org/prismatic/
[4] kskedlaya.org/papers/prismatic-ptx.pdf

triangulation of a surface). One can then extract the desired invariants by careful bookkeeping on the interactions between the pieces. The most robust version of this is **singular homology/cohomology** (also called **Betti homology/cohomology**).

2. The other is the **cohomology of differential forms**, which developed from the classical theorems in multivariable calculus about the relationship between integrals over a region and integrals over its boundary (and the physical laws from electromagnetism related to these), culminating in **Stokes's theorem**. The relationship between this and the singular theory was put on firm footing by the work of Georges de Rham, in whose honor the cohomology of differential forms is commonly referred to as **de Rham cohomology**.

   Note that one cannot speak of differential forms without some additional structure on a manifold, at least a smooth (C^∞) structure. For a complex manifold, one can do better: by Dolbeaut's theorem, one can compute de Rham cohomology using exclusively *holomorphic* forms (see [59], Chapter 3).

These two constructions are closely related via integration: for C a homology class of dimension k and ω a k-form on a complex manifold X, there is a well-defined integral $\int_C \omega$. Stokes's theorem then asserts that

$$\int_C d\omega = \int_{\delta(C)} \omega;$$

we thus obtain a pairing

$$H_i(X, \mathbb{C}) \times H^i_{\mathrm{dR}}(X) \to \mathbb{C}$$

which by the de Rham and Dolbeaut theorems is a perfect pairing; that is, the induced map

$$H^i_{\mathrm{dR}}(X) \cong H_i(X, \mathbb{C})^\vee \cong H^i(X, \mathbb{C})$$

is an isomorphism.

While one can think of this isomorphism as asserting that singular and de Rham cohomology are "the same", this is not the most useful conclusion to draw; it is better to interpret this as saying that "the whole is greater than the parts".

1. The space $H^i(X, \mathbb{C})$ is really the base extension to $\mathbb{C}$ of the $\mathbb{Q}$-vector space $H^i(X, \mathbb{Q})$. One can transport this rational subspace over to de Rham cohomology, but the result is rather mysterious! It can be described in terms of integrals of differential forms over rational homology classes (often called **periods** because the most basic example is the number π) but the arithmetic of these is quite subtle; there is a far-reaching conjecture about this due to Kontsevich and Zagier [87].

2. The singular cohomology depends only on the original manifold, whereas the de Rham cohomology depends on the extra data of a complex structure. For example, for Riemann surfaces of some genus $g \geq 2$, the underlying manifolds are all homeomorphic, so all of the variation comes from the complex structure.

3. By Hodge's theorem, every real cohomology class admits a unique harmonic representative. This then leads to the **Hodge decomposition** on the de Rham side.

4. When X is the base extension of an algebraic variety over a subfield K of $\mathbb{C}$, $H^i_{\mathrm{dR}}(X)$ can also be computed using *algebraic* differential forms, by an argument of Grothendieck [60] using Serre's GAGA theorem [114], and therefore is really the base extension to $\mathbb{C}$ of a certain K-vector space. One interesting consequence of this is that there is a strong relationship between the underlying topological spaces of the spaces obtained by taking different embeddings of K into $\mathbb{C}$; this becomes more interesting when you realize that these spaces need not in general be homeomorphic! For instance, Serre found examples for which these spaces have distinct fundamental groups [115]; see also [32], [98], [102].

   Note that the Hodge decomposition does not survive the descent to K, but one of the filtrations derived from it does: this is the **Hodge filtration**.

To retain all of this data at once, Deligne defined the notion of a **Hodge structure** consisting of a $\mathbb{C}$-vector space with a filtration plus a $\mathbb{Z}$-lattice. This captures much of the interesting data in the above picture; for example, from the Hodge structure of an abelian variety, one can recover the abelian variety by forming a complex torus (taking the quotient of the $\mathbb{C}$-vector space by the $\mathbb{Z}$-lattice).

1.2 The trouble with torsion

One thing that is missing from the previous discussion is the fact that singular homology can be defined over $\mathbb{Z}$, and is *not* in general a subspace of the singular homology over $\mathbb{Q}$; that is, the singular homology can have nontrivial torsion. This is true even for algebraic varieties.

Example 1.2.1 An **Enriques surface** (over an algebraically closed field) is a projective algebraic surface with irregularity 0 (in the sense of Riemann-Roch) for which the canonical bundle is nontrivial but its square is trivial (e.g., the quotient of a K3 surface by a fixed-point-free involution). The cycle class of the canonical bundle defines a nontrivial 2-torsion element of H^2. □

The comparison isomorphism as formulated cannot really say anything meaningful about torsion in homology; one of the goals of prismatic cohomology is to provide a mechanism for interpreting this torsion via reductions to positive characteristic. Here is a sample statement.

Theorem 1.2.2 *Let X be a smooth projective variety over $\mathbb{Q}$. Choose a prime number p for which X can be extended to a smooth proper scheme $\mathfrak{X}$ over $\mathbb{Z}_{(p)}$, and put $X_p = \mathfrak{X} \times_{\mathbb{Z}_{(p)}} \mathbb{F}_p$. Then*

$$\dim_{\mathbb{F}_p} H^i(X^{\mathrm{an}}, \mathbb{Z}/p\mathbb{Z}) \leq \dim_{\mathbb{F}_p} H^i_{\mathrm{dR}}(X_p).$$

Proof. See [22], Theorem 1.1. ■

One way to think of Theorem 1.2.2 is that while a nonzero rational homology class constitutes an obstruction to integrating differentials in characteristic 0, a nonzero p-torsion homology class constitutes an obstruction to integrating differentials in characteristic p.

Remark 1.2.3 For various reasons, the inequality in Theorem 1.2.2 can be strict. One reason is that X_p is not uniquely determined by X; this has to do with birational geometry in mixed characteristic (e.g., one can perform **flips** in the special fiber). Another is that the left-hand side is not uniquely determined by X_p; see [22], 2.1 for an example (a threefold which admits a nonsplit elliptic fibration over an Enriques surface, as compared with the split fibration).

1.3 The p-adic situation

With the previous discussion in mind, let us now transition to the analogous discussion for algebraic varieties over not $\mathbb{C}$ but a p-adic field (where p denotes a fixed prime).

The discussion we conducted in over $\mathbb{C}$ falls under the label of **Hodge theory**. There is a parallel discussion that happens for algebraic varieties over p-adic fields that is covered by the label of **p-adic Hodge theory**. In that context, there is no good analogue of singular (Betti) homology or cohomology, because the underlying topological spaces don't have the "right" homotopy type. (The homotopy type generally misses all of the "good reduction" information and only picks up "bad reduction" data. There is an extensive literature on this point; see [44] for an introduction.)

The best available replacement for singular homology/cohomology is **étale homology/cohomology** with p-adic coefficients, where crucially this is *the same* prime p as the residue characteristic. This choice of characteristic can be thought of as an "ugly duckling": underappreciated at first, but in fact a beautiful swan in the making.

One thing that étale cohomology with p-adic coefficients does not do gracefully is specialize to characteristic p; it does not give a Weil cohomology in that setting (that is, you cannot use it to keep track of zeta functions and L-functions with complete accuracy). There are various ways to control this, which all amount to switching over to de Rham cohomology and making that work better in characteristic p; a notable example is **crystalline cohomology**, which builds on Grothendieck's interpretation of de Rham cohomology via the infinitesimal site [61]). In any approach of this type, some effort is needed to overcome the fact that the Poincaré lemma doesn't hold in positive characteristic, the issue being that there are "too many constants": in a characteristic-p setting the formal derivative of any p-th power vanishes.

In this course we will consider an approach to p-adic cohomology via the mechanism of **prisms** (see Definition 5.3.1 for the definition). One benefit of this point of view is that almost everything we want to say appears already in a local setting, where we can talk very concretely about rings and complexes without having to keep track of too much fancy stuff (derived categories, simplicial objects, etc.). Another advantage is that it keeps track of "everything at once"; instead of constructing different cohomology theories and asserting **comparison isomorphisms** between them, we'll construct "one theory to rule them all", in the manner of the **universal coefficient theorem** of algebraic topology. That is, we will have a single functor which we can postcompose with various simple algebraic functors to recover more classical constructions.

1.4 The role of prisms

We give a representative statement of a prismatic cohomology isomorphism.

Definition 1.4.1 Fix a prime p and define the ring $A = \mathbb{Z}_p[\![u]\!]$; this is a regular noetherian local ring of dimension 2 with residue field $\mathbb{F}_p$. Let $\phi\colon A \to A$ be the continuous homomorphism with $\phi(u) = u^p$; this lifts the Frobenius endomorphism on $A/(p)$. Let I be the ideal $(u - p)$ of A; let $\theta\colon A \to \mathbb{Z}_p$ be the identification of A/I with $\mathbb{Z}_p$ taking u to p. The triple (A, ϕ, I) will later form a basic example of a **prism**; see Section 5 for the general definition.

The action of ϕ^* on $\operatorname{Spec} A$ fixes the axial points (u) and (p) and the closed point (u, p), but acts nontrivially on other points. For example, $(u - p)$ is carried to $(u - p^p)$. $\Diamond$

Theorem 1.4.2 *For R the p-adic completion of a smooth $\mathbb{Z}_p$-algebra, we can functorially define a complex $\mathbf{\Delta}_{R/A}$ consisting of (p, u)-adically complete A-modules, as an object in the derived category $D(A)$ of A-modules; together with a morphism $\phi_{R/A}\colon \phi^*\mathbf{\Delta}_{R/A} \to \mathbf{\Delta}_{R/A}$ in $D(A)$ (so in particular ϕ^* is an operation in $D(A)$). Moreover, the pair $(\mathbf{\Delta}_{R/A}, \phi_{R/A})$ will have the following additional properties.*

1. *The map $\phi_{R/A}$ becomes a quasi-isomorphism after inverting $u - p$.*

2. *There is a natural quasi-isomorphism*

$$\phi^*\mathbf{\Delta}_{R/A}\widehat{\otimes}^L_A\mathbb{F}_p \cong \Omega^*_{R_{\mathbb{F}_p}/\mathbb{F}_p}.$$

   That is, at the closed point of $\operatorname{Spec} A$, *$\phi^*\mathbf{\Delta}_{R/A}$ computes de Rham cohomology of $R_{\mathbb{F}_p}$. (There is also a version over $\mathbb{Z}_p$ using continuous differentials, which amounts to working at the point $(u - p^p)$.)*

3. *There is a quasi-isomorphism*

$$(\mathbf{\Delta}_{R/A}\widehat{\otimes}^L_A\overline{\mathbb{F}_p((u))})^{\phi_{R/A}\otimes\phi=1} \cong R\Gamma_{\mathrm{et}}(\operatorname{Spec} R_{\mathbb{C}_p}, \mathbb{F}_p)$$

   where $\mathbb{C}_p$ is a completed algebraic closure of $\mathbb{Q}_p$ and ϕ denotes the absolute Frobenius on $\overline{\mathbb{F}_p((u))}$; this becomes natural if we relate the algebraic closures of $\mathbb{Q}_p$ and $\mathbb{F}_p((u))$ using the field of norms isomorphism (Theorem 8.3.4). That is, at the point (p) of $\operatorname{Spec} A$, *$\mathbf{\Delta}_{R/A}$ computes the $\mathbb{F}_p$-étale cohomology of $R_{\mathbb{C}_p}$. (There is also a version with $\mathbb{Z}/(p^n)$-coefficients, which amounts to working in an infinitesimal thickening of $p = 0$, or if you prefer over* $\operatorname{Spec} A[p/u]$.)

4. *There is a natural identification*

$$H^i(\mathbf{\Delta}_{R/A}\widehat{\otimes}^L_{A,\theta}\mathbb{Z}_p) \cong \Omega^i_{R/\mathbb{Z}_p}.$$

   That is, at the point $(u - p)$ of $\operatorname{Spec} A$, *$\mathbf{\Delta}_{R/A}$ computes the Hodge cohomology of R. (A more robust version of this statement would include a twist; we'll come back to this later.)*

Remark 1.4.3 In case you know what this means, condition 1 in Theorem 1.4.2 is similar to a central restriction in the definition of a **shtuka**, or more precisely a **shtuka with one leg**. Recent developments in the Langlands correspondence over function fields, particularly the work of V. Lafforgue [91], makes heavy use of **shtukas with multiple legs**; while these have a geometric interpretation (see [47]), it is far from clear whether this can be integrated with the prismatic point of view.

Remark 1.4.4 One important aspect of Theorem 1.4.2 is that we are *not* asserting a functorial construction of a complex of A-modules "on the nose", but only in the derived category. This is in contrast with, say, de Rham cohomology, which is computed by a specific meaningful complex; it is more akin to the situation for étale cohomology in this respect.

However, in the local development one can mostly ignore derived aspects. They become unavoidable at the point when one wants to glue local structures together.

Remark 1.4.5 The positioning of different cohomological invariants at different points in $\operatorname{Spec} A$ is illustrated in Figure 1.4.6. One can also observe in this picture the metaphor behind the term **prism**: the prism is an object that "refracts" the information from the original space into a "spectrum" of cohomological invariants.

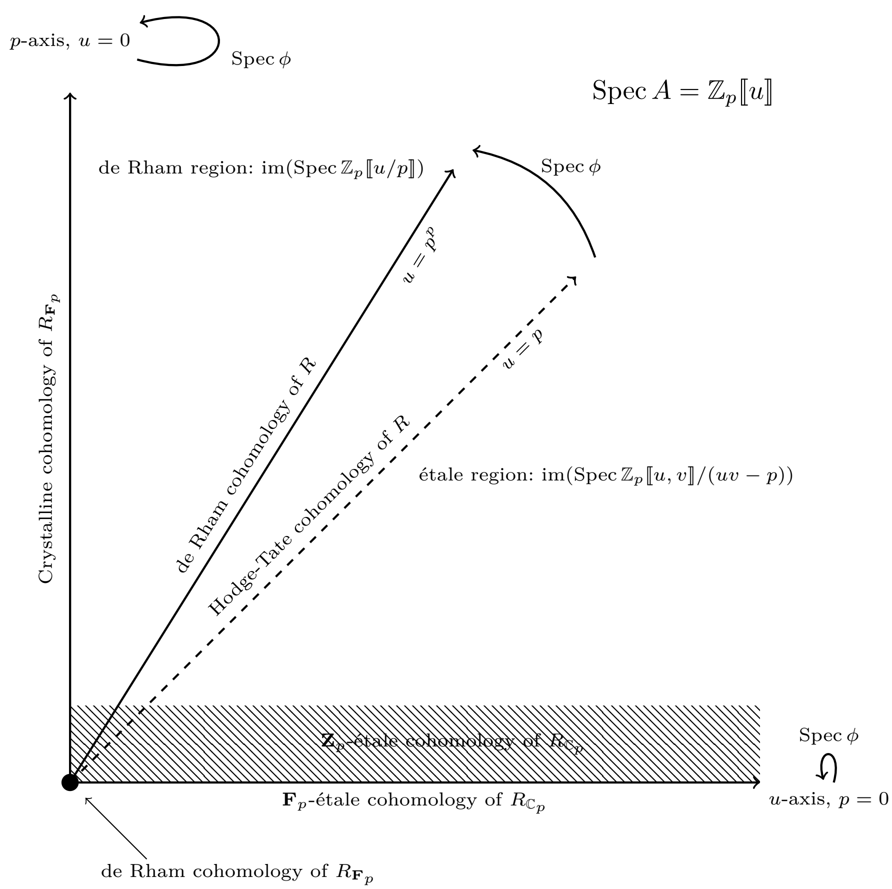

Figure 1.4.6 The "values" of $\mathbf{\Delta}_{R/A}$ at various points of $\mathrm{Spec}\, A = \mathbb{Z}_p[\![u]\!]$ as described by Theorem 1.4.2. The dashed arrow indicates where $\phi_{R/A}$ fails to be a quasi-isomorphism. Adapted from [18], Lecture I.

2 δ-rings

References. We have followed [18], Lecture II, fairly closely. Some of the exercises were taken from [30], others from [25], section 2.

In this section, we introduce the fundamental notion of a δ**-ring**. This definition was introduced by Joyal [74], [75] with a view towards applications in K-theory; it is closely related to the older notion of a λ**-ring**, which we will discuss briefly in the next section (Subsection 4.1). However, this development did not gain much attention until the same idea was rediscovered by Buium [34] under the guise of **arithmetic differentiation**. (Buium had the original goal of adapting Manin's proof of the finiteness of the set of torsion points on a hyperbolic algebraic curve embedded into its Jacobian from the function field case to the number field case; see [35] for a slightly later treatment presenting arithmetic differentiation on its own terms.)

Definition 2.0.1 For the remainder of the course (except as specified), fix a prime number p. Define the following standard categories:

- **Set**: sets.
- **Ab**: abelian groups.

- **Ring**: commutative unital rings.
- $\mathbf{Mod}_A$: modules over A (where $A \in \mathbf{Ring}$).

Let $\mathrm{Rad}(A)$ denote the Jacobson radical of $A \in \mathbf{Ring}$. For I an ideal of A, we say that A is ***I*-local** if $I \subseteq \mathrm{Rad}(A)$; if $I = (f)$, we also say that A is ***f*-local**. ♢

2.1 *p*-derivations and Frobenius lifts

We begin by formalizing the fundamental idea of "differentiation with respect to a prime number".

Definition 2.1.1 Following Joyal, we define a **δ-ring** to be a pair (A, δ) in which $A \in \mathbf{Ring}$ and $\delta \colon A \to A$ is a map of sets satisfying the following conditions for all $x, y \in A$.

$$\delta(1) = 0; \tag{2.1}$$

$$\delta(xy) = x^p \delta(y) + y^p \delta(x) + p\delta(x)\delta(y); \tag{2.2}$$

$$\delta(x+y) = \delta(x) + \delta(y) - \sum_{i=1}^{p-1} \frac{(p-1)!}{i!(p-i)!} x^i y^{p-i}. \tag{2.3}$$

The last condition implies that

$$p(\delta(x+y) - \delta(x) - \delta(y)) = x^p + y^p - (x+y)^p$$

and conversely if A is p-torsion-free. (In some sources, a map δ satisfying these conditions is called a ***p*-derivation**.)

We will habitually abuse notation and terminology and say that "A is a δ-ring" when it is meant to be clear from context what the map δ is supposed to be. We will also apply adjectives to a δ-ring (e.g., "p-torsion-free") when they are meant to apply to the underlying ring. ♢

Remark 2.1.2 Note that (2.3) also implies that $\delta(0) = 0$, so we don't need to include this condition separately. By contrast, (2.2) does not by itself imply that $\delta(1) = 0$; see Exercise 2.5.1.

The form of the previous definition is partly explained by the following construction.

Lemma 2.1.3 *Choose $A \in \mathbf{Ring}$.*

1. *Suppose that $\delta \colon A \to A$ is a p-derivation (that is, (A, δ) is a δ-ring). Then the map $\phi \colon A \to A$ given by*

   $$\phi(x) = x^p + p\delta(x)$$

   *is a ring homomorphism that induces the Frobenius endomorphism on A/pA (i.e., a **Frobenius lift** on A). We will refer to ϕ as the **associated Frobenius lift** on (A, δ).*

2. *If A is p-torsion-free, then this construction defines a bijection between p-derivations on A and Frobenius lifts on A.*

Proof. It was already pointed out in Definition 2.1.1 that δ satisfies (2.3) if and only if ϕ is additive, and conversely if A is p-torsion-free. Meanwhile, (2.1) implies that $\phi(1) = 1$, and conversely if A is p-torsion-free; while (2.2) implies that $\phi(xy) = \phi(x)\phi(y)$, and conversely if A is p-torsion-free. ■

Remark 2.1.4 It is possible for a p-torsion-free ring to admit no δ-ring structures, and hence no Frobenius lifts. A simple example (taken from [25], Lemma 2.35) is $A = \mathbb{Z}_{(p)}[x, x^p/p]$: if $\delta\colon A \to A$ were a p-derivation with associated Frobenius lift ϕ, we would have

$$\begin{aligned}
\frac{1}{p}(x^p/p)^p &= \frac{1}{p}(\phi(x^p/p) - p\delta(x^p/p)) \\
&= \frac{\phi(x)^p}{p^2} - \delta(x^p/p) \\
&= \frac{(x^p + p\delta(x))^p}{p^2} - \delta(x^p/p) \\
&= p^{p-2}(x^p/p + \delta(x))^p - \delta(x^p/p) \in A,
\end{aligned}$$

a contradiction. See Exercise 2.5.10 for a consequence of this calculation.

Definition 2.1.5 By (2.1) and (2.2), in any δ-ring A the elements $x \in A$ for which $\delta(x) = 0$ form a monoid under multiplication. By analogy with the case of an ordinary derivation, we call these the **δ-constant elements** of A. These elements also satisfy $\phi(x) = x^p$, and conversely if A is p-torsion free. (In [25] these elements are said to be **of rank** 1; this terminology will make more sense in the context of big Witt wectors, as in Remark 4.2.1.) $\Diamond$

Remark 2.1.6 One might ask to what extent the notion of a p-derivation is a "natural" modification of the definition of a usual derivation. One answer to this question can be found in [35]: one can define the notion of a **jet operator** δ on a local domain A of characteristic 0 and show that any such map is either an ordinary derivation, a π-difference operator for some $\pi \in A$ (i.e., $x \mapsto x + \pi\delta(x)$ is a ring homomorphism), or a π-derivation for some $\pi \in A$ (i.e., $x \mapsto x^q + \pi\delta(x)$ is a ring homomorphism for some prime power q). This can be thought of as a loose analogue of Ostrowski's classification of valuations.

2.2 Examples of δ-rings

Using Lemma 2.1.3, it is not difficult to generate examples of δ-rings. Here are a few illustrative cases.

Example 2.2.1 If p is invertible in the ring A, then *every* endomorphism of A is a Frobenius lift, and thus gives rise to a p-derivation. $\square$

Example 2.2.2 By Lemma 2.1.3, there is a unique way to equip $\mathbb{Z}$ with the structure of a δ-ring, namely via the map

$$\delta(x) = \frac{x - x^p}{p}. \tag{2.4}$$

The δ-constant elements are $\{0, 1\}$ if $p = 2$ and $\{0, 1, -1\}$ if $p > 2$.

By the same token, $\mathbb{Z}_p$ has trivial automorphism group (even if we ignore its topology!) and so admits a unique δ-ring structure. The δ-constant elements are $\{0\} \cup \mu_{p-1}$. This example is the first case of the general construction of rings of Witt vectors; see Subsection 3.1. $\square$

Remark 2.2.3 Using (2.3), we can start from the equality $\delta(0) = \delta(1) = 0$ and reconstruct the values of δ on arbitrary integers. Consequently, for any δ-ring A, the action of δ on integers is given by (2.4) *even if A is not p-torsion-free*. In particular, for any positive integer n,

$$\delta(p^n) = p^{n-1}(1 - p^{n(p-1)})$$

and the second factor is not divisible by p unless p is invertible in A (see Example 2.2.1); that is, δ "lowers the p-adic order of vanishing by 1". By the same token, for any $x \in A$,

$$\delta(p^n x) = p^{np}\delta(x) + x^p\delta(p^n) + p\delta(x)\delta(p^n) \equiv p^{n-1}x^p \pmod{p^n}.$$

See Exercise 2.5.5 for a related observation.

Example 2.2.4 Building on Example 2.2.2, take $A = \mathbb{Z}[\mu_n \colon \gcd(n,p) = 1]$. The automorphism $\phi\colon A \to A$ taking ζ_n to ζ_n^p for every positive integer n coprime to p is a Frobenius lift; for δ the corresponding p-derivation, the δ-constant elements are $\{0\} \cup \bigcup_{\gcd(n,p)=1} \mu_n$. □

Example 2.2.5 Take $A = \mathbb{Z}[x]$. For any $y \in A$, there is a unique Frobenius lift ϕ of A for which $\phi(x) = x^p + py$; consequently, there is a unique p-derivation δ on A with $\delta(x) = y$. It is tempting to interpret this as the statement that "the set of p-derivations on A is a free A-module of rank 1", but in fact there is no natural module structure on the set of p-derivations on a general ring. □

You may have noticed that none of these examples has p-torsion. That is not entirely an accident.

Lemma 2.2.6 *Let (A, δ) be a δ-ring such that for some nonnegative integer n, $p^n = 0$ in A. Then $A = 0$.*

Proof. We prove the claim by induction on n, with the base case $n = 0$ being vacuously true.

Suppose that $n > 0$. Then A is a $\mathbb{Z}_{(p)}$-algebra. Consequently, by (2.4) and Remark 2.2.3,

$$0 = \delta(0) = \delta(p^n) = p^{n-1}(1 - p^{np-n})$$

and the second factor is a unit in A. Hence $p^{n-1} = 0$ in A also, and the induction hypothesis applies. ■

Remark 2.2.7 Notwithstanding Lemma 2.2.6, there do exist examples of δ-rings in which the underlying ring is not p-torsion-free; it is difficult to write these down concretely (in part because it is not enough to specify the associated Frobenius lift), but they will be generated naturally by Definition 3.1.1. See for instance Exercise 3.6.1.

Fortunately, quite often we can ignore these examples when checking basic properties of δ-rings by appealing to the existence of free δ-rings; see again Definition 2.4.5. See also Exercise 2.5.4 for a variant of Lemma 2.2.6 that applies to this situation.

Lemma 2.2.8 *Let A be a δ-ring. For any $x \in A$ with $px = 0$, $\phi(x) = 0$. In particular, if ϕ is injective, then A is p-torsion-free.*

Proof. Since x maps to zero in $A[p^{-1}]$, so then does $\phi(x)$. It is thus sufficient to check the claim after localizing at (p), that is, we may assume that A is a $\mathbb{Z}_{(p)}$-algebra. Now apply (2.2) to write

$$0 = \delta(0) = \delta(px) = x^p\delta(p) + p^p\delta(x) + p\delta(x)\delta(p) = p^p\delta(x) + \phi(x)\delta(p).$$

By Remark 2.2.3, $\delta(p)$ is a unit in A; it will thus suffice to check that $p^p\delta(x) = 0$. This follows by writing

$$p^p\delta(x) = p^{p-1}(\phi(x) - x^p) = p^{p-2}(\phi(px) - px^p) = 0.$$

■

2.3 Truncated Witt vectors

Just as derivations can be naturally interpreted as giving first-order deformations of a ring, one can interpret p-derivations in the following manner.

Definition 2.3.1 For A a ring, let $W_2(A)$ be the set $A \times A$ equipped with binary operations $+, \times$ defined as follows:

$$(x_0, x_1) + (y_0, y_1) = (x_0 + y_0, x_1 + y_1 - \sum_{i=1}^{p-1} \frac{(p-1)!}{i!(p-i)!} x_0^i y_0^{p-i})$$
$$(x_0, x_1) \times (y_0, y_1) = (x_0 y_0, x_0^p y_1 + y_0^p x_1 + p x_1 y_1).$$

Note the relationship between these formulas to the definition of a δ-ring (Definition 2.1.1); see Remark 2.3.3. $\Diamond$

Lemma 2.3.2 *The set $W_2(A)$ with the operations $+, \times$ is a commutative ring. Moreover, the operation $A \mapsto W_2(A)$ defines a functor from* **Ring** *to* **Ring**.

Proof. One may see directly from the definitions that:

- addition is commutative and the element $0 = (0, 0)$ is an identity element;
- every element has an additive inverse;
- multiplication is commutative and the element $1 = (1, 0)$ is an identity element;
- the operation $A \mapsto W_2(A)$ defines a functor from the category of commutative rings to the category of sets equipped with two binary operations.

We thus need to check that addition and multiplication are associative and that multiplication distributes over addition. These are all conditions asserting the validity of certain polynomial identities in two arbitrary elements $x, y \in A$; thanks to the functoriality, these can be checked after lifting from A to some ring that surjects onto it. In particular, we may take A to be a polynomial ring over $\mathbb{Z}$, which in particular is p-torsion-free.

In this setting, the map

$$W_2(A) \to A \times A, \qquad (x_0, x_1) \mapsto (x_0, x_0^p + p x_1)$$

is a monomorphism (in the category of sets equipped with two binary operations) for the *usual* ring operations on $A \times A$. Consequently, we may deduce the desired properties by transferring the knowledge from $A \times A$. (This map is related to the **ghost map** on Witt vectors; see Subsection 3.2.) ■

Remark 2.3.3 There are two natural (in A) ring homomorphisms $\epsilon_1, \epsilon_2 \colon W_2(A) \to A$ given by

$$\epsilon_1((x_0, x_1)) = x_0, \qquad \epsilon_2((x_0, x_1)) = x_0^p + p x_1.$$

In this notation, a δ-ring structure on A corresponds to a ring homomorphism $w \colon A \to W_2(A)$ such that $\epsilon_1 \circ w = \mathrm{id}_A$. In this way, the ring $W_2(A)$ plays a role comparable to that of the **ring of dual numbers** $k[\epsilon]/(\epsilon^2)$ over a field k.

On a related note, for $A \in \mathbf{Ring}_\delta$, $B \in \mathbf{Ring}$, and $f \colon A \to B$ a morphism in **Ring**, the formula

$$a \mapsto (f(a), f(\delta(a)))$$

defines a homomorphism $A \to W_2(B)$ in **Ring**: namely, this is the composition of $w \colon A \to W_2(A)$ with the functorial map $W_2(A) \to W_2(B)$. This map will reappear later via the adjunction property of Witt vectors (Definition 3.1.1).

Remark 2.3.4 For A a p-torsion-free ring, $W_2(A)$ can also be described as the fiber product of the reduction map $A \to A/(p)$ with the composition $A \to A/(p) \xrightarrow{\phi} A/(p)$ where ϕ denotes the Frobenius on $A/(p)$; the two projection maps $W_2(A) \to A$ are the ones from Remark 2.3.3. That is, $\operatorname{Spec} W_2(A)$ consists of two copies of $\operatorname{Spec} A$ glued along $\operatorname{Spec} A/(p)$ via Frobenius.

In [18], Lecture II, Remark 3.4, Bhatt also suggests a version of this statement without the p-torsion-free condition: for a general ring A, a δ-structure on A corresponds to an endomorphism $\phi\colon A \to A$ which is a "derived Frobenius lift". That is, for $\overline{A} = A \otimes^L_{\mathbb{Z}} \mathbb{Z}/(p)$, there exists (and is specified!) a homotopy between the composition $A \xrightarrow{\phi} A \to \overline{A}$ and the composition $A \to \overline{A} \xrightarrow{\text{Frob}} \overline{A}$. This description follows from the previous discussion by interpreting $W_2(A)$ as the fiber product of $A \to \overline{A}$ and $A \to \overline{A} \xrightarrow{\text{Frob}} \overline{A}$.

2.4 The category of δ-rings

Definition 2.4.1 A **morphism** of δ-rings $(A, \delta) \to (A', \delta')$ is a homomorphism $f\colon A \to A'$ of rings such that $f \circ \delta = \delta' \circ f$ (again as maps of sets only). It is evident that with this definition, δ-rings form a category, denoted $\mathbf{Ring}_\delta$. $\Diamond$

Lemma 2.4.2 *Let A be a δ-ring. Let I be an ideal of A such that $\delta(I) \subseteq I$. Then A/I admits a unique δ-ring structure compatible with the map $A \to A/I$.*

Proof. All we need to check is that if $x \equiv y \pmod{I}$, then $\delta(x) \equiv \delta(y) \pmod{I}$. This is apparent from (2.3), which implies that $\delta(x) \equiv \delta(y) + \delta(x-y) \pmod{x-y}$. ■

Lemma 2.4.3 Limits and colimits. *The category $\mathbf{Ring}_\delta$ admits arbitrary limits and colimits. Moreover, the formation of these commutes with the forgetful functor to* $\mathbf{Ring}$.

Proof. For limits, this is pretty straightforward. For colimits, it is perhaps easiest to use Remark 2.3.3: if A is the colimit of a diagram $\{A_i\}$, then we get maps $\operatorname{colim} A_i \to \operatorname{colim} W_2(A_i) \to W_2(\operatorname{colim} A_i)$ whose composition splits the projection map, and then we recover a δ-structure on $\operatorname{colim} A_i$. ■

Remark 2.4.4 The analogue of Lemma 2.4.3 fails for the category of rings with a Frobenius lift; see Exercise 2.5.9. This is one reason why to prefer the category of δ-rings as a basic object of study.

Definition 2.4.5 By Lemma 2.4.3 plus Freyd's adjoint functor theorem (see [94], section V.8 or [117], tag 0AHM) and a set-theoretic consideration (see Remark 2.4.12), the forgetful functor $\mathbf{Ring}_\delta \to \mathbf{Ring}$ admits both a left adjoint and a right adjoint. (More precisely, existence of limits gives the left adjoint, existence of colimits gives the right adjoint.) The right adjoint gives rise to Witt vectors; see Section 3.

The left adjoint can be described concretely in the case of the polynomial ring $\mathbb{Z}[S]$ (for S an arbitrary set, not necessarily finite); it produces the **free δ-ring** on S which we denote by $\mathbb{Z}\{S\}$. Concretely, the underlying ring of $\mathbb{Z}\{S\}$ is given by $\mathbb{Z}[S_0, S_1, \dots]$ where each S_i is a copy of S. The map δ acts on these elements as follows: for $s \in S$ corresponding to $s_i \in S_i$, we have $\delta(s_i) = s_{i+1}$. (To evaluate the left adjoint on an arbitrary ring, we can write it as a quotient of $\mathbb{Z}[S]$ for some S, say by using the adjunction between commutative rings and sets, and then take a quotient of the resulting free δ-ring using Lemma 2.4.2.)

An important corollary of this observation is that every δ-ring can be written as a quotient of some p-torsion-free δ-ring (e.g., if $R \in \mathbf{Ring}_\delta$ then it is a quotient of $Z\{R\}$). This will allow us to reduce many computations to the p-torsion-free case. $\Diamond$

Remark 2.4.6 Note that in Definition 2.4.5, applying the left adjoint to a finitely generated polynomial ring over $\mathbb{Z}$ produces a δ-ring whose underlying ring is *not noetherian*. This suggests that we cannot entertain any hope of staying within the noetherian realm as we go along. This is similar to what happens in **difference algebra**, the study of rings equipped with an endomorphism.

Lemma 2.4.7 *Let (A, δ) be a δ-ring and let ϕ be the associated Frobenius lift. Then ϕ is an endomorphism of (A, δ).*

Proof. We are claiming that $\delta \circ \phi = \phi \circ \delta$; this is a polynomial identity on a single element $x \in A$ and its image under δ, so by lifting to a free δ-ring using Definition 2.4.5, we may reduce to the case where A is p-torsion-free. In this case, the claim reduces to checking that ϕ is an endomorphism of the difference ring (A, ϕ), which is obviously true. ∎

Lemma 2.4.8 *Let A be a δ-ring. Let S be a multiplicative subset of A such that $\phi(S) \subseteq S$. Then there is a unique way to equip $S^{-1}A$ with the structure of a δ-ring so that the localization homomorphism $A \to S^{-1}A$ becomes a morphism of δ-rings.*

Proof. Suppose first that A is p-torsion-free; then so is $S^{-1}A$. Since $\phi(S) \subseteq S$, the map $\phi\colon A \to A$ extends uniquely to a morphism $\phi\colon S^{-1}A \to S^{-1}A$ which is again a Frobenius lift. We thus recover a unique δ-structure on $S^{-1}A$.

In the general case, form a surjection $F \to A$ of δ-rings with F being p-torsion-free. Let T be the preimage of S in F; it is again a multiplicative subset such that $\phi(T) \subseteq T$, so we may uniquely promote $T^{-1}F$ to a δ-ring over T. Since Figure 2.4.9 is a pushout diagram in **Ring**, the pushout in $\mathbf{Ring}_\delta$ (which is a colimit, and hence covered by Lemma 2.4.3) gives us the unique δ-ring structure on $S^{-1}A$. (One can also argue more explicitly using Lemma 2.4.2.)

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & T^{-1}F \arrow[d] \\
A \arrow[r] & S^{-1}A
\end{tikzcd}$$

Figure 2.4.9

∎

Remark 2.4.10 An important special case of Lemma 2.4.8 is localization at a closed subscheme on which p vanishes, taking S to be the complement of the radical ideal defining this subscheme. In this case, the hypothesis $\phi(S) \subseteq S$ is automatically satisfied.

Remark 2.4.11 We do not know whether the following natural generalization of Lemma 2.4.8 holds: if A is a δ-ring and $A \to B$ is an étale morphism of rings, then the ways to promote this to a morphism of δ-rings correspond precisely to the ways to extend the action of ϕ on A to B. (This is true if A is p-torsion-free, but the reduction to this case is a bit subtle.)

Remark 2.4.12 We fill in the missing set-theoretic consideration from Definition 2.4.5. In order to apply the adjoint functor theorem to a functor $F\colon \mathcal{C} \to \mathcal{C}'$ to get the left adjoint, one must know that for every $y \in \mathcal{C}'$, there is a set of elements $x_i \in \mathcal{C}$ such that for any $x \in \mathcal{C}$, any morphism $f\colon y \to F(x)$ factors as $F(g) \circ f_i$ for some i, some $f_i\colon y \to F(x_i)$, and some $g\colon x_i \to x$. This is needed to ensure that when we construct the image of an object under the adjoint functor, we are not trying to take a limit indexed by a class which is too large to be a set.

Similarly, to get the right adjoint, one must know that for every $y \in \mathcal{C}'$, there is a set of elements $x_i \in \mathcal{C}$ such that for any $x \in \mathcal{C}$, any morphism $f: F(x) \to y$ factors as $f_i \circ F(g)$ for some i, some $g: x \to x_i$, and some $f_i: F(x_i) \to F(y)$.

Typically conditions like these are established by taking all of the pairs (x_i, f_i) which satisfy some cardinality bound. For the case of $\mathbf{Ring}_\delta \to \mathbf{Ring}$, see Exercise 2.5.12 and Exercise 2.5.13.

2.5 Exercises

1. Show that in Definition 2.1.1, condition (2.1) cannot be omitted.

Hint. Check that the map $\delta(x) = -x^p/p$ satisfies the other conditions when it is well-defined. (Note that it corresponds to ϕ being the zero map, which is additive and multiplicative but not a homomorphism of unital rings.)

2. Suppose that $p = 2$. For x, y in a δ-ring, compute $\delta(\delta(xy))$ and $\delta(\delta(x+y))$.

3. Assume that $p > 2$. Show that in Example 2.2.4, if we replace A with $A[\zeta_p]$ and extend the map so that it fixes ζ_p, we get an endomorphism which is not a Frobenius lift. (There is a more permissive definition of Frobenius lifts that would allow this, but it does not integrate neatly with the theory of δ-rings.)

4. Prove the following refinement of Lemma 2.2.6: in any δ-ring, every p-power-torsion element is nilpotent. In particular, any reduced δ-ring is p-torsion-free.

Hint. Adapt the proof of Lemma 2.2.8 to show that if $p^n x = 0$, then $p^{n-1}\phi(x) = 0$. See also [25], Lemma 2.28.

5. (Emerton) Prove that for any δ-ring A and any $x \in A$,

$$\phi(x) = p^{p-1}x^p + \delta(px).$$

Hint. Reduce to the p-torsion-free case, then use the fact that ϕ is a ring homomorphism.

6. Let A be a p-adically separated δ-ring. Show that any element of A which admits p^n-th roots in A for all positive integers n must be δ-constant.

Hint. It is enough to show that for each positive integer n, for any $y \in A$ we have $\delta(y^{p^n}) \in p^n A$. This can be checked by reducing to the p-torsion-free case and computing in terms of ϕ.

7. Let A be a δ-ring and let $x \in A$ be an element. Prove that there exists a faithfully flat map $A \to B$ of δ-rings such that the image of x in B belongs to the image of ϕ. That is, "ϕ is fpqc-locally surjective."

Hint. See [25], Corollary 2.12.

8. Let A be a δ-ring. Let I be a finitely generated ideal of A containing p (we do not assume any compatibility with δ). Then the I-adic completion of A admits a unique δ-structure compatible with A.

Hint. See [25], Lemma 2.17.

9. Let $\mathbf{Ring}_\phi$ be the category of rings equipped with a Frobenius lift.

(a) Show that $\mathbf{Ring}_\phi$ admits arbitrary colimits and products, and that these commute with the forgetful functor to rings.

(b) Show that $\mathbf{Ring}_\phi$ also admits equalizers (and hence arbitrary limits), but these do *not* commute with the forgetful functor to rings. This is a reason why to prefer the category $\mathbf{Ring}_\delta$ over $\mathbf{Ring}_\phi$.

10. Let A be a p-torsion-free δ-ring over $\mathbb{Z}_{(p)}$. Define the **divided power operations** $\gamma_n \colon A \to A[p^{-1}]$ by

$$\gamma_n(x) = \frac{x^n}{n!}.$$

Show that if $x \in A$ satisfies $\gamma_p(x) \in A$, then $\gamma_n(x) \in A$ for all $n \geq 0$. In particular, this conclusion does not depend on the δ-ring structure, so it can be used to exhibit an obstruction to the existence of Frobenius lifts on some rings, as in Remark 2.1.4.

Hint. First adapt the calculation from Remark 2.1.4 to show that $\gamma_{p^2}(x) \in A$. Then use this as the basis for an induction on n (uniformly over all A and x), by comparing $\gamma_{kp}(x)$ to $\gamma_k(\gamma_p(x))$. See also [25], Lemma 2.35.

11. Let $A = \mathbb{Z}_{(p)}\{x\}$ and let D be the **divided power envelope** of A with respect to the ideal (x), that is, the smallest subring of $A[p^{-1}]$ with the property that the divided power operations (Exercise 2.5.10) carry every element of the ideal xA into D. (They also carry every element of the ideal xD into D.) Prove that $D = \mathbb{Z}_{(p)}\{x, \frac{\phi(x)}{p}\}$.

Hint. See Corollary 14.3.3.

12. Show that for the forgetful functor $\mathbf{Ring}_\delta \to \mathbf{Ring}$, the set-theoretic condition for the left adjoint in Remark 2.4.12 is satisfied by taking all objects x_i with $|x_i| \leq \max\{|x|, \aleph_0\}$.

Hint. Let $f \colon A \to B$ be a morphism in $\mathbf{Ring}$ with $B \in \mathbf{Ring}_\delta$. Then the δ-subring of B generated by $f(A)$ has cardinality at most $\max\{|A|, \aleph_0\}$.

13. Show that for the forgetful functor $\mathbf{Ring}_\delta \to \mathbf{Ring}$, the set-theoretic condition for the right adjoint in Remark 2.4.12 is satisfied by taking all objects x_i with $|x_i| \leq 2^{\max\{|y|, \aleph_0\}}$.

Hint. Let $f \colon A \to B$ be a morphism in $\mathbf{Ring}$ with $A \in \mathbf{Ring}_\delta$. Let I be the set of $x \in A$ for which $\delta^m(x) \in \ker(f)$ for all $m \geq 0$. Then I is a δ-stable ideal of A and the map

$$A/I \to B \times B \times \cdots, \qquad x \mapsto (f(x), f(\delta(x)), f(\delta^2(x)), \dots)$$

is injective.

14. Show that the functor W_2 on $\mathbf{Ring}$ commutes with filtered colimits, but not with coequalizers.

3 Witt vectors

References. As in Section 2. The original paper of Witt is [127]. See also [116], chapter II, section 6.

We now relate the discussion of δ-rings to the older construction of p**-typical Witt vectors**. Our main goal is to relate this construction to perfect δ-rings Proposition 3.3.6; this only involves evaluating the Witt functor on perfect rings of characteristic p, but to develop the theory it is easier to remember that it defines a functor on *arbitrary* commutative rings.

3.1 p-typical Witt vectors via adjunction

We introduce the p-typical Witt vectors, building upon our work with truncated Witt vectors in Subsection 2.3 and the adjunction between rings and δ-rings.

However, to make this unorthodox development compatible with the more standard treatment (and the big Witt vectors to follow), we must introduce a key change of coordinates.

Definition 3.1.1 As indicated in Definition 2.4.5, the forgetful functor $\mathbf{Ring}_\delta \to \mathbf{Ring}$ admits a right adjoint W. To identify the image of a ring A under this functor, we use the set-theoretic identifications

$$\begin{aligned} W(A) &= \mathrm{Hom}_{\mathbf{Ring}}(\mathbb{Z}[y], W(A)) \\ &= \mathrm{Hom}_{\mathbf{Ring}_\delta}(\mathbb{Z}\{y\}, W(A)) \\ &= \mathrm{Hom}_{\mathbf{Ring}}(\mathbb{Z}[y_0, y_1, \dots], A) \\ &= A \times A \times \cdots . \end{aligned}$$

This means that each element of $W(A)$ has a unique expansion $(y_0, y_1, \dots)$ with each $y_n \in A$; we call the y_n the **y-coordinates** (or **Joyal coordinates**) of this element of $W(A)$. (This presentation does not directly describe the ring structure on $W(A)$; see Remark 4.2.6.)

In Lemma 3.1.3 below. we will give a second set of generators $x_0, x_1, \dots$ of the polynomial ring $\mathbb{Z}[y_0, y_1, \dots]$. This means that each element of $W(A)$ has a unique expansion $(x_0, x_1, \dots)$ with each $x_n \in A$; we call the x_n the **x-coordinates** (or **Witt coordinates**) of this element of $W(A)$. In these coordinates, $W(A)$ will become none other than the ring of **p-typical Witt vectors** over A via the translation described in Definition 3.2.1. $\Diamond$

Remark 3.1.2 Before continuing, we record a statement which will come up repeatedly: for elements x, y of a commutative ring,

$$(x + py)^p \equiv x^p \pmod{p^2 y}. \tag{3.1}$$

In particular,

$$x \equiv y \pmod{p^m} \Rightarrow x^p \equiv y^p \pmod{p^{m+1}}. \tag{3.2}$$

Lemma 3.1.3 *With notation as in Definition 3.1.1, in the ring $\mathbb{Z}\{y\}$ there exist elements*

$$x_n \in y_n + (y_1, \dots, y_{n-1})\mathbb{Z}[y_0, \dots, y_{n-1}] \qquad (n = 0, 1, \dots)$$

such that $x_0 = y_0, x_1 = y_1$, and

$$\phi^n(x_0) = x_0^{p^n} + p x_1^{p^{n-1}} + \cdots + p^n x_n \qquad (n = 0, 1, \dots). \tag{3.3}$$

Proof. Given $x_0, \dots, x_{n-1} \in \mathbb{Z}\{y\}$ satisfying (3.3) with n replaced by $n-1$, use (3.1) to write

$$\begin{aligned} \phi^n(x_0) &= \phi(\phi^{n-1}(x_0)) \\ &= \phi(x_0)^{p^{n-1}} + \cdots + p^{n-1}\phi(x_{n-1}) \\ &= (x_0^p + p*)^{p^{n-1}} + \cdots + p^{n-2}(x_{n-2}^p + p*)^p + p^{n-1}\phi(x_{n-1}) \\ &= x_0^{p^n} + \cdots + p^{n-2}x_{n-2}^{p^2} + p^{n-1}(x_{n-1}^p + p\delta(x_{n-1})) + p^n * \end{aligned}$$

where each $*$ denotes a quantity in $(y_1, \dots, y_{n-1})\mathbb{Z}[y_0, \dots, y_{n-1}]$. We thus can (and must) take

$$x_n = \delta(x_{n-1}) + *.$$

Since $x_{n-1} - y_{n-1} \in \mathbb{Z}[y_0, \dots, y_{n-2}]$, we have

$$\delta(x_{n-1}) = \delta(y_{n-1}) + \delta(x_{n-1} - y_{n-1}) + * = y_n + *$$

and so $x_n - y_n \in (y_1, \dots, y_{n-1})\mathbb{Z}[y_0, \dots, y_{n-1}]$. ∎

Corollary 3.1.4 *For $w_n = \sum_{m=0}^{n} p^m x_m^{p^{n-m}} \in \mathbb{Z}\{y\}$, we have*

$$\phi^n(w_m) = w_{n+m} \qquad (m, n \geq 0).$$

Proof. By Lemma 3.1.3, both sides are equal to $\phi^{n+m}(x_0)$. ∎

Corollary 3.1.5 *In the ring $\mathbb{Z}\{y\}$, we have*

$$\phi(x_n) \equiv x_n^p \pmod{p}.$$

Consequently, for A a ring of characteristic p, the map ϕ on $W(A)$ coincides with the map induced by functoriality by the Frobenius on A. (This is also true in the y-coordinates.)

Proof. The first assertion is a consequence of the fact that ϕ is a Frobenius lift (because $\mathbb{Z}\{y\}$ is a δ-ring). The second assertion is a direct consequence of the first, but let us spell this out for clarity as the mechanism of the argument will recur in what follows. For a general ring A, each of the elements $x_0, x_1, \dots \in \mathbb{Z}\{x\}$ defines a function $W(A) \to A$ which is natural in A. Similarly, every element of $\mathbb{Z}\{x\}$ can be viewed as a "polynomial function" on $W(A)$ valued in A which is again natural in A; that is, we have a map of sets $h \colon \mathbb{Z}\{x\} \to \mathrm{Hom}_{\mathbf{Set}}(W(A), A)$. This map has the property that

$$h(\phi(t))(u) = \phi(h(t)(u)) \qquad (t \in \mathbb{Z}\{x\}, u \in W(A)).$$

In the case where A is of characteristic p, we have $\phi(x_n) = x_n^p + p\delta(x_n)$ and so for any $u \in W(A)$,

$$\begin{aligned} \phi(h(x_n)(u)) &= h(\phi(x_n))(u) \\ &= h(x_n^p + p\delta(x_n))(u) \\ &= h(x_n^p)(u) + ph(\delta(x_n))(u) \\ &= h(x_n)(u)^p. \end{aligned}$$

This shows that the ϕ acts via the functorial Frobenius. ∎

We record some consequences of the adjunction between **Ring** and $\mathbf{Ring}_\delta$.

Definition 3.1.6 The identity map in $\mathrm{Hom}_{\mathbf{Ring}_\delta}(W(A), W(A))$ corresponds via adjunction to a morphism $W(A) \to A$ of rings. In coordinates, this is the map $(x_0, x_1, \dots) \mapsto x_0$.

The identity map in $\mathrm{Hom}_{\mathbf{Ring}}(W(A), W(A))$ corresponds via adjunction to a morphism $\Delta \colon W(A) \to W(W(A))$ in $\mathbf{Ring}_\delta$ which is moreover functorial in A. This map is sometimes called the **diagonal**. ♢

Definition 3.1.7 Recall that the action of δ on $\mathbb{Z}[y_0, y_1, \dots]$ satisfies $\delta(y_n) = y_{n+1}$; consequently, if we express an element of $W(A)$ in the y-coordinates as $(y_0, y_1, \dots)$, it is δ-constant if and only if $y_1 = y_2 = \cdots = 0$. By Lemma 3.1.3, in the usual coordinates, an element $(x_0, x_1, \dots)$ of $W(A)$ is δ-constant if and only if $x_1 = x_2 = \cdots = 0$.

That is, the δ-constants are the image of the multiplicative (but *not additive*; see Exercise 3.6.3) section $[\bullet] \colon A \to W(A)$ of the projection $W(A) \to A$ given by $[x] = (x, 0, 0, \dots)$. We call $[x]$ the **constant lift** (or the **multiplicative lift**) of $x \in A$. ♢

Remark 3.1.8 The constant lift is also known as the **Teichmüller lift**. While this terminology is (fairly) historically accurate, in light of Oswald Teichmüller's role in the Nazi Party and in denunciation of Jewish mathematicians in Germany during the 1930s[5], I prefer to use a non-eponymous terminology in this instance.

3.2 Ghost coordinates

The description of $W(A)$ we are using does not make it especially clear how the addition and multiplication operations work. To clarify this, we relate back to the more standard presentation of Witt vectors.

Definition 3.2.1 Define the elements $w_n \in \mathbb{Z}\{y\}$ as in Corollary 3.1.4. These define a set-theoretic map

$$w\colon W(A) \to A \times A \times \cdots, \qquad (x_n)_{n=0}^{\infty} \mapsto \left(\sum_{m=0}^{n} p^m x_m^{p^{n-m}} \right)_{n=0}^{\infty}.$$

which we call the **ghost map**.

Note that in general, this map is neither injective (unless A is p-torsion-free) nor surjective (unless A is p-divisible). Nonetheless, for $x \in W(A)$, it will be convenient to refer to the terms of $w(x) = (w_0, w_1, \dots)$ as the **ghost coordinates** of x. By Corollary 3.1.4, the ghost coordinates of $\phi^n(x)$ are $(w_n, w_{n+1}, \dots)$.

Now recall the map $x \mapsto w_0 = x_0$ is the homomorphism $W(A) \to A$ obtained by adjunction. It follows that the map $W(A) \xrightarrow{\phi^n} W(A) \to A$ is given by $x \mapsto w_n$. That is, the ghost map is a natural transformation of functors of rings! ♢

Remark 3.2.2 Using the ghost map, we can now see that $W(A)$ agrees with the usual definition of the ring of p-typical Witt vectors of A, in which the arithmetic operations on Witt vectors are given by certain universal polynomials in the entries of the Witt vectors. We may read off properties of these polynomials using functoriality; this is similar to a more typical proof of the existence of the functor W (see for example [73], section 8.10), except that now we don't need to worry about its existence! This means that we can freely pass from general rings to p-torsion-free rings to $\mathbb{Z}[p^{-1}]$-algebras.

Definition 3.2.3 For any ring A, the **Verschiebung** map $V\colon W(A) \to W(A)$ is defined by

$$V(x_0, x_1, \dots) = (0, x_0, x_1, \dots).$$

Note that this corresponds to the map on ghost coordinates given by

$$(w_0, w_1, \dots) \mapsto (0, pw_0, pw_1, \dots).$$

Using the ghost map as per Remark 3.2.2, we may deduce that V is additive (but not multiplicative) and that $\phi \circ V$ acts via multiplication by p. ♢

Definition 3.2.4 Using the method of Remark 3.2.2, we may show that for each positive integer n, there is a natural transformation from W to another functor W_n on **Ring** which on sets corresponds to the projection

$$(x_0, x_1, \dots) \mapsto (x_0, \dots, x_{n-1})$$

(and similarly for ghost components). The ring $W_n(A)$ is called the ring of **truncated p-typical Witt vectors of length n** over A; for $n = 1$ we get A itself, while for $n = 2$ we recover the construction of Definition 2.3.1. Note that

[5] `mathshistory.st-andrews.ac.uk/Biographies/Teichmuller/`

the natural transformation $W \to \lim_n W_n$ is an isomorphism.

The action of ϕ on $W(A)$ does not induce an endomorphism of $W_n(A)$ in general (unless $p = 0$ in A, in which case Corollary 3.1.5 applies). However, it does induce a homomorphism $W_{n+1}(A) \to W_n(A)$ (the **Witt vector Frobenius**), from which we can recover ϕ as the induced map

$$\lim_n W_n(A) = \lim_n W_{n+1}(A) \to \lim_n W_n(A).$$

$\lozenge$

Remark 3.2.5 In [127], what we call **ghost coordinates** were instead called **Nebenkomponenten**, or **secondary components**. The terminology we use here is quite commonplace but its origins are unclear; the earliest reference we were able to find is Barsotti's Mathematical Reviews synopsis of [130], but it seems likely that the terminology was in circulation before that.

3.3 Witt vectors and perfect δ-rings

We now focus more closely on Witt vectors valued in a perfect ring of characteristic p, and obtain their more familiar ring-theoretic properties.

Definition 3.3.1 A δ-ring A is **perfect** if ϕ is an isomorphism. By the same token, a ring of characteristic p is perfect if ϕ is an isomorphism; in this case, injectivity of ϕ is equivalent to A being reduced. $\lozenge$

Lemma 3.3.2 *Let A be a perfect ring of characteristic p. Then the ring $W(A)$ is p-torsion-free and p-adically complete and $W(A)/(p) \cong A$.*

Proof. By Corollary 3.1.5, ϕ is an automorphism of $W(A)$. By Lemma 2.2.8, the ring $W(A)$ is p-torsion-free. Since $\phi \circ V$ is multiplication by p (Definition 3.2.3) and ϕ is bijective, the ideal $pW(A)$ coincides with the image of V, which in turn equals the kernel of the map $W(A) \to A$; hence $W(A)/(p) \cong A$. By similar logic, for each positive integer n, the ideal $p^n W(A)$ coincides with the image of V^n; from this, we see that $W(A)$ is p-adically complete. ■

Example 3.3.3 We have $W(\mathbb{F}_p) \cong \mathbb{Z}_p$. More generally, for any finite extension $\mathbb{F}_q$ of $\mathbb{F}_p$, $W(\mathbb{F}_q) = \mathbb{Z}_p[\zeta_{q-1}]$. □

Definition 3.3.4 For A a perfect ring of characteristic p, Lemma 3.3.2 implies that each element x of $W(A)$ can be written uniquely as a p-adically convergent sum $\sum_{n=0}^{\infty} p^n [x_n]$ with $x_n \in A$, where $[x_n]$ denotes the constant lift (Definition 3.1.7). We call this the **series representation** of x. $\lozenge$

Lemma 3.3.5 *Let R be a perfect ring of characteristic p. Let S be a p-adically complete ring. Then any morphism $R \to S/(p)$ lifts uniquely to a morphism $W(R) \to S$.*

Proof. We first use (3.2) to lift $R \to S/(p)$ to a multiplicative map $R \to S$. Using the series representations from Definition 3.3.4, we then obtain a set-theoretic map $W(R) \to S$ which we must show is an homomorphism; it is enough to check that it induces a homomorphism $W(R) \to S/p^n$ for each n. This is not too onerous to prove by direct computation; see for example [80], Lemma 1.1.7.

A second, more conceptual approach is to apply the fact that if A is a perfect ring of characteristic p, then the cotangent complex $L_{A/\mathbb{F}_p}$ vanishes; we will revisit this comment once we have introduced the cotangent complex in Subsection 17.1. See Exercise 17.5.1 (and [18], Lecture II, Lemma 3.5). ■

Proposition 3.3.6 *The following categories are equivalent (via the functors described below).*

1. *The category of p-adically complete, perfect δ-rings.*

2. *The category of p-torsion-free, p-adically complete rings whose reductions modulo p are perfect.*

3. *The category of perfect rings of characteristic p.*

The functor from (1) to (2) is the forgetful functor; the functor from (2) to (3) is $A \mapsto A/pA$; the functor from (3) to (1) is W.

Proof. The composition from (3) to (1) to (2) to (3) is an equivalence by Lemma 3.3.2. In particular, the functor from (2) to (3) is essentially surjective. By Lemma 3.3.5, the composition from (3) to (1) to (2) is also essentially surjective; hence (2) and (3) are equivalent. We can now use Lemma 2.2.8 and Corollary 3.1.5 to add (1) to the loop. ■

3.4 Beyond the perfect case in characteristic p

It is *not* the case that Proposition 3.3.6 can be extended to relate p-torsion-free, p-adically complete rings whose reductions modulo p are not perfect with the image of the functor W on nonperfect rings of characteristic p. We record some assorted remarks here.

Definition 3.4.1 The inclusion of the full subcategory of perfect rings of characteristic p into arbitrary rings of characteristic p has both left and right adjoints. The left adjoint maps A to $\operatorname{colim}_\phi A$, which we call the **coperfection** of A. The right adjoint maps A to $\lim_\phi A$, which we call the **perfection** of A. ♢

The following examples show that the relationship between the perfection and coperfection can be a bit subtle.

Example 3.4.2 For $A = \mathbb{F}_p[x]$, the coperfection equals $\mathbb{F}_p[x^{p^{-\infty}}]$ while the perfection equals $\mathbb{F}_p$. □

Example 3.4.3 For $A = \mathbb{F}_p[x^{p^{-\infty}}]/(x)$, the coperfection equals $\mathbb{F}_p$ while the perfection equals the x-adic completion of $\mathbb{F}_p[x^{p^{-\infty}}]$. □

Remark 3.4.4 Let A be a p-torsion-free, p-adically complete δ-ring. Let R be the coperfection of $A/(p)$ (Definition 3.3.1). Using Proposition 3.3.6, we obtain a morphism $A \to W(R)$ in $\mathbf{Ring}_\delta$; this map is injective if $A/(p)$ is reduced. If we fix A as an underlying ring while varying its δ-ring structure, the target $W(R)$ remains fixed while the morphism $A \to W(R)$ varies.

Example 3.4.5 Put $A = \mathbb{Z}[x]$. As in Example 2.2.5, for each $y \in A$ there is a unique δ-ring structure on A for which $\delta(x) = y$. Each of these gives rise to an injective morphism $A \to W(\operatorname{colim}_\phi \mathbb{F}_p[x])$ of δ-rings. □

Lemma 3.4.6 *Let $A \to B$ be a morphism of p-torsion-free, p-adically complete rings. Suppose that A is equipped with a δ-ring structure and that $A/(p) \to B/(p)$ is étale. Then B admits a unique δ-ring structure compatible with A.*

Proof. See [18], Lecture II, Lemma 2.9. See also [104] for a supplemental argument that can be used to eliminate the p-torsion-free hypothesis. ■

Remark 3.4.7 Corollary 3.1.5 implies that when A is a reduced ring of characteristic p, the map ϕ on $W(A)$ is injective. By contrast, if A is a nonreduced ring of characteristic p, then ϕ is not injective: for any nonzero $x \in A$ with $x^p = 0$, we have $[x] \neq 0$ but $\phi([x]) = [x^p] = 0$.

If A is a ring not of characteristic p, then the map ϕ on $W(A)$ is not injective either, but this is somewhat more subtle. See Exercise 3.6.7. (One case which is not subtle: if p is invertible in $W(A)$, then the ghost map is an isomorphism and so we may see the kernel of ϕ on the ghost side, remembering that ϕ acts here as the left shift.)

Remark 3.4.8 For A of characteristic p, the map ϕ on $W(A)$ is surjective if and only if it is surjective on A, i.e., if and only if A is **semiperfect**. However, by contrast with Remark 3.4.7, there are many rings A not of characteristic p for which ϕ is surjective on $W(A)$. There are even more rings for which $\phi\colon W_{n+1}(A) \to W_n(A)$ is surjective for each n; these rings are said to be **Witt-perfect** in [40], which see for additional characterizations.

3.5 Additional remarks

Proposition 3.5.1 *For any etale morphism $f\colon A \to B$ and any positive integer n, the map $W_n(f)\colon W_n(A) \to W_n(B)$ is etale.*

Proof. This was originally shown by van der Kallen ([122], (2.4)); see also [28], Theorem B. (Both of these references also cover the truncated big Witt vector functors; see Definition 4.1.3.) For the case of a localization, see also Exercise 3.6.5. ■

Remark 3.5.2 By Proposition 3.5.1, we may apply the functors W_n also to schemes. See [29] for some discussion of this construction.

3.6 Exercises

1. Describe the ring $W(A)$ explicitly for $A = \mathbb{F}_p[x]/(x^p)$, and show that it is a δ-ring with nontrivial p-torsion. (This provides a nontrivial example of Lemma 2.2.8.)

Hint. Use the fact that $\phi \circ V$ acts as multiplication by p.

2. Let A be a p-torsion-free, p-adically complete ring. Let R be the perfection of $A/(p)$ (Definition 3.3.1). Show that the natural maps

$$\lim_{\phi} W(A) \to \lim_{\phi} W(A/(p)) \to \lim_{\phi} W(R) \to W(R)$$

are all isomorphisms.

3. Show that for any ring A, the map $[\bullet]\colon A \to W(A)$ is multiplicative.

Hint. Use the fact that the δ-constant elements both form a multiplicative subset *and* coincide with the image of $[\bullet]$.

4. Let R be a perfect ring of characteristic p. Prove that R is noetherian if and only if R is a finite (possibly empty) direct product of fields. Consequently, $W(R)$ is noetherian if and only if the same conditions hold.

5. Let A be a ring and let S be a multiplicative subset. Let $[S]$ be the image of S under the constant section. Prove that for each positive integer n, there is a natural isomorphism $[S]^{-1}W_n(A) \to W_n(S^{-1}A)$. By contrast, the natural map $[S]^{-1}W(A) \to W(S^{-1}A)$ is not an isomorphism.

Hint. The natural map exists because elements of $[S]$ become units in $W_n(S^{-1}A)$. To show that it is surjective, first use the ghost map (and naturality) to figure out how multiplication by a constant lift acts on the Witt components of a general vector.

6. Show that in Example 3.4.5, the image of $\mathbb{Z}[x]$ in $W(\operatorname{colim}_\phi \mathbb{F}_p[x])$ need not be generated (as a $\mathbb{Z}$-algebra) by multiplicative lifts.

7. Let A be a ring. Show that if $\phi\colon W(A) \to W(A)$ is injective, then $p = 0$ in A.

Hint. Show that any multiple of p occurs as x_0 in some $x = (x_0, x_1, \dots) \in$

$W(A)$ with $\phi(x) = 0$. For more details, see [40], Corollary 2.6.

8. Let R be a perfect ring of characteristic p. Show that $V \circ \phi$ acts on $W(R)$ by multiplication by p (just as $\phi \circ V$ does for arbitrary R).

Hint. Use the fact that ϕ agrees with the functorial Frobenius to see that it commutes with multiplication by p.

9. Prove that for any nonzero ring A, the characteristic of the ring $W(A)$ is either 0 or not divisible by p.

Hint. If any maximal ideal of A has characteristic p, then $W(A)$ maps to a ring of characteristic 0. Otherwise, $W(A)$ splits as a product of copies of A.

10. Prove that Figure 3.6.1 is a pullback square.

$$\begin{tikzcd} W(\mathbb{Z}) \arrow[r] \arrow[d, "w"] & W(\mathbb{Z}_p) \arrow[d, "w"] \\ \mathbb{Z} \times \mathbb{Z} \times \cdots \arrow[r] & \mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \end{tikzcd}$$

Figure 3.6.1

Hint. The corresponding statement with $\mathbb{Z}_p$ replaced by $\widehat{\mathbb{Z}}$ holds because the latter is faithfully flat over $\mathbb{Z}$. Now rewrite $\widehat{\mathbb{Z}}$ as the product of $\mathbb{Z}_p$ with a $\mathbb{Z}[p^{-1}]$-algebra and recall that for the latter, the ghost map is an isomorphism.

11. Show that for $A \in \mathbf{Ring}$, the map

$$N \colon W(A) \to W(A), \qquad N(x) = x - V_p(\delta(x))$$

is multiplicative and satisfies

$$(F \circ N)(x) = x^p \qquad (x \in W(A)).$$

This is called the **norm map** in [3]; see also Exercise 4.3.8.

Hint. Show that the effect on ghost coordinates is given by

$$(w_0, w_1, w_2, \dots) \mapsto (w_0, w_0^p, w_1^p, w_2^p, \dots).$$

4 Big Witt vectors and λ-rings

References. In addition to the references given in Section 2, see [30] and [12] for the perspective of λ-rings and [128] for a comprehensive treatment. (An interesting historical reference, oriented towards characteristic classes of vector bundles, is [14].)

We take a bit of a digression to relate the p-typical Witt vector functor to the **big Witt vector** functor and to the theory of $\boldsymbol{\lambda}$**-rings**. This is not used anywhere in [18] or [25], but we prefer to provide a broader context with an eye towards potential future developments.

In this section, we *do not* fix a prime p.

4.1 The big Witt vector functor

We start with some context from [31]. See Remark 4.2.6 for more of the story.

Definition 4.1.1 We started our description of the p-typical Witt vector functor with the fact that the underlying functor to sets is represented by the ring $A = \mathbb{Z}\{y\}$, but crucially we *already* had produced a functor valued in rings (and even in δ-rings). If we had needed to construct from scratch a functor valued in rings, we would have needed structures on A giving rise to the addition and multiplication maps. These structures are:

- a **coaddition** morphism $\Delta^+ \colon A \to A \otimes_{\mathbb{Z}} A$; and
- a **comultiplication** morphism $\Delta^\times \colon A \to A \otimes_{\mathbb{Z}} A$.

A ring A equipped with these structures represents a functor from rings to sets equipped with two binary operations $+, \times$. A **biring** is a ring equipped with coaddition and comultiplication operators which are further subject to the axioms that correspond to the ring axioms on $+, \times$. Namely, the coaddition map is cocommutative, coassociative, and admits a counit and an antipode (giving rise to additive inverses); the comultiplication map is cocommutative, coassociative, codistributive over coaddition, and admits a counit.

A shorter way to say this is that a biring is a commutative ring object in the category of affine schemes. (Remember that the functor $\operatorname{Spec}\colon \mathbf{Ring} \to \mathbf{Sch}$ is contravariant!) $\diamondsuit$

Proposition 4.1.2 *There is a unique functor $\mathbb{W}$ from* **Ring** *to* **Ring** *characterized by the following conditions.*

- *The underlying functor to sets is*
$$\mathbb{W}(A) = A \times A \times \cdots.$$
- *There is a natural transformation w from $\mathbb{W}$ to the ordinary product $A \mapsto A^{\mathbb{N}}$ given by the* ***ghost map****:*
$$(x_n)_{n=1}^\infty \mapsto (w_n)_{n=1}^\infty, \qquad w_n = \sum_{d|n} d x_d^{n/d}.$$
(Again, the individual factors of this map are called ***ghost components****.)*

The ring $\mathbb{W}(A)$ is called the ***ring of big Witt vectors*** *over A.*

Proof. It suffices to produce a unique biring structure on $\mathbb{Z}[x_1, x_2, \dots]$ representing the desired functor. To begin with, since the ghost map is an isomorphism whenever A is a $\mathbb{Q}$-algebra, we obtain a biring structure on $\mathbb{Q}[w_0, w_1, \dots] = \mathbb{Q}[x_0, x_1, \dots]$; this already implies uniqueness. For existence, it suffices to check that for each prime p, this biring structure descends to $\mathbb{Z}_{(p)}[x_1, x_2, \dots]$; this will imply that the coaddition and comultiplication maps act on $\bigcap_p \mathbb{Z}_{(p)}[x_1, x_2, \dots] = \mathbb{Z}[x_1, x_2, \dots]$.

Define the family of elements y_n of $\mathbb{Q}[x_1, x_2, \dots]$ as follows: for each positive integer m coprime to p and each nonnegative integer i,

$$w_{mp^i} = \sum_{j=0}^{i} p^j y_{mp^j}^{p^{i-j}}.$$

By a calculation which we omit (see Exercise 4.3.1), we see that $\mathbb{Z}_{(p)}[y_1, y_2, \dots] = \mathbb{Z}_{(p)}[x_1, x_2, \dots]$. In the y-coordinates, $\mathbb{W}(A)$ splits into a collection of copies of $W(A)$ indexed by positive integers coprime to p; hence we obtain a biring structure on $\mathbb{Z}_{(p)}[y_1, y_2, \dots] = \mathbb{Z}_{(p)}[x_1, x_2, \dots]$ as needed. ■

Definition 4.1.3 By analogy with Remark 3.2.2, we can detect various additional structures on $\mathbb{W}(A)$ using the ghost map. We leave the details to the reader. (Another approach is to use the **splitting principle**; see Exercise 4.3.5.)

- For any nonempty subset S of the positive integers which is closed under taking divisors, there is a natural transformation from $\mathbb{W}$ to another functor $\mathbb{W}_S$ on **Ring** (the **S-truncated Witt vectors**) which on sets corresponds to the projection

$$(x_1, x_2, \dots) \mapsto (x_n)_{n \in S}$$

  (and similarly for ghost components). In the case where $S = \{1, p, p^2, \dots\}$ for some prime p, this yields a projection $\mathbb{W}(A) \to W(A)$.

- There is a family of commuting endomorphisms $\phi_n \colon \mathbb{W}(A) \to \mathbb{W}(A)$ indexed by positive integers n, which are natural in A and correspond via the ghost map to

$$(w_1, w_2, \dots) \mapsto (w_n, w_{2n}, \dots).$$

  The map ϕ_n induces a map $\mathbb{W}_S(A) \to \mathbb{W}_{S'}(A)$ on truncated Witt vectors whenever $nS' \subseteq S$.

- The map $[\bullet] \colon A \to \mathbb{W}(A)$ given by $[x] = (x, 0, 0, \dots)$ is multiplicative; it corresponds via the ghost map to $x \mapsto (x, x^2, x^3, \dots)$. We again refer to $[x]$ as the **constant lift** of $x \in A$ (see Exercise 4.3.4).

- The **Verschiebung** maps $V_n \colon \mathbb{W}(A) \to \mathbb{W}(A)$, for n a positive integer, defined by

$$V_n((x_m)_{m=1}^{\infty}) = (y_m)_{m=1}^{\infty}, \qquad y_m = \begin{cases} x_{m/n} & m \equiv 0 \pmod{n} \\ 0 & m \not\equiv 0 \pmod{n} \end{cases}$$

  form a commuting family of additive maps such that $\phi_n \circ V_n$ acts by multiplication by n.

- There is a natural transformation $\Delta \colon \mathbb{W} \to \mathbb{W} \circ \mathbb{W}$ (the **diagonal**) such that $\Delta([x]) = [[x]]$ for all $x \in A$,.

$\lozenge$

4.2 λ-rings

Remark 4.2.1 Another interpretation of $\mathbb{W}(A)$ (also due to Witt) can be given starting with the bijection of $\mathbb{W}(A)$ with $1 + TA[\![T]\!]$ given by

$$(x_1, x_2, \dots) \mapsto \prod_{n=1}^{\infty} (1 - x_n T^n)^{-1}. \tag{4.1}$$

When A is a $\mathbb{Q}$-algebra, the addition operation on $\mathbb{W}(A)$ imposed by the ghost map corresponds to the multiplication of formal power series in $1 + TA[\![T]\!]$. This gives us the underlying additive group on $\mathbb{W}(A)$. One then shows that there is a T-adically continuous map $\otimes$ which is natural in A, distributes over addition, and satisfies

$$(1 - aT)^{-1} \otimes (1 - bT)^{-1} = (1 - abT)^{-1}. \tag{4.2}$$

A more conceptual way to express (4.2) is that given two finite projective A-modules M_1, M_2 equipped with A-linear endomorphisms S_1, S_2,

$$\det(1 - TS_1, M_1)^{-1} \otimes \det(1 - TS_2, M_2)^{-1} = \det(1 - T(S_1 \otimes S_2), M_1 \otimes M_2)^{-1}.$$

This point of view appears in work of Almkvist [2] and Grayson [57], [58] in the context of K-theory of endomorphisms. See also Exercise 4.3.6.

Definition 4.2.2 The interpretation from Remark 4.2.1 leads naturally to the related notion of a **λ-ring**. This consists of a ring A together with operations $\lambda^n \colon A \to A$ for $n = 0, 1, \dots$ satisfying various conditions. To begin with,

$$\lambda^0(x) = 1, \lambda^1(x) = x \quad (x \in A).$$

To state the remaining conditions, define the object

$$\Lambda(x) = (1 - \lambda_1(x)T + \lambda_2(x)T^2 - \cdots)^{-1} \in 1 + TA[\![T]\!].$$

In this notation, we impose the conditions

$$\begin{aligned}
\Lambda(x + y) &= \Lambda(x)\Lambda(y) \\
\Lambda(xy) &= \Lambda(x) \otimes \Lambda(y) \\
\Lambda(\lambda^m(x)) &= \wedge^m \Lambda(x)
\end{aligned}$$

where $\otimes$ is the map described in Remark 4.2.1 and $\wedge^m$ is similar; it is the T-adically continuous map characterized by

$$\wedge^m \det(1 - TS, M)^{-1} = \det(1 - T(\wedge^m S), \wedge^m M)^{-1}.$$

(The last condition implies that $\lambda^n(1) = 0$ for all $n \geq 2$.) We define in the obvious way a morphism of λ-rings (as a morphism of underlying rings which commute with the maps λ^n), and hence the category $\mathbf{Ring}_\lambda$ of λ-rings. (One can also express the conditions on the λ^n in terms of certain operations on symmetric functions.)

With this definition, we can show that there is a unique way to promote $\mathbb{W}$ to a functor from $\mathbf{Ring}$ to $\mathbf{Ring}_\lambda$ such that $\Lambda([x]) = (1 - xT)^{-1}$ for every ring A and every element $x \in A$. We omit details here.

The analogue of the adjunction property of the functor W is that $\mathbb{W}$ is a right adjoint of the forgetful functor from $\mathbf{Ring}_\lambda$ to $\mathbf{Ring}$. This follows from the existence of the diagonal transformation $\Delta \colon \mathbb{W} \to \mathbb{W} \circ \mathbb{W}$. $\Diamond$

Remark 4.2.3 In any λ-ring, we can define additional ring homomorphisms ψ^n for $n = 0, 1, \dots$ called **Adams operations**. In the case of $\mathbb{W}(A)$, these are characterized by T-adic continuity and the property

$$\psi^n \det(1 - TS, M)^{-1} = \det(1 - TS^n, M)^{-1};$$

this implies that

$$\psi^n([x]) = [x^n],$$

from which we can deduce that in fact $\psi^n = \phi_n$.

In general, the maps ψ^p for p prime form a family of pairwise commuting Frobenius lifts; moreover, a λ-ring is a δ-ring for every prime p. Conversely (and analogously to Lemma 2.1.3), for a $\mathbb{Z}$-torsion-free ring any family of pairwise commuting Frobenius lifts gives rise to a unique λ-ring structure (see [126]).

Example 4.2.4 Equip the ring $A = \mathbb{Z}[q]$ with the endomorphisms ψ^p sending q to q^p for each prime p. By the criterion of Remark 4.2.3, these occur as the Adams operations for a unique λ-ring structure on A. Similarly, the rings $\mathbb{Z}[\![q-1]\!]$ and $\mathbb{Z}[\![q-1]\!][(q-1)^{-1}]$ admit λ-ring structures.

If one wishes to avoid the $(q-1)$-adic completion, the ring

$$\mathbb{Z}[q, (q-1)^{-1}, (q^2-1)^{-1}, \dots]$$

also admits a λ-ring structure. □

Remark 4.2.5 Some additional examples of λ-rings occurring "in nature" include:

- the ring of symmetric polynomials over $\mathbb{Z}$ (see Remark 4.2.7);
- the representation ring of a finite group (see [86] for more on the relationship with the previous example);
- the Grothendieck ring of the category of finite projective modules over a commutative ring;
- the K-theory of a topological space (or a connective spectrum).

Remark 4.2.6 In [31] one finds the concept of a **plethory**, which is a monoid in the category of birings; the functors W and $\mathbb{W}$ are represented by such objects. (The name comes from the operation of **plethysm** from representation theory or the corresponding operation in the theory of symmetric polynomials.) The systematic study of plethories, which builds upon ideas from the subject of **universal algebra** (see especially [120]), provides a natural context in which to talk about variant constructions. For example, for a prime p and a finite extension E of $\mathbb{Q}_p$, one can define a functor of **ramified Witt vectors** valued in $\mathfrak{o}_E$-algebras. (See any of [41], section 1; [62], (18.6.13); or [36]. See also [34] for the corresponding version of p-derivations.)

Remark 4.2.7 The category $\mathbf{Ring}_\lambda$ admits all limits and colimits, and these are compatible with the forgetful functor to **Ring** (either by direct calculation, or using the interpretation from [31]). Consequently, the forgetful functor from λ-rings to rings admits a left adjoint; as in Definition 2.4.5, the value of the left adjoint on the free polynomial ring $\mathbb{Z}[S]$ is the **free λ-ring** on S. (The free λ-ring on a single element is the λ-ring of symmetric polynomials over $\mathbb{Z}$.)

Remark 4.2.8 Circling back to the original interpretation of a δ-ring as a ring in which one can "differentiate with respect to p", one can think of a λ-ring as a ring equipped with descent data from $\operatorname{Spec} \mathbb{Z}$ to something "below". That putative object shares some of the expected characteristics of a mythical object called the **field with one element**; another (nonmythical) object that does likewise is the **sphere spectrum** in algebraic topology.

4.3 Exercises

1. Complete the proof of Proposition 4.1.2 by proving that

$$\mathbb{Z}_{(p)}[y_1, y_2, \dots] = \mathbb{Z}_{(p)}[x_1, x_2, \dots].$$

Hint. Using the equality

$$\sum_{j=0}^{i} p^j y_{mp^j}^{p^{i-j}} = w_{mp^i} = \sum_{j=0}^{i} p^j \sum_{d|m} d x_{dp^j}^{p^{i-j} m/d},$$

show by induction on i that

$$y_{mp^i} = \sum_{d|m} d x_{dp^i}^{m/d} + *$$

where $* \in \mathbb{Z}_{(p)}[x_{dp^j} : d|m, j < i]$.

2. Check that the map (4.1) defines a homomorphism between the additive group of $\mathbb{W}(A)$ and the multiplicative group $1 + TA[\![T]\!]$.

3. Let X, Y be two schemes of finite type over a finite field $\mathbb{F}_q$. Let $Z(X/\mathbb{F}_q, T)$ and $Z(Y/\mathbb{F}_q, T)$ be the zeta functions of X and Y, respectively.

(a) Prove that

$$Z((X \times_{\mathbb{F}_q} Y)/\mathbb{F}_q, T) = Z(X/\mathbb{F}_q, T) \otimes Z(Y/\mathbb{F}_q, T)$$

where $\otimes$ is the operation on $1+T\mathbb{Z}[\![T]\!]$ corresponding to multiplication in $\mathbb{W}(\mathbb{Z})$ via the isomorphism (4.1).

(b) Prove that for any positive integer n,

$$Z((X \times_{\mathbb{F}_q} \mathbb{F}_{q^n})/\mathbb{F}_{q^n}) = \psi^n(Z(X, T))$$

where ψ^n is the n-th Adams operation (Remark 4.2.3).

Hint. Note that the second statement is a special case of the first. To prove the first, write $Z(X, T)$ as the product of $(1 - T^{\deg(x/\mathbb{F}_q)})^{-1}$ as x varies over closed points of X, and similarly for Y; then describe the closed points of $X \times_{\mathbb{F}_q} Y$ and appeal to (4.2).

4. Prove the following analogue of Definition 3.1.7: for any ring A, the elements of $\mathbb{W}(A)$ in the kernel of the p-derivation for all primes p are precisely the constant lifts. (Combined with Remark 4.2.1, this explains the terminology **elements of rank 1** in [25] for what we call **δ-constant elements** of a δ-ring.)

Hint. First show that the elements in the kernels of all of the p-derivations form a set stable under the Frobenius maps to reduce to checking the vanishing of the Witt components for all nontrivial prime powers. Then use the projection maps $\mathbb{W}(A) \to W(A)$ to reduce to the p-typical case.

5. Let A be a ring and let $x \in \mathbb{W}(A)$ be an element. Prove that for each positive integer n, there exists a faithfully flat ring map $A \to B$ such that the image of x in $\mathbb{W}(B) \cong 1 + TB[\![T]\!]$ is congruent modulo T^{n+1} to a sum of constant elements. This is sometimes called the **splitting principle**, as it allows various algebraic properties of the big Witt vectors (or more generally of λ-rings) to be verified using arithmetic on constant elements. (This occurs frequently in the theory of **characteristic classes** of vector bundles, as in [119].)

6. Let A be a ring. Show that under the identification $\mathbb{W}(A) \cong 1 + TA[\![T]\!]$, the power series which represent rational functions of T form a subring of $\mathbb{W}(A)$. (Compare Remark 4.2.1.)

7. Let $p_1, \dots, p_n$ be distinct primes and let S be the set of positive integers of the form $p_1^{e_1} \cdots p_n^{e_n}$ for some nonnegative integers $e_1, \dots, e_n$. Let W_{p_i} denote the p_i-typical Witt vector functor. Show that there exists a natural isomorphism

$$W_{p_1} \circ \cdots \circ W_{p_n} \cong \mathbb{W}_S$$

of functors from **Ring** to **Ring**.

8. For $A \in \mathbf{Ring}$ and p prime, define the **p-norm map** $N_p \colon \mathbb{W}(A) \to \mathbb{W}(A)$ by

$$N_p(x) = x - V_p(\delta_p(x)),$$

where δ_p is the p-derivation associated functorially to the Frobenius lift ψ^p (see Remark 4.2.3). Prove that the maps N_p are multiplicative, commute with each other, and satisfy

$$(\psi_p \circ N_p)(x) = x^p \qquad (x \in \mathbb{W}(A)).$$

As in Exercise 3.6.11, see [3] for some discussion of the role of this construction in algebraic topology.

5 Distinguished elements and prisms

Reference. [18], Lecture III. The underlying reference is [25], section 2.

Using the framework of δ-rings, we now set up the formalism of **prisms**, modulo a key technical detail: the difference between **classical completion** and **derived completion** with respect to an ideal. We postpone discussion of the latter until Section 6.

Notation. For $A \in \mathbf{Ring}$, let $\mathbf{Mod}_A$ denote the category of A-modules. For an ideal I of A, and an object $M \in \mathbf{Mod}_A$, write $M[I]$ for the I-torsion submodule of M and $M[I^\infty]$ for the union $\bigcup_n M[I^n]$; if $I = (f)$ is principal, we also notate these as $M[f]$ and $M[f^\infty]$.

5.1 Distinguished elements and examples

We begin by singling out elements of a δ-ring which behave as if they "vanish to order 1", as indicated by the p-derivation.

Definition 5.1.1 Let A be a δ-ring. An element $d \in A$ is **distinguished** if $(p, d, \delta(d))$ is the unit ideal of A. That is, the intersection of the zero loci of $p, d, \delta(d)$ on $\operatorname{Spec} A$ is empty.

If A is (p, d)-local, then $d \in A$ is distinguished if and only if $\delta(d)$ is a unit in A; in fact this is the definition used in [18] and [25]. The discrepancy will not affect the definition of a prism because the latter already includes a completeness hypothesis (see Definition 5.3.1). One confusing aspect of our definition is that units in A are always distinguished.

In many arguments that follow, we can reduce to the (p, d)-local case by localizing A at (p, d). By Remark 2.4.10, the result is still a δ-ring. $\lozenge$

Remark 5.1.2 Any morphism in $\mathbf{Ring}_\delta$ carries distinguished elements to distinguished elements. The converse holds for the map ϕ; see Exercise 5.5.1.

We describe a series of examples which will be related to various preexisting p-adic cohomology theories. We will promote these examples to prisms in Remark 5.3.4.

Example 5.1.3 Crystalline cohomology. Take $A = \mathbb{Z}_p$ with $d = p$. Then $\delta(d) = 1 - p^{p-1} \equiv 1 \pmod p$, so p is distinguished. By the same token, by Remark 2.2.3, p is distinguished in any δ-ring. $\square$

Example 5.1.4 q-de Rham cohomology and Wach modules. Take $A = \mathbb{Z}_p[\![q-1]\!]$ with the δ-structure for which $\phi(q) = q^p$, and define d to be the

"q-analogue of p":

$$d = [p]_q = \frac{q^p - 1}{q - 1} = \sum_{i=0}^{p-1} q^i.$$

Under the map $A \to \mathbb{Z}_p$ taking q to 1, d maps to p which is distinguished in the target; it follows that d is itself distinguished.

This example is closely related to Fontaine's theory of **(φ, Γ)-modules**. The original construction of Fontaine [49] described an equivalence of categories between the continuous representations of the absolute Galois group of $\mathbb{Q}_p$ on finite $\mathbb{Z}_p$-modules and a certain category of finite modules over the p-adic completion of $A[(q-1)^{-1}]$, in which the continuous action of the monoid $\mathbb{Z}_p \setminus \{0\}$ on A characterized by $\gamma(q) = q^\gamma$ is extended to the module. The elements of the p-adic completion can be viewed as formal Laurent series in $q - 1$ with coefficients in $\mathbb{Z}_p$; it was later shown by Cherbonnier and Colmez [39] that the base ring can be shrunk down to the subring consisting of Laurent series whose negative tails converge on some region (see also [80]).

The ring A itself is the base ring of the theory of **Wach modules** [124], [13]; the (φ, Γ)-module associated to a Galois representation descends to a Wach module if and only if the representation is **crystalline** in Fontaine's sense. Similar considerations apply if we enlarge A by replacing $\mathbb{Z}_p$ with an unramified extension $\mathfrak{o}_K$, where now the Galois group in question is that of K. $\square$

Example 5.1.5 Breuil-Kisin cohomology. Let $K/\mathbb{Q}_p$ be a finite extension. Let π be a uniformizer of K. Let $W \subseteq \mathfrak{o}_K$ be the maximal unramified subring (i.e., the ring $W(k)$ where k is the residue field of $\mathfrak{o}_K$). Take $A = W[\![u]\!]$ with the δ-structure extending the canonical one on W for which $\phi(u) = u^p$. Take d to be a generator of the kernel of the map $A \to \mathfrak{o}_K$ taking u to π; by projecting along the map $u \mapsto 0$ as in Example 5.1.4, we see that d is distinguished.

The ring A is the base ring of the theory of **Breuil-Kisin modules** ([85]), which provides an alternative to Wach modules that can be used to classify crystalline representations of the Galois group of a *ramified* extension of $\mathbb{Q}_p$. See [37] for more on the parallel between the two constructions. $\square$

Example 5.1.6 $\mathbf{A}_{\mathrm{inf}}$-cohomology. Let A be the $(p, q-1)$-adic completion of $\mathbb{Z}_p[q^{p^{-\infty}}]$. By Proposition 3.3.6, we have an isomorphism $A \cong W(R)$ where R is the $(q-1)$-adic completion of the coperfection of $\mathbb{F}_p[q-1]$. In particular, A has a unique δ-ring structure, for which $\phi(q) = q^p$; note that in this case ϕ is an automorphism. By Example 5.1.4, $d = [p]_q$ is a distinguished element, as is $\phi^n(d)$ for any $n \in \mathbb{Z}$.

Let K be the p-adic completion of the p-cyclotomic extension $\mathbb{Q}_p(\mu_{p^\infty})$. The ring R can then be identified with the perfection of $\mathfrak{o}_K/(p)$ by fixing a choice of a coherent sequence (ζ_{p^n}) of p-power roots of unity and identifying q with this sequence; this identifies R with the **tilt** of $\mathfrak{o}_K$ (see Section 7 for further discussion). In this context, the ring A arises in Fontaine's notation as the value of the functor $\mathbf{A}_{\mathrm{inf}}$ evaluated at the valuation ring $\mathfrak{o}_K$. $\square$

5.2 Properties of distinguished elements

We collect some lemmas about distinguished elements. See also Lemma 7.1.2 for a precise characterization of distinguished elements in $W(R)$ when R is a perfect ring of characteristic p.

We first show that "distinguished elements are locally irreducible."

Lemma 5.2.1 *Let A be a δ-ring and choose $f, h \in A$. Then fh is distinguished if and only if f and h are both distinguished and (p, f, h) is the unit ideal of A.*

Proof. Suppose first that fh is distinguished. By (2.2),

$$\delta(fh) = h^p\delta(f) + f^p\delta(h) + p\delta(h)\delta(f) \equiv h^p\delta(f) \pmod{(p, f)}. \tag{5.1}$$

If fh is distinguished, then $\delta(fh)$ is a unit modulo (p, fh) and hence also modulo (p, f); we deduce that both $\delta(f)$ and h are invertible modulo (p, f). This means that f is distinguished and (p, f, h) is the unit ideal; by symmetry, h is also distinguished.

Conversely, suppose that f and h are both distinguished and that (p, f, h) is the unit ideal. To check that $(p, fh, \delta(fh))$ is the unit ideal, we may work in the localizations at (p, f) and (p, h); without loss of generality, we may then assume that $p, f \in \mathrm{Rad}(A)$. In this case, $\delta(f)$ and h are both units, and so (5.1) implies that $\delta(fh)$ is a unit modulo $(p, f) = (p, fh)$; hence fh is distinguished. ■

Remark 5.2.2 While Lemma 5.2.1 is written in a symmetric manner, in practice we will use it in the case where $p, f \in \mathrm{Rad}(A)$. We again reiterate that according to our conventions, any unit is a distinguished element.

We now see that the property of an element being distinguished depends only on the principal ideal generated by that element.

Lemma 5.2.3 *Let A be a δ-ring. Then an element $f \in A$ is distinguished if and only if $p \in (p^2, f, \phi(f))$. (If A is p-local, this is equivalent to $p \in (f, \phi(f))$.)*

Proof. If f is distinguished, then $ap + bf + c\delta(f) = 1$ for some $a, b, c \in A$. Since $\phi(f) - f^p = p\delta(f)$, we can write $ap^2 + bfp + c\phi(f) - cf^p = p$, yielding $p \in (p^2, f, \phi(f))$. Conversely, suppose that $p \in (p^2, f, \phi(f))$ and (by way of contradiction) $(p, f, \delta(f))$ is not the unit ideal; using Remark 2.4.10, we may localize A to reduce to the case where $p, f, \delta(f) \subseteq \mathrm{Rad}(A)$ (and $A \neq 0$). In this case, $p \in (f, \phi(f))$, so there exist $a, b \in A$ such that $p = af + b\phi(f)$; that is,

$$p(1 - b\delta(f)) = af + bf^p = f(a + bf^{p-1}).$$

Since p is distinguished (Example 5.1.3), so is f by two applications of Lemma 5.2.1 (one in each direction); this yields the desired contradiction. ■

Corollary 5.2.4 *For A a δ-ring, the property of $d \in A$ being distinguished depends only on the image of d in A/p^2.*

Proof. This is immediate from Lemma 5.2.3. ■

It will be convenient later to globalize the notion of an ideal generated by a distinguished element. Fortunately, the resulting condition still has a convenient characterization.

Lemma 5.2.5 *Let A be a p-local δ-ring. Let I be a locally principal ideal of A contained in $\mathrm{Rad}(A)$. Then the following conditions are equivalent.*

1. *We have $p \in I + \phi(I)A$.*
2. *There exists a faithfully flat map $A \to A'$ of p-local δ-rings which is an ind-Zariski localization, such that $IA' = (f)$ for some distinguished element f of A' contained in $\mathrm{Rad}(A')$.*

Moreover, if these conditions hold, then $p \in I^p + \phi(I)A$.

Proof. The equivalence of (1) and (2) is a consequence of Remark 2.4.10 (which allows us to construct A' such that IA' is principal) and Lemma 5.2.3. Compare [18], Lecture III, Corollary 1.9 or [25], Lemma 3.1.

To check that (1) and (2) imply $p \in I^p + \phi(I)A$, we may reduce to the case where $I = (f)$ for some distinguished element f of A. In this case, the equation $\phi(f) = f^p + p\delta(f)$ shows that $p \in (f^p, \phi(f))$ because $\delta(f)$ is a unit. ■

5.3 Prisms

A prism will consist of a δ-ring A and an ideal I such that the closed subschemes of $\operatorname{Spec} A$ defined by I and $\phi^{-1}(I)$ intersect "as transversely as possible" along the closed subscheme defined by p.

Definition 5.3.1 A **δ-pair** consists of a pair (A, I) in which A is a δ-ring and I is an ideal.

A **prism** is a δ-pair (A, I) satisfying the following conditions.

- The ideal I defines a Cartier divisor on $\operatorname{Spec} A$ (i.e., I is an invertible A-module, or equivalently I is locally principal generated by a non-zerodivisor). In most of our examples, I will be principal; see Exercise 6.7.14 for a restriction that applies otherwise.
- The ring A is derived (p, I)-complete (as a module over itself). We will define this condition a bit later (see Definition 6.2.1); for the moment, note that it implies $(p, I) \subseteq \operatorname{Rad}(A)$ (see Corollary 6.3.2) and hence also $\phi(I) \subseteq \operatorname{Rad}(A)$. See also Remark 5.3.2.
- We have $p \in I + \phi(I)A$. By Lemma 5.2.3, this holds if I is generated by a distinguished element.

A prism (A, I) is **orientable** if the ideal I is principal. A prism (A, I) is **oriented** if it is orientable *and* we have fixed the choice of a generator d, which by Lemma 5.2.3 is a distinguished element (and a non-zerodivisor).

A prism (A, I) is **bounded** if A/I has bounded p^∞-torsion; that is, there is a positive integer n such that $(A/I)[p^n] = (A/I)[p^\infty]$. $\Diamond$

Remark 5.3.2 Definition 5.3.1 includes a condition on derived completeness that we have not yet defined. We insert a few remarks in order to maintain the narrative flow.

If A is classically (p, I)-complete, then A is derived (p, I)-complete. The converse holds if A is (p, I)-adically separated; this will be true in particular if (A, I) is a bounded prism (see Lemma 6.4.2).

For these reasons, on first reading it is safe to pretend that Definition 5.3.1 requires A to be classically (p, I)-complete rather than derived (p, I)-complete. However, when proving theorems it will be problematic to take completions due to the bad behavior of this functor in some situations (Remark 6.1.2). The notion of derived completeness will help mitigate this, as will the odd definition of flatness for morphisms of prisms (Definition 5.4.3).

Example 5.3.3 A δ-pair (A, I) with $I = (p)$ is a prism if and only if A is p-torsion-free and classically p-complete. We say that such a prism is **crystalline**. $\square$

Remark 5.3.4 By Lemma 5.2.3 (and the fact that the rings in question are all integral), all of the examples of distinguished elements enumerated in Subsection 5.1 give rise to prisms (taking $I = (d)$). These examples are all bounded.

Example 5.1.3 is an example of a crystalline prism. Example 5.1.6 is an example of a perfect prism; we will describe these in terms of perfectoid rings in Section 7.

Example 5.3.5 The universal oriented prism. Let $A_0 = \mathbb{Z}_{(p)}\{d\}$ be the free δ-ring in a single variable d over $\mathbb{Z}_{(p)}$. Let S be the multiplicative subset of A_0 generated by $\phi^n(\delta(d))$ for all $n \geq 0$. By Lemma 2.4.8, the localization $A_1 = S^{-1}A_0$ is also a δ-ring. Let A be the derived (p, d)-completion of A_1; since A_1 is p-torsion-free, A is classically p-complete. By construction, d is a distinguished element of A and (A, dA) is a bounded prism. Moreover, p, d is a regular sequence in A and $\phi\colon \overline{A} \to \overline{A}$ is p-completely flat (see Definition 6.5.1). $\square$

Lemma 5.3.6 *Let (A, I) be a prism. Then the ideal $\phi(I)A$ is principal, and any generator of it is a distinguished element.*

Proof. It will be enough to produce a single generator of $\phi(I)A$, as then Lemma 5.2.3 (which applies because $pA + \phi(I) \subseteq \operatorname{Rad}(A)$) will imply that any other generator is also distinguished.

By definition, we have $p = a + b$ with $a \in I^p, b \in \phi(I)A$; we will show that b generates $\phi(I)A$ and is distinguished. Choose a faithfully flat map $A \to A'$ of δ-rings as per Lemma 5.2.5; it will suffice to show that b generates $\phi(I)A'$ and is distinguished in A'. By construction, IA' is generated by a distinguished element $d \in A'$. Write $a = xd^p, b = y\phi(d)$ for some $x, y \in A'$. Since $\phi(d)$ is also distinguished, it will suffice to show that y is a unit in A'. Since $pA+I \subseteq \operatorname{Rad}(A)$, it will further suffice to show that $pA' + IA' + yA' = A'$.

Suppose the contrary; using Remark 2.4.10, we may choose a further localization $A' \to A''$ of δ-rings such that $pA'' + IA'' + yA'' \subseteq \operatorname{Rad}(A'')$. The equation $p = a + b = xd^p + y\phi(d)$ yields

$$p(1 - y\delta(d)) = a + (b - py\delta(d)) = d^p(x + y) = d(d^{p-1}(x + y)).$$

Since $1 - y\delta(d)$ is a unit in A'' and p is distinguished, we may apply Lemma 5.2.1 twice to deduce that d is distinguished in A'' and $d^{p-1}(x + y)$ is a unit; this is impossible because $d \in \operatorname{Rad}(A'')$. (Compare [18], Lecture III, Lemma 3.5 or [25], Lemma 3.6.) $\blacksquare$

Remark 5.3.7 Let (A, I) be a prism. Since I is an invertible A-module, $I \otimes_A A/I = I/I^2$ is an invertible A/I-module, as is I^n/I^{n+1} for any nonnegative integer n. These will appear in the discussion of Hodge-Tate cohomology.

5.4 The category of prisms

Definition 5.4.1 The category of δ-pairs is defined so that a morphism $(A, I) \to (B, J)$ is a morphism $f\colon A \to B$ of δ-rings such that $f(I) \subseteq J$. The category of prisms, denoted **Prism**, is defined as the full subcategory of the category of δ-pairs consisting of prisms. $\Diamond$

Lemma 5.4.2 Rigidity of prisms. *Let $(A, I) \to (B, J)$ be a morphism in* **Prism**. *Then the natural map $I \otimes_A B \to J$ is an isomorphism of B-modules. In particular, $J = IB$.*

Proof. Since the map in question is between invertible B-modules, it is enough to check that it is surjective. Using Lemma 5.2.5, we may reduce to the case where $I = (f)$ and $J = (g)$ are both principal ideals generated by distinguished elements. Then f is a multiple of g in B, so we may apply Lemma 5.2.1 to conclude. (Compare [18], Lecture III, Lemma 3.7 or [25], Lemma 3.5.) $\blacksquare$

Definition 5.4.3 A map $(A, I) \to (B, J)$ in **Prism** is **(faithfully) flat** if B is I-completely (faithfully) flat in the sense of Definition 6.5.1. This holds in particular if $A \to B$ is (faithfully) flat. $\Diamond$

5.5 Exercises

1. Let A be a δ-ring. Prove that an element $d \in A$ is distinguished if and only if $\phi(d)$ is distinguished.

6 Derived completeness

Reference. [117], tag 091N, 0BKF. See also [117], tag 0BKH for the case of a noetherian ring, where some simplifications occur.

We fill in a missing detail from Section 5, namely the distinction between classical and derived completeness of a module with respect to an ideal. As you might imagine, the latter is best treated within the framework of **derived categories**; we will do as much as we can in classical language, and end with a few statements which one may want to postpone until after reading Section 10.

6.1 The trouble with classical completion

Definition 6.1.1 For $A \in \mathbf{Ring}$ and I a *finitely generated* ideal of A, an A-module M is **classically I-complete** if the natural map $M \to \lim_n M/I^n M$ is an isomorphism. In particular, this means that M is I-adically separated: $\bigcap_n I^n M = 0$. ◊

Remark 6.1.2 Completions behaving badly. Much of our intuition about completion of modules with respect to an ideal is derived from the case of finitely generated modules over a noetherian ring. A few pitfalls to keep in mind include the following.

- Classically I-complete modules do not form an abelian subcategory of the category of all A-modules. For example, it is possible for the quotient of A by a principal ideal to be noncomplete; see [117], tag 05JD. This is remedied by using derived I-complete modules instead; see Proposition 6.3.1.

- The completion functor $M \mapsto \lim_n M/I^n M$ preserves surjections, but it is not even right exact even on finitely presented modules. (This is arguably not surprising because completion is the composition of a right exact functor with a left exact functor.) See [117], tag 05JF and also Example 6.1.4.

- The completion of a flat module (or even a flat A-algebra) need not be flat; see Example 6.1.3. This phenomenon will force us to adopt Definition 6.5.1.

If we drop the restriction that I be finitely generated, then things get even stranger. For example, completion is no longer an idempotent operation; see [117], tag 05JA.

Example 6.1.3 For $A \in \mathbf{Ring}$, $A[\![x]\!]$ is the x-adic completion of the flat A-algebra $A[x]$, but is flat over A if and only if A is **coherent** (every finitely generated ideal is finitely presented); see [117], tag 0ALB. See [117], tag 0AL8 for a concrete example.

On a related note, one can construct a ring A and an element $f \in A$ such that $A_f[\![x]\!]$, which is the x-adic completion of the flat $A[\![x]\!]$-algebra $A[\![x]\!]_f$, is not itself flat over $A[\![x]\!]$. See again [117], tag 0AL8. □

One can see some additional issues with classical completion from the following basic example (adapted from [129], Example 3.20; see also [117], tag 0G3F).

Example 6.1.4 Take $A = \mathbb{Z}_p, I = pA$. Let M_0 be the set of sequences $(x_n)_{n=0}^\infty$ over $\mathbb{Z}_p$ with $\lim_{n\to\infty} x_n = 0$. Let $M_1 \subset M_0$ be the set of sequences $(x_n)_{n=0}^\infty$ with $x_n \equiv 0 \pmod{p^n}$. Let $M_2 \subset M_1$ be the set of sequences $(x_n)_{n=0}^\infty$ with $\lim_{n\to\infty} x_n/p^n = 0$.

The modules $M_0, M_1, M_2, M_0/M_1, M_1/M_2$ are all classically I-complete (note that M_0 is isomorphic to M_2 via the map $(x_n) \mapsto (x_n p^n)$). However, M_0/M_2 is not I-adically separated: we have $\bigcap_n p^n(M_0/M_2) = M_1/M_2$. (In other words, the closure of M_2 in M_0 is equal to M_1.) Consequently, applying the completion functor to the exact sequence

$$0 \to M_2 \to M_0 \to M_0/M_2 \to 0$$

yields the sequence

$$0 \to M_2 \to M_0 \to M_0/M_1 \to 0$$

which is not exact in the middle. □

6.2 Derived completeness

Definition 6.2.1 For $A \in \mathbf{Ring}$ and I a finitely generated ideal of A, an A-module M is **derived I-complete** if for each $f \in I$,

$$\operatorname{Hom}_A(A_f, M) = 0 \text{ and } \operatorname{Ext}^1_A(A_f, M) = 0. \tag{6.1}$$

By Lemma 6.2.3, it will suffice to check this condition for f running over a generating set of I (or of any ideal with the same radical as I). ♢

Remark 6.2.2 When working with (6.1), note that A_f admits the following free resolution as an A-module:

$$0 \to A[T] \stackrel{\times(1-Tf)}{\to} A[T] \stackrel{T \mapsto f^{-1}}{\to} A_f \to 0$$

Consequently, for any A-module M, $\operatorname{Ext}^n_A(A_f, M) = 0$ for $n \geq 2$. Since $\operatorname{Hom}_A = \operatorname{Ext}^0_A$, this means that (6.1) can be reformulated as

$$\operatorname{Ext}^n_A(A_f, M) = 0 \qquad (n \geq 0). \tag{6.2}$$

For example, this makes it clear that if any two terms in a short exact sequence are derived I-complete, then so is the third.

Lemma 6.2.3 *For $A \in \mathbf{Ring}$ and $M \in \mathbf{Mod}_A$, the set of $f \in A$ for which (6.1) holds is a radical ideal of A.*

Proof. Let I be the set in question. For $f \in I, g \in A$, the functor $\operatorname{Hom}_A(A_{fg}, \bullet)$ factors as

$$M \mapsto \operatorname{Hom}_A(A_f, M) \mapsto \operatorname{Hom}_A(A_g, \operatorname{Hom}_A(A_f, M)).$$

From the spectral sequence for a composition of functors (or more elementary considerations), we see that if $\operatorname{Ext}^n_A(A_f, M) = 0$ for all $n \geq 0$, then $\operatorname{Ext}^n_A(A_{gf}, M) = 0$ for all $n \geq 0$; hence $fg \in I$.

For $f, g \in I$, the sequence

$$0 \to A_{f+g} \to A_{f(f+g)} \oplus A_{g(f+g)} \to A_{fg(f+g)} \to 0$$

is exact because f, g generate the unit ideal in A_{f+g}. (As an aside, see [81], Lemma 1.6.12 for another application of this observation.) Since $f(f+g), g(f+g), fg(f+g) \in I$ by the previous paragraph, using the snake lemma we obtain $f + g \in I$. Consequently, I is an ideal of A.

For any $f \in I$ and any positive integer n, $A_f = A_{f^n}$. Hence I is a radical ideal of A. (Compare [117], tag 091Q.) ■

Lemma 6.2.4 *Let I be a finitely generated ideal of $A \in \mathbf{Ring}$ and choose $M \in \mathbf{Mod}_A$.*

1. *If M is classically I-complete, then (6.1) holds for all $f \in I$.*
2. *Conversely, if (6.1) holds for all $f \in I$, then $M \to \lim_n M/I^n M$ is surjective.*

Proof. If M is classically I-complete, we have

$$\operatorname{Hom}_A(A_f, M) = \operatorname{Hom}_A(A_f, \lim_n M/I^n M) = 0.$$

To show that $\operatorname{Ext}^1_A(A_f, M) = 0$, consider an extension

$$0 \to M \to E \to A_f \to 0. \tag{6.3}$$

For each $n \geq 0$, pick $e_n \in E$ mapping to $f^{-n} \in A_f$ and set $\delta_n = fe_{n+1} - e_n \in M$. Since M is complete, we may define the elements

$$e'_n = e_n + \delta_n + f\delta_{n+1} + f^2\delta_{n+2} + \cdots$$

which satisfy $fe'_{n+1} = fe'_n$; we thus obtain a map $A_f \to E$ splitting the sequence by mapping f^{-n} to e'_n.

In the converse direction, by an elementary argument (Exercise 6.7.1) we can reduce to the case where $I = (f)$. That, we must show that if (6.1) holds, then for any $x_0, x_1, \ldots \in M$ there exists $x \in M$ such that $x \equiv x_0 + fx_1 + \cdots + f^{n-1}x_{n-1} \pmod{f^n M}$ for each n. To this end, form an extension as in (6.3) by taking

$$E = M \oplus \bigoplus_n Ae_n/(x_n - fe_{n+1} + e_n)$$

with e_n mapping to f^{-n} in A_f; note that by the snake lemma, $M/f^n M = E/f^n E$ for all n. Since the extension splits by hypothesis, there is an element $x + e_0 \in E$ which generates a copy of A_f in E. We then have

$$x + e_0 = x - x_0 + fe_1 = x - x_0 + x_1 + f^2 e_2 = \cdots;$$

this yields the desired result. (Compare [117], tag 091R.) ■

Corollary 6.2.5 *Let I be a finitely generated ideal of $A \in \mathbf{Ring}$ and choose $M \in \mathbf{Mod}_A$. Then M is classically I-complete if and only if M is I-adically separated and derived I-complete.*

Proof. This is immediate from Lemma 6.2.4. ■

The following can be viewed as an algebraic version of the **open mapping theorem** from functional analysis.

Proposition 6.2.6 *Let I be a finitely generated ideal of $A \in \mathbf{Ring}$ and let M be a derived I-complete A-module. If M is I-power torsion, then $I^n M = 0$ for some integer n.*

Proof. See [117], tag 0CQY. ■

6.3 The category of derived-complete modules

We next introduce the category of derived I-complete modules and its basic properties. This leads naturally to the operation of **derived completion**.

Proposition 6.3.1 *Let I be a finitely generated ideal of $A \in \mathbf{Ring}$.*

1. *Derived Nakayama's lemma.*

Let M be a derived I-complete A-module. Then $M = 0$ if and only if $M/IM = 0$.

2. *The inclusion functor from derived I-complete A-modules to A-modules admits a left adjoint $M \mapsto \widehat{M}$, called* ***derived I-completion****. (We will often write $M_I^\wedge$ for the derived I-completion so that we can specify the ideal I in the notation.)*

3. *The full subcategory of the category of A-modules consisting of derived I-complete A-modules is an abelian category. More precisely, it is closed under formation of kernels, cokernels, and images in the ambient category. (It is moreover a* ***weak Serre subcategory*** *of $\mathbf{Mod}_A$ in the sense of [117], tag 02M0.)*

Proof. See [117], tags 0G1U, 091V, 091U. ∎

Corollary 6.3.2 *Let I be a finitely generated ideal of $A \in$ **Ring**. Suppose that A is derived I-complete (as a module over itself).*

1. *We have $I \subseteq \mathrm{Rad}(A)$.*

2. *Every finitely presented A-module is derived I-complete.*

3. *The pair (A, I) is henselian.*

Proof. To prove (1), choose any $u \in 1 + I$, then apply derived Nakayama (Proposition 6.3.1 to $M = A/(u)$ to deduce that $u \in A^\times$.

To prove (2), apply part (3) of Proposition 6.3.1.

For (3), see [117], tag 0G3H. ∎

Remark 6.3.3 With notation as in Example 6.1.4, Proposition 6.3.1 implies that the A-module M_0/M_2, which is not I-adically separated, is nonetheless derived I-complete.

Remark 6.3.4 The category of derived I-complete A-modules does not have the property that filtered colimits are exact ([117], tag 0ARC). In particular, it is not a **Grothendieck abelian category**.

On the other hand, a *countably filtered* colimit of derived I-complete A-modules is again derived I-complete. The point is that any potential witness to the failure of completeness can be expressed using only countably many module elements.

Remark 6.3.5 For $I = (f_1, \dots, f_n)$, the derived I-completion functor from Proposition 6.3.1 can be described as the composition of the derived f_i-completions for $i = 1, \dots, n$ (in any order). These individual functors can be described concretely using Lemma 6.4.1. For an alternate description in the language of derived categories, see Proposition 6.6.2.

Definition 6.3.6 Let I be a finitely generated ideal of $A \in$ **Ring**. We may then promote the derived I-completion functor from A-modules to A-algebras as follows.

For any A-algebra B, the multiplication map on B defines a morphism $B \otimes_A B \to B$ of A-modules. Let $\widehat{B}$ denote the derived I-completion of B. Now consider the composition

$$\widehat{B} \otimes_A \widehat{B} \to \widehat{B \otimes_A B} \to \widehat{B}$$

where the first map is induced by the individual maps $B \to B \otimes_A B$ and the second map is induced by the multiplication morphism. This gives us a multiplication map on $\widehat{B}$ which gives it the structure of an A-algebra. In particular, the derived I-completion $\widehat{A}$ of A is itself an A-algebra, and the ring $\widehat{B}$ is also an $\widehat{A}$-algebra. ◊

6.4 Derived f-completion

The following lemma gives an explicit recipe for derived completion with respect to a principal ideal.

Lemma 6.4.1 *Choose $A \in \mathbf{Ring}$ and $f \in A$.*

1. *The derived f-completion functor (Proposition 6.3.1) is given by*

$$M \mapsto \widehat{M} = \operatorname{Ext}^1_A(A_f/A, M).$$

2. *For any A-module M, we have a natural (in M) exact sequence*

$$0 \to R^1 \lim_n M[f^n] \to \widehat{M} \to \lim_n M/f^n M \to 0, \tag{6.4}$$

in which the modules $M[f^n]$ form a projective system via multiplication by f; compare [117], tag 0BKG. (Note that if $M = \widehat{M}$, then the last map is the surjection from Lemma 6.2.4.)

Proof. From the exact sequence

$$0 \to A \to A_f \to A_f/A \to 0$$

we obtain a morphism $M \to \operatorname{Ext}^1_A(A_f/A, M)$ which is an isomorphism whenever M is derived I-complete; we claim that the target of this map is in fact $\widehat{M}$. Namely, if $M \to N$ is another morphism with N derived I-complete, then by functoriality of Ext^1_A in the second argument we obtain a unique morphism $\operatorname{Ext}^1_A(A_f/A, M) \to \operatorname{Ext}^1_A(A_f/A, N) \cong N$ through which $M \to N$ factors. This yields (1); by writing $A_f/A = \operatorname{colim}_n f^{-n}A/A$, we may then deduce (2). ■

As an application of Lemma 6.4.1, we obtain a criterion that lets us forget about the difference between classical and derived completions in many cases of interest.

Lemma 6.4.2 *Let I be a finitely generated ideal of $A \in \mathbf{Ring}$ and choose $M \in \mathbf{Mod}_A$. If M has bounded f^∞-torsion (that is, there exists a positive integer n such that $M[f^n] = M[f^\infty]$), then the derived I-completion of M is classically I-complete.*

Proof. The derived completion $\widehat{M}$ fits into an exact sequence given in (6.4). By the assumption about bounded torsion, the projective system formed by the $M[f^n]$, in which the transition maps are multiplicaton by f, is essentially zero (that is, any sufficiently long composition is the zero map). Hence the R^1 term vanishes and we obtain the desired conclusion. (Compare [18], Lecture III, Lemma 2.4.) ■

A key application of Lemma 6.4.2 is the following.

Lemma 6.4.3 *Let R be a perfect ring of characteristic p. Then for any $f \in R$, $R[f^\infty] = R[f]$. Consequently (by Lemma 6.4.2), the derived f-completion of R coincides with the classical f-adic completion.*

Proof. For $x \in R[f^\infty]$, we have $f^{p^n} x = 0$ for some nonnegative integer n. We then have $f^{p^n} x^{p^n} = 0$, and then $fx = 0$ because R is perfect. ■

Remark 6.4.4 One way to make a ring which is derived f-complete but not classically f-complete is to start with a ring A, an element $f \in A$, and a module M which is derived f-complete but not classically f-complete (e.g., see Remark 6.3.3), and then form the square-zero extension $A \oplus M$.

6.5 Flatness and smoothness

Definition 6.5.1 Let I be a finitely generated ideal of $A \in \mathbf{Ring}$. We say that $M \in \mathbf{Mod}_A$ is **I-completely flat** (resp. **I-completely faithfully flat**) if M/IM is a flat (resp. faithfully flat) A/I-module and $\mathrm{Tor}_i^A(A/I, M) = 0$ for $i > 0$. Equivalently, for any $N \in \mathbf{Mod}_{A/I}$, $\mathrm{Tor}_i^A(N, M) = 0$ for $i > 0$. If M is (faithfully) flat, then it is I-completely (faithfully) flat (Exercise 6.7.6), but not conversely (see Remark 6.1.2).

More generally, we say that M has **finite I-complete Tor amplitude** if there exists some $c > 0$ such that $\mathrm{Tor}_i^A(A/I, M) = 0$ for $i \geq -c$. Equivalently, for any $N \in \mathbf{Mod}_{A/I}$, $\mathrm{Tor}_i^A(N, M) = 0$ for $i \geq c$. $\lozenge$

Definition 6.5.2 Let I be a finitely generated ideal of $A \in \mathbf{Ring}$. A derived I-complete A-algebra R is **I-completely étale** (resp. **I-completely smooth**, **I-completely ind-smooth**) if $R \otimes_A^L A/I$ is concentrated in degree 0 where it is an étale (resp. smooth, ind-smooth) A/I-algebra. That is, $R \otimes_A^L A/I$ is an étale (resp. smooth, ind-smooth) A/I-algebra and $\mathrm{Tor}_i^A(R, A/I) = 0$ for $i > 0$. $\lozenge$

Proposition 6.5.3 *Let I be a finitely generated ideal of $A \in \mathbf{Ring}$. Let R be a derived I-complete A-algebra. Then R is I-completely étale (resp. I-completely smooth) if and only if it is the derived I-completion of some étale (resp. smooth) A-algebra.*

Proof. This follows from an algebraization theorem originally due to Elkik [45]. For a more modern treatment, see [9]. ■

Remark 6.5.4 An I-completely smooth morphism is sometimes called a "formally smooth" morphism, but this is not entirely accurate: the latter simply means that the infinitesimal lifting criterion is satisfied ([117], tag 00TI). An I-completely smooth morphism is formally smooth, but formal smoothness is a meaningful condition without any reference to derived completeness.

By contrast, a **smooth** morphism of rings is one which is formally smooth and of finite presentation (see [117], tag 02H6). In general an I-completely smooth morphism is not of finite type and hence not smooth; for instance, $\mathbb{Z}_p \to \mathbb{Z}_p[x]^\wedge_{(p)}$ is p-completely smooth but not of finite type.

6.6 Derived completeness in the derived category

If you don't yet know what derived categories are, skip this discussion for now and return after you have read Section 10.

Definition 6.6.1 Let I be a finitely generated ideal of $A \in \mathbf{Ring}$. An object K in the derived category $D(A)$ of A-modules is **derived I-complete** if for each $f \in I$,

$$R\operatorname{Hom}_A(A_f, K) = 0.$$

The argument of Lemma 6.2.3 carries over to show that it suffices to check this condition at a generating set of I, or of any other ideal with the same radical.

One immediate consequence of this definition is that if any two terms of a distinguished triangle are derived I-complete, then so is the third. In particular, the mapping cone of a morphism between derived I-complete complexes is derived I-complete; see Remark 6.6.6. $\lozenge$

Proposition 6.6.2 *Let I be a finitely generated ideal of $A \in \mathbf{Ring}$.*

1. *Derived Nakayama's lemma.*

   Let $K \in D(A)$ be a derived I-complete object. Then $K = 0$ if and only if $K \otimes_A^L A/I = 0$.

2. *The derived I-complete objects of $D(A)$ form a full triangulated subcategory $D_{\mathrm{comp}}(A)$ which is closed under derived inverse limits. This inclusion has a left adjoint $K \mapsto \widehat{K}$ (the* ***derived I-completion****) which can be computed as follows: for $I = (f_1, \dots, f_r)$,*

$$\widehat{K} = R\lim_n (K \otimes^L_{\mathbb{Z}[x_1,\dots,x_r]} \mathbb{Z}[x_1,\dots,x_r]/(x_1^n,\dots,x_r^n))$$

for the action of $\mathbb{Z}[x_1,\dots,x_r]$ on K with x_i acting via f_i.

3. *An object $K \in D(A)$ is derived I-complete if and only if $H^i(K)$ is derived I-complete for each $i \in \mathbb{Z}$.*

Proof. See [117], tags 091V, 091Z, 0G1U. ■

Remark 6.6.3 In general, an A-module admits derived I-completions "as a module", as an object in $\mathbf{Mod}_A$ as per Proposition 6.3.1; and also "as a complex", i.e., by identifying the module with a singleton complex in $D(A)$ concentrated in degree 0 and then applying Proposition 6.6.2. The former is H^0 of the latter, but the other cohomology groups of the latter carries some extra information that is invisible at the module-theoretic level; see Example 6.6.5 for a typical example.

However, if an A-module M is derived I-complete "as a module", it is also derived I-complete "as a complex": Proposition 6.6.2 says that the latter condition applied to $M[0]$ can be tested at the level of cohomology groups.

Remark 6.6.4 In the case where $I = (f)$ is a principal ideal, we may adapt the proof of Lemma 6.4.1 to show that the derived f-completion of an A-module M as a complex is concentrated in degrees -1 and 0, with the cohomology in degree -1 being $\lim_n M[f^n]$. Consequently, Lemma 6.4.2 can be upgraded to assert that the derived f-completion as a complex is also isomorphic to the classical f-completion.

Example 6.6.5 Take $A = \mathbb{Z}, I = (p), M = \mathbb{Q}/\mathbb{Z}$. Then the ordinary completion of M vanishes, as does the derived completion of M *as a module*, but the derived completion of M *as a complex* is the group $\mathbb{Z}_p$ concentrated in degree -1. □

Remark 6.6.6 We will frequently used the derived Nakayama's lemma (Proposition 6.6.2) in the following form: for $f \colon K^\bullet \to L^\bullet$ a morphism of derived I-complete complexes, f is a quasi-isomorphism if and only if the induced morphism $K^\bullet \otimes^L_A A/I \to L^\bullet \otimes^L_A A/I$ of complexes over A/I is a quasi-isomorphism. To be precise, this will follow from applying Proposition 6.6.2 to the mapping cone of f (Definition 10.2.2).

Definition 6.6.7 Let I be a finitely generated ideal of $A \in \mathbf{Ring}$. We say that a complex K of A-modules is ***I*-completely flat** if for any I-torsion A-module N, the derived tensor product $K \otimes^L_A N$ is concentrated in degree 0. Equivalently, $K \otimes^L_A A/I$ is concentrated in degree 0 where it is a flat A/I-module. If in addition $K \otimes^L_A A/I$ is faithfully flat as an A/I-module, we say that K is ***I*-completely faithfully flat**. Note that these definitions agree with the corresponding notions for modules (Definition 6.5.1).

Similarly, we say that K has **finite *I*-complete Tor amplitude** if there exists some $c \geq 0$ such that for any I-torsion A-module N, $K \otimes^L_A N$ is concentrated in degrees $\geq -c$. ◊

6.7 Exercises

1. Let I be a finitely generated ideal of $A \in \mathbf{Ring}$ and choose $M \in \mathbf{Mod}_A$. Suppose that for each $f \in I$, the map $M \to \lim_n M/f^n M$ is surjective. Prove that the map $M \to \lim_n M/I^n M$ is surjective.

Hint. See [117], tag 090S.

2. Let I be a finitely generated ideal of $A \in \mathbf{Ring}$. Let $M \to N$ be a morphism of derived I-complete A-modules such that $M/IM \to N/IN$ is surjective. Prove that $M \to N$ is surjective.

3. Let A be a ring which is derived f-complete, but not classically f-complete, for some $f \in A$ (e.g., see Remark 6.4.4). Let I be the kernel of the surjective (by Lemma 6.2.4) map from A to its classical f-completion $\lim_n A/f^n$.

(a) Show that $I^2 = 0$.

(b) Show that any classically f-complete A-module is also a module over $\lim_n A/f^n$ (that is, it is annihilated by I).

(c) Deduce that A, as a module over itself, cannot be written as the quotient of a classically f-complete A-module.

Hint. See [117], tag 0G3G.

4. Let I be a finitely generated ideal of $A \in \mathbf{Ring}$.

(a) Show that the kernel of any morphism between classically I-complete A-modules is classically I-complete.

(b) Let M be an A-module which can be written as the cokernel of a morphism between classically I-complete A-modules. Show that for any classically I-complete A-module N and any surjective morphism $f\colon N \to M$ of A-modules, $\ker(f)$ is classically I-complete.

(c) Deduce that the abelian closure of the category of classically I-complete A-modules consists of the cokernels of morphisms between classically I-complete A-modules. By Exercise 6.7.3, this can be strictly smaller than the category of derived I-complete A-modules. (See however Exercise 6.7.5.)

5. Let f be a non-zerodivisor in some $A \in \mathbf{Ring}$. Show that $M \in \mathbf{Mod}_A$ is derived f-complete if and only if there exists a short exact sequence

$$0 \to K \to L \to M \to 0$$

in which $K, L \in \mathbf{Mod}_A$ are f-torsion-free and f-adically complete.

Hint. [117], tag 09AT.

6. Let I be a finitely generated ideal of $A \in \mathbf{Ring}$. Show that for any flat A-module M, the derived I-completion of M as a complex is I-completely flat.

Hint. Use the adjunction property of derived I-completion.

7. Let $A \in \mathbf{Ring}$ be noetherian and let I be an ideal of A. Suppose that $M \in D(R)$ is I-completely flat. Then the derived I-completion of M is concentrated in degree 0, where it is a flat A-module.

Hint. See [17], Lemma 5.15.

8. Suppose that $A \in \mathbf{Ring}$ is derived I-complete for some finitely generated ideal I. Prove that a map $M_1 \to M_2$ between finite projective A-modules is surjective (resp. bijective) if and only if the induced map $M_1/IM_1 \to M_2/IM_2$ is surjective (resp. bijective). In particular, a finite projective A-module M is free if and only if M/IM is a free A/I-module.

Hint. For surjectivity, apply derived Nakayama (Proposition 6.3.1) to the cokernel. For bijectivity, first verify that M_1 and M_2 have the same rank everywhere (using the fact that $I \subseteq \mathrm{Rad}(A)$), then recall that a surjective map between projective modules of the same finite rank is a bijection.

9. Suppose that $A \in \mathbf{Ring}$ is derived I-complete for some finitely generated ideal I. Prove that the base extension functor from finite projective A-modules to finite projective A/I-modules is essentially surjective.

Hint. Start with a finite projective A/I-module $\overline{M}$. Choose a finite free A/I-module $\overline{F}$ and a projector $\overline{U}\colon \overline{F} \to \overline{F}$ whose image is isomorphic to $\overline{M}$. View $\overline{F}$ as the reduction of a finite free A-module F and choose an endomorphism $U_0\colon F \to F$ lifting $\overline{U}$. Show by induction that for each positive integer n, U_0 is congruent modulo I to an endomorphism U_n such that $U_n^2 \equiv U_n \pmod{I^{n+1}}$. Take the limit to lift to the classical I-completion of A, then use the fact that the kernel of the map from A to the classical completion has square zero (Exercise 6.7.3) to lift to A. Conclude that we have produced a projector on F whose image has base extension isomorphic to $\overline{M}$.

10. Suppose that $A \in \mathbf{Ring}$ is derived I-complete for some finitely generated ideal I. Prove that the natural map $\mathrm{Pic}(A) \to \mathrm{Pic}(A/I)$ is an isomorphism.

Hint. Apply Exercise 6.7.8 and Exercise 6.7.9.

11. Let I be a finitely generated ideal of $A \in \mathbf{Ring}_\delta$ containing p. Let $A \to B$ be a morphism in $\mathbf{Ring}$ such that B is derived I-complete and $A \to B$ is I-completely etale. Then $A \to B$ promotes uniquely to a morphism in $\mathbf{Ring}_\delta$.

Hint. Using Remark 2.3.3, it suffices to lift $A \to B$ to a morphism $W_2(A) \to W_2(B)$. Achieve this by combining Proposition 6.5.3 with Proposition 3.5.1. (Compare [25], Lemma 2.18.)

12. Let I be a finitely generated ideal of $A \in \mathbf{Ring}$ containing p. Prove that the derived I-completion of A (as a module, viewed as an A-algebra using Definition 6.3.6) admits a unique δ-ring structure compatible with A. (That is, Exercise 2.5.8 remains true when the classical completion is replaced by the derived completion.)

Hint. Use the characterization of δ-structures on A in terms of $W_2(A)$ (Remark 2.3.3).

13. Let (A, I) be a bounded prism. Show that for any flat A-module M, the derived (p, I)-completion of M as a complex is concentrated in degree 0 and is both classically (p, I)-complete and (p, I)-completely flat.

14. Let (A, I) be a prism. Prove that the class of I in $\mathrm{Pic}(A)$ is p-torsion.

Hint. First use Exercise 6.7.10 to argue that $\mathrm{Pic}(A) \to \mathrm{Pic}(A/p)$ is an isomorphism. Then show that ϕ induces the p-power map on $\mathrm{Pic}(A/p)$, and apply Lemma 5.3.6 to conclude. See also [25], Lemma 3.6.

7 Perfect prisms

Reference. [18], Lecture IV.

In this lecture, we study perfect prisms (i.e., prisms with a bijective Frobenius map) in detail. These end up being closely related to **perfectoid rings**, which appear frequently in p-adic Hodge theory; however, we will not use too much of the existing theory of perfectoid rings, and in fact we will end up recovering some of it via a different approach.

Notation. For I an ideal in a commutative ring, write $\sqrt{I}$ for its radical.

7.1 Distinguished elements in perfect δ-rings

Recall that the condition of an element of a δ-ring being distinguished is meant to capture the idea that "the p-adic order of vanishing equals 1". For perfect δ-rings, we can further develop this metaphor to assert that "the linear coefficient in p is a unit".

The following will be used later in the discussion of perfect prisms (see Lemma 7.1.2).

Lemma 7.1.1 *Let A be a p-local, p-torsion-free, p-adically separated δ-ring in which A/p is reduced (e.g., $W(R)$ where R is a perfect $\mathbb{F}_p$-algebra). Suppose that $d \in \operatorname{Rad}(A)$ is a distinguished element.*

1. *In the ring A, d is a non-zerodivisor.*
2. *We have $(A/d)[p^\infty] = (A/d)[p]$.*

Proof. To prove (1), suppose by way of contradiction that $df = 0$ for some nonzero $f \in A$. Since A is p-torsion-free and p-adically separated, we may divide f by a suitable power of p to reduce to the case where $f \notin pA$. Now

$$0 = \delta(df) = f^p\delta(d) + \delta(f)\phi(d).$$

Multiplying by $\phi(f)$ and using that ϕ is a ring homomorphism, we obtain

$$0 = f^p\phi(f)\delta(d) + \delta(f)\phi(df) = f^p\phi(f)\delta(d).$$

Since A is (p, d)-local, $\delta(d)$ is a unit in A, so $f^p\phi(f) = 0$. Reducing modulo p, we obtain $f^{2p} \equiv 0 \pmod{p}$. Since A/p is reduced, this implies $f \equiv 0 \pmod{p}$, contradicting our earlier choice of f and thus proving the claim.

To prove (2), it is enough to show that $(A/d)[p^2] = (A/d)[p]$. That is, given $f, g \in A$ with $p^2 f = gd$, we must have $pf \in dA$. Since $gd \in p^2A$, we have $\delta(gd) \in pA$ and hence $\phi(g)\delta(gd) \in pA$. Rewriting this as $\delta(d)g^p\phi(g) + \delta(g)\phi(gd)$, we see that $\delta(d)g^p\phi(g) \in pA$. Since A is (p, d)-local, $\delta(d)$ is a unit in A, so $g^p\phi(g) \in pA$ and so $g^{2p} \in pA$. Because A/p is reduced, this implies $g \in pA$; since A is p-torsion-free, this implies that $pf \in dA$ as desired. (Compare [25], Lemma 2.34.) ∎

Lemma 7.1.2 *Let R be a perfect $\mathbb{F}_p$-algebra. Then $d = \sum_{n=0}^\infty [x_n]p^n \in W(R)$ is distinguished if and only if (x_0, x_1) is the trivial ideal of R. In particular, if $d \in \operatorname{Rad}(W(R))$ (which means $x_0 \in \operatorname{Rad}(R)$), then d is distinguished if and only if x_1 is a unit.*

Proof. Use (3.1) to write $d^p \equiv [x_0]^p \pmod{p^2}$ and

$$p\delta(d) = \phi(d) - d^p \equiv p[x_1]^p \pmod{p^2}$$

to deduce that the ideals $(p, d, \delta(d))$ and $(p, [x_0], [x_1])$ coincide. ∎

Remark 7.1.3 The criterion for distinguished elements in Lemma 7.1.2 coincides with Fontaine's notion of a **primitive element of degree** 1. While this terminology was introduced in [50], it echoes similar constructions found elsewhere (e.g., [79]).

7.2 Perfect prisms

Definition 7.2.1 A prism (A, I) is **perfect** if A is a perfect δ-ring. $\Diamond$

Theorem 7.2.2 *Let (A, I) be a perfect prism.*

1. *The ideal I is principal, and any generator d of I is a distinguished element and a non-zerodivisor.*
2. *The ring A is p-torsion-free and classically (p, I)-complete.*
3. *We have a canonical isomorphism $A \cong W(A/(p))$ of δ-rings.*
4. *We have $(A/I)[p^\infty] = (A/I)[p]$ and $(A/p)[I^\infty] = (A/p)[I]$. In particular, (A, I) is a bounded prism.*

Proof. By Lemma 5.3.6, the ideal I is principal and any generator d of I is a distinguished element. By Lemma 2.2.8, A is p-torsion-free.

The ring $A/(p)$ is perfect (by functoriality) and derived I-complete (by Proposition 6.3.1, it being the cokernel of $A \stackrel{\times p}{\to} A$). By Lemma 6.4.3, $A/(p)$ is also classically I-complete. By induction on n using the exact sequence

$$0 \to p^{n-1}A/p^nA \to\to A/p^n \to A/p^{n-1} \to 0$$

and the isomorphism $A/p \cong p^{n-1}A/p^nA$ of A-modules (a consequence of A being p-torsion-free), we deduce that each quotient $A/(p^n)$ is classically I-complete.

Since A is p-torsion-free and derived p-complete, it is also classically p-complete by Lemma 6.4.2. By the previous paragraph, it is also classically (p, I)-complete.

By Proposition 3.3.6, $A \cong W(A/p)$. By Lemma 7.1.2, any generator d of I is a non-zerodivisor. By Lemma 6.4.3, $(A/p)[I^\infty] = (A/p)[I]$. By Lemma 7.1.2, $(A/I)[p^\infty] = (A/I)[p]$. ∎

Proposition 7.2.3 *The inclusion of the category of perfect prisms into* **Prism** *admits a left adjoint. Given a prism (A, I), the left adjoint is obtained by taking the classical (p, I)-completion of the coperfection of A (which we call the **coperfection** of (A, I)).*

Proof. Let A' be the coperfection of A; by Lemma 2.2.8, A' is p-torsion-free. Let A'' be the classical p-completion of A'; by Lemma 6.4.2, A'' is also the derived p-completion. By Exercise 2.5.8, A'' can be canonically promoted to a δ-ring over A. Now Proposition 3.3.6 implies $A'' \cong W(A''/p)$.

For each positive integer n, we may now argue as in the proof of Theorem 7.2.2 that the derived I-completion of A/p^n coincides with the classical completion. Consequently, if we take A''' to be the classical (p, I)-completion of A'' (or equivalently of A'), then A''' also equals the derived (p, I)-completion of either A' or A''. By Exercise 2.5.8, A''' can be canonically promoted to a δ-ring over A''. Again, Proposition 3.3.6 implies $A''' \cong W(A'''/p)$.

At this point, (A''', IA''') is a prism (the conditions on the ideal IA''' are implied by the corresponding conditions on I) and A''' is universal for maps of A to derived (p, I)-complete δ-rings. Thus the proof is complete. (Compare [18], Lecture IV, Lemma 1.3 or [25], Lemma 3.9.) ∎

7.3 Tilting and slicing

Definition 7.3.1 For any prism (A, I) (perfect or not), define the **slice** (or **face**) of (A, I) as the ring $\overline{A} = A/I$. Define the **tilt** of (A, I) (or of $\overline{A}$), denoted $\overline{A}^\flat$, as the perfection of $\overline{A}/p$.

Suppose that (A, I) is bounded, so that $\overline{A}$ is classically p-complete. Using Lemma 3.3.5, we may lift the projection map $\overline{A}^\flat \to \overline{A}/p$ uniquely to a map $\theta_A \colon W(\overline{A}^\flat) \to \overline{A}$. $\Diamond$

Remark 7.3.2 The term **slice** is not standard terminology. Another reasonable name would be the **special fiber**, in the sense that the prism is some sort of "thickening" of the slice.

Proposition 7.3.3 *Let (A, I) be a perfect prism with slice $\overline{A}$ and tilt $\overline{A}^\flat$. We then have a commutative diagram as in Figure 7.3.4 in which the horizontal arrows are all surjective, the vertical arrows are all reductions modulo p, and the diagonal arrows are all isomorphisms. Moreover, this diagram is natural in (A, I).*

$$\begin{tikzcd}
A \arrow[rr, two heads] \arrow[dr, "\sim"] \arrow[dd] & & \overline{A} \arrow[dr, equal] \arrow[dd] & \\
& W(\overline{A}^\flat) \arrow[rr, two heads, "\theta_A"] \arrow[dd] & & \overline{A} \arrow[dd] \\
A/p \arrow[rr, two heads] \arrow[dr, "\sim"] & & \overline{A}/p \arrow[dr, equal] & \\
& \overline{A}^\flat \arrow[rr, two heads] & & \overline{A}/p
\end{tikzcd}$$

Figure 7.3.4

Proof. Everything will follow once we construct a natural isomorphism $A \cong W(\overline{A}^\flat)$. By Theorem 7.2.2, it will suffice to construct a natural isomorphism $A/p \cong \overline{A}^\flat$.

By Theorem 7.2.2, I admits a generator d which is a distinguished element. By definition, we have $\overline{A}/p = A/(p, d)$. For each positive integer n, the n-fold Frobenius $A/(p, d) \to A/(p, d)$ identifies with the canonical map $A/(p, d^{p^n}) \to A/(p, d)$ compatibly with n, so the limit $\lim_\phi \overline{A}/p$ gets identified with $\lim_\phi A/(p, d^{p^n})$. The latter is naturally isomorphic to $A/(p)$ because the latter is clasically d-complete (Lemma 6.4.3). ■

Theorem 7.3.5 *The slice functor $(A, I) \mapsto \overline{A}$ restricts to a fully faithful functor from perfect prisms to* **Ring**.

Proof. It will suffice to explain how to recover A and I functorially from $\overline{A}$. Since $\overline{A}$ is in the essential image of the functor, $\phi \colon \overline{A}/p \to \overline{A}/p$ is surjective and so $\overline{A}^\flat \to \overline{A}/p$ is surjective. Consequently, $\theta_A \colon W(\overline{A}^\flat) \to \overline{A}$ is also surjective. We can now reconstruct the diagram of Figure 7.3.4 to recover $A = W(\overline{A}^\flat)$ and $I = \ker(A \to \overline{A})$. ■

We will study the essential image of this functor in more detail in Section 8.

7.4 Exercises

1. Show that the category of perfect $\mathbb{F}_p$-algebras is closed under arbitrary limits and colimits in **Ring**.

2. Let R be a p-adically complete ring and set $R^\flat = \lim_\phi R/p$. Prove that the natural map

$$\lim_{x \mapsto x^p} R \to \lim_\phi R/p$$

is a multiplicative bijection. This gives the set on the left a ring structure; can you describe the addition law explicitly?

3. Let R be a perfect $\mathbb{F}_p$-algebra. Choose $f \in R$ and define the ideal $I = \sqrt{(f)}$ of R. Prove that $R/I \in \mathbf{Mod}_R$ has Tor-dimension at most 1.

Hint. First check that $I = (f^{p^{-\infty}})$. Then verify that the map

$$\operatorname{colim}(R \stackrel{f^{1-1/p}}{\to} R \stackrel{f^{1/p-1/p^2}}{\to} \cdots) \to I$$

is an isomorphism. (See also [18], Lecture IV, Exercise 2.4.)

4. Let $A \to B, A \to C$ be morphisms of perfect $\mathbb{F}_p$-algebras. Show that $\operatorname{Tor}_i^A(B, C) = 0$ for all $i > 0$.

Hint. Reduce to the case where $A \to B$ is the quotient by an ideal of the form $\sqrt{(f)}$, then apply Exercise 7.4.3.

8 Lenses

Reference. [18], Lecture IV. The theory of perfectoid fields, rings, and spaces has been described in numerous sources; instead of recapping this history here, see [81] (especially Remark 2.3.18).

In Section 7, we showed that a perfect prism (A, I) can be recovered from the ring $\overline{A} = A/I$. Here, we study the rings of this form in more detail. These end up being closely related to **perfectoid rings**, which appear frequently in p-adic Hodge theory; however, we will not use too much of the existing theory of perfectoid rings, and in fact we will end up recovering some of it via a different approach.

8.1 The category of lenses

Definition 8.1.1 A **lens** is a ring which occurs as the slice of some perfect prism. We define the category of lenses to be the full subcategory of **Ring** consisting of lenses; by Theorem 7.3.5, the slice functor from perfect prisms to lenses is an equivalence of categories.

For $\overline{A} = A/I$ a lens, we say that $\overline{A}$ is an **untilt** of $\overline{A}^\flat$. ♢

Example 8.1.2 For any perfect ring R of characteristic p, the pair $(W(R), (p))$ is a perfect prism with slice and tilt both equal to R. In particular, R is a lens. □

Example 8.1.3 Let R be the t-adic completion of $\mathbb{F}_p[t^{p^{-\infty}}]$ and put $A = W(R)$. We can construct multiple perfect prisms (A, I) with tilt R, such as the following.

- Take $I = (d)$ with $d = \sum_{i=0}^{p-1} [t+1]^i$. The lens A/I is isomorphic to the p-adic completion of $\mathbb{Z}_p[\mu_{p^\infty}]$ via a map with $[t+1]^{p^{-n}} \mapsto \zeta_{p^n}$. The prism (A, I) is isomorphic to the prism from Example 5.1.6 and is the coperfection of the prism from Example 5.1.4.

- Take $I = (d)$ with $d = p - [t]$. The lens A/I is isomorphic to the p-adic completion of $\mathbb{Z}_p[p^{p^{-\infty}}]$ via a map with $[t]^{p^{-n}} \mapsto p^{p^{-n}}$. The prism (A, I) is the coperfection of the prism from Example 5.1.5 in the special case $K = \mathbb{Q}_p, \pi = p$.

□

Remark 8.1.4 While the terminology of **tilting** and **untilting** is now quite commonly used, our references to the **category of lenses** is highly nonstandard; in [22] and [25], objects of this category are called **perfectoid rings**. However, that usage is incompatible with most prior literature; in older terminology these would be **integral perfectoid rings**. To minimize confusion, we sidestep this issue by using a nonce terminology based on the metaphor of prisms.

8.2 On the structure of lenses

Definition 8.2.1 A ring R of characteristic p is **semiperfect** if the Frobenius automorphism of R is surjective. Note that R is perfect if and only if it is both reduced and semiperfect. ◊

Example 8.2.2 A basic example of a semiperfect ring that is not perfect is the ring $\mathbb{F}_p[t^{p^{-\infty}}]/(t)$. □

Lemma 8.2.3 *Let R be a lens.*

1. *The ring R/p is semiperfect.*
2. *There exists an element $\varpi \in R$ admitting a compatible system ϖ^{1/p^n} of p-power roots, such that $\varpi = pu$ for some unit $u \in R^\times$ and the kernel of the Frobenius map on R/p is generated by $\varpi^{1/p}$. (Note that $\varpi = 0$ is possible.)*
3. *The ideal $\sqrt{pR}$ is an increasing union of principal ideals and satisfies $(\sqrt{pR})^2 = \sqrt{pR}$. (It is also flat as an R-module; see Lemma 8.2.4.)*
4. *We have $R[p] = R[\sqrt{pR}]$.*

Proof. Let (A, I) be the perfect prism with $A/I \cong R$ and let $R^\flat$ be the tilt. Then $R^\flat$ is perfect and $R/p \cong R^\flat/p$, so R/p is semiperfect. This proves (1).

For (2), apply Theorem 7.2.2 to write $I = (d)$ with $d \in A$ a distinguished element. By Lemma 7.1.2, we have $d = [a_0] - pu$ for some $a_0 \in R^\flat$ and some unit $u \in A^\times$. We may then take ϖ to be the image of $[a_0]$ to obtain (2).

To check (3), we show that $\sqrt{pR} = \bigcup_n (\varpi^{p^{-n}})$. Since the left side contains the right side, it suffices to observe that the quotient $\overline{R} = R/\bigcup_n (\varpi^{p^{-n}})$ is itself perfect, and hence reduced.

To check (4), keep notation as above; it suffices to check that $R[p]$ is killed by $[a_0^{p^{-n}}]$ for each n (since these elements generate $\sqrt{pR}$). To show that $R[p]$ is killed by $[a_0^{p^{-n}}]$ for some particular n, note that neither p nor d is a zerodivisor in A (by Lemma 7.1.2 and Theorem 7.2.2), so we may write

$$R[p] = (A/d)[p] = (A/p)[d] = R^\flat[d]$$

(see Exercise 8.5.3). By Lemma 6.4.3, the latter is annihilated by $[a_0^{p^{-n}}]$, as claimed. (Compare [18], Lecture IV, Lemma 2.6.) ■

Lemma 8.2.4 *With notation as in Lemma 8.2.3, the ideal $\sqrt{pR}$ is a flat R-module.*

Proof. We must check that for any $M \in \mathbf{Mod}_R$, $\mathrm{Tor}_i^R(M, \sqrt{pR}) = 0$ for all $i > 0$, or equivalently $\mathrm{Tor}_i^R(M, \overline{R}) = 0$ for all $i > 1$. By tensoring M with the exact sequence

$$0 \to \mathrm{Tor}_1^{\mathbb{Z}_p}(M, \mathbb{Q}_p) \to M \to M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to 0$$

and using the fact that $\overline{R} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = 0$, we may further reduce to the case of a module M which is p^∞-power torsion. By the compatibility of tensor products with colimits, we may reduce to the case of a module which is killed by some power of p; by devissage, we may further reduce to the case where M is killed by p.

Since d is a non-zerodivisor in both $W(R^\flat)$ and $W(\overline{R})$ (Lemma 7.1.2),

$$\mathrm{Tor}_i^R(M, \overline{R}) = \mathrm{Tor}_i^{W(R^\flat)}(M, W(\overline{R})).$$

Similarly, since p is a non-zerodivisor in both $W(R^\flat)$ and $W(\overline{R})$ and $pM = 0$,

$$\mathrm{Tor}_i^{W(R^\flat)}(M, W(\overline{R})) = \mathrm{Tor}_i^{R^\flat}(M, \overline{R}).$$

By Exercise 7.4.3, $\overline{R} \in \mathbf{Mod}_{R^\flat}$ has Tor-dimension at most 1, proving the claim. (Compare [18], Lecture IV, Lemma 2.6.) ■

We can now give an intrinsic characterization of lenses, without reference to perfect prisms.

Proposition 8.2.5 *A commutative ring R is a lens if and only if the following conditions hold.*

1. *The ring R is classically p-complete and and R/p is semiperfect.*
2. *The kernel of the map $\theta_R \colon W(R^\flat) \to R$ is principal. (Recall that $R^\flat = \lim_\phi R/p$.)*
3. *There exists some $\varpi \in R$ such that $\varpi^p = pu$ for some unit $u \in R$.*

Proof. If $R = A/I$ is a lens, then all of the stated conditions follow directly from Theorem 7.2.2 and R/p is semiperfect by Lemma 8.2.3. Conversely, suppose that these conditions hold; we will show that for $A = W(R^\flat), I = \ker(\theta_R)$, the pair (A, I) is a perfect prism with $A/I \cong R$. Since θ_R induces a surjective map mod p, it is in fact surjective, so $A/I \cong R$. The ring A is classically p-complete; we may check that it is classically (p, I)-complete by checking that $A/p \cong R^\flat$ is classically I-complete, which is straightforward.

At this point, we must show that I admits a distinguished generator. To this end, choose $x, v \in A$ lifting ϖ, u and put $g = pv - x^p \in \ker(I)$. The series expansion of x has zero coefficient of p; since v is a unit (because A is I-local), we deduce from Lemma 7.1.2 that g is distinguished. Since I is principal, we may apply Lemma 5.2.1 to deduce that g is in fact a generator. ■

In the p-torsion-free case we can make this description even simpler.

Proposition 8.2.6 *A p-torsion-free commutative ring R is a lens if and only if the following conditions hold.*

1. *The ring R is classically p-complete and and R/p is semiperfect.*
2. *The ring R is p-**normal**: every $x \in R[p^{-1}]$ with $x^p \in R$ belongs to R.*
3. *There exists some $\varpi \in R$ such that $\varpi^p = pu$ for some unit $u \in R$.*

Proof. Suppose first that R is a lens. In light of Proposition 8.2.5, we only need to check that R is p-normal. Take ϖ as in Lemma 8.2.3. Given $x \in R[p^{-1}]$ with $x^p \in R$, let n be the smallest nonnegative integer such that $\varpi^n x \in R$. If $n > 0$, then by writing

$$(\varpi^n x)^p = \varpi^{np} x^p \in \varpi^{np} R \subset \varpi^p R$$

and using that the Frobenius map $R/\varpi \to R/\varpi^p$ is a bijection, we see that $\varpi^n x \in \varpi R$ and so $\varpi^{n-1} x \in R$, a contradiction. Hence $n = 0$ and $x \in R$, as desired.

Conversely, suppose that the given conditions hold. It will suffice to show that the kernel of θ_R is principal, as then we can apply Proposition 8.2.5. We first show that the kernel of the (surjective) Frobenius map $R/p \to R/p$ is generated by ϖ. Given $x \in R$ with $x^p \in pR$, write $x^p = \varpi^p y$ for some $y \in R$. Then $(x/\varpi)^p = y \in R$ and so $x \in \varpi R$.

Now the Frobenius on R/p factors as $R/p \to R/\varpi \to R/p$ where the first map is the canonical projection and the second map is an isomorphism. Since this composite is surjective, the image of ϖ in R/p admits a compatible system of p-power roots $\overline{\varpi}^{p^{-n}}$. By induction on n, the kernel of ϕ^n on R/p is generated by $\overline{\varpi}^{p^{-n}}$. Hence the kernel of $\overline{\theta}_R \colon R^\flat \to R/p$ is generated by the element $\overline{\varpi}$ of $R^\flat$ corresponding to the coherent sequence $(\overline{\varpi}^{p^{-n}})_n$. Since both $W(R^\flat)$ and R are p-torsion-free and classically p-complete, the kernel of θ_R is generated by any element of the kernel lifting $\overline{\varpi}$, and in particular is principal. (Compare [18], Lecture IV, Proposition 2.10.) ∎

Remark 8.2.7 The example $R = \mathbb{Z}_p$ shows that condition 3 of Proposition 8.2.5 does not follow from the others (and likewise for Proposition 8.2.6).

Crucially, Proposition 8.2.5 enables us to produce many lenses in cases where it is not so obvious how to give a direct construction of the corresponding perfect prism. In particular, we can make some prisms without specifying either a δ-ring structure or a Frobenius lift!

8.3 Perfectoid fields

Definition 8.3.1 A **perfectoid field** is a field K satisfying the following conditions.

1. The field K is complete for the topology induced by some nonarchimedean valuation with *nondiscrete* value group.

2. The valuation ring $\mathfrak{o}_K$ of K has residue characteristic p, and the ring $\mathfrak{o}_K/p$ is semiperfect.

By Lemma 8.3.3, the valuation ring of a perfectoid field is a lens. Its tilt is also the valuation ring of a perfectoid field (of characteristic p), denoted $K^\flat$. ♢

Remark 8.3.2 We report an observation from [81], Remark 2.1.8: perfectoid fields are the same as the **hyperperfect fields** of [97].

Lemma 8.3.3 *For any perfectoid field K, the valuation ring $\mathfrak{o}_K$ is a lens.*

Proof. If K is of characteristic p, then $\mathfrak{o}_K/p = \mathfrak{o}_K$ is reduced and semiperfect, hence perfect. We thus assume hereafter that K is characteristic 0; we may then check the conditions of Proposition 8.2.6.

It is clear that $\mathfrak{o}_K$ is classically p-complete and p-normal (since it is integrally closed), and by hypothesis $\mathfrak{o}_K/p$ is semiperfect. Since K is not discretely valued, we can choose an element $x \in \mathfrak{o}_K$ of positive valuation such that x^p divides p. Since $\mathfrak{o}_K/p$ is semiperfect, there exists $y \in \mathfrak{o}_K$ such that $y^p \equiv p/x^p \pmod{p}$; put $\varpi = xy$. Then $\varpi^p/p \equiv 1 \pmod{x^p}$ and so $u = \varpi^p/p$ is a unit in $\mathfrak{o}_K$. ∎

The following result generalizes the **field of norms isomorphism** of Fontaine and Wintenberger [52]. We will later give an independent "prismatic" proof; see Remark 23.1.2.

Theorem 8.3.4 Tilting correspondence for perfectoid fields. *Let K be a perfectoid field. Then for every finite extension L of K, L is a perfectoid field and $[L : K] = [L^\flat : K^\flat]$. Consequently, the categories of finite etale algebras over K and $K^\flat$ are canonically isomorphic; in particular, the absolute Galois groups of K and $K^\flat$ are isomorphic.*

Proof. See [80] or [107] (or other references as given in [81], Remark 2.1.8). ■

For a continuation of this discussion, see Subsection 22.4.

8.4 Glueing of lenses

Most familiar examples of lenses are either p-torsion-free or of characteristic p. We can prove a result that shows that this accounts for all possibilities up to a "glueing" construction.

Lemma 8.4.1 *Let R be a perfect $\mathbb{F}_p$-algebra. Let J be a radical ideal of R and let $J' = R[J]$. Then J' and $J + J'$ are both radical ideals and the square in Figure 8.4.2 is both a pullback square and a pushout square of commutative rings.*

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R/J' \arrow[d] \\ R/J \arrow[r] & R/(J+J') \end{tikzcd}$$

Figure 8.4.2

Proof. We first check that J' is radical. If $x \in R$ with $x^p \in J'$, then $x^p J = 0$; since R is perfect, it follows that $xJ = 0$ and so $x \in J'$.

We next check that $J + J'$ is radical. The ideal $J + J'$ is the kernel of $R \to R/J \otimes_R R/J'$; the target is a colimit of perfect rings and hence is itself perfect.

The square in question is already a pushout square at the level of groups, hence also at the level of rings. To check that it is a pullback square, we must check that $J \cap J' = 0$. To this end, choose $x \in J \cap J'$; since $x \in J'$ we have $xJ = 0$, but since $x \in J$ this implies $x^2 = 0$ and finally $x = 0$ because R is perfect. ■

Proposition 8.4.3 *Let (A, I) be a perfect prism and put $R = A/I$. Put $\overline{R} = R/\sqrt{pR}$, $S = R/R[\sqrt{pR}]$, $\overline{S} = S/\sqrt{pS}$. Then $\overline{R}, S, \overline{S}$ are all lenses and the square in Figure 8.4.4 is both a pullback square and a pushout square of commutative rings.*

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \arrow[d] \\ R' \arrow[r] & S' \end{tikzcd}$$

Figure 8.4.4

In addition, the following statements hold.

1. *The ring S is p-torsion-free.*

2. *The ideal $\sqrt{pR}$ maps isomorphically onto $\sqrt{pS}$ (and hence is also p-torsion-free).*

3. *The map $R \to \overline{R}$ induces an isomorphism $R[\sqrt{pR}] \to \ker(\overline{R} \to \overline{S})$, and thus $x \mapsto x^p$ is bijective on $R[\sqrt{pR}]$.*

Proof. We first show that the square is a pullback. By Theorem 7.2.2 we can write $I = (d)$ with d distinguished. By Lemma 7.1.2 we can write $d = [a_0] - pu$ with $a_0 \in R^\flat$, $u \in A^\times$. Consider the perfect ideals $J = (a_0^{p^{-\infty}})$ and $J' = R^\flat[J]$ of the perfect ring $R^\flat$. The square Figure 8.4.5 consists of p-torsion-free, p-adically complete rings and its reduction modulo p is the pullback square from Figure 8.4.2; hence by devissage it is a pullback square.

$$\begin{tikzcd} W(R^\flat) \arrow[r] \arrow[d] & W(R^\flat/J') \arrow[d] \\ W(R^\flat/J) \arrow[r] & W(R^\flat/(J+J')) \end{tikzcd}$$

Figure 8.4.5

Since d is a non-zerodivisor in each of the rings in Figure 8.4.5 by Lemma 7.1.2, we may reduce modulo d to get another pullback square (Figure 8.4.6). Let S' be the top right and bottom right entry of the new square.

$$\begin{tikzcd} W(R^\flat)/d \cong R \arrow[r] \arrow[d] & S' = W(R^\flat/J')/d \arrow[d] \\ W(R^\flat/J)/d \arrow[r] & W(R^\flat/(J+J'))/d \end{tikzcd}$$

Figure 8.4.6

We now show that Figure 8.4.6 coincides with Figure 8.4.4. Inside $W(R^\flat/J)$ we have $(d) = (p)$ since $a_0 \in I$, so by Lemma 8.2.3,

$$W(R^\flat/J)/d = R^\flat/J = R^\flat/(a_0^{p^{-\infty}}) \cong \overline{R}.$$

Since both d and p are non-zerodivisors on S', by Exercise 8.5.3 we have $(S'/(d))[p] \cong (S'/(p))[d] = (R^\flat/J')[d]$. The latter vanishes because the element $d = a_0$ of $R^\flat/J'$ is a non-zerodivisor (by Lemma 6.4.2). We deduce that $S'/(d)$ is p-torsion-free, and so the surjection $R \to S'$ from the top row factors through $R/R[p^\infty] = S$. As in the previous paragraph, we may identify the bottom right entry with $S'/\sqrt{pS'}$.

Let K be the kernel of $R \to S'$. Since Figure 8.4.6 is a pullback square, K embeds into $W(R^\flat)/J)/d = \overline{R}$ and hence is p-torsion. Hence $K \subseteq R[p^\infty]$ and so the induced map $R/R[p^\infty] = S \to S'$ is injective. Since it is also surjective (because $R \to S'$ is) it is an isomorphism; this proves that Figure 8.4.4 and Figure 8.4.6 are the same square, and hence the former is a pullback square.

To conclude, note that the first numbered assertion is included in Lemma 8.2.3; the second and third assertions follow from the fact that Figure 8.4.4 is now a pullback square; and these in turn imply that the square is a pushout. (Compare [18], Lecture IV, Proposition 3.2.) ∎

Corollary 8.4.7 *Any lens is a reduced ring.*

Proof. By Proposition 8.4.3, we may reduce to the cases of a perfect ring of characteristic p and of a p-torsion-free untilted ring. In the former case, it is evident that any perfect ring is reduced. in the latter case, let R be the lens in question. Apply Lemma 8.2.3 to choose an element $\varpi \in R$ such that $\varpi^p = pu$ for some unit $u \in R^\times$. It will suffice to check that any $x \in R$ with $x^p = 0$ vanishes, or (because R is p-adically separated) that any such x is divisible by ϖ^n for any positive integer n. We prove this by induction starting with the base case $n = 0$. Given that $x = \pi^n y$ for some nonnegative integer n and some $y \in R$, we have $\varpi^{np} y^p = 0$ and hence $y^p = 0$ because R is p-torsion-free. By Lemma 8.2.3 again, the kernel of the Frobenius on R/p is generated by ϖ; hence $y \in \varpi R$ and $x \in \varpi^{n+1} R$. (Compare [18], Lecture IV, Corollary 3.3.) ■

The following argument makes a mild use of derived categories; see Section 10.

Proposition 8.4.8 *Let $A \to B$, $A \to C$ be morphisms of lenses. Then the derived p-completion of $B \otimes_A^L C$ has cohomology only in degree 0, which is a lens.*

Proof. It is clear that $R = B^\flat \otimes_{A^\flat} C^\flat$ is perfect. By Exercise 7.4.4, we also have an isomorphism in $D(A^\flat)$

$$R \cong B^\flat \otimes_{A^\flat}^L C^\flat.$$

Applying the Witt vector functor, we obtain an isomorphism

$$W(R) \cong W(B^\flat) \widehat{\otimes}_{W(A^\flat)}^L W(C^\flat)$$

where $\widehat{\otimes}^L$ denotes the derived p-completion of the derived tensor product. Write $A \cong W(A^\flat)/d$ with $d \in W(A^\flat)$ distinguished (Theorem 7.2.2). Since d is a non-zerodivisor in $W(A^\flat), W(B^\flat), W(C^\flat)$ (Lemma 7.1.2), we get an isomorphism

$$W(R)/(d) \cong B \widehat{\otimes}_A^L C.$$

This proves the claim. (Compare [18], Lecture IV, Proposition 2.11.) ■

8.5 Exercises

1. Show that the category of lenses is closed under arbitrary colimits and products in the category of all derived p-complete rings. However, this does *not* imply closure under arbitrary limits; see Exercise 8.5.2.

2. Show that the category of lenses is not closed under formation of equalizers in the category of rings.

Hint. One approach is to use the theorem of Ax-Sen-Tate (see [11]); this implies for example that if K is a (possibly infinite) Galois algebraic extension of $\mathbb{Q}_p$ with Galois group G, then the invariant subfield of the completion of K under the action of G is equal to $\mathbb{Q}_p$.

3. For $A \in \mathbf{Ring}$ and $x, y \in A$ two non-zerodivisors, prove that the A-modules $(A/x)[y]$ and $(A/y)[x]$ are isomorphic.

Hint. As per [18], Lecture IV, Lemma 2.7, compute both modules from the homology of the Koszul complexes $\operatorname{Kos}(A; x, y) \cong \operatorname{Kos}(A; y, x)$.

9 Homotopy categories

Reference. [117], tag 05QI or [125], Chapter 10.

In this section, we fill in some background material about **homotopy categories**. Our immediate need for this is to define derived functors of complexes. We will assume (as we have already done up to now) that the reader is familiar with a more classical treatment of homological algebra. (We postpone the introduction of **derived categories** until Section 10.)

Throughout this section, let $\mathcal{A}$ be a fixed abelian category, e.g., $\mathbf{Mod}_A$ for some $A \in \mathbf{Ring}$. (This is not the weakest hypothesis possible on $\mathcal{A}$, but will suffice for our purposes.)

9.1 A bit of motivation

By way of motivation, we recall the way that chain complexes appear in the construction of derived functors. We discuss only right derived functors, the story for left derived functors being the symmetric image of this.

Definition 9.1.1 Let $\mathcal{A}'$ be a second abelian category. A covariant functor $F\colon \mathcal{A} \to \mathcal{A}'$ is **left exact** if every exact sequence

$$0 \to M_1 \to M \to M_2$$

yields an exact sequence

$$0 \to F(M_1) \to F(M) \to F(M_2).$$

Under suitable conditions (namely, that $\mathcal{A}$ **has enough injectives**), we can "fill in the gap" on the right: if the original sequence extends to an exact sequence

$$0 \to M_1 \to M \to M_2 \to 0,$$

then we get a long exact sequence

$$0 \to R^0F(M_1) \to R^0F(M) \to R^0F(M_2) \to R^1F(M_1) \to R^1F(M) \to R^1F(M_2) \to R^1F(M_1) \to \cdots$$

where R^iF are the **right derived functors** of F (with $R^0F = F$). These functors can be evaluated at M by forming an **injective resolution** of M, i.e., a complex

$$0 \to I^0 \to I^1 \to \dots \tag{9.1}$$

in which each object $I^j \in \mathcal{A}$ is **injective** (that is, $\mathrm{Hom}(N, I^j) \to \mathrm{Hom}(N', I^j)$ is surjective whenever $N' \to N$ is a monomorphism) and the augmented sequence

$$0 \to M \to I^0 \to I^1 \to \cdots$$

is exact; then RF^i is the cohomology at i of the complex

$$0 \to F(I^0) \to F(I^1) \to \cdots .$$

However, there is some work to be done to confirm that these are well-defined functors. $\lozenge$

Remark 9.1.2 What we want to do here is create a larger category than $\mathcal{A}$ in which the injective resolution (9.1) is itself an object *and* any two injective resolutions are canonically isomorphic in this new category. This will have various practical advantages when trying to work with derived functors.

For example, when composing left exact functors, if one only keeps track of the derived functors individually, the most one can say about the derived functors of the composition is that they are the limit of the **Grothendieck spectral sequence**, but this leaves some ambiguity if you do not know what the differentials are (and anyway being the limit of a spectral sequence does not determine the final objects exactly, only the successive quotients of some filtration. By contrast, when working in the homotopy category this ambiguity is completely eliminated by the fact that we can define derived functors not just for objects, but also for complexes.

9.2 Categories of chain complexes

As a first step, we construct a category of chain complexes.

Definition 9.2.1 A **chain complex** in $\mathcal{A}$ is a sequence

$$\cdots \to K^{n-1} \stackrel{d^{n-1}}{\to} K^n \stackrel{d^n}{\to} K^{n+1} \to \cdots$$

in $\mathcal{A}$ such that any two consecutive morphisms compose to zero. The morphisms d^n are commonly called the **differentials** of the complex. (Note that here I am using **cohomological numbering** rather than **homological numbering**.)

A complex $K^\bullet$ is:

- **bounded below** if $K^n = 0$ for all $n \ll 0$;
- **bounded above** if $K^n = 0$ for all $n \gg 0$;
- **bounded** if both of these hold.

$\Diamond$

Definition 9.2.2 We view chain complexes as forming a category $\mathbf{Comp}(\mathcal{A})$ in which a morphism $f^\bullet \colon K_1^\bullet \to K_2^\bullet$ is given by a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & K_1^{n-1} \arrow[r, "d_1^{n-1}"] \arrow[d, "f^{n-1}"] & K_1^n \arrow[r, "d_1^n"] \arrow[d, "f^n"] & K_1^{n+1} \arrow[r, "d_1^{n+1}"] \arrow[d, "f^{n+1}"] & \cdots \\
\cdots \arrow[r] & K_2^{n-1} \arrow[r, "d_2^{n-1}"] & K_2^n \arrow[r, "d_2^n"] & K_2^{n+1} \arrow[r, "d_2^{n+1}"] & \cdots
\end{tikzcd}$$

Figure 9.2.3

such a morphism induces morphisms on cohomology groups $h^n(K_1^\bullet) \to h^n(K_2^\bullet)$. Let $\mathbf{Comp}^+(\mathcal{A}), \mathbf{Comp}^-(\mathcal{A}), \mathbf{Comp}^b(\mathcal{A})$ be the full subcategories of $\mathbf{Comp}(\mathcal{A})$ consisting of bounded below complexes, bounded above complexes, or bounded complexes, respectively. Any functor $F \colon \mathcal{A} \to \mathcal{A}'$ of abelian categories induces functors $\mathbf{Comp}^*(\mathcal{A}) \to \mathbf{Comp}^*(\mathcal{A}')$ for each of $* \in \{\emptyset, +, -, b\}$. $\Diamond$

Definition 9.2.4 For each integer $i \in \mathbb{Z}$, we have a functor $[i] \colon \mathcal{A} \to \mathbf{Comp}^b(\mathcal{A})$ taking $M \in \mathcal{A}$ to the complex $K^\bullet$ with

$$K^n = \begin{cases} M & n = -i \\ 0 & n \neq i. \end{cases}$$

(note the minus sign). This extends to a functor $[i] \colon \mathbf{Comp}^*(\mathcal{A}) \to \mathbf{Comp}^*(\mathcal{A})$ given by

$$K[i]^n = K^{n+i}.$$

In the other direction, we have a functor $H^i \colon \mathbf{Comp}(\mathcal{A}) \to \mathcal{A}$ (the i**-th cohomology** given by

$$H^i(K^\bullet) = \ker(d^i)/\operatorname{im}(d^{i-1}).$$

The composition $H^i \circ [-i]$ is an equivalence of categories. In particular, each functor $[i] \colon \mathcal{A} \to \mathbf{Comp}^b(\mathcal{A})$ defines a full embedding.

A morphism in $\mathbf{Comp}(\mathcal{A})$ is a **quasi-isomorphism** if its image under each h^i is an isomorphism. In general, such a morphism need not have an inverse. $\Diamond$

Remark 9.2.5 Returning to Remark 9.1.2, we now see that an injective resolution $I^\bullet$ of an object $M \in \mathcal{A}$ does define an object of $\mathbf{Comp}^+(\mathcal{A})$ and the augmentation defines a morphism $M[0] \to I^\bullet$ in $\mathbf{Comp}^+(\mathcal{A})$. However, it is not the case that different injective resolutions define the **same** object (or more precisely, canonically isomorphic objects) in $\mathbf{Comp}^+(\mathcal{A})$. The construction of the homotopy category will resolve this issue.

9.3 Split exact sequences

Continuing by way of motivation, we recall another basic construction in homological algebra.

Definition 9.3.1 A short exact sequence

$$0 \to M \stackrel{f}{\to} N \stackrel{g}{\to} P \to 0$$

in $\mathcal{A}$ is **split** if any of the following equivalent conditions hold.

1. There exists a morphism $t\colon N \to M$ such that $t \circ f = \mathrm{id}_M$.
2. There exists a morphism $s\colon P \to N$ such that $g \circ s = \mathrm{id}_P$.

The existence of a splitting guarantees that for any functor $F\colon \mathcal{A} \to \mathcal{A}'$, the sequence

$$0 \to F(M) \stackrel{F(f)}{\to} F(N) \stackrel{F(g)}{\to} F(P) \to 0$$

is again split exact *whether or not* F is an exact functor. $\Diamond$

Remark 9.3.2 In Definition 9.3.1, the equivalence from the two conditions comes from the fact that in either condition, reversing the order of composition yields an idempotent endomorphism (i.e., a **projector**) on N. In particular, we can always choose the two splittings s, t so that $f \circ t + s \circ g = \mathrm{id}_N$. The maps s, t together provide a "robust witness" to the exactness of the sequence.

9.4 Chain complexes and the homotopy category

The discussion of split exact sequences can be generalized as follows.

Definition 9.4.1 A **chain homotopy** for a morphism $f\colon K_1^\bullet \to K_2^\bullet$ in $\mathbf{Comp}(\mathcal{A})$ is a collection of morphisms $h_n\colon K_1^n \to K_2^{n-1}$ such that

$$d_2^{n-1} \circ h_n + h_{n+1} \circ d_1^n = f^n \qquad (n \in \mathbb{Z}).$$

This implies that f maps to the zero morphism via each cohomology functor h^n. If such a homotopy exists, we say that f is **homotopic to** 0; similarly, if $f, g\colon K_1^\bullet \to K_2^\bullet$ are two morphisms and $f - g$ is homotopic to 0, we say that f and g are **homotopic (to each other)**.

Note that morphisms homotopic to zero form a two-sided ideal under composition. We may thus define the **homotopy category** of $\mathcal{A}$, denoted $K(\mathcal{A})$, to be the category with the same objects as $\mathbf{Comp}(\mathcal{A})$ but where the group of morphisms from $K_1^\bullet$ to $K_2^\bullet$ is the quotient of the group of morphisms in $\mathbf{Comp}(\mathcal{A})$ by the ideal of morphisms homotopic to 0. We may similarly define $K^+(\mathcal{A}), K^-(\mathcal{A}), K^b(\mathcal{A})$ as quotients of $\mathbf{Comp}^+(\mathcal{A}), \mathbf{Comp}^-(\mathcal{A}), \mathbf{Comp}^b(\mathcal{A})$; these are the **bounded below homotopy category**, the **bounded above homotopy category**, and the **bounded homotopy category** of $\mathcal{A}$.

The functors $H^i\colon \mathbf{Comp}(\mathcal{A}) \to \mathcal{A}$ factor through $K(\mathcal{A})$ and satisfy

$$H^i = H^0 \circ [i].$$

In particular, any isomorphism in $K(\mathcal{A})$ induces isomorphisms of cohomology groups. Any functor $\mathcal{A} \to \mathcal{A}'$ induces a corresponding functor $K^*(\mathcal{A}) \to K^*(\mathcal{A}')$. $\lozenge$

Returning to Remark 9.1.2, using the universal property of injective objects, it is straightforward to check the following.

Lemma 9.4.2 *For $M \in \mathcal{A}$, let $I^\bullet$ and $J^\bullet$ be two injective resolutions of M.*

1. *There exists a morphism $I^\bullet \to J^\bullet$ in $\mathbf{Comp}^+(\mathcal{A})$ which commute with the augmentations $M[0] \to I^\bullet$, $M[0] \to J^\bullet$.*
2. *Any two such morphisms are homotopic to each other.*
3. *In particular, the classes of $I^\bullet$ and $J^\bullet$ in $K^+(\mathcal{A})$ are canonically isomorphic.*

Proof. Left to the reader, or see [117], tag 013P. ■

9.5 Derived functors revisited

Definition 9.5.1 Suppose that the abelian category $\mathcal{A}$ has enough injectives (e.g., $\mathcal{A} = \mathbf{Mod}_A$). Then every object $M \in \mathcal{A}$ admits an injective resolution. In fact, by Lemma 9.4.2, there is a canonical morphism $M[0] \to I^\bullet$ in $K^+(\mathcal{A})$ in which the target is an injective resolution.

Now let $F\colon \mathcal{A} \to \mathcal{A}'$ be a left exact (covariant) functor. The right derived functors R^iF can then be evaluated on M by taking cohomology of the object $F(I^\bullet)$; in fact, we can think of $M \mapsto F(I^\bullet)$ as defining a single **derived functor** $RF\colon \mathcal{A} \to K^+(\mathcal{A})$.

In fact, every object of $K^+(\mathcal{A})$ can likewise be resolved by injectives, so we can extend the derived functor to $RF\colon K^+(\mathcal{A}) \to K^+(\mathcal{A})$. For example, we have the **derived Hom** $R\operatorname{Hom}_A(M, \bullet)$ which computes Ext groups.

Similarly, if $\mathcal{A}$ has enough projectives and F is a right exact covariant functor, we can define a left derived functor $LF\colon K^-(\mathcal{A}) \to K^-(\mathcal{A})$. For example, we have the **derived tensor product** $M \otimes^L_A \bullet$ which computes Tor groups. $\lozenge$

Remark 9.5.2 We can still improve upon the situation: the morphism from an object $M \in \mathcal{A}$ to an injective resolution is a **quasi-isomorphism**, in that it induces isomorphisms of cohomology, but is not itself an isomorphism in $K^+(\mathcal{A})$. To form the **derived category** we would like to resolve this issue by formally inverting all quasi-isomorphisms; this involves a localization process similar to, but more complicated than, the localization of rings in commutative (or noncommutative) algebra. See Section 10 for further discussion.

10 Derived categories

Reference. [117], tag 05QI or [125], Chapter 10; we are skipping a *lot* of details. We do not develop the point of view of triangulated categories very thoroughly; for more on that point of view, see [84], Chapter II.

We pick up from Section 9 (retaining notation) and introduce the **derived category** associated to an abelian category. This amounts to checking that we can perform the localization of the homotopy category $K(\mathcal{A})$ at the family of quasi-isomorphisms.

Here we take a "classical" point of view on derived categories; however, it is better in the long run to express the construction in the language of **∞-categories**. See Subsection 16.5 and Remark 13.4.1 for further remarks in this vein.

10.1 Localization in a category

Remark 10.1.1 Recall from Remark 9.5.2 that we are in the situation of having to construct one category from another by "formally inverting" some morphisms. We are familiar with processes of these type from algebra, such as the group completion of a monoid (e.g., passage from positive integers to arbitrary integers) or the localization of a ring at a multiplicative subset (e.g., passage from integers to rational numbers). The category-theoretic situation is similar but rather fraught with arrows, and somewhat complicated by the fact that composition of morphisms is not commutative. Similar (but a bit less fraught) considerations apply to localization in a *noncommutative* ring.

To isolate a key difficulty, imagine trying to define a morphism in the localization category as a formal composition $g^{-1} \circ f$ where f is a morphism and g^{-1} is the "formal inverse" of another morphism. Then the composition of two such morphisms would have the form $g_1^{-1} \circ f_1 \circ g_2^{-1} \circ f_2$ and we would then need to rewrite the inner composition $f_1 \circ g_2^{-1}$ as a composition $g_3^{-1} \circ f_3$ in the opposite order. Then the total composition would become

$$g_1^{-1} \circ g_3^{-1} \circ f_3 \circ f_2 = (g_3 \circ g_1)^{-1} \circ (f_2 \circ f_3)$$

which has the right form.

We give only a brief summary of the formalism needed to make this idea work. See [117], tag 04VB for further details.

Definition 10.1.2 Let $\mathcal{C}$ be a category (not necessarily abelian or even additive). Let S be a collection of morphisms in $\mathcal{C}$. We say that S is a **left multiplicative system** if the following conditions hold.

1. The collection S contains all identity morphisms and is closed under composition (of composable pairs).

2. Given the solid arrows as in Figure 10.1.3 with $t \in S$, for some choice of Y' there exist dashed arrows with $s \in S$ forming a commutative square.

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "t"] & Y \arrow[d, dashed, "s"] \\ X' \arrow[r, dashed, "f"] & Y' \end{tikzcd}$$

   Figure 10.1.3

   You should think of this as saying that the "formal composition" $g \circ t^{-1} \colon Z \to Y$ can be refactored as $s^{-1} \circ f$, with the formal inverse moved from the right to the left.

3. For every pair of morphisms $f, g \colon X \to Y$ and every $t \in S$ with target X such that $f \circ t = g \circ t$, there exists a morphism $s \in S$ with source Y (and unspecified target) such that $s \circ f = s \circ g$. (In this case, the morphisms f and g are going to be conflated in the localization, and we want that to make sense with respect to composition on both sides.)

If $\mathcal{C}$ is an additive category, it is equivalent to require that for every morphism $f\colon X \to Y$ and every $t \in S$ with target X such that $f \circ t = 0$, there exists a morphism $s \in S$ with source Y (and unspecified target) such that $s \circ f = 0$.

Similarly, a **right multiplicative system** is a collection of morphisms of $\mathcal{C}$ that constitutes a left multiplicative system in the opposite category. A **multiplicative system** is a collection of morphisms of $\mathcal{C}$ which is simultaneously a left multiplicative system and a right multiplicative system.

We say that a multiplicative system is **saturated** if for any three composable morphisms f, g, h with $f \circ g, g \circ h \in S$, we also have $g \in S$. For example, the collection of all isomorphisms has this property. ◊

Definition 10.1.4 Let $\mathcal{C}$ be a category and let S be a multiplicative system. We define the category $S^{-1}\mathcal{C}$ as follows. (There are some steps to verify that this is a well-posed definition of a category; see [117], tag 04VD.)

1. The objects of $S^{-1}\mathcal{C}$ are the objects of $\mathcal{C}$.

2. For $X, Y \in \mathcal{C}$ two objects, the morphisms $X \to Y$ in $\mathcal{C}$ are given by pairs $(f\colon X \to Y', s\colon Y \to Y')$ where $Y' \in \mathcal{C}$ is a third object modulo the following equivalence relation: two pairs

$$(f_i\colon X \to Y_i, s_i\colon Y \to Y_i)$$

for $i = 1, 2$ are equivalent if there is a third pair with $i = 3$ fitting into a diagram as in Figure 10.1.5 for some morphisms $Y_i \to Y_3$ in $\mathcal{C}$ (not necessarily in S).

$$\begin{tikzcd} & Y_1 \arrow[d] & \\ X \arrow[ur, "f_1"] \arrow[r, "f_3"] \arrow[dr, "f_2"'] & Y_3 & Y \arrow[ul, "s_1"'] \arrow[l, "s_3"'] \arrow[dl, "s_2"] \\ & Y_2 \arrow[u] & \end{tikzcd}$$

Figure 10.1.5

You should think of a pair (f, s) as corresponding to the formal composition $s^{-1} \circ f$.

3. The composition of a pair $(f\colon X \to Y', s\colon Y \to Y')$ with a pair $(g\colon Y \to Z', t\colon Z \to Z')$ is defined to be the equivalence class of a pair $(h \circ f\colon X \to Z'', u \circ t\colon Z \to Z'')$ where h and $u \in S$ are chosen (using the definition of a left multiplicative system) to fill in the commutative square Figure 10.1.6.

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "s"] & Z' \arrow[d, dashed, "u"] \\ Y' \arrow[r, dashed, "h"] & Z'' \end{tikzcd}$$

Figure 10.1.6

The identity morphism on X is the class of $(\mathrm{id}_X, \mathrm{id}_X)$.

One can similarly form the localization of the opposite category, then take the opposite category of the result (using the definition of a right multiplicative system). This gives the same answer; see [117], tag 04VL.

The morphisms of $\mathcal{C}$ which become isomorphisms in $S^{-1}\mathcal{C}$ also form a multiplicative system; in fact, this is the smallest saturated multiplicative system containing S ([117], tag 04VB), and so equals S if and only if S is itself saturated. $\diamondsuit$

Remark 10.1.7 In Definition 10.1.4, we have glossed over a serious set-theoretic difficulty; since the definition of a morphism is quantified over an unspecified third object Y' of $\mathcal{C}$, it is not clear that the collection of morphisms between two fixed objects is a set, as is required in the definition of a category.

One way to avoid this issue is to only consider localizations of categories which are **small**, meaning that there is a set of objects which meets every isomorphism class. Then one can instead quantify Y' over this set of representatives without losing anything.

A more robust mechanism is to use the **Gabriel-Zisman theorem** which gives a criterion for constructing localizations even when the ambient category is not small. See [125], Theorem 10.3.7.

Remark 10.1.8 There is a way to interpret ring-theoretic localization as a special case of localization of categories. See [117], tag 0BM1.

10.2 Distinguished triangles

Recall that a short exact sequence of complexes gives rise to a long exact sequence of cohomology groups. This serves as inspiration for the following discussion of **triangles** in the homotopy category.

Definition 10.2.1 A **triangle** in $\mathbf{Comp}(\mathcal{A})$ is a tuple $(A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta)$ coming from a diagram of the form

$$A^\bullet \xrightarrow{\alpha} B^\bullet \xrightarrow{\beta} C^\bullet \xrightarrow{\delta} A^\bullet[1]$$

which is a complex; that is, the compositions

$$\begin{gathered} A^\bullet \to B^\bullet \to C^\bullet \\ B^\bullet \to C^\bullet \to A^\bullet[1] \\ C^\bullet \to A^\bullet[1] \to B^\bullet[1] \end{gathered}$$

are zero. We can then consider morphisms of triangles in either $\mathbf{Comp}(\mathcal{A})$ or $K(\mathcal{A})$.

We can define an operation called **forward rotation** on the set of triangles:

$$(A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta) \mapsto (B^\bullet, C^\bullet, A[1]^\bullet, \beta, \delta, -\alpha[1])$$

(note the minus sign). The inverse operation is **backward rotation**. $\diamondsuit$

Here is a key family of examples.

Definition 10.2.2 For a morphism $f\colon K^\bullet \to L^\bullet$ in $\mathbf{Comp}(\mathcal{A})$, the **cone** (or **mapping cone**) of f is the complex

$$\mathrm{Cone}(f)^n = L^n \oplus K^{n+1}, \qquad d^n_{\mathrm{Cone}(f)} = \begin{pmatrix} d^n_L & f^{n+1} \\ 0 & -d^{n+1}_K \end{pmatrix}.$$

This complex fits into a triangle

$$K^\bullet \xrightarrow{f} L^\bullet \to \mathrm{Cone}(f)^\bullet \to K^\bullet[1]$$

where the maps in and out of $\mathrm{Cone}(f)^\bullet$ are the obvious ones. Any triangle isomorphic to one of this form is said to be **distinguished**.

The previous triangle can be reinterpreted as

$$L^\bullet[-1] \to \mathrm{Cone}(f)^\bullet[-1] \to K^\bullet \xrightarrow{f} L^\bullet.$$

That is, we can interpret $\mathrm{Cone}(f)^\bullet[-1]$ as the **cocone** (or **mapping cocone**) of f. $\Diamond$

Lemma 10.2.3 *For any distinguished triangle* $(A^\bullet, B^\bullet, C^\bullet, \alpha, \beta, \delta)$*, the sequence*

$$\cdots \to H^i(A^\bullet) \xrightarrow{\alpha} H^i(B^\bullet) \xrightarrow{\beta} H^i(C^\bullet) \xrightarrow{\delta} H^{i+1}(A^\bullet) \to \cdots \tag{10.1}$$

is exact. That is, H^0 *is a **homological functor**.*

Proof. We may as well start with the triangle associated to a mapping cone. In this case, the morphism δ coincides with the family of connecting homomorphisms coming from the short exact sequence of complexes

$$0 \to L^\bullet \to \mathrm{Cone}(f)^\bullet \to K^\bullet[1] \to 0$$

and so the sequence in question is just the long exact sequence in cohomology. Alternatively, we can first prove Lemma 10.2.4 and then use this to reduce to checking exactness at $H^i(C^\bullet)$. ■

Lemma 10.2.4 *The set of distinguished triangles is preserved by forward and backward rotation.*

Proof. The set of distinguished triangles is preserved by the shift operators, so it will be enough to check preservation by forward rotation. That is, given a triangle of the form

$$K^\bullet \xrightarrow{f} L^\bullet \xrightarrow{g} \mathrm{Cone}(f)^\bullet \xrightarrow{h} K^\bullet[1]$$

we must produce a commutative diagram in $K(\mathcal{A})$ of the form of Figure 10.2.5 in which the dashed arrow is an isomorphism in $K(\mathcal{A})$.

$$\begin{tikzcd}
L^\bullet \arrow[r, "g"] \arrow[d, equal] & \mathrm{Cone}(f)^\bullet \arrow[r, "h"] \arrow[d, equal] & K[1]^\bullet \arrow[r, "f[1]"] \arrow[d, dashed] & L[1]^\bullet \arrow[d, equal] \\
L^\bullet \arrow[r, "g"] & \mathrm{Cone}(f)^\bullet \arrow[r] & \mathrm{Cone}(g)^\bullet \arrow[r] & L[1]^\bullet
\end{tikzcd}$$

Figure 10.2.5

We construct the arrow $K[1]^\bullet \to \mathrm{Cone}(g)^\bullet = \mathrm{Cone}(f)^\bullet \oplus L[1]^\bullet$ so that the first factor is the injection $K^{n+1} \to L^n \oplus K^{n+1}$ and the second factor is $-f^{n+1}$. We construct the arrow $\mathrm{Cone}(f)^\bullet \oplus L[1]^\bullet = \mathrm{Cone}(g)^\bullet \to K[1]^\bullet$ as the projection onto $\mathrm{Cone}(f)^n \oplus L^{n+1} \to \mathrm{Cone}(f)^n$ followed by h^n. One may check as in [117], tag 014I that these maps are inverses in $K(\mathcal{C})$. ■

Corollary 10.2.6 *Any morphism in* $K(\mathcal{C})$ *can be included into a distinguished triangle (in any position).*

Proof. Any morphism can be included as the first morphism of a distinguished triangle using the mapping cone. For the other positions, apply Lemma 10.2.4. ■

Lemma 10.2.7 *Given any collection of solid arrows in Figure 10.2.8 forming a commutative diagram in $K(\mathcal{A})$, in which the rows form distinguished triangles, there exists a dashed morphism such that the vertical arrows form a morphism of triangles in $K(\mathcal{A})$.*

$$\begin{tikzcd} A^\bullet \arrow[r, "f"] \arrow[d, "a"] & B^\bullet \arrow[r, "g"] \arrow[d, "b"] & C^\bullet \arrow[r, "h"] \arrow[d, dashed] & A[1]^\bullet \arrow[d, "a[1]"] \\ A'^\bullet \arrow[r, "f'"] & B'^\bullet \arrow[r, "g'"] & C'^\bullet \arrow[r, "h'"] & A'[1]^\bullet \end{tikzcd}$$

Figure 10.2.8

Proof. We may assume at once that $C = \mathrm{Cone}(f), C' = \mathrm{Cone}(f')$. In this case, commutativity of the square in $K(\mathcal{A})$ implies the existence of a homotopy h for the map $b \circ f - f' \circ a$. We may then write down a morphism $c\colon \mathrm{Cone}(f) \to \mathrm{Cone}(f')$ by the formula

$$c^n = \begin{pmatrix} b^n & h^{n+1} \\ 0 & a^{n+1} \end{pmatrix} : B^n \oplus A^{n+1} \to B'^n \oplus A'^{(n+1)}$$

and verify that this yields a morphism of triangles. (Compare [117], tag 014F.) ■

The following result is akin to the universal property of kernels and cokernels.

Corollary 10.2.9 *Let $f\colon X^\bullet \to Y^\bullet$ be a morphism in $K(\mathcal{C})$. Then every morphism $h\colon Y^\bullet \to Z^\bullet$ such that $h \circ f = 0$ in $K(\mathcal{C})$ can be factored through $Y^\bullet \to \mathrm{Cone}(f)^\bullet$.*

Proof. Apply Lemma 10.2.7 to the diagram in Figure 10.2.10.

$$\begin{tikzcd} X^\bullet \arrow[r, "f"] \arrow[d, "0"] & Y^\bullet \arrow[r] \arrow[d, "h"] & \mathrm{Cone}(f)^\bullet \arrow[r] \arrow[d, dashed] & X[1]^\bullet \arrow[d, "a[1]"] \\ 0 \arrow[r] & Z^\bullet \arrow[r, "\mathrm{id}"] & Z^\bullet \arrow[r] & 0^\bullet \end{tikzcd}$$

Figure 10.2.10

■

10.3 Localization at quasi-isomorphisms

We return to our unfinished business from Remark 9.5.2, namely modifying the homotopy category so as to force every quasi-isomorphism to acquire an inverse. Thanks to the cone construction, we can relate quasi-isomorphisms to acyclic objects, which are easier to handle.

Definition 10.3.1 An object $K^\bullet$ of $\mathbf{Comp}(\mathcal{A})$ is **acyclic** if $H^n(K^\bullet) = 0$ for all $n \in \mathbb{Z}$; this property is preserved under isomorphisms in $K(\mathcal{A})$. By Lemma 10.2.3, if two of the three complexes in a distinguished triangle are acyclic, then so is the third. From this (and the preservation of the acyclic property under shifts) we may deduce that the full subcategory of $K(\mathcal{A})$ consisting of acyclic objects is also a triangulated category. ♢

Lemma 10.3.2 *A morphism $f\colon K^\bullet \to L^\bullet$ in $K(\mathcal{A})$ is a quasi-isomorphism if and only if there exists a distinguished triangle $(K^\bullet, L^\bullet, M^\bullet, f, g, h)$ in which $M^\bullet$ is acyclic.*

Proof. This is immediate from Lemma 10.2.3, using the mapping cone for the "only if" direction. ■

Proposition 10.3.3 *The set of quasi-isomorphisms in $K(\mathcal{A})$ is a saturated multiplicative system in the sense of Definition 10.1.2.*

Proof. It suffices to check the conditions for a left multiplicative system, as the symmetric argument will imply the conditions for a right multiplicative system. The first condition in Definition 10.1.2 is evidently satisfied: every identity morphism is a quasi-isomorphism, and any composition of quasi-isomorphisms is a quasi-isomorphism.

To check the second condition, apply Corollary 10.2.6 to fit g into a distinguished triangle (X, Y, Z, g, h, i), then set $Y' = \mathrm{Cone}(i[-1])$; we obtain the map s by filling the diagram Figure 10.3.4 using Lemma 10.2.4 (to rotate) and Lemma 10.2.7. (We deduce from Lemma 10.2.3 that s is a quasi-isomorphism.)

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "s"] & Y \arrow[r, "h"] \arrow[d, dashed, "t"] & Z \arrow[r, "i"] \arrow[d, equal] & X[1] \arrow[d, "s[1]"] \\
X' \arrow[r] & Y' \arrow[r] & Z \arrow[r] & X'[1]
\end{tikzcd}$$

Figure 10.3.4

To check the third condition, start with a morphism $f : X \to Y$ and a quasi-isomorphism $t : Z \to X$ such that $f \circ t = 0$, Apply Corollary 10.2.6 to fit t into a distinguished triangle (Z, X, Q, t, d, h). By Corollary 10.2.9, we can choose a morphism $i : Q \to Y$ such that $i \circ d = f$. Apply Corollary 10.2.6 again to fit i into a distinguished triangle (Q, Y, W, i, j, k); then $j \circ f = j \circ i \circ d = 0 \circ d = 0$. By Lemma 10.3.2, t being a quasi-isomorphism implies that Q is acyclic, which in turn implies that j is a quasi-isomorphism. (Compare [117], 05RG.)

$$\begin{tikzcd}
Z \arrow[r, "t"] & X \arrow[r, "d"] \arrow[d, equal] & Q \arrow[r, "h"] \arrow[d, "i"] & Z[1] \\
& X \arrow[r, "f"] & Y \arrow[d, "j"] & \\
& & W &
\end{tikzcd}$$

Figure 10.3.5

■

Definition 10.3.6 Suppose that $\mathcal{A}$ is a small abelian category. By Proposition 10.3.3, we may apply Definition 10.1.4 to construct the localization of $K(\mathcal{A})$ at the saturated multiplicative system of quasi-isomorphisms. The result is called the **derived category** of $\mathcal{A}$, denoted $D(\mathcal{A})$. Similarly, we may define the **bounded below derived category** $D^+(\mathcal{A})$, the **bounded above derived category** $D^-(\mathcal{A})$, and the **bounded derived category** $D^b(\mathcal{A})$.

An important case is when $\mathcal{A}$ is the category of modules over a ring A. This is not a small category, but modulo set-theoretic issues (see Remark 10.3.8) we can still define $D^*(\mathcal{A})$ for $* \in \{\emptyset, -, +, b\}$; we denote this also by $D^*(A)$.

As in the homotopy category, we say that a triangle in $D(\mathcal{A})$ is **distinguished** if it is isomorphic to the triangle associated to some mapping cone. ♢

The following example shows that Lemma 10.2.3 does not admit a converse.

Example 10.3.7 Let $f : \mathbb{Z} \to \mathbb{Z}$ be the multiplication-by-p map for some prime p. In $D(\mathbf{Ab})$ the cone of f is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ placed in degree 0, so we obtain a distinguished triangle of the form

$$\mathbb{Z} \xrightarrow{f} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \xrightarrow{\delta} \mathbb{Z}[1].$$

By contrast, the triangle

$$\mathbb{Z} \xrightarrow{f} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \xrightarrow{0} \mathbb{Z}[1]$$

gives rise to the same long exact sequence

$$\cdots \to 0 \to \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0 \to \cdots$$

but is not distinguished in $D(\mathbf{Ab})$: otherwise we could apply Lemma 10.2.7 to compare the two triangles, yielding a contradiction. □

Remark 10.3.8 To work around the fact that the derived category construction requires a small abelian category as input, one can view the full category $\mathbf{Mod}_A$ as a 2-colimit of full subcategories consisting of modules of increasingly larger cardinalities. This works because $\mathbf{Mod}_A$ is not just an abelian category but a **Grothendieck abelian category**; see [117], tag 09PA. For a more general abelian category $\mathcal{A}$, however, the set-theoretic difficulty becomes a genuine obstruction; see [117], tag 07JS.

Remark 10.3.9 Just as the properties of the category $\mathbf{Mod}_A$ are abstracted by the notion of an **abelian category**, the properties of homotopy categories and derived categories are abstracted by the notion of a **triangulated category**. A triangulated category is an additive category equipped with a collection of **distinguished triangles** and **shift functors** subject to various conditions analogous to some of the properties we have seen above (especially Lemma 10.2.4 and Lemma 10.2.7). See [117], tag 05QI for further discussion.

Proposition 10.3.10 *Assume that $\mathcal{A}$ has enough injectives and let $F\colon \mathcal{A} \to \mathcal{A}'$ be a left exact functor. Then the right derived functor $RF\colon K^+(\mathcal{A}) \to K^+(\mathcal{A}')$ (see Definition 9.5.1) takes acyclic objects to acyclic objects, and so induces a functor $RF\colon D^+(\mathcal{A}) \to D^+(\mathcal{A}')$.*

Proof. The right derived functor preserves distinguished triangles, so using the criterion of Lemma 10.3.2 it is enough to check that RF takes acyclic objects to acyclic object. For this, see [117], tag 05TA. ■

10.4 Truncation

Definition 10.4.1 For $K^\bullet \in \mathcal{C}$ and any $n \in \mathbb{Z}$, the **canonical truncation** $\tau^{\geq n} K^\bullet$ is the complex given by the second row of Figure 10.4.2, equipped with the morphism $K^\bullet \to \tau^{\geq n} K^\bullet$ defined by the vertical arrows.

$$\begin{tikzcd}
\cdots \arrow[r] & K^{n-1} \arrow[r, "d^{n-1}"] \arrow[d, "0"] & K^n \arrow[r, "d^n"] \arrow[d] & K^{n+1} \arrow[r, "d^{n+1}"] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & \operatorname{coker}(d^{n-1}) \arrow[r, "d^0"] & K^{n+1} \arrow[r, "d^{n+1}"] & \cdots
\end{tikzcd}$$

Figure 10.4.2

Similarly, the **canonical truncation** $\tau^{\leq n} K^\bullet$ is the complex given by the first row of Figure 10.4.3, equipped with the morphism $\tau^{\leq n} K^\bullet \to K^\bullet$ defined by the vertical arrows.

$$\begin{tikzcd}
\cdots \arrow[r] & K^{n-1} \arrow[r, "d^{n-1}"] \arrow[d, equal] & \ker(d^n) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d, "0"] & \cdots \\
\cdots \arrow[r] & K^{n-1} \arrow[r, "d^{n-1}"] & K^n \arrow[r, "d^n"] & K^{n+1} \arrow[r, "d^{n+1}"] & \cdots
\end{tikzcd}$$

Figure 10.4.3

$\diamondsuit$

Lemma 10.4.4 *For any interval I, the following conditions on an object $K^\bullet \in D(A)$ are equivalent.*

1. *We have $H^i(K^\bullet) = 0$ for all $i \notin I$.*
2. *There exists an isomorphism $K^\bullet \to L^\bullet$ in $D(A)$ such that $L^i = 0$ for all $i \notin I$.*

Proof. Suppose for simplicity that $I = [0, \infty)$, the other cases being similar. In this case, if $H^i(K^\bullet) = 0$ for all $i < 0$, then the morphism $K^\bullet \to \tau^{\geq 0} K^\bullet$ is a quasi-isomorphism. ■

Corollary 10.4.5 *An object $K^\bullet$ of $D(\mathcal{A})$ belongs to $D^+(\mathcal{A}), D^-(\mathcal{A}), D^b(\mathcal{A})$ if and only if $H^i(K^\bullet) = 0$ for respectively $i \gg 0, i \ll 0, |i| \gg 0$.*

Proof. This is immediate from Lemma 10.4.4. ■

Remark 10.4.6 It also follows from Lemma 10.4.4 that $D^+(\mathcal{A}), D^-(\mathcal{A}), D^b(\mathcal{A})$ are all *full* subcategories of $D(\mathcal{A})$. For example, for $D^+(\mathcal{A})$ this holds because if we have two bounded-below complexes, any morphism between them is *automatically* zero at all sufficiently small indices (because any map between two zero objects is zero).

Remark 10.4.7 By Lemma 10.4.4 applied with $I = \{0\}$, the essential image of the functor $[0]\colon \mathcal{A} \to D(\mathcal{A})$ is precisely the intersection of the essential images of the functors $\tau^{\geq 0}, \tau^{\leq 0}\colon D(\mathcal{A}) \to D(\mathcal{A})$.

In the more general framework of triangulated categories, one can define a **t-structure** (short for **truncation structure**) to be a pair of functors $\tau^{\geq 0}, \tau^{\leq 0}$ satisfying suitable conditions, and then define the **heart** of the t-structure as the intersection of the essential images of these functors. This gives us a way to start with a triangulated category and realize it as a derived category; in fact, by varying the t-structure we can sometimes realize the same triangulated category as a derived category in multiple ways! (The motivating example of this is the construction of **perverse sheaves** in connection with the Weil conjectures; see [84].)

10.5 Pseudocoherent and perfect complexes

Let us now specialize to the category of modules over a ring and introduce some additional boundedness conditions.

Definition 10.5.1 For $A \in \mathbf{Ring}, \mathcal{A} = \mathbf{Mod}_A$, an object $K^\bullet$ of $D(A) = D(\mathcal{A})$ is **pseudocoherent** (resp. **perfect**) if it is isomorphic to a bounded above (resp. bounded) complex of finite projective A-modules. An object of $\mathbf{Mod}_A$ is **pseudocoherent** (resp. **perfect**) if $M[0]$ is so as an object of $D(A)$. $\diamondsuit$

Lemma 10.5.2 *For $A \in$ **Ring**, an object $K^\bullet$ of $D(A)$ is perfect if and only if it is pseudocoherent and has finite Tor dimension.*

Proof. See [117], tag 0658. ■

Remark 10.5.3 If A is a noetherian ring, then a module is pseudocoherent if and only if it is finitely presented, but such a module need not be perfect.

That said, there do exist many noetherian rings over which every pseudocoherent module is perfect. For example, the ring A is said to have **finite global dimension** if there exists an integer n such that every A-module admits a resolution by projective A-modules of length n. If A is of finite projective dimension, then every pseudocoherent A-module is perfect. (See [117], tag 00O2.)

An important special case is the ring $A = k[x_1, \dots, x_n]$ where k is a field. This ring has finite global dimension (bounded by n, the number of variables) by the Hilbert syzygy theorem ([117], tag 00OQ).

10.6 Exercises

1. Let A be a commutative ring. Show that if any two terms of a distinguished triangle in $D(A)$ are pseudocoherent (resp. perfect), then so is the third.

Hint. See [117], tag 066R.

2. Let A be a commutative ring. Prove that if $K^\bullet \in D^b(A)$ has the property that $H^i(K^\bullet)$ is perfect for all i, then $K^\bullet$ is perfect.

Hint. See [117], tag 066U.

3. Let J be a finitely generated ideal of $A \in \mathbf{Ring}$. Let $A \to A'$ be a morphism in $\mathbf{Ring}$ with finite J-complete Tor amplitude (see Definition 6.5.1).

(a) Show that there exists some $c \geq 0$ such that for any $K \in D^{\geq 0}(A)$, $K \widehat{\otimes}^L_A A' \in D^{\geq -c}(A')$.

(b) Show that the derived J-completed base change functor $M^\bullet \mapsto M^\bullet \widehat{\otimes}^L_A A'$ commutes with totalizations in $D^{\geq 0}$.

Hint. For (1), choose generators $x_1, \dots, x_r$ of J and use derived Nakayama (Proposition 6.6.2 to reduce to checking that for some c,

$$K \mapsto (K \otimes^L_A A') \otimes^A_{A'} \mathrm{Kos}(C'; x_1, \dots, x_r)$$

takes $D^{\geq 0}(A)$ to $D^{\geq -c}(A)$ (where now the tensor products are uncompleted). For more details, see [25], Lemma 4.20.

11 The prismatic site

Reference. [18], lecture V.

In this section, we introduce the **prismatic site** of an affine scheme in the sense of [18]. This should perhaps be called the **naive prismatic site** because it does not give correct answers when one drops the affine hypothesis; see Remark 11.7.2.

Note that while we make some basic definitions in a rather expansive degree of generality, we will be unable to compute anything except under some smoothness hypotheses. We impose those starting in Section 12.

Remark 11.0.1 **Warning.** Our definition of the prismatic cohomology $\mathbf{\Delta}_{R/A}$ is preliminary; it will be overridden later by the construction of **derived prismatic cohomology** in Section 18.

11.1 Indiscrete Grothendieck topologies

Definition 11.1.1 For a topological space X, let $|X|$ be the (small) category consisting of the open subsets of X, where the set of morphisms from U to V is a singleton set if $U \subseteq V$ and 0 otherwise.

A presheaf on X valued in some category $\mathcal{C}$ is nothing but a contravariant functor $|X| \to \mathcal{C}$. A presheaf F is a sheaf if and only F preserves the colimit of the diagram $\coprod_{i,j \in I} V_i \cap V_j \rightrightarrows \coprod_i V_i \to U$ for any covering of an open subset U by open subsets $\{V_i\}_{i \in I}$. (Since the functor is contravariant, that means we get a limit in $\mathcal{C}$.) $\lozenge$

Remark 11.1.2 Building upon this idea, one can define the notion of a **Grothendieck topology** on any category; the key point is to specify which families of morphisms to a given target are **coverings** of that target, and then the sheaf property on a presheaf is formulated in terms of diagrams as above. (A **site** means a category equipped with a Grothendieck topology.)

To deal with the naive prismatic site, we only need the case of an **indiscrete** (or **chaotic**) Grothendieck topology, in which *no* families of morphisms are coverings except isomorphisms, and there is consequently no distinction between presheaves and sheaves. What makes this interesting is that we do not assume that our category has a final object!

Definition 11.1.3 Let $\mathcal{C}$ be a small category; we can then form the category $\mathbf{Pshv}(\mathcal{C})$ of presheaves of abelian groups on $\mathcal{C}$. The functor $\mathbf{Pshv}(\mathcal{C}) \to \mathbf{Ab}$ given by

$$F \mapsto H^0(\mathcal{C}, F) = \lim_{X \in \mathcal{C}} F(X)$$

is left exact; we can then form its derived functor $R\Gamma(C, \bullet)$. $\lozenge$

Example 11.1.4 Let $\mathcal{C}$ be the category $\{0 \to 1 \to 2 \to \cdots\}$; that is, the objects are nonnegative integers, and the morphisms from i to j form a singleton set if $i \leq j$ and the empty set otherwise. In this case, we have

$$H^0(\mathcal{C}, F) = \lim_n F(n), \qquad H^1(\mathcal{C}, F) = R^1 \lim_n F(n)$$

and $H^i(\mathcal{C}, F) = 0$ for all $i \geq 2$ (see Exercise 11.8.1). $\square$

While the indiscrete Grothendieck topology on a small category becomes trivial if the category admits a final object (i.e., an object to which every other object maps uniquely), we will be interested in a slightly less rigid situation where the topology becomes both interesting and computable.

Definition 11.1.5 Recall that a **final object** in a category $\mathcal{C}$ is an object $X \in \mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(Y, X)$ is a singleton set for every $Y \in \mathcal{C}$. By contrast, a **weakly final object** in $\mathcal{C}$ is an object $X \in \mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(Y, X) \neq \emptyset$ for every $Y \in \mathcal{C}$. (That is, the natural map from the representable functor h_X to the forgetful functor $\mathcal{C} \to \mathbf{Set}$ is a bijection if X is final, but only surjective if X is weakly final.) $\lozenge$

Example 11.1.6 Let $\mathcal{C}$ be the category of algebraic field extensions of a fixed field F, viewed as a full subcategory of $\mathbf{Ring}_F$. Then every algebraic closure of F is a weakly final object of $\mathcal{C}$, but not a final object unless F is itself separably closed. $\square$

Lemma 11.1.7 Čech-Alexander resolution. *Let $\mathcal{C}$ be a small category admitting finite nonempty products and containing a weakly final object X. Then for any $F \in \mathbf{Pshv}(\mathcal{C})$, $R\Gamma(\mathcal{C}, F)$ is computed by applying F to the **Čech nerve** of X (see Example 11.2.5); that is, it is given by the **Čech-Alexander***

complex

$$0 \to F(X) \to F(X \times X) \to F(X \times X \times X) \to \cdots \tag{11.1}$$

in which the differentials are given by alternating sums as per Definition 11.2.2.

Proof. Using the fact that the map from h_X to the forgetful functor is a surjection (and that h_{X^n} is a sheaf for all n because we are using the indiscrete topology), this reduces to the general Čech spectral sequence for a (not necessarily indiscrete) Grothendieck topology. See [117], tag 07JM. ∎

11.2 A word on (co)simplicial objects

In preparation to use this language more extensively later, we introduce a bit of terminology that relates naturally to the previous discussion. Our conventions on simplicial sets and objects are taken to match [117], tag 0162.

Definition 11.2.1 Let Δ be the **category of finite ordered sets**. That is, the objects of Δ are the sets $[n] = \{0, \dots, n\}$ for $n = 0, 1, \dots$ and a morphism $f\colon [n] \to [m]$ is a nondecreasing map of sets (i.e., $i \leq j$ implies $f(i) \leq f(j)$).

For $n \geq 1$ and $0 \leq j \leq n$, let $\delta_j^n\colon [n-1] \to [n]$ be the injective morphism in Δ with $(\delta_j^n)^{-1}(\{j\}) = \emptyset$. For $n \geq 0$ and $0 \leq j \leq n$, let $\sigma_j^n\colon [n+1] \to [n]$ be the surjective morphism in Δ with $(\sigma_j^n)^{-1}(\{j\}) = \{j, j+1\}$. Every morphism in Δ can be factored into morphisms of these forms; see Exercise 11.8.2. ◊

Definition 11.2.2 A **simplicial object** of a category $\mathcal{C}$ is a covariant functor $U\colon \Delta^{\mathrm{op}} \to \mathcal{C}$. A **cosimplicial object** of a category $\mathcal{C}$ is a covariant functor $U\colon \Delta \to \mathcal{C}$ (i.e., a simplicial object of $\mathcal{C}^{\mathrm{op}}$). See Figure 11.2.3 and Figure 11.2.4 for graphical representations of simplicial and cosimplicial objects, respectively.

We will frequently consider **(co)simplicial abelian groups**, **(co)simplicial (commutative) rings**, and **(co)simplicial modules** over a (co)simplicial ring. Any cosimplicial abelian group U gives rise to a complex in which the differential are alternating sums of the maps δ_j^n:

$$d^n = \sum_{j=0}^{n+1} (-1)^j U(\delta_j^{n+1}).$$

◊

$$\begin{tikzcd}
U([0]) \arrow[r, "\sigma_0^0"] & U([1]) \arrow[l, shift right=6, "\delta_1^1"'] \arrow[l, shift left=6, "\delta_0^1"] \arrow[r, shift left=6, "\sigma_0^1"'] \arrow[r, shift right=6, "\sigma_1^1"] & U([2]) \cdots \arrow[l, shift right=12, "\delta_2^2"'] \arrow[l, "\delta_1^2"'] \arrow[l, shift left=12, "\delta_0^2"]
\end{tikzcd}$$

Figure 11.2.3 A simplicial object.

$$
\begin{tikzcd}
U([0]) \arrow[r, shift left=4, "\delta_0^1"] & U([1]) \arrow[l, "\sigma_0^0"'] \arrow[l, phantom] \arrow[r, shift left=8, "\delta_0^2"] \arrow[r, shift right=0, "\delta_1^2"] \arrow[r, shift right=8, "\delta_2^2"'] & U([2]) \cdots \arrow[l, shift left=4, "\sigma_0^1"'] \arrow[l, shift right=4, "\sigma_1^1"]
\end{tikzcd}
$$
$$
U([0]) \underset{\delta_1^1}{\rightarrow} U([1])
$$

Figure 11.2.4 A cosimplicial object.

Example 11.2.5 Suppose that the category $\mathcal{C}$ admits finite nonempty products. Then for any $X \in \mathcal{C}$, we can make a simplicial object U in $\mathcal{C}$ by taking $U([n])$ to be the product of copies of X indexed by the elements of $[n]$. This gives the **Čech nerve** of X, as in Lemma 11.1.7; for $F: \mathcal{C} \to \mathbf{Ab}$ a contravariant functor, the complex associated to the cosimplicial abelian group $F(U)$ is the Čech-Alexander complex (Lemma 11.1.7). $\square$

11.3 The prismatic site and "oppo-site"

Definition 11.3.1 Let (A, I) be a prism with slice $\overline{A} = A/I$, and let R be an $\overline{A}$-algebra. The **prismatic oppo-site** of R relative to A, denoted $(R/A)^{\mathrm{op}}_{\mathbf{\Delta}}$, will be the category in which an object consists of a morphism $(A, I) \to (B, IB)$ together with a morphism of $\overline{A}$-algebras $R \to B/IB$. (Recall that by Lemma 5.4.2, for any morphism $(A, I) \to (B, J)$ of prisms we must have $J = IB$.) We will typically notate such an object as $(R \to B/IB \leftarrow B)$ and depict such an object as a diagram as in Figure 11.3.2 (where δ indicates a morphism in $\mathbf{Ring}_\delta$); a morphism of objects will consist of a morphism between the corresponding diagrams. Taking the opposite category yields the **prismatic site** $(R/A)_{\mathbf{\Delta}}$.

$$
\begin{tikzcd}
B/IB & & B \arrow[ll] \\
R \arrow[u] & & \\
\overline{A} = A/I \arrow[u] & & A \arrow[ll] \arrow[uu, "\delta"']
\end{tikzcd}
$$

Figure 11.3.2

$\lozenge$

Remark 11.3.3 Note that the category $(R/A)^{\mathrm{op}}_{\mathbf{\Delta}}$ depends on the whole prism (A, I) and not just on the underlying ring A. However, to keep the notation under control we leave I out, to be inferred from context (as in [18]).

Example 11.3.4 For $R = \overline{A}$, $(R/A)^{\mathrm{op}}_{\mathbf{\Delta}}$ is simply the category of prisms over (A, I), and thus has the initial object $(R \cong A/I \leftarrow A)$. That is, the prismatic site in this case has a final object, and so cohomology on it is trivial. $\square$

Example 11.3.5 Take $R = \overline{A}\langle X \rangle$ to be the classical p-completion of $\overline{A}[X]$. In this case, the prismatic site does not have a final object; however, there are some useful test objects. For instance, let B be the (p, I)-completion of $A[X]$, viewing the latter as a δ-ring with $\delta(X) = 0$; the isomorphism $B/IB \cong R$ gives us an object of $(R/A)_{\mathbf{\Delta}}$. (Compare [18], Lecture V, Example 2.7.) $\square$

One may generalize Example 11.3.5 as follows.

Proposition 11.3.6 *Let (A, I) be a bounded prism and let R be a p-completely smooth A/I-algebra. Then there exists a prism (B, J) over (A, I) with $B/IB \cong R$.*

Proof. Using Remark 2.3.3, we may reduce the claim to Proposition 6.5.3. ■

11.4 The case of a perfect prism

Definition 11.4.1 Let (A, I) be a perfect prism. Define the **perfect prismatic site** to be the subcategory of $(R/A)_{\mathbf{\Delta}}$ consisting of objects of the form $(R \to B/IB \leftarrow B)$ in which (B, IB) is a perfect prism. Recall by Theorem 7.3.5 that these objects are in one-to-one correspondence with lenses over R. ♢

Example 11.4.2 Let (A, I) be a perfect prism. Take $R = \overline{A}\langle X \rangle$ as in Example 11.3.5. Take $S = \overline{A}\langle X^{p^{-\infty}} \rangle$; we then have $S = B/IB$ where B is the (p, I)-completion of $A[X^{p^{-\infty}}]$ for the δ-structure under which $X^{p^{-n}}$ is δ-constant for all n. Note that $R \to S$ is p-completely faithfully flat. □

Remark 11.4.3 If we further reduce the perfect prismatic site by considering only perfect prisms (B, I) in which B/I is p-normal, we end up with the **diamond** of R in the sense of [110].

11.5 Prismatic and Hodge-Tate cohomology

Definition 11.5.1 With notation as in Definition 11.3.1, define the functors $\mathcal{O}_{\mathbf{\Delta}}$ and $\overline{\mathcal{O}}_{\mathbf{\Delta}}$ from $(R/A)_{\mathbf{\Delta}}^{\mathrm{op}}$ taking $(R \to B/IB \leftarrow B)$ to B and B/IB respectively. We will think of these as the **structure (pre)sheaf** and the **reduced structure (pre)sheaf**.

The **prismatic complex** of R relative to A (or more precisely, relative to (A/I)) is the object $\mathbf{\Delta}_{R/A} = R\Gamma((R/A)_{\mathbf{\Delta}}, \mathcal{O}_{\mathbf{\Delta}}) \in D(A)$. This is a derived (p, I)-complete commutative algebra object in $D(A)$; the Frobenius action on $\mathcal{O}_{\mathbf{\Delta}}$ induces a ϕ-semilinear map $\mathbf{\Delta}_{R/A} \to \mathbf{\Delta}_{R/A}$.

The **Hodge-Tate complex** of R relative to A is the object $\overline{\mathbf{\Delta}}_{R/A} = R\Gamma((R/A)_{\Delta}, \overline{\mathcal{O}}_{\Delta}) \in D(\overline{A})$. By construction, we have $\overline{\mathbf{\Delta}}_{R/A} = \mathbf{\Delta}_{R/A} \otimes_A^L \overline{A}$ (with no completion in the tensor product). ♢

Remark 11.5.2 To reiterate a point made in Section 1, the objects $\mathbf{\Delta}_{R/A}$ and $\overline{\mathbf{\Delta}}_{R/A}$ are by their nature intrinsic only in $D(A)$ and $D(\overline{A})$, respectively; they do not come with distinguished representations as complexes.

11.6 More on the prismatic site

We now verify the properties of the prismatic site needed in order to compute cohomology on it via the Čech resolution (Lemma 11.1.7).

Lemma 11.6.1 *Let (A, I) be a prism. Then the forgetful functor from prisms over (A, I) to δ-pairs over (A, I) admits a left adjoint (the* ***prismatic envelope****).*

Proof. We may check this locally on A, so by Lemma 5.2.5 we may assume that I is principal generated by a distinguished element d. Let $(A, I) \to (B, J)$ be a morphism of δ-pairs. Let B' be the free δ-ring over A in the generators x/d for $x \in J$. Let B_1 be the derived (p, d)-completion of B (viewed as a δ-ring using Exercise 6.7.12). If B_1 is d-torsion-free, then (B_1, dB_1) has the desired universal property. Otherwise, we transfinitely iterate the operations of taking the maximal d-torsion-free quotient and taking the derived (p, d)-completion;

this terminates because a countably filtered colimit of derived (p, d)-complete rings is again derived (p, d)-complete (Remark 6.3.4), so we can stop taking the completions once we get to an uncountable ordinal. ■

Remark 11.6.2 The proof of Lemma 11.6.1 gives very little insight into the structure of the resulting objects. See Lemma 14.4.2 for an example where we can make this construction explicit.

Remark 11.6.3 For some purposes, it is more natural to modify the definition of a prism to replace the ideal I with a "virtual Cartier divisor", to provide some missing stability under base change. In this context Lemma 11.6.1 becomes much more straightforward, as the issue with taking torsion-free quotients becomes irrelevant.

Lemma 11.6.4 *Let (A, I) be a prism with slice $\overline{A} = A/I$, and let R be an $\overline{A}$-algebra. Then the category $(R/A)_{\Delta}$ admits finite nonempty products.*

Proof. It is equivalent to show that $(R/A)_{\Delta}^{\mathrm{op}}$ admits finite nonempty coproducts. Let $(R \to B/IB \leftarrow B)$ and $(R \to C/IC \leftarrow C)$ be two objects of $(R/A)_{\Delta}^{\mathrm{op}}$. Form the δ-ring $D_0 = B \otimes_A C$ using Lemma 2.4.3. Let J be the kernel of the natural map

$$D_0 \to B/IB \otimes_{A/IA} C/IC \to B/IB \otimes_R C/IC;$$

that is, J is generated by elements of the form $x \otimes 1 - 1 \otimes y$ where $x \in B, y \in C$ have the property that there is some $z \in R$ mapping to $x \in B/IB$ and to $y \in C/IC$. Apply Lemma 11.6.1 to the pair (D_0, J) to obtain a prism (D, ID); the object $(R \to D/ID \leftarrow D) \in (R/A)_{\Delta}^{op}$ is the desired coproduct. ■

Proposition 11.6.5 *Let (A, I) be a prism with slice $\overline{A} = A/I$, and let R be an $\overline{A}$-algebra. Then the category $(R/A)_{\Delta}$ admits a weakly final object.*

Proof. Let F_0 be the free δ-ring over A on the set R, so that there is a surjection of A-algebras $F_0 \to R$; let J be the kernel of this map. Applying Lemma 11.6.1 to the δ-pair (F_0, J) gives a prism (F, IF) over (A, I).

$$\begin{tikzcd}
F/IF & F \arrow[l] \\
F_0/J \cong R \arrow[u] & F_0 \arrow[l] \arrow[u] \\
A/I \arrow[u] & A \arrow[l] \arrow[u]
\end{tikzcd}$$

Figure 11.6.6

We will check that (F, IF) is a weakly initial object in $(R/A)_{\Delta}^{\mathrm{op}}$. By the adjunction property from Lemma 11.6.1, it suffices to check that for any object $(R \to B/IB \leftarrow B)$ of $(R/A)_{\Delta}$, there exists a morphism $F_0 \to B$ of δ-rings compatible with the map $R \to B/IB$; this holds because F_0 is a free δ-ring over A. ■

Remark 11.6.7 To summarize, with notation as in Definition 11.3.1, we can compute the cohomology of either $\mathcal{O}_{\Delta}$ or $\overline{\mathcal{O}}_{\Delta}$ on $(R/A)_{\Delta}$ by choosing a weakly final object (F, IF) and forming the cosimplicial A-algebra $F^\bullet$ from Lemma 11.1.7; that is, F^n is the $(n+1)$-fold completed tensor product of F over A.

11.7 Additional remarks

Remark 11.7.1 One awkward feature is that a morphism $Y \to X$ does not give rise to a pullback functor $(X/A)_{\boldsymbol{\Delta}} \to (Y/A)_{\boldsymbol{\Delta}}$, because there is no natural way to perform base change for prisms along a morphism at the level of slices. At the level of rings, this is saying that given an object $(R \to B/IB \leftarrow B)$ of $(R/A)_{\boldsymbol{\Delta}}$ and a morphism $R \to S$ of rings, there is no natural way to promote the map $B/IB \to B/IB \widehat{\otimes}_R S$ to a morphism $B \to *$. This is in fact a rather common issue with Grothendieck topologies; it also arises for the infinitesimal and crystalline sites.

The standard fix for this is to replace the **prismatic site** with its associated category of sheaves of sets, the **prismatic topos**. In this language, one can show ([25], Remark 4.3) that the functor $h_X : (B, IB) \mapsto \mathrm{Hom}_{\overline{A}}(\mathrm{Spf}(B/IB), X)$ is a sheaf on the site $(\overline{A}/A)_{\boldsymbol{\Delta}}$ and the **slice topos** over this functor is naturally equivalent to the topos of $(R/A)_{\boldsymbol{\Delta}}$. (This also applies if we replace the indiscrete Grothendieck topology with the one in Remark 11.7.2.)

Remark 11.7.2 As pointed out above, what we are calling the **prismatic site** here (following [18]) should really be called the **naive prismatic site**. The site defined in [25], Definition 4.1 has a different Grothendieck topology: a morphism $(B, IB) \to (C, IC)$ of prisms corresponds to a covering if and only if it is I-completely faithfully flat. This changes the resulting topos, but not the prismatic or Hodge-Tate cohomology; it also gives better results when replacing the ring R with a (usually smooth) p-adic formal scheme X, now with an object being given by a diagram as in Figure 11.7.3 (where Spf is always taken with respect to the p-adic topology) to obtain the site $(X/A)_{\boldsymbol{\Delta}}$.

$$
\begin{tikzcd}
\mathrm{Spf}(B/IB) \arrow[r] \arrow[d] & \mathrm{Spf}(B) \arrow[dd] \\
X \arrow[d] & \\
\mathrm{Spf}(\overline{A}) \arrow[r] & \mathrm{Spf}(A)
\end{tikzcd}
$$

Figure 11.7.3

An alternate foundational treatment based on the **prismatization functor** on p-adic formal schemes and the **absolute prismatic site** (in which one does not fix a base prism, only the formal scheme X; for $X = \mathrm{Spf}\,\mathbb{Z}_p$ this is just the category $\boldsymbol{\Delta}$ itself) can be found in work of Bhatt–Lurie (in preparation), Bhatt–Scholze (in preparation), and Drinfeld [42].

11.8 Exercises

1. Verify the claim of Example 11.1.4.
2. Show that any morphism in Δ can be factored as a composition of morphisms each of the form δ^n_j or σ^n_j for some n, j.

12 The Hodge-Tate comparison map

Reference. [18], lecture V.

In this section, we formulate our first application of prismatic cohomology, the **Hodge-Tate comparison theorem**. The proof will be sketched in Section 15.

12.1 Graded commutativity for graded rings

Definition 12.1.1 Let $E^\bullet$ be a (not necessarily commutative) graded ring. We say that $E^\bullet$ is **graded commutative** if

$$ab = (-1)^{mn} ba \qquad (a \in E^n, b \in E^m).$$

$\Diamond$

Lemma 12.1.2 *For $A \in \mathbf{Ring}$, let $K^\bullet$ be a commutative A-algebra object in $D(A)$. Then $\bigoplus_{n \geq 0} H^n(K^\bullet)$ carries a natural graded ring structure, with respect to which it is graded commutative.*

Proof. The multiplication map on $K^\bullet$ is given by a morphism $K^\bullet \otimes_A^L K^\bullet \to K^\bullet$ in $D(A)$. We may directly read off the multiplication in $\bigoplus_{n \geq 0} H^n(K^\bullet)$ and its properties (associativity, distributivity over addition) to obtain the graded ring structure. It remains to check graded commutativity; this follows from the **Koszul sign rule** appearing in the Alexander-Whitney construction (see [83], tag 00P4). We will see this concretely in Remark 13.2.3. ■

12.2 The de Rham complex

Definition 12.2.1 For $A \in \mathbf{Ring}$, a **differential graded algebra** over A (also known as a A**-dga**) is a complex $(E^\bullet, d)$ of A-modules in which $E^\bullet$ is also equipped with the structure of a (not necessarily commutative) graded A-algebra subject to the **signed Leibniz rule**

$$d^{n+m}(ab) = d^n(a)b + (-1)^n a d^m(b) \qquad (a \in E^n, b \in E^m).$$

We say that an A-dga $(E^\bullet, d)$ is **commutative** if $E^\bullet$ is graded commutative. We say that it is **strictly commutative** if it is commutative and moreover $a^2 = 0$ for any a of odd degree. (This last condition is redundant if E is 2-torsion-free.) $\Diamond$

The prototypical example of this definition is the following construction.

Definition 12.2.2 Let $A \to B$ be a morphism in $\mathbf{Ring}$. The **de Rham complex**

$$(\Omega^\bullet_{B/A}, d_{\mathrm{dR}}) = \left(B \to \Omega^1_{B/A} \to \Omega^2_{B/A} \to \cdots\right),$$

in which $\Omega^i_{B/A} = \wedge^i_B \Omega^1_{B/A}$, is a strictly commutative A-dga with multiplication given by the wedge product. $\Diamond$

The universal property of the module of Kähler differentials can be reinterpreted as follws.

Lemma 12.2.3 Universal property of the de Rham complex. *Let $(E^\bullet, d)$ be a graded commutative A-dga supported in degrees ≥ 0. Let $\eta\colon B \to E^0$ be a map of A-algebras such that for each $x \in B$, the element $y = d(\eta(x)) \in E^1$ satisfies $y^2 = 0$ (note that this is automatic if $E^\bullet$ is* strictly *commutative). Then η extends uniquely to a map $\Omega^\bullet_{B/A} \to E^\bullet$ of A-dgas.*

Proof. See [18], Lecture V, Lemma 3.3. ■

Definition 12.2.4 The completed de Rham complex. For I a finitely generated ideal in $A \in \mathbf{Ring}$ and R a derived I-complete A-algebra, we may define the module of completed Kähler differentials $\widehat{\Omega}^1_{R/A}$ as the derived I-completion of the usual module $\Omega^1_{R/A}$. If A is derived I-complete and R is the derived I-completion of a finitely generated A-algebra, then $\widehat{\Omega}^1_{R/A}$ is a finitely generated A-module.

Now suppose that A is derived I-complete. Then the **completed de Rham complex** $\widehat{\Omega}^i_{R/A}$ is a strictly commutative A-dga, and in Lemma 12.2.3, if $E^\bullet$ is derived I-complete, then η extends uniquely to a map $\widehat{\Omega}^\bullet_{B/A} \to E^\bullet$ of A-dgas. ♢

12.3 Construction of the Hodge-Tate comparison map

Definition 12.3.1 Let (A, I) be a prism and let R be an $\overline{A}$-algebra (writing $\overline{A} = A/I$). For $M \in \mathbf{Mod}_{\overline{A}}$ and n an integer, define the **Breuil-Kisin twist** $M\{n\} = M \otimes_{\overline{A}} (I/I^2)^{\otimes n}$; note that this makes sense even if $n < 0$ because I/I^2 is an invertible $\overline{A}$-module (from the definition of a prism).

For $n \geq 0$, consider the exact sequence

$$0 \to I^{n+1}\mathcal{O}_{\mathbf{\Delta}}/I^{n+2} \to I^n\mathcal{O}_{\mathbf{\Delta}}/I^{n+2} \to I^n\mathcal{O}_{\mathbf{\Delta}}/I^{n+1} \to 0$$

of $\mathcal{O}_{\mathbf{\Delta}}$-modules on $(R/A)_{\mathbf{\Delta}}$, then take a connecting homomorphism to obtain the **Bockstein differential**

$$\beta_I \colon H^n(\overline{\mathbf{\Delta}}_{R/A})\{n\} \to H^{n+1}(\overline{\mathbf{\Delta}}_{R/A})\{n+1\}.$$

It will follow from Lemma 12.3.2 that these indeed form the differentials in a complex $(H^\bullet(\overline{\mathbf{\Delta}}_{R/A})\{\bullet\}, \beta_I)$.

As per Definition 11.3.1, the object $\overline{\mathbf{\Delta}}_{R/A} \in D(\overline{A})$ carries the structure of a commutative ring object over $\overline{A}$. From Lemma 12.1.2, we deduce that the graded group $\bigoplus_{n\geq 0} H^n(\overline{\mathbf{\Delta}}_{R/A})\{n\}$ carries the structure of a commutative $\overline{A}$-dga. It is also strictly commutative, but this requires some extra verification; see Lemma 12.3.4.

Suppose now that R is derived p-complete. Then the universal property of the completed de Rham complex gives us a morphism of $\overline{A}$-dgas

$$\eta^\bullet_R \colon (\widehat{\Omega}^\bullet_{R/\overline{A}}, d_{\mathrm{dR}}) \to (H^\bullet(\overline{\mathbf{\Delta}}_{R/A})\{\bullet\}, \beta_I). \tag{12.1}$$

♢

To see that the Bockstein differentials are indeed the differentials of a complex, we make the following general observation.

Lemma 12.3.2 *Let I be an invertible ideal of a ring A (e.g., the principal ideal generated by a non-zerodivisor). Given $M^\bullet \in D(A)$, let*

$$\beta^n \colon H^n(M^\bullet \otimes^L_A I^n/I^{n+1}) \to H^{n+1}(M^\bullet \otimes^L_A I^{n+1}/I^{n+2})$$

be the connecting homomorphism in the exact sequence obtained by applying $M^\bullet \otimes^L_A *$ to the sequence

$$0 \to I^{n+1}/I^{n+2} \to I^n/I^{n+2} \to I^n/I^{n+1} \to 0.$$

Then the composition $\beta^{n+1} \circ \beta^n$ vanishes for all n.

Proof. Consider the commutative diagram in Figure 12.3.3 in which the rows are exact.

$$\begin{tikzcd} 0 \arrow[r] & I^{n+1}/I^{n+3} \arrow[r] \arrow[d] & I^n/I^{n+3} \arrow[r] \arrow[d] & I^n/I^{n+1} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & I^{n+1}/I^{n+2} \arrow[r] & I^n/I^{n+2} \arrow[r] & I^n/I^{n+1} \arrow[r] & 0 \end{tikzcd}$$

Figure 12.3.3

By applying $M^\bullet \otimes_A^L \star$ to the terms and comparing the two rows, we see that the map β^n factors as

$$H^n(M^\bullet \otimes_A^L I^n/I^{n+1}) \to H^{n+1}(M^\bullet \otimes_A^L I^{n+1}/I^{n+3}) \to H^{n+1}(M^\bullet \otimes_A^L I^{n+1}/I^{n+2})$$

where the first map is the connecting homomorphism obtained from the upper row of Figure 12.3.3. By applying $M^\bullet \otimes_A^L *$ to the exact sequence

$$0 \to I^{n+2}/I^{n+3} \to I^{n+1}/I^{n+3} \to I^{n+1}/I^{n+2} \to 0$$

we deduce that the composition

$$H^{n+1}(M^\bullet \otimes_A^L I^{n+1}/I^{n+3}) \to H^{n+1}(M^\bullet \otimes_A^L I^{n+1}/I^{n+2}) \stackrel{\beta^{n+1}}{\to} H^{n+2}(M^\bullet \otimes_A^L I^{n+2}/I^{n+3})$$

vanishes. Combining these two observations proves the claim. (Compare [117], tag 0F7N.) ∎

To check strict commutativity, we make an explicit computation. Remember that there is nothing to check here unless $p = 2$. For a more conceptual approach, see Proposition 15.3.2.

Lemma 12.3.4 *For any $t \in R$, the class $\beta_I(\eta(t)) \in H^1(\overline{\mathbf{\Delta}}_{R/A})$ squares to zero in $H^2(\overline{\mathbf{\Delta}}_{R/A})$.*

Proof. We may assume $p = 2$ as otherwise this follows from ordinary commutativity; this will allow us to use the universal formula (for a, b in any δ-ring)

$$\delta(a - b) = \delta(a) - \delta(b) + b(a - b). \tag{12.2}$$

Using Lemma 5.2.5, we may also reduce to the case where $I = (f)$ with $f \in A$ distinguished (this is mostly just to simplify notation).

We use the fact that $(R/A)_{\mathbf{\Delta}}$ contains a weakly final object (F, IF) which moreover is f-torsion-free (Proposition 11.6.5) to compute Hodge-Tate cohomology using the cosimplicial ring $(F^\bullet, d^\bullet)$as per Remark 11.6.7. Lift $\eta(t) \in F/IF$ to $T \in F^0$. Let $U, V \in F^1$ and $X, Y, Z \in F^2$ be the images of T under the various maps $F^0 \to F^1$ and $F^0 \to F^2$ in the cosimplicial ring $F^\bullet$, so that

$$d^0(T) = U - V, \qquad d^1(U) = X - Y + Z.$$

Since $U - V \in F^1$ vanishes modulo f (the reductions of U and V modulo f are the two images of t) and $F^\bullet$ is f-torsion-free, the unique element $\alpha \in F^1$ with $U - V = f\alpha$ is also a cocycle. Tracing through the construction of the Bockstein differential, we see that $\beta_I(t)$ equals the image of α in $H^1(F^\bullet/fF^\bullet)$, so we need to check that the latter squares to zero.

Multiplying by f again, we may instead check that $U - V$ squares to zero in $H^2(f^2F^\bullet/f^3F^\bullet)$. The square is represented by $(X - Y)(Y - Z) \in f^2F^2$; we will check that this is the boundary of $f^2\delta(\alpha) \in f^2F^1$. To begin, note that $\phi(\alpha)$ is a cocycle because ϕ commutes with the differential in $F^\bullet$. Hence on one hand,

$$\delta(U - V) = \delta(f\alpha) = f^2\delta(\alpha) + \phi(\alpha)\delta(f)$$

and so

$$d^1(\delta(U-V)) = d^1(f^2\delta(\alpha)).$$

On the other hand, by (12.2),

$$\delta(U-V) = \delta(U) - \delta(V) + V(U-V) = d^1(\delta(T)) + V(U-V)$$

and so

$$d^1(\delta(U-V)) = d^1(V(U-V)) = Y(X-Y)+Z(Y-Z)-Z(X-Z) = (X-Y)(Y-Z).$$

(Compare [18], Lecture V, Lemma 5.4.) ■

12.4 The Hodge-Tate comparison theorem

Theorem 12.4.1 Hodge-Tate comparison theorem. *Let (A, I) be a bounded prism. Let R be a p-completely smooth $\overline{A}$-algebra. Then the Hodge-Tate comparison map (12.1) is an isomorphism.*

Proof. See the discussion in Section 15, particularly Proposition 15.3.1 and Proposition 15.3.2. ■

Corollary 12.4.2 *With notation as in Theorem 12.4.1, the object $\overline{\Delta}_{R/A} \in D(R)$ is perfect.*

Proof. By Proposition 6.5.3, we may view R as the p-completion of a smooth $\overline{A}$-algebra; consequently, $\widehat{\Omega}^1_{R/A}$ is a finite projective R-module, so the completed de Rham complex consists of finite projective R-modulse. We may thus deduce the claim from Theorem 12.4.1. ■

Example 12.4.3 The Hodge-Tate isomorphism in q-de Rham cohomology. Take $(A, I) = (\mathbb{Z}_p[\![q-1]\!], ([p]_q))$. We identify $\overline{A} = A/I$ with $\mathbb{Z}_p[\zeta_p]$ via $q \mapsto \zeta_p$.

Take $R = \overline{A}\langle X^{\pm}\rangle$, i.e., the p-adic completion of the Laurent polynomial ring $\overline{A}[X^{\pm}]$, so that

$$R = \widehat{\bigoplus_{i\in\mathbb{Z}}} \overline{A}X^i.$$

We will show that

$$\begin{aligned}\Delta_{R/A} &\cong \left(A\langle X^{\pm}\rangle \xrightarrow{\nabla_q} A\langle X^{\pm}\rangle \frac{dX}{X}\right) \\ &\cong \widehat{\bigoplus_{i\in\mathbb{Z}}} \left(AX^i \xrightarrow{[i]_q} AX^i \frac{dX}{X}\right)\end{aligned}$$

where $[i]_q = (q^i-1)/(q-1)$ is the q-analogue of i, where the hat denotes the $(p, [p]_q)$-completion. We now distinguish between the case where $i \not\equiv 0 \pmod p$, in which case $[i]_q$ maps to a unit in $\overline{A}$, and the case where $i \equiv 0 \pmod p$, in which case $[i]_q$ maps to zero in $\overline{A}$. Thus reduction modulo I yields a quasi-isomorphism

$$\overline{\Delta}_{R/A} \cong \widehat{\bigoplus_{k\in\mathbb{Z}}} \left(\overline{A}X^{kp} \xrightarrow{0} \overline{A}X^{kp}\right).$$

□

12.5 Exercises

1. Give an example of a $\mathbb{Z}$-dga which is commutative but not strictly commutative.

13 Double complexes

Reference. [117], tag 0FNB. Other references may use different sign conventions.

We gather a few key facts about double complexes that will come up in our cohomology computations. Throughout, let $\mathcal{A}$ be a fixed abelian category.

13.1 Double complexes and totalization

Definition 13.1.1 A **double complex** in $\mathcal{A}$ consists of a collection of objects $K^{p,q}$ together with morphisms $d_1^{p,q}\colon K^{p,q} \to K^{p+1,q}$ and $d_2^{p,q}\colon K^{p,q} \to K^{p,q+1}$ such that the resulting diagram commutes and each row and column is itself a complex. ♢

$$\begin{tikzcd}
 & \vdots & \vdots & \\
\cdots \arrow[r] & K^{p,q+1} \arrow[u] \arrow[r, "d_1^{p,q+1}"] & K^{p+1,q+1} \arrow[u] \arrow[r] & \cdots \\
\cdots \arrow[r] & K^{p,q} \arrow[u, "d_2^{p,q}"] \arrow[r, "d_1^{p,q}"] & K^{p+1,q} \arrow[u, "d_2^{p+1,q}"] \arrow[r] & \cdots \\
 & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

Figure 13.1.2

Remark 13.1.3 A double complex can itself be viewed as a complex in the category $\mathbf{Comp}(\mathcal{A})$. There are of course two different ways to do this, which for the moment are symmetric; we will have to break symmetry to discuss totalization. While this symmetry break will have some curious side effects (e.g., the graded commutativity of cohomology, as per Remark 13.2.3), most of the statements we make asymmetrically will have straightforward counterparts with the orientation reversed.

Definition 13.1.4 Totalization. Given a double complex $K^{\bullet,\bullet}$, the **associated total complex** (or for short the **totalization**) is the complex $\mathrm{Tot}(K^{\bullet,\bullet})$ with

$$\mathrm{Tot}^n(K^{\bullet,\bullet}) = \bigoplus_{n=p+q} K^{p,q}$$

with differential

$$d^n = \sum_{n=p+q} (d_1^{p,q} + (-1)^p d_2^{p,q}). \tag{13.1}$$

More precisely, this should be called the **direct sum totalization** as distinct from the **direct product totalization**, in which we take the product rather than the sum. The two coincide if the original complex is bounded above in both directions, or bounded below in both directions. However, we will later (in Section 18) have reason to consider the mixed situation, in which the complex is bounded above in one direction and bounded below in the other direction, and in this case we must pay attention to this distinction. $\diamond$

13.2 Interchanging the rows and columns

Remark 13.2.1 Let $K^{\bullet,\bullet}$ be a double complex in $\mathcal{A}$. Let $L^{\bullet,\bullet}$ be the **transposed complex** with $L^{p,q} = K^{p,q}$ (and similarly for differentials). Then there are natural isomorphisms

$$\mathrm{Tot}(K^{\bullet,\bullet})^n \cong \mathrm{Tot}(L^{\bullet,\bullet})^n$$

for each n, but we have to choose these carefully to make this an isomorphism of complexes: the identification of $K^{p,q} \subset \mathrm{Tot}(K^{\bullet,\bullet})^n$ with $L^{q,p} \subset \mathrm{Tot}(L^{\bullet,\bullet})^n$ is given by multiplication by $(-1)^{pq}$.

Example 13.2.2 Suppose that $\mathcal{A}$ is a symmetric monoidal category (e.g., $\mathbf{Mod}_A$ using the tensor product) and let $K^\bullet$ and $L^\bullet$ be two bounded-below complexes. Then $K^\bullet \otimes L^\bullet$ and $L^\bullet \otimes K^\bullet$ are transposed complexes of each other, so we may use Remark 13.2.1 to identify their totalizations; in the case $\mathcal{A} = \mathbf{Mod}_A$, both of these are quasi-isomorphic to $K^\bullet \otimes_A^L L^\bullet$. $\square$

Remark 13.2.3 We can use this to explain the signs in Lemma 12.1.2 as follows. For $A \in \mathbf{Ring}$, let $K^\bullet$ be a commutative A-algebra object in $D(A)$. The multiplication map can be interpreted as a map $\mathrm{Tot}(L_1^\bullet \otimes L_2^\bullet) \to L_3^\bullet$ for some complexes L_1, L_2, L_3 which are quasi-isomorphic to $K^\bullet$. (Note that we cannot necessarily take the same representative and get a genuine map of complexes; that is, we did not assume that $K^\bullet$ is a commutative ring object at the level of complexes.) Given classes $a \in H^n(L_1^\bullet)$, $b \in H^m(L_2^\bullet)$, we compute their product in $H^{m+n}(L_3^\bullet)$ by choosing representatives of a and b in their respective complexes, taking the product, putting that into the totalization, and then applying the map to L_3^{m+n}. From this, it is clear that switching the order of the terms should introduce a sign of $(-1)^{mn}$ in conformance with Remark 13.2.1.

13.3 The spectral sequence(s) of a double complex

Rather than giving an axiomatic treatment of spectral sequences, we give a narrow treatment centered around a bounded-below double complex, this being the case of most pressing interest for prismatic cohomology. Our goal is to present the key ideas without drowning the reader in the notation needed to make everything completely precise.

Proposition 13.3.1 *Let $K^{\bullet,\bullet}$ be a double complex concentrated in nonnegative degrees (in both directions). Then there exist objects $E_i^{p,q}$ for $i, p, q \geq 0$ with the following properties (where $E_i^{p,q} = 0$ if p, q are not both nonnegative).*

1. *We have $E_0^{p,q} = K^{p,q}$ for all p, q.*
2. *For each i, there exist maps $d_{(i)}^{p,q}\colon E_i^{p,q} \to E_i^{p+i,q+1-i}$ such that the maps in and out of $E_i^{p,q}$ compose to zero and the cohomology of the resulting complex there is $E_{i+1}^{p,q}$. In particular, for any given p, q, the terms $E_i^{p,q}$ for $i \gg 0$ stabilize to an object we call $E_\infty^{p,q}$.*

3. *For $i = 0$, $d_{(i)}^{p,q}$ equals the differential $(-1)^p d_2^{p,q}$ of $K^{\bullet,\bullet}$.*

4. *For $i = 1$, $d_{(i)}^{p,q}$ is the map induced by $d_1^{p,q}$.*

5. *For $n \geq 0$, there is a filtration on $H^n(\mathrm{Tot}(K^{\bullet,\bullet}))$ whose successive quotients are the objects $E_\infty^{p,q}$ for $p + q = n$.*

Moreover, the construction is natural in $K^{\bullet,\bullet}$.

Proof. We define a filtration on $\mathrm{Tot}(K^{\bullet,\bullet})^n$ by taking

$$\mathrm{Fil}^p \,\mathrm{Tot}(K^{\bullet,\bullet})^n = \bigoplus_{i+j=n, i\geq p} K^{i,j}.$$

We then construct the spectral sequence associated to this filtration as per [117], tag 012K. ■

Definition 13.3.2 In Proposition 13.3.1, the $E_i^{\bullet,\bullet}$ is commonly called the i-th **page** (or **sheet** or **stage**) of the spectral sequence. See Figure 13.3.3, Figure 13.3.4, Figure 13.3.5, and Figure 13.3.6 for illustrations of the first four pages. ♢

$$\begin{tikzcd}
E_0^{0,3} & E_0^{1,3} & E_0^{2,3} & E_0^{3,3} \\
E_0^{0,2} \arrow[u, "d_{(0)}^{0,2}"] & E_0^{1,2} \arrow[u, "d_{(0)}^{1,2}"] & E_0^{2,2} \arrow[u, "d_{(0)}^{2,2}"] & E_0^{3,2} \arrow[u, "d_{(0)}^{3,2}"] \\
E_0^{0,1} \arrow[u, "d_{(0)}^{0,1}"] & E_0^{1,1} \arrow[u, "d_{(0)}^{1,1}"] & E_0^{2,1} \arrow[u, "d_{(0)}^{2,1}"] & E_0^{3,1} \arrow[u, "d_{(0)}^{3,1}"] \\
E_0^{0,0} \arrow[u, "d_{(0)}^{0,0}"] & E_0^{1,0} \arrow[u, "d_{(0)}^{1,0}"] & E_0^{2,0} \arrow[u, "d_{(0)}^{2,0}"] & E_0^{3,0} \arrow[u, "d_{(0)}^{3,0}"]
\end{tikzcd}$$

Figure 13.3.3

$$
\begin{tikzcd}
E_1^{0,3} \arrow[r, "d_{(1)}^{0,3}"] & E_1^{1,3} \arrow[r, "d_{(1)}^{1,3}"] & E_1^{2,3} \arrow[r, "d_{(1)}^{2,3}"] & E_1^{3,3} \\
E_1^{0,2} \arrow[r, "d_{(1)}^{0,2}"] & E_1^{1,2} \arrow[r, "d_{(1)}^{1,2}"] & E_1^{2,2} \arrow[r, "d_{(1)}^{2,2}"] & E_1^{3,2} \\
E_1^{0,1} \arrow[r, "d_{(1)}^{0,1}"] & E_1^{1,1} \arrow[r, "d_{(1)}^{1,1}"] & E_1^{2,1} \arrow[r, "d_{(1)}^{2,1}"] & E_1^{3,1} \\
E_1^{0,0} \arrow[r, "d_{(1)}^{0,0}"] & E_1^{1,0} \arrow[r, "d_{(1)}^{1,0}"] & E_1^{2,0} \arrow[r, "d_{(1)}^{2,0}"] & E_1^{3,0}
\end{tikzcd}
$$

Figure 13.3.4

$$
\begin{tikzcd}
E_2^{0,3} \arrow[drr, "d_{(2)}^{0,3}"] & E_2^{1,3} \arrow[drr, "d_{(2)}^{1,3}"] & E_2^{2,3} & E_2^{3,3} \\
E_2^{0,2} \arrow[drr, "d_{(2)}^{0,2}"] & E_2^{1,2} \arrow[drr, "d_{(2)}^{1,2}"] & E_2^{2,2} & E_2^{3,2} \\
E_2^{0,1} \arrow[drr, "d_{(2)}^{0,1}"] & E_2^{1,1} \arrow[drr, "d_{(2)}^{1,1}"] & E_2^{2,1} & E_2^{3,1} \\
E_2^{0,0} & E_2^{1,0} & E_2^{2,0} & E_2^{3,0}
\end{tikzcd}
$$

Figure 13.3.5

$$
\begin{tikzcd}
E_3^{0,3} \arrow[ddrrr, "d_{(3)}^{0,3}"] & E_3^{1,3} & E_3^{2,3} & E_3^{3,3} \\
E_3^{0,2} \arrow[ddrrr, "d_{(3)}^{0,2}"] & E_3^{1,2} & E_3^{2,2} & E_3^{3,2} \\
E_3^{0,1} & E_3^{1,1} & E_3^{2,1} & E_3^{3,1} \\
E_3^{0,0} & E_3^{1,0} & E_3^{2,0} & E_3^{3,0}
\end{tikzcd}
$$

Figure 13.3.6

Corollary 13.3.7 *Let $K^{\bullet,\bullet} \to L^{\bullet,\bullet}$ be a morphism of double complexes. If the induced maps $E_\infty^{p,q}(K) \to E_\infty^{p,q}(L)$ is an isomorphism, then the map $\mathrm{Tot}(K^{\bullet,\bullet}) \to \mathrm{Tot}(L^{\bullet,\bullet})$ is a quasi-isomorphism.*

Proof. By Proposition 13.3.1 (and in particular the naturality), the map $H^n(\mathrm{Tot}(K^{\bullet,\bullet})) \to H^n(\mathrm{Tot}(L^{\bullet,\bullet}))$ has the property that it induces isomorphisms on the successive quotients of some filtration. By the five lemma, this implies that it is itself an isomorphism. ■

Corollary 13.3.8 *Let $K^{\bullet,\bullet}$ be a double complex in which the single complexes $K^{\bullet,q}$ are acyclic for all $q > 0$. Then the morphism $\mathrm{Tot}(K^{\bullet,\bullet}) \to K^{\bullet,0}$ is a quasi-isomorphism.*

Proof. Apply Corollary 13.3.7 after filling $K^{\bullet,0}$ out to a double complex by adding zeroes. ■

13.4 Totalization in the derived category

Remark 13.4.1 Let $K^0 \to K^1 \to \cdots$ be a sequence of morphisms in $D(A)$ for some $A \in \mathbf{Ring}$, with every two consecutive arrows composing to zero; that is, it is a "complex consisting of objects of $D(A)$".

In order to work with this sequence, one would like to choose representatives in $K(A)$ so that the terms $K^\bullet$ fit into a double complex. In practice, this is obstructed by the construction of **Toda brackets**. To illustrate this point, suppose that we have managed to represent each K^i as a complex and each morphism $K^i \to K^{i+1}$ as a morphism of complexes (without localization). We then have a diagram

$$\begin{tikzcd}
K^0 \arrow[r, "d^0"] \arrow[rr, bend right, "0"'] & K^1 \arrow[r, "d^1"] \arrow[d, phantom, "\Uparrow\alpha"] \arrow[rr, bend left, "0"] & K^2 \arrow[r, "d^2"] \arrow[u, phantom, "\Downarrow\beta"] & K^3
\end{tikzcd}$$

Figure 13.4.2

in which α represents some homotopy witnessing the vanishing of $d^1 \circ d^0$ in $K(A)$ and β represents some homotopy witnessing the vanishing of $d^2 \circ d^1$ in $K(A)$. Then $d^2 \circ \alpha$ and $\beta \circ d^0$ are both homotopies that witness the vanishing of $d^2 \circ d^1 \circ d^0$ in $K(A)$, but it may not be possible to choose α and β to make them equal. In fact, these two homotopies together define a loop in the π_1 of the space of maps between simplicial realizations of K^0 and K^3; the Toda bracket is the isotopy class of this loop, whose nonvanishing provides an obstruction to choosing the morphisms so that the compositions $d^1 \circ d^0$ and $d^2 \circ d^1$ vanish on the nose. (One can similarly make **higher Toda brackets** by considering longer chunks of the sequence, conditionally on the vanishing of the lower-order brackets.)

This gives an example of why it is easier in the long run to work with $D(A)$ in the framework of stable ∞-categories. See [10] in particular for a description of totalization in this framework that properly accounts for the Toda brackets.

14 Hodge-Tate comparison for crystalline prisms

Reference. [18], lecture VI; [25], sections 5, 6.

In this section, we prove the Hodge-Tate comparison theorem (Theorem 12.4.1) in the special case where the base prism (A, I) is crystalline (meaning that $I = (p)$) and the ring R is a polynomial ring over $\overline{A} = A/p$. This simultaneously shows off some key ideas and provides a crucial base case for the general argument.

14.1 de Rham cohomology in characteristic p

We first recall how de Rham cohomology works in characteristic p, focusing on the key case of an affine space (polynomial ring). The key point is that even in this case the cohomology is quite large, and in fact the cohomology groups reflect the structure of the original complex via the **Cartier isomorphism**.

Lemma 14.1.1 *For any morphism $R \to S$ in $\mathbf{Ring}_{\mathbb{F}_p}$, the map $\phi_S \colon \Omega^i_{S/R} \to \Omega^i_{S/R}$ is zero for all $i > 0$.*

Proof. For any $x \in S$, we have

$$\phi_S(dx) = d\phi_S(x) = d(x^p) = px^{p-1}\, dx = 0$$

because $p = 0$ in S. This proves the claim for $i = 1$, from which the rest follows at once. ■

Definition 14.1.2 For $R \to S$ a morphism in $\mathbf{Ring}_{\mathbb{F}_p}$, the map $\phi_S \colon S \to S$ factors through an R-linear map $\phi_{S/R} \colon S^{(1)} \to S$ where $S^{(1)} = S \otimes_{R,\phi_R} R$. We call $\phi_{S/R}$ the **relative Frobenius** for the map $R \to S$.

In geometric language, $\phi_{S/R}$ is the linearization of ϕ_S^* over $\operatorname{Spec} R$, obtained by factoring ϕ_S^* through a fiber product. See Figure 14.1.3. $\Diamond$

$$\begin{tikzcd}
\operatorname{Spec} S \arrow[dr, dashed, "\operatorname{Spec}\phi_{S/R}"] \arrow[drr, bend left, "\operatorname{Spec}\phi_S"] \arrow[ddr, bend right] & & \\
& \operatorname{Spec} S^{(1)} \arrow[r] \arrow[d] & \operatorname{Spec} S \arrow[d] \\
& \operatorname{Spec} R \arrow[r, "\operatorname{Spec}(\phi_R)"] & \operatorname{Spec} R
\end{tikzcd}$$

Figure 14.1.3

Corollary 14.1.4 *For any morphism $R \to S$ in $\mathbf{Ring}_{\mathbb{F}_p}$, the map $\phi_{S/R} \colon \Omega^i_{S^{(1)}/R} \to \Omega^i_{S/R}$ is zero for all $i > 0$.*

Proof. It suffices to check the claim for $i = 1$. Moreover, we may assume $S = R[x_1, \dots, x_n]$, as we may then take colimits to deduce the case where S is a polynomial ring in any number of variables, and then take quotients to deduce the case where S is arbitrary.

When $S = R[x_1, \dots, x_n]$, we may identify $S^{(1)}$ with a second copy of $R[x_1, \dots, x_n]$ with the map $S^{(1)} \to S$ being given by the R-linear substitution $x_i \mapsto x_i^p$. In particular, $\phi_{S/R}(dx_i) = d(x_i^p) = 0$ as per Lemma 14.1.1. ■

Remark 14.1.5 As indicated in Figure 14.1.3, the construction of relative Frobenius extends to an arbitrary morphism of schemes $f \colon Y \to X$ in characteristic p. The example of a polynomial ring, and its description in the proof of Corollary 14.1.4, may be misleading: in general Y and $Y^{(1)}$ will not be isomorphic over X. For example, if X is the spectrum of an algebraically closed field k and Y is an elliptic curve over k, then the j-invariants $j(Y)$ and $j(Y^{(1)})$ will differ in general (the latter being the image of the former under ϕ_k).

Lemma 14.1.6 Cartier isomorphism for affine space. *Choose $R \in \mathbf{Ring}_{\mathbb{F}_p}$, put $S = R[x_1, \dots, x_r]$, and let $\phi_{S/R} \colon S^{(1)} \to S$ be the relative Frobenius. Then there is a quasi-isomorphism*

$$(\Omega^\bullet_{S^{(1)}/R}, 0) \to (\Omega^\bullet_{S/R}, d_{\mathrm{dR}})$$

of $S^{(1)}$-dga's acting as $\phi_{S/R}$ in degree 0 and taking dx_j to $x_j^{p-1} dx_j$.

Proof. The map $\phi_{S/R}$ induces a morphism of complexes thanks to Corollary 14.1.4. To check that this map is a quasi-isomorphism, we form a decomposition

$$\Omega^i_{S/R} = \bigoplus_{e_1, \dots, e_r \in \{0, \dots, p-1\}} \bigoplus_{1 \leq j_1 < \cdots < j_i \leq r} x_1^{e_1} \cdots x_r^{e_r} S^{(1)} dx_{j_1} \wedge \cdots \wedge dx_{j_i}. \tag{14.1}$$

We obtain a morphism $(\Omega^\bullet_{S/R}, d_{\mathrm{dR}}) \to (\Omega^\bullet_{S^{(1)}/R}, 0)$ of complexes (not respecting the multiplicative structure) taking $x_1^{e_1} \cdots x_r^{e_r} dx_{j_1} \wedge \cdots \wedge dx_{j_i}$ to $dx_{j_1} \wedge \cdots \wedge dx_{j_i}$ for

$$e_j = \begin{cases} p-1 & j \in \{j_1, \dots, j_i\} \\ 0 & j \notin \{j_1, \dots, j_i\} \end{cases}.$$

We must show that the composition of these maps is homotopic to the identity on $\Omega^\bullet_{S/R}$. By proceeding by induction, we may reduce to the case $r = 1$. In this case, for $e_1 = 1, \dots, p-1$, d_{dR} maps $x_1^{e_1} S^{(1)}$ to $x_1^{e_1-1} S^{(1)}\, dx_1$ taking $x_1^{e_1} f$ to $e_1 x_1^{e_1-1} f\, dx$, and this map is evidently invertible. ■

Remark 14.1.7 While the Cartier map described in Lemma 14.1.6 is defined in terms of coordinates on the polynomial ring S, the construction is *canonical up to homotopy* in that the resulting map in $D(S^{(1)})$ is well-defined independently of the way that S is expresesd as a polynomial ring. For example, making the change of variables $x_1 \mapsto x_1 + x_2$ does not change the map because

$$(x_1+x_2)^{p-1}\, d(x_1+x_2) - x_1^{p-1}\, dx_1 - x_2^{p-2}\, dx_2 = \sum_{i=1}^{p-2} \binom{p-1}{i} x_1^i x_2^{p_1-i}(dx_1+dx_2)$$

contributes only to summands in (14.1) that get killed off by the homotopy.

Yet another construction can be given (for $R = \mathbb{F}_p$, then deducing the general case by base change) by lifting from $\mathbb{F}_p[x_1, \dots, x_r]$ to $\mathbb{Z}_p[x_1, \dots, x_r]$. Given an element $f \in \mathbb{Z}_p[x_1^p, \dots, x_r^p]$ lifting $\overline{f} \in \mathbb{F}_p[x_1^p, \dots, x_r^p] \cong S^{(1)}$, the element $p^{-1} df$ reduces to an element of $\Omega^1_{S/R}$ independent of the choice of f, and this is the image of $d\overline{f}$ under the Cartier map.

This last construction is quite similar to how the Cartier isomorphism will appear in the proof of the Hodge-Tate comparison for crystalline prisms (Proposition 14.4.12). In fact that result will itself establish the canonicality of the Cartier isomorphism, so we don't need to worry much about it right now.

In any case, once canonicality is established by some means, we can easily promote Lemma 14.1.6 to a similar statement for any smooth morphism $R \to S$ in $\mathbf{Ring}_{\mathbb{F}_p}$. We omit the details here, as we will see the same argument again soon (Lemma 15.1.2).

14.2 Divided powers

We next recall an algebraic construction that will help us study de Rham cohomology in mixed characteristic. See [15], section 3 for a detailed development, which also covers cases where the ring can have $\mathbb{Z}$-torsion.

Definition 14.2.1 For $R \in \mathbf{Ring}$ flat over $\mathbb{Z}$, the **divided power operations** $\gamma_n \colon R \to R \otimes_{\mathbb{Z}} \mathbb{Q}$ are the maps

$$\gamma_n(x) = \frac{x^n}{n!} \qquad (x \in R, n \geq 0).$$

From the identities

$$\gamma_n(x+y) = \sum_{i=0}^{n} \gamma_i(x)\gamma_{n-i}(y), \qquad \gamma_n(xy) = x^n \gamma_n(y),$$

we see that the set of $x \in R$ for which $\gamma_n(x) \in R$ for all $n \geq 0$ is an ideal of R. For J an ideal contained in this ideal, we say that R **admits divided powers** on J. ♢

Example 14.2.2 The ring $\mathbb{Z}_{(p)}$ admits divided powers on (p) because

$$\frac{p^{n-1}}{n!} \in \mathbb{Z}_{(p)} \qquad (n \geq 1). \tag{14.2}$$

□

Remark 14.2.3 Our definition of "admits divided powers" is not quite the usual one: normally one also requires that γ_n maps J into J for each $n \geq 1$. For example, this occurs in Example 14.2.2 because (14.2) includes p^{n-1} rather than p^n. However, the last statement in Definition 14.2.4 ensures that this discrepancy doesn't affect anything later.

Definition 14.2.4 The **divided power envelope** of (R, J) is the subring D of $R \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by R and $\gamma_n(x)$ for all $x \in J$. Using the identities

$$\gamma_m(x)\gamma_n(x) = \binom{m+n}{n}\gamma_{m+n}(x) \tag{14.3}$$

$$\gamma_m(\gamma_n(x)) = \frac{(mn)!}{m!(n!)^m}\gamma_{mn}(x) \tag{14.4}$$

and the fact that $(mn)!/(m!(n!)^m) \in \mathbb{Z}$ (it counts unordered partitions of $\{1, \dots, mn\}$ into n-element subsets), we see that D admits divided powers on the ideal generated by $\gamma_n(J)$ for all $n \geq 1$ (even in the stronger sense of Remark 14.2.3). $\Diamond$

Remark 14.2.5 When studying divided powers, it is common to use the initialism **pd** for the French phrase **puissances divisées**. For example, the divided power envelope is also called the **pd-envelope**.

One key motivation for introducing divided powers is to formulate the **Poincaré lemma**.

Proposition 14.2.6 *Suppose that $A \in$ **Ring** is $\mathbb{Z}$-flat. Set $P = A[x]$ and let D be the divided power envelope of $(P, (x))$. Then the morphism*

$$d \colon D \to D \otimes_P \Omega^1_{P/A} = D\,dx$$

is surjective with kernel A; the same remains true if we replace D with its p-adic completion.

Proof. Exercise (see Exercises 14.5). ∎

Remark 14.2.7 Proposition 14.2.6 amounts to the computation of the **crystalline cohomology** of a point. We will see in Subsection 14.4 that the proof of the Hodge-Tate comparison isomorphism for a crystalline prism naturally passes through crystalline cohomology.

One potentially confusing point is that unlike de Rham cohomology in characteristic 0, the crystalline cohomology of a higher-dimensional affine space is not the same as that of a point! In fact, the crystalline cohomology of $\operatorname{Spec} \mathbb{F}_p[x_1, \dots, x_r]$ is computed by the complex $\widehat{\Omega}^\bullet_{P/\mathbb{Z}_p}$ where $P = \mathbb{Z}_p[x_1, \dots, x_r]^\wedge_{(p)}$ (this ring already admits divided powers on (p) by Example 14.2.2); taking the derived base change from $\mathbb{Z}_p$ to $\mathbb{F}_p$ yields $\Omega_{R/\mathbb{F}_p}$, whose cohomology we already know is quite large (Lemma 14.1.6). What Proposition 14.2.6 tells us is that the answer does not change if we include some extra "divided power variables"; see Proposition 14.2.8 for a concrete statement.

In any case, none of this has much meaning without an actual definition of crystalline cohomology itself. For that, see [15].

Proposition 14.2.8 Crystalline and de Rham cohomology of affine space. *Put $R = \mathbb{F}_p[x_1, \dots, x_r]$, $P_0 = \mathbb{Z}_p[x_1, \dots, x_r]$, $P = \mathbb{Z}_p[x_1, \dots, x_r, y_1, \dots, y_s]$, and let D be the p-adic completion of the divided power envelope of $(P, (p, y_1, \dots, y_s))$. Then there is a natural quasi-isomorphism*

$$\widehat{\Omega}^\bullet_{P_0/\mathbb{Z}_p} \cong D \widehat{\otimes}_P \widehat{\Omega}^\bullet_{P/\mathbb{Z}_p}$$

and hence a quasi-isomorphism

$$\Omega_{R/\mathbb{F}_p} \cong (D\widehat{\otimes}_P \widehat{\Omega}^\bullet_{P/\mathbb{Z}_p}) \otimes^L_{\mathbb{Z}_p} \mathbb{F}_p.$$

Proof. By repeated application of Proposition 14.2.6 we may reduce to the case $s = 0$, in which case this is evident. ∎

14.3 Divided powers in δ-rings

We next make a crucial link between δ-rings and divided powers.

Remark 14.3.1 If R is a $\mathbb{Z}_{(p)}$-algebra which admits a δ-ring structure, then R admits divided powers on some ideal J if and only if $\gamma_p(x) \in R$ for all $x \in J$. See Exercise 2.5.10.

Lemma 14.3.2 *For $R = \mathbb{Z}_{(p)}\{x\}$ and $J = xR$, the map from R to the divided power envelope of (R, J) promotes to a morphism of δ-rings.*

Proof. Let D be the divided power envelope; it is the smallest subring of $\mathbb{Z}_{(p)}\{x\}[p^{-1}]$ containing $\mathbb{Z}_{(p)}\{x\}$ and $\gamma_n(x)$ for all $n \geq 1$. The maximal ideal on which D admits divided powers includes both x (by construction) and p (by Example 14.2.2), and hence also $\phi(x)$; consequently, for all $n \geq 1$,

$$\phi(\gamma_n(x)) = \gamma_n(\phi(x)) \in D.$$

Hence ϕ induces an endomorphism of D.

We next check that ϕ induces a Frobenius lift on D; this amounts to checking that for all $n \geq 1$,

$$\phi(\gamma_n(x)) \equiv \gamma_n(x)^p \pmod{pD}.$$

We will see that in fact both sides are divisible by p. For $\phi(\gamma_n(x)) = \gamma_n(\phi(x))$, this holds by writing $\phi(x) = p(x^p/p + \delta(x)) \in pD$ and invoking (14.2). For $\gamma_n(x)^p$, this holds by writing $\gamma_n(x)^p = p!\gamma_p(\gamma_n(x))$ and applying (14.4).

Since D is p-torsion-free, by Lemma 2.1.3 we obtain a δ-structure compatible with R, as desired. ∎

Corollary 14.3.3 *In Lemma 14.3.2, the divided power envelope equals $\mathbb{Z}_{(p)}\{x, \frac{\phi(x)}{p}\}$, or more precisely the quotient of $\mathbb{Z}_{(p)}\{x, z\}$ by the δ-ideal generated by $\phi(x) - pz$.*

Proof. Let D be the divided power envelope and put $D' = \mathbb{Z}_{(p)}\{x, \frac{\phi(x)}{p}\}$; there is a natural map $D' \to D[p^{-1}]$ which one checks is injective. Within $D[p^{-1}]$, we then have $D' \subseteq D$ by Lemma 14.3.2 and $D \subseteq D'$ by Remark 14.3.1. (Compare [25], Lemma 2.36.) ∎

Corollary 14.3.4 *Let $A \in$ $\mathbf{Ring}_\delta$ be p-torsion-free. Choose $f_1, \dots, f_r \in A$ which form a regular sequence in A/p and set $I = (f_1, \dots, f_r)$. Then the divided power envelope of (A, I) is a δ-ring, and can be written as $A\{\phi(f_1)/p, \dots, \phi(f_r)/p\}$ (viewing the latter as a subring of $A[p^{-1}]$).*

Proof. By induction this reduces to the case $r = 1$, in which case we write f for f_1. In this case, we may deduce the claim from Lemma 14.3.2 and Corollary 14.3.3 by base change. (Compare [25], Corollary 2.38.) ∎

Remark 14.3.5 Corollary 14.3.4 implies that the subring $A\{\phi(f_1)/p, \dots, \phi(f_r)/p\}$ of $A[p^{-1}]$ is independent of the choice of δ-structure on A, as the characterization via the divided power envelope of (A, I) makes no reference to δ or ϕ. By contrast, $A\{f_1/p, \dots, f_r/p\}$ is not independent of this choice; see [25], Warning 2.40.

Remark 14.3.6 Lemma 14.3.2 asserts that the divided power envelope of (R, J) as a ring is also the divided power envelope as a δ-ring. The corresponding statement for λ-rings is false: for A the free λ-ring on x (over $\mathbb{Z}$), the divided power envelope of $(A,(x))$ is not a λ-subring of $A \otimes_{\mathbb{Z}} \mathbb{Q}$. The issue here is with the use of Example 14.2.2: the ring $\mathbb{Z}$ does not admit divided powers on (p) for any prime p.

This then leads to the question of describing the smallest λ-subring of $A\otimes_{\mathbb{Z}}\mathbb{Q}$ containing A which admits divided powers on (x). We do not know the answer, but as a partial result we note that this ring contains the elements $\delta_p(x)^n/m$ where p is a prime, n is a positive integer, and m is the prime-to-p factor of $n!$.

A related question is whether Remark 14.3.5 admits a λ-ring analogue. That is, if $A \in \mathbf{Ring}_\lambda$ is $\mathbb{Z}$-flat and $f_1, \dots, f_r \in A$ form a regular sequence in A/p for every prime p, does the minimal λ-subring of $A \otimes_{\mathbb{Z}} \mathbb{Q}$ containing A which admits divided powers on $(f_1, \dots, f_r)$ depend only on the underlying ring structure of A and not its λ-structure?

14.4 Prismatic cohomology for a crystalline prism

We next use divided powers to explicitly compute the cohomology of affine space over a crystalline prism. To begin with, we make the construction of weakly final objects of the prismatic site (Proposition 11.6.5) more explicit in some cases of interest.

Lemma 14.4.1 *For P a polynomial ring over $\mathbb{Z}_p$, for every $i > 0$, the complex*

$$\Omega^i_P \to \Omega^i_{P\otimes P} \to \Omega^i_{P\otimes P \otimes P} \to \cdots$$

vanishes in the homotopy category $K(\mathbb{Z}_p)$. (More precisely, this is witnessed by a homotopy at the level of $P^\bullet$-cosimplicial modules; see Definition 16.2.5 for the meaning of this statement.)

Proof. We may reduce to the case of a polynomial ring in finitely many variables by taking colimits. We may further reduce the case $i = 1$ using exterior powers. We may further reduce to the case $P = \mathbb{Z}_p[x]$ using base change and induction on the number of variables. At this point, we can write down the homotopy h explicitly: if we write the $(n+1)$-fold tensor product of P as $P^n = \mathbb{Z}[x_{n0}, \dots, x_{nn}]$, then the homotopy carries $\Omega^1_{P^n}$ to $\Omega_{P^{n-1}}$ taking dx_{ni} to $dx_{(n-1)i}$ for $i = 0, \dots, n-1$ and to 0 for $i = n$. We leave it to the reader to check that h is a homotopy for the identity map (Exercise 14.5.2). (Compare [19], Example 2.16.) ∎

Lemma 14.4.2 *Let (A, I) be the prism $(\mathbb{Z}_p, (p))$ and put $R = \mathbb{F}_p[x_1, \dots, x_r]$. Let P be the classical p-completion of $\mathbb{Z}_p\{x_1, \dots, x_r\}$. Let J be the kernel of the map $P \to R$ taking x_i to x_i and $\delta^m(x_i)$ to 0 for all $m > 0$. Write $P\{J/p\}^\wedge_{(p)}$ for the classical p-completion of $P\{f/p \colon f \in J\}$. Then $(P\{J/p\}^\wedge_{(p)}, (p))$ is a weakly final object of $(R/A)_{\mathbf{\Delta}}$.*

Proof. By Exercise 2.5.8, (P, J) is a δ-pair. As in the proof of Proposition 11.6.5. we may apply Lemma 11.6.1 to (P, J) to obtain a weakly final object of $(R/A)_{\mathbf{\Delta}}$. To identify the result explicitly, we step through the proof of Lemma 11.6.1. We first take the derived p-completion of $P\{f/p \colon f \in J\}$; as this object is p-adically separated this is in fact a classical p-completion. In addition the result is p-torsion-free, so there is no need to iterate the construction. (Compare [18], Lecture VI, Corollary 2.3.) ∎

Remark 14.4.3 Note that $P^\bullet$ is itself a Čech-Alexander complex, namely the one associated to the covering $\operatorname{Spf} P \to \operatorname{Spf} A$ in the category of p-adic formal schemes. In particular, $A \to P^\bullet$ is an isomorphism in $K(A)$ by the Čech-Alexander construction in the category of p-formal schemes (and even a homotopy equivalence of cosimplicial A-algebras, as per Definition 16.2.5).

Corollary 14.4.4 *With notation as in Lemma 14.4.2, for $n \geq 0$, identify*

$$P^n = \mathbb{Z}_p\{x_{ij} : i = 1, \ldots, r; j = 0, \ldots, n\}^\wedge_{(p)}$$

with the $(n+1)$-fold completed tensor product of P over $\mathbb{Z}_p$. Let J^n be the kernel of the morphism $P^n \to R$ in $\mathbf{Ring}_\delta$ taking x_{ij} to x_i and $\delta^m(x_{ij})$ to 0 for all $m > 0$, and write $P^n\{J^n/p\}^\wedge_{(p)}$ for the classical p-completion of $P^n\{f/p : f \in J^n\}$. Then $\mathbf{\Delta}_{R/A}$ is quasi-isomorphic to the Čech-Alexander complex

$$0 \to P^0\{J^0/p\}^\wedge_{(p)} \to P^1\{J^1/p\}^\wedge_{(p)} \to P^2\{J^2/p\}^\wedge_{(p)} \to \cdots .$$

Proof. By Lemma 14.4.2, $P^0\{J^0/p\}^\wedge_{(p)}$ is a weakly final object of $(R/A)_{\mathbf{\Delta}}$. Now note that $(P^n\{J^n/p\}^\wedge_{(p)}, (p))$ is the $(n+1)$-fold product of $(P^0\{J^0/p\}^\wedge_{(p)}, (p))$ in $\mathbf{\Delta}_{R/A}$. Hence we are in the setting described in Remark 11.6.7. ■

Remark 14.4.5 Note that while J^0 is generated by $\delta^m(x_i)$ for all $m > 0$, J^n is not generated by $\delta^m(x_{ij})$ for all $m > 0$; we must also add the generators $x_{ij} - x_{ij'}$ for $j \neq j'$.

Remark 14.4.6 With notation as in Lemma 14.4.2 and Corollary 14.4.4, the map ϕ on $A = \mathbb{Z}_p$ is an isomorphism. By Remark 14.4.3, $\phi_{P*} : P^\bullet \to P^\bullet$ is a quasi-isomorphism (and a homotopy equivalence), yielding isomorphisms in $K(\mathbb{Z}_p)$ of the form

$$P^\bullet\{J^\bullet/p\}^\wedge_{(p)} \to \phi^*_{P^\bullet}(P^\bullet\{J^\bullet/p\})^\wedge_{(p)} = P^\bullet\{\phi(J^\bullet)/p\}^\wedge_{(p)}.$$

By Corollary 14.3.4, the latter coincides with the p-completed divided power envelope $D_{J^\bullet}(P^\bullet)$ of $(P^\bullet, J^\bullet)$. (More precisely, these are homotopy equivalences of cosimplicial $\mathbb{Z}_p$-algebras; see again Definition 16.2.5.)

To summarize, the rows of Figure 14.4.7 are quasi-isomorphic to each other and to $\mathbf{\Delta}_{R/A}$.

$$\begin{tikzcd}
0 \arrow[r] & P^0\{J^0/p\}^\wedge_{(p)} \arrow[r] \arrow[d, "\phi"] & P^1\{J^1/p\}^\wedge_{(p)} \arrow[r] \arrow[d, "\phi"] & P^2\{J^2/p\}^\wedge_{(p)} \arrow[r] \arrow[d, "\phi"] & \cdots \\
0 \arrow[r] & P^0\{\phi(J^0)/p\}^\wedge_{(p)} \arrow[r] \arrow[d, equal] & P^1\{\phi(J^1)/p\}^\wedge_{(p)} \arrow[r] \arrow[d, equal] & P^2\{\phi(J^2)/p\}^\wedge_{(p)} \arrow[r] \arrow[d, equal] & \cdots \\
0 \arrow[r] & D_{J^0}(P^0) \arrow[r] & D_{J^1}(P^1) \arrow[r] & D_{J^2}(P^2) \arrow[r] & \cdots
\end{tikzcd}$$

Figure 14.4.7

Lemma 14.4.8 *Let (A, I) be the prism $(\mathbb{Z}_p, (p))$ and put $R = \mathbb{F}_p[x_1, \ldots, x_r]$. Let (P, IP) be the weakly final object of $(R/A)_{\mathbf{\Delta}}$ given by Lemma 14.4.2. Then the totalization of the double complex displayed in Figure 14.4.9 is quasi-isomorphic to both its first row and its first column via the inclusion maps.*

$$\begin{tikzcd}
D_{J^0}(P^0) \arrow[r] \arrow[d] & D_{J^1}(P^1) \arrow[r] \arrow[d] & D_{J^2}(P^2) \arrow[r] \arrow[d] & \cdots \\
D_{J^0}(P^0)\widehat{\otimes}_{P^0}\widehat{\Omega}^1_{P^0/\mathbb{Z}_p} \arrow[r] \arrow[d] & D_{J^1}(P^1)\widehat{\otimes}_{P^1}\widehat{\Omega}^1_{P^1/\mathbb{Z}_p} \arrow[r] \arrow[d] & D_{J^2}(P^2)\widehat{\otimes}_{P^2}\widehat{\Omega}^1_{P^2/\mathbb{Z}_p} \arrow[r] \arrow[d] & \cdots \\
D_{J^0}(P^0)\widehat{\otimes}_{P^0}\widehat{\Omega}^2_{P^0/\mathbb{Z}_p} \arrow[r] \arrow[d] & D_{J^1}(P^1)\widehat{\otimes}_{P^1}\widehat{\Omega}^2_{P^1/\mathbb{Z}_p} \arrow[r] \arrow[d] & D_{J^2}(P^2)\widehat{\otimes}_{P^2}\widehat{\Omega}^2_{P^2/\mathbb{Z}_p} \arrow[r] \arrow[d] & \cdots \\
\vdots & \vdots & \vdots &
\end{tikzcd}$$

Figure 14.4.9

Proof. We can compute the cohomology of the total complex using the "first" spectral sequence, in which we first compute the cohomology of the columns. In this case, $H^m(D_{J^n}(P^n)\widehat{\otimes}_{P^n}\widehat{\Omega}^\bullet_{P^n/\mathbb{Z}_p})$ is independent of n: by Proposition 14.2.8 each column computes the crystalline cohomology of $\operatorname{Spec}\mathbb{F}_p[x_1,\dots,x_r]$, and for this identification each map $P^{n-1}\to P^n$ represents the identity map on cohomology. Hence the horizontal differentials between the columns (which are alternating sums of the induced maps) are represented by

$$H^m \xrightarrow{0} H^m \xrightarrow{1} H^m \xrightarrow{0} H^m \xrightarrow{1} \cdots .$$

At the next page of the spectral sequence, we end up with the groups $H^m(D_{J^0}(P^0)\widehat{\otimes}_{P^0}\widehat{\Omega}^\bullet_{P^0/\mathbb{Z}_p})$ in column 0 and zeroes elsewhere. By Corollary 13.3.8, the map from the first column to the totalization is a quasi-isomorphism. (Compare Example 16.2.4 for a similar phenomenon.)

Meanwhile, by Lemma 14.4.1, each row except the first is homotopic to zero. Consequently, by Corollary 13.3.8 again, the natural map from the first row to the totalization is also a quasi-isomorphism. (Compare [19], Theorem 2.12.) ■

Corollary 14.4.10 *Let (A,I) be the prism $(\mathbb{Z}_p,(p))$ and put $R=\mathbb{F}_p[x_1,\dots,x_r]$. Then $\phi^*\mathbf{\Delta}_{R/A}$ is quasi-isomorphic to the crystalline cohomology of the affine space $\operatorname{Spec}\mathbb{F}_p[x_1,\dots,x_r]$ in the sense of Remark 14.2.7, i.e., to $\widehat{\Omega}^\bullet_{P/\mathbb{Z}_p}$ for $P=\mathbb{Z}_p[x_1,\dots,x_r]^\wedge_{(p)}$.*

Proof. By Remark 14.4.6, we obtain a quasi-isomorphism of $\phi^*\mathbf{\Delta}_{R/A}$ with the first row of Figure 14.4.9. By Lemma 14.4.8, this is in turn quasi-isomorphic to the left column of Figure 14.4.9. By Proposition 14.2.8, the latter is quasi-isomorphic to $\widehat{\Omega}^\bullet_{P/\mathbb{Z}_p}$ for $P=\mathbb{Z}_p[x_1,\dots,x_r]^\wedge_{(p)}$ (note that this amounts to the same use of the Poincaré lemma as was needed to compare columns in Lemma 14.4.8). ■

Remark 14.4.11 In Corollary 14.4.10, we write $\phi^*\mathbf{\Delta}_{R/A}$ instead of $\mathbf{\Delta}_{R/A}$ to keep track of the fact that prismatic cohomology computes not crystalline cohomology *per se*, but rather a canonical Frobenius descent of it.

Proposition 14.4.12 *Let (A,I) be the prism $(\mathbb{Z}_p,(p))$ and put $R=\mathbb{F}_p[x_1,\dots,x_r]$. Then the Hodge-Tate comparison map (12.1) is an isomorphism.*

Proof. By Lemma 14.4.2, we may compute the object $\mathbf{\Delta}_{R/A}\in D(A)$ using the Čech-Alexander complex associated to the weakly final object (P,IP) as described in Corollary 14.4.4, and then obtain $\overline{\mathbf{\Delta}}_{R/A}\in D(\overline{A})$ by applying $\otimes^L_A A/p$. By Remark 14.4.6, the object $\mathbf{\Delta}_{R/A}$ (or more correctly $\phi^*\mathbf{\Delta}_{R/A}$) is represented by the top row of the double complex Figure 14.4.9, which by Lemma 14.4.8 is isomorphic in $D(A)$ to the first column of the double complex. By Corollary 14.4.10, that column computes the crystalline cohomology of affine space. By applying $\otimes^L_{\mathbb{Z}_p}\mathbb{F}_p$, we obtain an isomorphism

$$\phi^*\overline{\mathbf{\Delta}}_{R/A} \cong (\Omega^\bullet_{R/\mathbb{F}_p}, d_{\mathrm{dR}}) \tag{14.5}$$

of $\mathbb{F}_p$-dga's which in degree 0 is the identity map on R.

To check that η_R is an isomorphism, it will suffice to deduce from (14.5) that $\phi^*(\eta_R)$ corresponds to the Cartier isomorphism; it suffices to do this in degree 1. As in Definition 12.3.1, consider the exact sequence

$$0 \to pT/p^2T \to T/p^2T \to T/pT \to 0, \qquad T = P^0\{J^0/p\}^\wedge_{(p)}.$$

Viewing the element $x_i \in P^0\{J^0/p\}^\wedge_{(p)}$ as representing a class in $H^0(\overline{\mathbf{\Delta}}_{R/A})$, we find that its image under the Bockstein differential is represented by $(x_{i0}-x_{i1})/p$. This is then the image of $dx_i \in \Omega_{R/\mathbb{F}_p}$ under η_R, and it remains to transfer the answer via (14.5).

Applying ϕ to $(x_{i0}-x_{i1})/p$ yields $(x_{i0}^p - x_{i1}^p)/p \in D_{J^1}(P^1)$. Going down the vertical arrow $D_{J^1}(P^1) \to D_{J^1}(P^1)\widehat{\otimes}_{P^1}\widehat{\Omega}^1_{P^1/\mathbb{Z}_p}$ in Figure 14.4.9 yields

$$d((x_{i0}^p - x_{i1}^p)/p) = x_{i0}^{p-1}dx_{i0} - x_{i1}^{p-1}dx_{i1}.$$

This is the image of $x_i^{p-1}\,dx_i$ along the horizontal arrow $D_{J^0}(P^0)\widehat{\otimes}_{P^0}\widehat{\Omega}^1_{P^0/\mathbb{Z}_p} \to D_{J^1}(P^1)\widehat{\otimes}_{P^1}\widehat{\Omega}^1_{P^1/\mathbb{Z}_p}$ in Figure 14.4.9. When we reduce mod p, we get exactly the image of dx_i under the Cartier map, proving the claim. ■

14.5 Exercises

1. Prove Proposition 14.2.6.
2. Complete the proof of Lemma 14.4.1 by confirming that h is indeed a homotopy for the identity map.

15 Proof of the Hodge-Tate comparison

Reference. [18], lecture VI; [25], sections 5, 6.

In this section, we prove the Hodge-Tate comparison theorem (Theorem 12.4.1). Our strategy will be to build up from the special case treated in Section 14, in which we used the crystalline prism $(\mathbb{Z}_p, (p))$ as the base and the ring $R = \mathbb{F}_p[x_1, \dots, x_r]$.

We also assert the crystalline and de Rham comparison theorems. These are technically a bit more involved, so we do not include all of the details here.

Throughout, we fix a bounded prism (A, I) and denote its slice A/I by $\overline{A}$.

15.1 Étale localization and base change

Remark 15.1.1 Recall that a morphism $R \to S$ of rings is smooth if and only if locally on $\mathrm{Spec}(S)$, it can be written in the form $R \to R[x_1, \dots, x_r] \to S$ where the second map is étale (see [117], tag 054L). Similarly, if $R \to S$ is a p-completely smooth map, then locally on $\mathrm{Spec}(S/p)$ it can be written in the form $R \to R\langle x_1, \dots, x_r\rangle \to S$ where the second map is p-completely étale (use Proposition 6.5.3 to reduce to the previous statement).

If $\overline{A} \to R$ is p-completely smooth, then $\widehat{\Omega}^1_{R/\overline{A}}$ is a finite projective R-module (again by Proposition 6.5.3 to reduce to the corresponding statement about differentials for a smooth morphism). Consequently, if $R \to S$ is p-completely étale, then $\widehat{\Omega}^i_{R/\overline{A}} \widehat{\otimes}_R S \cong \widehat{\Omega}^i_{S/\overline{A}}$ for all i.

This suggests the strategy of proving the Hodge-Tate comparison for a general p-completely smooth $\overline{A}$-algebra R by proving the corresponding compatibility with étale maps, and then using this to reduce to the case $R = \overline{A}\langle X_1, \dots, X_r \rangle$. The first step in this program is executed by Lemma 15.1.2.

Lemma 15.1.2 Étale localization for Hodge-Tate cohomology. *Let $R \to S$ be a p-completely étale map of p-completely smooth $\overline{A}$-algebras. Then the natural map $\overline{\mathbf{\Delta}}_{R/A} \widehat{\otimes}^L_R S \to \overline{\mathbf{\Delta}}_{S/A}$ is an isomorphism.*

Proof. The restriction functor $(S/A)_{\mathbf{\Delta}} \to (R/A)_{\mathbf{\Delta}}$ admits a right adjoint F taking $(B \to B/IB \leftarrow R)$ to $(B_S \to B_S/IB_S \leftarrow S)$ where $B_S/IB_S = B/IB \widehat{\otimes}^L_R S$ and $B \to B_S$ is the unique lift of the étale morphism $B/IB \to B/IB \widehat{\otimes}^L_R S$ given by the the henselian property of derived completions (see Corollary 6.3.2), promoted from **Ring** to $\mathbf{Ring}_\delta$ using Exercise 6.7.11. Applying F to a weakly final object of $(R/A)_{\mathbf{\Delta}}$ (Proposition 11.6.5), we obtain a weakly final object of $(S/A)_{\mathbf{\Delta}}$; since F also preserves finite products, we can take a complex computing $\overline{\mathbf{\Delta}}_{R/A}$ and apply F term by term to obtain a complex computing $\overline{\mathbf{\Delta}}_{S/A}$. It thus remains to compare this with $\overline{\mathbf{\Delta}}_{R/A} \widehat{\otimes}^L_R S$; we have a natural isomorphism at the level of simplicial rings, and (since $R \to S$ is I-completely flat and thus has finite I-complete Tor amplitude) we may now deduce the claim from Exercise 10.6.3. (Compare [25], Lemma 4.19.) ∎

Another tool we will use is a base-change assertion. This will allow us to simplify the base ring $\overline{A}$ in some cases.

Lemma 15.1.3 Base change for prismatic and Hodge-Tate cohomology. *Let R be a p-completely smooth $\overline{A}$-algebra. Let $(A, I) \to (A', I')$ be a map of bounded prisms such that $A \to A'$ has finite (p, I)-complete Tor amplitude (e.g., a faithfully flat morphism). For $R' = R \widehat{\otimes}_A A'$, we have natural isomorphisms*

$$\mathbf{\Delta}_{R/A} \widehat{\otimes}^L_A A' \cong \mathbf{\Delta}_{R'/A'}, \qquad \overline{\mathbf{\Delta}}_{R/A} \widehat{\otimes}^L_A A' \cong \overline{\mathbf{\Delta}}_{R'/A'}.$$

Proof. Let (P, IP) be a weakly final object of $(R/A)_{\mathbf{\Delta}}$, then apply Remark 11.6.7 to construct a Čech-Alexander complex computing $\mathbf{\Delta}_{R/A}$. Then $\mathbf{\Delta}_{R'/A'}$ is computed by the complex obtained by applying $\bullet \widehat{\otimes}_A A'$ termwise. Under the hypothesis on the Tor amplitude, we may apply Exercise 10.6.3 to conclude. (Compare [25], Lemma 4.18.) ∎

As an immediate application of étale localization and base change, we upgrade our previous statement about the Hodge-Tate comparison for crystalline prisms (Proposition 14.4.12).

Lemma 15.1.4 *Suppose that (A, I) is a crystalline prism and R is a smooth $\overline{A}$-algebra. Then the Hodge-Tate comparison map (12.1) is an isomorphism.*

Proof. By Proposition 14.4.12 and Lemma 15.1.3, the claim holds when $R = \overline{A}[x_1, \dots, x_r]$. We may then deduce the general case using Lemma 15.1.2. (Compare [25], Corollary 5.5.) ∎

15.2 Comparing a universal prism to a crystalline prism

Remark 15.2.1 We reproduce [18], Lecture VI, Remark 2.2, in order to justify why we can't directly transpose the proof of the Hodge-Tate comparison from crystalline prisms to more general prisms. Suppose that $I = (d)$. Consider the object $P = A[x] \in \mathbf{Ring}_\delta$ with $\delta(x) = 0$. Then the derived (p, d)-completion of $P\{\phi(x)/d\}$ does *not* equal the derived (p, d)-completion of the divided power envelope of $(P, (x))$. A typical example of this is the case $(A, (d)) = (\mathbb{Z}_p[\![q -$

$1]\!], ([p]_q))$; in this case, the derived (p,d)-completion of $P\{\phi(x)/d\}$ will end up coinciding with the derived (p,d)-completion of the **q-divided power envelope** of $(P,(x))$.

Lemma 15.2.2 *Let $(A,(d))$ be the universal oriented prism (Example 5.3.5) and put $B = A\{\phi(d)/p\}^\wedge$ (the completion being the derived p-completion).*

- *The ring B is classically (p,d)-complete.*
- *The ring B equals the classical p-completion of the divided power envelope of $(A,(d))$ (and is p-torsion-free).*

Proof. Since A is p-torsion-free, B is not just derived p-complete but also classically p-complete. Since $d^p = p(\phi(d)/p - \delta(p))$ is divisible by p, B is also classically (p,d)-complete. By Corollary 14.3.4, B equals the classical p-completion of the divided power envelope of $(A,(d))$. (Compare [25], Construction 6.1.) ■

Remark 15.2.3 In the notation of Lemma 15.2.2, B is again a δ-ring and both $\phi(d)$ and p are distinguished elements. Since $\phi(d)$ is divisible by p, we may apply Lemma 5.2.1 to deduce that $\phi(d)$ and p generate the same ideal in B. In other words, the composition of maps of δ-rings

$$A \to B \stackrel{\phi_B}{\to} B$$

promotes to a composition of maps of prisms

$$(A,(d)) \to (B,(d)) \stackrel{\phi_B}{\to} (B,(\phi(d))) = (B,(p))$$

in which the target is crystalline! This will ultimately allow us to transfer information from the crystalline case of the Hodge-Tate comparison to the universal case, and then from there to the general case.

Lemma 15.2.4 *With notation as in Lemma 15.2.2, let $\alpha\colon A \to B$ be the composition of $\phi\colon A \to A$ with the natural map $A \to B$.*

- *The map $A/p \to B/p$ induced by α factors as a composition*

$$A/p \to A/(p,d) \stackrel{\phi}{\to} A/(p,d^p) \to B/p$$

  in which the first map has finite (p,d)-complete Tor amplitude and the second and third maps are faithfully flat.

- *The functor $\widehat{\alpha^*}\colon D_{\mathrm{comp}}(A) \to D_{\mathrm{comp}}(B)$ is conservative (i.e., reflects isomorphisms). Here $D_{\mathrm{comp}}(*)$ denotes the subcategory of derived (p,d)-complete objects of $D(*)$.*

- *For any p-completely smooth $\overline{A}$-algebra R, writing $R_B = R\widehat{\otimes}_A B$, the map $\widehat{\alpha^*}\mathbf{\Delta}_{R/A} \to \mathbf{\Delta}_{R_B/B}$ is an isomorphism.*

Proof. In (1), the first map has finite (p,d)-complete Tor amplitude because d is not a zero-divisor in A/p; the second and third maps are faithfully flat by construction. (Compare [25], Construction 6.1.) ■

Proposition 15.2.5 *Let $(A,(d))$ be the universal oriented prism (Example 5.3.5) and put $R = \overline{A}\langle x_1,\dots,x_r\rangle$. Then the Hodge-Tate comparison map (12.1) is an isomorphism.*

Proof. Let $(A,(d)) \to (B,(d))$ be the morphism from Lemma 15.2.2. By Lemma 15.2.4, we may apply Lemma 15.1.3 to reduce the claim from the prism $(A,(d))$ to $(B,(d))$. The latter is a crystalline prism by Remark 15.2.3, so Lemma 15.1.4 applies. ■

15.3 Hodge-Tate comparisons

We finally treat the Hodge-Tate comparison theorem (Theorem 12.4.1) in general. Before treating the general case, we give an easier argument that covers many cases of interest.

Proposition 15.3.1 *Suppose that $(A, I) = (A, (d))$ is an oriented (bounded) prism and the map from the universal oriented prism has finite (p, d)-complete Tor-amplitude. (For example, this last condition holds if d is a non-zerodivisor in A/p.) Let R be any p-completely smooth $\overline{A}$-algebra. Then the Hodge-Tate comparison map (12.1) is an isomorphism.*

Proof. By Lemma 15.1.2, we may reduce to the case $R = \overline{A}\langle x_1, \dots, x_r\rangle$. In this case, since we assumed the map $(A_0, (d)) \to (A, (d))$ from the universal oriented prism has finite (p, d)-complete Tor amplitude, we can use Lemma 15.1.3 to transfer the desired result from $(A_0, (d))$ (to which Proposition 15.2.5 applies) to $(A, (d))$. (Compare [25], Proposition 6.2.) ∎

Proposition 15.3.2 *In the full generality of Theorem 12.4.1, the Hodge-Tate comparison map (12.1) is an isomorphism.*

Proof. Again by Lemma 15.1.2, we may reduce to the case $R = \overline{A}\langle x_1, \dots, x_r\rangle$. Using Lemma 5.2.5, we can further reduce to the case where $(A, I) = (A, (d))$ is an oriented prism. Let $(A_0, (d)) \to (A, (d))$ be the morphism from the universal oriented prism. Form a diagram as in Proposition 15.2.5 in which α is the map from Lemma 15.2.2 and the square is a pushout of (p, d)-complete simplicial commutative rings. (The key technical complication here is that E is not necessarily an ordinary ring.)

$$\begin{tikzcd}
& A_0 \arrow[r] \arrow[d, "\alpha"] & A \arrow[d, "\beta"] \\
\mathbb{Z}_p \arrow[r] & D_0 \arrow[r] & E = A \widehat{\otimes}^L_{A_0} D_0.
\end{tikzcd}$$

Figure 15.3.3

The arrow $\mathbb{Z}_p \to D_0$ promotes to a map $(\mathbb{Z}_p, (p)) \to (D_0, (p))$ of prisms, so we also have a map $\gamma\colon (\mathbb{Z}_p, (p)) \to (E, (p))$ of oriented prisms. Using the explicit description of prismatic cohomology given in Corollary 14.4.4, we may produce a natural isomorphism $\widehat{\beta^*}\mathbf{\Delta}_{R/A} \cong \widehat{\gamma^*}\mathbf{\Delta}_{\mathbb{F}_p[x_1,\dots,x_r]/\mathbb{Z}_p}$. By Proposition 14.4.12, we know that the Hodge-Tate comparison map is an isomorphism for the prism $(\mathbb{Z}_p, (p))$ and the ring $\mathbb{F}_p[x_1, \dots, x_r]$; combining this with the previous isomorphism, we deduce that the original Hodge-Tate comparison map becomes an isomorphism after applying $\widehat{\beta^*}$. Since this last functor is conservative (because $\widehat{\alpha^*}$ is conservative by Lemma 15.2.4), it is itself an isomorphism. (Compare [25], Proposition 6.2.) ∎

15.4 The crystalline and de Rham comparisons

On a related note, we describe the comparison between prismatic, crystalline, and de Rham cohomology under some mild restrictions (in addition to our running condition that (A, I) is bounded). One complication is that we do not have an analogue of Lemma 15.1.2 for prismatic cohomology: for $R \to S$ an I-completely etale morphism of $\overline{A}$-algebras, there is no obvious base change functor to relate $\mathbf{\Delta}_{R/A}$ to $\mathbf{\Delta}_{S/A}$.

Theorem 15.4.1 *Assume that* $(A,(p))$ *is a crystalline prism and let* J *be an ideal of* A *containing* p *on which* A *admits divided powers valued in* I *(that is,* J *is a* ***pd-ideal*** *of* A*). Let* $\psi\colon A/J \to A/p$ *be the morphism induced by the Frobenius on* A/p*. Let* R *be a smooth* A/J*-algebra and put* $R^{(1)} = R \otimes_{A/I,\psi} A/p$*. Then there is a canonical isomorphism of* $\mathbf{\Delta}_{R^{(1)}/A}$ *with the crystalline cohomology of* $\mathrm{Spf}(R)$ *relative to the pd-thickening* $\mathrm{Spec}(A/J) \subset \mathrm{Spf}(A)$*; more precisely, this is an isomorphism of* $E_\infty - A$*-algebras compatible with Frobenius.*

Proof. The key point is to construct the map, as thereafter one can compute in local coordinates as in the proof of Lemma 14.4.8. See [25], Theorem 5.2. ■

Remark 15.4.2 Before continuing, we make an observation that will explain a somewhat odd condition in Theorem 15.4.3. Recall that by construction, the p-typical Witt vector functor W is a right adjoint to the forgetful functor $\mathbf{Ring}_\delta \to \mathbf{Ring}$ (Definition 3.1.1). We may thus apply adjunction to the canonical map $A \to \overline{A}$ to obtain a morphism $A \to W(\overline{A})$ in $\mathbf{Ring}_\delta$.

Now let $\psi\colon A \to W(\overline{A})$ be the composition of the resulting map with the Frobenius ϕ on $W(\overline{A})$. This map carries I into (p): the original map $A \to W(\overline{A})$ carries I into the image of the Verschiebung V on $W(\overline{A})$, and the composition $\phi \circ V$ is multiplication by p (Definition 3.2.3). Hence the map ψ induces a morphism $(A,I) \to (W(\overline{A}),(p))$ of prisms provided that $W(\overline{A})$ is p-torsion-free (so that the target is actually a prism).

Theorem 15.4.3 *Assume that the prism* (A,I) *is bounded and that* $W(\overline{A})$ *is* p*-torsion-free. Let* R *be a* p*-completely smooth* $\overline{A}$*-algebra. Then there is a natural isomorphism* $\mathbf{\Delta}_{R/A}\widehat{\otimes}^L_{A,\phi}\overline{A} \cong \Omega^\bullet_{R/\overline{A}}$ *of commutative ring objects in* $D(\overline{A})$ *(where the completion is the derived* p*-adic completion).*

Proof. In light of Remark 15.4.2, it is enough to construct a functorial isomorphism of $\mathbf{\Delta}_{R/A}\widehat{\otimes}^L_{A,\psi}W(\overline{A}) \cong \mathbf{\Delta}_{R'/W(\overline{A})}$, where $R' = R\widehat{\otimes}_{\overline{A},\psi}W(\overline{A})/p$ (and the stated isomorphism is given by Lemma 15.1.3), with the crystalline cohomology of R/p with coefficients in $W(\overline{A})$. This amounts to an application of Theorem 15.4.1. (Compare [25], Theorem 6.4.) ■

Remark 15.4.4 In Theorem 15.4.3, the condition that $W(\overline{A})$ is p-torsion-free holds in two natural cases of interest: when A/I is p-torsion-free, or when $I=(p)$ and $\overline{A}$ is reduced. The result remains true without this condition, but this is more difficult and falls outside the scope of these notes; see [25], Corollary 15.4.

16 Nonabelian derived functors

Reference. [18], lecture VII. The underlying reference is [101]; the definitive modern treatment is [93], section 5.5.8. However, we generally follow conventions from [117], tag 0162.

For a concise introduction to simplicial commutative rings, see [96].

In this section, we describe a natural analogue of **derived functors** for categories which are not necessarily additive. Putting this theory in its proper level of generality involves addressing a *lot* of technicalities which we elide here.

For $A \in \mathbf{Ring}$, let $\mathbf{Ring}_A$ be the arrow category (i.e., commutative A-algebras).

16.1 More on simplicial objects

To introduce this section, we start with a motivating remark.

Remark 16.1.1 Suppose one is trying to write down a functor F from $\mathbf{Mod}_A$ to some abelian category which is right exact and commutes with filtered colimits. Then it is enough to specify the values of F on arbitrary *finite free* A-modules: every module is a cokernel of a morphism between two free modules, each of which is itself a filtered colimit of finite free modules. Furthermore, using projective resolutions by free modules, we can compute the left derived functors of F from this.

The construction of nonabelian derived functors allows us to do something similar starting from the category $\mathbf{Ring}_A$. The free objects in this case (i.e., the essential image of the left adjoint of the forgetful functor to $\mathbf{Set}$) are polynomial rings. In order to replace modules to rings, we need to reconceptualize some familiar constructions without reference to the additive structure of the category $\mathbf{Mod}_A$; for example, in $\mathbf{Mod}_A$ we can form the equalizer of two maps $f_1, f_2 \colon M \to N$ as the kernel of the difference $f_1 - f_2$, but now we need to forgo this shortcut.

The resulting process amounts to the transition from **homological algebra** to **homotopical algebra** in the sense of Quillen [101]. Nowadays this is usually done in the framework of ∞-categories, as in [93]; we will keep ourselves in a very limited part of the picture so as to keep the prerequities for the discussion under control.

To begin, we recall the definition of **simplicial objects** from Definition 11.2.2, filling some extra details.

Definition 16.1.2 Let Δ be the category of finite ordered sets (Definition 11.2.1). Recall (from Definition 11.2.2) that for $\mathcal{C}$ an arbitrary category, a **simplicial object** of a category $\mathcal{C}$ is a covariant functor $U \colon \Delta^{\mathrm{op}} \to \mathcal{C}$, while a **cosimplicial object** of a category $\mathcal{C}$ is a covariant functor $U \colon \Delta \to \mathcal{C}$. For a simplicial object U, we will usually write U_n as a shorthand for the image object $U([n])$. $\diamondsuit$

The shift operators on derived categories have the following simpicial analogue.

Definition 16.1.3 For $n \geq 0$, let $\Delta[n]$ denote the simplicial set

$$\Delta^{\mathrm{op}} \to \mathbf{Set}, \qquad [k] \mapsto \mathrm{Hom}_\Delta([k], [n]).$$

For any simplicial set U, the morphisms of simplicial sets from $\Delta[n]$ to U are naturally in bijection with U_n. $\diamondsuit$

Definition 16.1.4 Let V be a simplicial set such that each V_n is finite and nonempty. Then for any category $\mathcal{C}$ admitting finite coproducts and any simplicial object U of $\mathcal{C}$, we define the product $U \times V$ to be the simplicial object of $\mathcal{C}$ with

$$(U \times V)_n = \coprod_{v \in V_n} U_n$$

such that the map

$$\coprod_{v \in V_n} U_n \to \coprod_{v' \in V_m} U_m$$

corresponding to $\phi \colon [m] \to [n]$ carries the component indexed by v to the component indexed by $v' = V(\phi)(v)$ via $U(\phi)$. (Compare [117], tag 017C.) $\diamondsuit$

Example 16.1.5 In Definition 16.2.5, we will consider the special case of Definition 16.1.4 in which $V = \Delta[1]$. In this case, the two maps $e_0, e_1 \colon \Delta[0] \to \Delta[1]$ corresponding to the two morphisms $[0] \to [1]$ induce morphisms

$$e_0, e_1 \colon U \to U \times \Delta[1].$$

□

Remark 16.1.6 By way of motivation, you should imagine that $\Delta[n]$ represents an n-dimensional simplex and the product $U \times \Delta[n]$ represents taking the product of some geometric object corresponding to U with this simplex. This motivates the definition of homotopies between maps of simplicial objects, as in Definition 16.2.5.

16.2 Simplicial resolutions

Definition 16.2.1 A **simplicial resolution** of an object $X \in \mathcal{C}$ is a simplicial object $U \colon \Delta^{\mathrm{op}} \to \mathcal{C}$ with colimit X. A **cosimplicial resolution** of X is a cosimplicial object $U \colon \Delta \to \mathcal{C}$ with limit X. ♢

Let us see how the previous construction, specialized to the case of modules over a ring, gives resolutions in the homological sense.

We now give a simplicial analogue of a **resolution** in homological algebra.

Example 16.2.2 Take $\mathcal{C} = \mathbf{Mod}_A$ for some $A \in \mathbf{Ring}$. Let U be a cosimplicial resolution of $M \in \mathcal{C}$. Then the associated complex $U([\bullet])$ (Definition 11.2.2) is a resolution of M; that is, $M[0] \to U([\bullet])$ is a quasi-isomorphism. □

We describe a trivial example which is not quite so trivial after all.

Definition 16.2.3 For any object $X \in \mathcal{C}$, the simplicial object U with $U([n]) = X$ for all n, is a resolution of X. We call this the **trivial resolution** of X. ♢

Example 16.2.4 Take $\mathcal{C} = \mathbf{Mod}_A$ with $A \in \mathbf{Ring}$. Then the trivial resolution of M has associated complex

$$\cdots M \xrightarrow{0} M \xrightarrow{1} M \xrightarrow{0} M$$

which is homotopy equivalent to $M[0]$. Compare this to the proof of Lemma 14.4.8. □

When working with resolutions, we would like to be able to compare these, in the same way that we can show that any two injective/projective resolutions of an object of $\mathbf{Mod}_A$ are homotopy equivalent. Here is the key definition.

Definition 16.2.5 Suppose that the category $\mathcal{C}$ has finite coproducts. Let U, V be simplicial objects of $\mathcal{C}$ and let $a, b \colon U \to V$ be two morphisms. A **homotopy** from a to b is a morphism

$$h \colon U \times \Delta[1] \to V$$

(interpreting the source as per Definition 16.1.4) such that $a = h \circ e_0$ and $b = h \circ e_1$. The property that such a homotopy exists, for a given pair a, b is reflexive but not necessarily symmetric or transitive.

We say that a and b are **homotopic** if they belong to the same equivalence class under the equivalence relation generated by homotopies. We say that a single morphism $a \colon U \to V$ is a **homotopy equivalence** if there exists a second morphism $b \colon V \to U$ such that $b \circ a$ is homotopic to id_U and $a \circ b$ is homotopic to id_V. ♢

Example 16.2.6 In Definition 16.2.5, the maps $e_0, e_1 \colon U \times \Delta[1] \to U$ are themselves homotopy equivalences. See Exercise 16.6.2. $\square$

Example 16.2.7 Take $\mathcal{C} = \mathbf{Mod}_A$ with $A \in \mathbf{Ring}$. Then a homotopy between morphisms $a, b \colon U \to V$ of simplicial objects gives rise to a homotopy of the corresponding complexes in $\mathbf{Comp}(A)$. In particular, if two simplicial objects U, V are homotopy equivalent, then the corresponding objects in $K(A)$ are isomorphic (and similarly for cosimplicial objects). For a converse to this assertion, see [117], tag 01A1. $\square$

Remark 16.2.8 Just as in homological algebra, one would like to work in the derived category to enforce that any object is "interchangeable" with a sufficiently nice resolution, in the simplicial realm one wants to to replace objects with simplicial objects that are more flexible (in the sense of being **fibrant** or **cofibrant**). The general story is out of scope for these notes (in part due to the need to develop robust combinatorial formalism, as in the language of ∞-categories, to keep track of homotopy coherence); here we limit ourselves to a few critical examples, such as Example 16.2.9.

Example 16.2.9 Let $A \to B$ be a morphism in **Ring**. Choose a simplicial resolution U of B by free A-algebras (e.g., the standard resolution; see Example 16.3.4). Then for any morphism $A \to C$ of rings, we may define the **simplicial tensor product** $B \otimes_A^L C$ to be the simplicial ring $U \otimes_A C$; any two choices of U will give rise to homotopy equivalent objects. Similarly, we may define the simplicial tensor product of two simplicial A-algebras. $\square$

16.3 Standard resolution

The following construction gives a *functorial* construction of simplicial resolutions; see [117], tag 08N8.

Definition 16.3.1 Let $V \colon \mathcal{C}_1 \to \mathcal{C}_2$ be a functor with a left adjoint $U \colon \mathcal{C}_2 \to \mathcal{C}_1$. By definition, this means we have natural transformations

$$\eta\colon\ \mathrm{id}_{C_2} \to V \circ U, \qquad \epsilon\colon U \circ V \to \mathrm{id}_{C_1}$$

(the **unit** and **counit**).

For $n \geq 0$, let X_n be the $(n+1)$-fold composition of $U \circ V$, with $X_{-1} = \mathrm{id}_{C_1}$; note that we have a natural identification $X_{n+m+1} = X_n \circ X_m$. Define the natural transformations

$$U(\delta_j^n) = \mathrm{id}_{X_{j-1}} \star \epsilon \star \mathrm{id}_{X_{n-j-1}}, \qquad U(\sigma_j^n) = \mathrm{id}_{X_{j-1} \circ V} \star \eta \star \mathrm{id}_{U \circ X_{n-j-1}}.$$

(writing $\star$ for composition of natural transformations to distinguish it from $\circ$ for composition of functors).

By Lemma 16.3.2, for any $Y \in \mathcal{C}_1$, the objects $X_n(Y)$ form a simplicial resolution of Y. We call this the **standard resolution** of Y with respect to the functor V. $\lozenge$

Lemma 16.3.2 *In Definition 16.3.1, X is a simplicial resolution of the constant functor $\mathrm{id}_{\mathcal{C}_1}$ via ϵ. Consequently, for any $Y \in \mathcal{C}$, the objects $X_n(Y)$ form a simplicial resolution of Y in $\mathcal{C}_1$.*

Proof. See [117], tag 08NC. $\blacksquare$

For example, this construction can be used to construct functorial projective resolutions of modules over a ring.

Example 16.3.3 In Definition 16.3.1, take V to be the forgetful functor $\mathbf{Mod}_A \to \mathbf{Set}$ for some $A \in \mathbf{Ring}$; we may then take U to be the functor taking $S \in \mathbf{Set}$ to the free A-module A^S. For $M \in \mathbf{Mod}_A$, we obtain a simplicial resolution P_n with $P_{-1} = M$ and $P_{n+1} = A^{P_n}$ for $n \geq -1$. This in particular gives rise to a projective resolution of M using the dual construction of the one in Definition 11.2.2. □

Here is the natural analogue for algebras over a ring.

Example 16.3.4 In Definition 16.3.1, take V to be the forgetful functor $\mathbf{Ring}_A \to \mathbf{Set}$ for some $A \in \mathbf{Ring}$; we may then take U to be the functor taking $S \in \mathbf{Set}$ to the free polynomial ring $A[S]$. For $B \in \mathbf{Ring}_A$, we obtain a simplicial resolution P_n with $P_{-1} = B$ and $P_{n+1} = A[P_n]$ for $n \geq -1$. □

Lemma 16.3.5 *In Lemma 16.3.2, the maps*

$$\mathrm{id}_V \star \epsilon \colon V \circ X \to V, \qquad \epsilon \star \mathrm{id}_U \colon X \circ U \to U$$

are homotopy equivalences.

Proof. See [117], tag 08ND. ■

Corollary 16.3.6 *The standard resolution of any object in a category is homotopy equivalent to the trivial resolution.*

Proof. See [117], tag 08NE. ■

Remark 16.3.7 It should be stressed that while the standard resolution is a "natural" (and functorial) way to construct simplicial resolutions, the resulting resolutions are not *preferred* in any mathematical sense. In particular, if one starts performing operations one quickly ends up with simplicial resolutions that are not the standard ones but are homotopy equivalent, and the distinction will carry no value (if anything it is more of a hindrance).

16.4 Nonabelian derived functors

We now ready to define an analogue of derived functors for algebras over a given ring. For this, the following definition will be useful.

Definition 16.4.1 Given covariant functors $F \colon \mathcal{C}_1 \to \mathcal{C}_2, G \colon \mathcal{C}_1 \to \mathcal{C}_3$, the **left Kan extension** of G along F consists of a covariant functor $L \colon \mathcal{C}_2 \to \mathcal{C}_3$ and a natural transformation $\alpha \colon G \to L \circ F$ which are universal for this property: that is, if $M \colon \mathcal{C}_2 \to \mathcal{C}_3$ is another functor and $\beta \colon G \to M \circ F$ is a natural transformation, then there is a unique natural transformation $\sigma \colon L \to M$ making the second diagram in Figure 16.4.2 commute.

$$\begin{tikzcd}
\mathcal{C}_1 \arrow[r, "F"] \arrow[d, "G"'] \arrow[dr, phantom, "\overset{\alpha}{\Longrightarrow}"] & \mathcal{C}_2 \arrow[d, dashed, "L"] \\
\mathcal{C}_3 \arrow[r, equal] & \mathcal{C}_3
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & L \circ F \arrow[d, dashed, "\sigma \circ F"] \\
& M \circ F
\end{tikzcd}$$

Figure 16.4.2

◊

Remark 16.4.3 Note that in Definition 16.4.1, both the commutativity of the second diagram in Figure 16.4.2 and the uniqueness of σ are well-posed because a natural transformation is specified by a collection of morphisms between prescribed sources and targets, so the comparison of these is set-theoretic and not category-theoretic.

As usual with a definition via a universal property, the use of the definite article is justified by the observation that any two objects satisfying the definition are uniquely isomorphic. However, α is not itself guaranteed to be an isomorphism of functors; that is, G is not necessarily isomorphic to the restriction of L along F.

Here is the motivating example.

Example 16.4.4 Let $G\colon \mathbf{Mod}_A \to \mathcal{A}$ be a right exact covariant functor to an abelian category. Let $\mathcal{C}$ be the subcategory of $K^-(A)$ consisting of complexes of projective modules. Using the fact that simplicial resolutions by projective modules give rise to projective resolutions (Example 16.3.3), we may check that the usual left derived functor of G is the left Kan extension of $G\colon \mathcal{C} \to K^-(\mathcal{A})$ along the inclusion $\mathcal{C} \to K^-(A)$. The point is that the formation of projective resolutions corresponds to replacing general objects of $K^-(A)$ by **cofibrant objects**. □

With Example 16.4.4 in mind, it is now clear how to proceed with modules replaced by rings.

Definition 16.4.5 Let $\mathbf{Poly}_A$ be the full subcategory of $\mathbf{Ring}_A$ consisting of polynomial rings over A in finitely many variables (i.e., the essential image of the restriction to finite sets of the left adjoint of the forgetful functor from $\mathbf{Ring}_A$ to sets). Note that objects in $\mathbf{Poly}_A$ do *not* come with a specified choice of polynomial generators, and so morphisms in $\mathbf{Poly}_A$ are not required to respect these generators. ◊

Proposition 16.4.6 *For $A \in \mathbf{Ring}$ and $F\colon \mathbf{Poly}_A \to D(\mathbf{Ab})$ a covariant functor, the functor F admits a left Kan extension $LF\colon \mathbf{Ring}_A \to D(\mathbf{Ab})$ along the inclusion $\mathbf{Poly}_A \to \mathbf{Ring}_A$, which moreover has the following properties. (We call LF the **left derived functor** of F.)*

1. *The natural transformation α is an equivalence: that is, LF restricts to F on $\mathbf{Poly}_A$.*

2. *LF commutes with filtered colimits. In particular, if $A[S]$ is a polynomial algebra on a possibly infinite set S, we can compute $LF(A[S])$ as the colimit of $F(A[T])$ over all finite subsets T of S.*

3. *Given a simplicial resolution $P_\bullet \to B$ of an object $B \in \mathbf{Ring}_A$, $LF(B)$ is the colimit of $LF(P^\bullet)$ (see Remark 16.4.7). (For example, this means we can evaluate LF using the standard resolution, as per Example 16.3.4.)*

Proof. See the references given in [18], lecture VII, section 1. ■

Remark 16.4.7 In practice, we will be considering cases in which F can be lifted to a functor $\tilde{F}\colon \mathbf{Poly}_A \to \mathbf{Comp}(\mathbf{Ab})$, in which case the colimit in part (2) of Proposition 16.4.6 can be interpreted as the totalization of a double complex made out of the terms $L\tilde{F}(P^\bullet)$. Otherwise, one should replace the derived category $D(\mathbf{Ab})$ with its ∞-categorical analogue and take the colimit there (where it can be reinterpreted as the **geometric realization**).

To give a concrete example of the effect of the colimit, note that if B is the coequalizer of two maps $f_0, f_1\colon P_1 \to P_0$, then $LF(B) = \mathrm{Cone}(f_0 - f_1)$.

Remark 16.4.8 An important basic example will be given by the exterior power $\wedge^i\colon \mathbf{Mod}_A \to \mathbf{Mod}_A$, which will give us the derived exterior power $L\wedge^i\colon \mathbf{Mod}_A \to D(A)$. This in turn extends to a functor $L\wedge^i\colon D^{\leq 0}(A) \to D(A)$. (As indicated in [18], Lecture VII, Remark 1.4, this is a point at which we are forced to be a bit sloppy by not working in the language of ∞-categories, but never mind.)

Remark 16.4.9 In what follows we will frequently use the following construction without explicit comment. Let $G' : \mathcal{C}_1 \to \mathcal{C}_3$ be another functor admitting a left Kan extension $(L' : \mathcal{C}_2 \to \mathcal{C}_3, \alpha : G' \to L' \circ F)$, and suppose that $\gamma : G \to G'$ is a natural transformation. Then we obtain a natural transformation $\alpha \circ \gamma : G \to L' \circ F$ to which we may apply the universal property of the left Kan extension of L, so as to obtain a natural transformation $\sigma : L \to L'$. That is, a natural transformation between two functors from $\mathcal{C}_2$ to $\mathcal{C}_3$ can be uniquely specified by giving its restriction (along F) to $\mathcal{C}_1$.

16.5 Under the hood: ∞-categories

Remark 16.5.1 It was mentioned in passing earlier that the derived category of A-modules, for some $A \in \mathbf{Ring}$, is more robust to work with in the language of (stable) ∞-categories. This allows us to be more careful about making identifications "up to homotopy"; rather than simply declaring two morphisms of complexes to be equal if there is a homotopy between them, in the homotopical approach one *records* the data of the homotopy witness and keeps track of it as one performs further operations.

One reason this is advantageous is that the formation of mapping cones is not functorial in the derived category as we have described it, but it becomes functorial in the stable ∞-category (because of the retention of the homotopy data). A minimal example is given by the map from $A \to 0$ to $0 \to A$.

Another reason is that one cannot perform any reasonable descent on the functor $A \mapsto D(A)$ without the homotopical data: for instance, for a Zariski covering of three or more opens, it is not generally possible to lift descent data from objects in derived categories to chain complexes. Again, recording the homotopy data makes it possible to perform this lifting.

16.6 Exercises

1. Prove that for any $n \geq 0$, the unique morphism $\Delta[n] \to \Delta[0]$ is a homotopy equivalence, with a homotopy inverse given by the map $\Delta[0] \to \Delta[n]$ induced by the map $[0] \to [n]$ taking 0 to n.

Hint. See [117], tag 08Q3.

2. Let $\mathcal{C}$ be a category admitting finite nonempty coproducts. Prove that for any simplicial object U in $\mathcal{C}$, the maps $e_0, e_1 : U \times \Delta[1] \to U$ are homotopy equivalences.

17 Derived de Rham cohomology

Reference. [18], lecture VII.

In this section, we apply the formalism of nonabelian derived functors (Section 16) to the cohomology of differential forms, starting with the cotangent complex and then moving to derived de Rham cohomology. This will set up a paradigm of leveraging our knowledge about polynomial rings (or their completions) which will persist in the discussion of derived prismatic cohomology in Section 18.

17.1 The cotangent complex

We illustrate the formalism with Illusie's construction of the "derived cotangent bundle" [68], [69].

Definition 17.1.1 For $A \in \mathbf{Ring}$, the **cotangent complex** is the functor $L_{\bullet/A}\colon \mathbf{Ring}_A \to D(A)$ taking obtained by taking the left derived functor of the functor $\mathbf{Poly}_A \to D(A)$ given by $B \mapsto \Omega^1_{B/A}[0]$. It is straightforward to check that in fact $L_{B/A} \in D^{\leq 0}(B)$.

Note that it also makes sense to talk about $L_{B/A}$ when B is itself a simplicial object in $\mathbf{Ring}_A$. This will be useful when stating the base change property in Proposition 17.1.2. $\Diamond$

Proposition 17.1.2 *For $A \to B$ a morphism in $\mathbf{Ring}$, the cotangent complex $L_{B/A} \in D(B)$ has the following properties. (These also hold when B is a simplicial object in $\mathbf{Ring}_A$.)*

1. *We have a natural (in B) isomorphism $H^0(L_{B/A}) \cong \Omega^1_{B/A}$.*

2. *If $A \to B$ is smooth, then $L_{B/A} \cong \Omega^1_{B/A}[0]$. In particular, if $A \to B$ is étale then $L_{B/A} \cong 0$ (but not conversely; see for example Exercise 17.5.1).*

3. *For any morphisms $A \to B \to C$, we have a distinguished triangle in $D(C)$*
$$L_{B/A} \otimes^L_B C \to L_{C/A} \to L_{C/B} \to . \tag{17.1}$$
This extends the usual low-degree exact sequence for differentials.

4. *For any surjective morphism $A \to B$ with kernel I, we have $H^{-1}(L_{B/A}) \cong I/I^2$. Moreover, if I is generated by a regular sequence, then $H^i(L_{B/A}) = 0$ for $i \neq -1$. (This generalizes the assertion that a closed immersion of schemes is unramified.)*

5. *For any morphisms $A \to B, A \to C$, we have a natural base change isomorphism*
$$L_{C/A} \otimes^L_A B \cong L_{C \otimes^L_A B/B}$$
*where $C \otimes^L_A B$ is to be interpreted as a simplicial ring as per Example 16.2.9. We can replace $C \otimes^L_A B$ with $C \otimes_A B$ if one of $A \to B$ or $A \to C$ is flat, or more generally if $A \to B$ and $A \to C$ are **Tor independent** (meaning that $\operatorname{Tor}^A_i(B, C) = 0$ for all $i > 0$).*

Proof. See [117], tag 08P5. ∎

The following is an analogue of the flatness of completion for noetherian rings, but without a noetherian hypothesis.

Lemma 17.1.3 *Let $A \in \mathbf{Ring}$ be classically p-complete with bounded p-power torsion. Let $B \in \mathbf{Ring}_A$ be flat (so that B also has bounded p-power torsion). Let $\widehat{B}$ be the classical p-completion of B. Then the derived p-completion of the cotangent complex of $B \to \widehat{B}$ is zero in $D(\widehat{B})$.*

Proof. The complex in question vanishes after applying $\otimes^L_{\mathbb{Z}} \mathbb{Z}/p$ by the base change formula (Proposition 17.1.2), and then derived Nakayama (Proposition 6.6.2) yields the claim. ∎

17.2 Derived de Rham cohomology

We first prepare for the Hodge-Tate comparison by introducing derived de Rham cohomology, picking up the thread from our previous discussion of the cotangent complex (Subsection 17.1).

Definition 17.2.1 For $k \in \mathbf{Ring}_{\mathbb{F}_p}$, the **derived de Rham cohomology** functor $\mathrm{dR}_{\bullet/k}\colon \mathbf{Ring}_k \to D(k)$ is the left derived functor of the functor $\mathbf{Poly}_k \to D(k)$ given by $R \mapsto \Omega^\bullet_{R/k}$.

Let us unwind this for a given ring $R \in k$. Let $P_\bullet \to R$ be the standard simplicial resolution of R (Example 16.3.4) Then $\mathrm{dR}_{R/k}$ is the totalization of the double complex $\Omega^\bullet_{P_\bullet/k}$. Note that this double complex is bounded below in one direction (coming from the de Rham complex) and bounded above in the other (coming from the simplicial resolution), so we must handle this totalization with some care (see Remark 17.2.8). $\diamondsuit$

To state the derived analogue of the Cartier isomorphism, we need to explain what we mean by a "filtration" in a derived category. (This is also the correct context in which to construct the spectral sequence associated to a filtered complex, as arose in the proof of Proposition 13.3.1.)

Definition 17.2.2 For $k \in \mathbf{Ring}$, by an **increasing exhaustive filtration** of an object $K \in D(k)$, we will mean a sequence $\mathrm{Fil}_0 \to \mathrm{Fil}_1 \to \cdots$ in $D(k)$ with colimit K. The **associated graded quotients** are the mapping cones $\mathrm{gr}_i(\mathrm{Fil}_\bullet) = \mathrm{Cone}(\mathrm{Fil}_{i-1} \to \mathrm{Fil}_i)$. $\diamondsuit$

Remark 17.2.3 You might initially find it confusing that Definition 17.2.2 does not specify that the maps $\mathrm{Fil}_i \to \mathrm{Fil}_{i+1}$ are injections. The point is that this is not a meaningful concept in $D(k)$!

Take note of the level of generality in the following proposition; there is no restriction on R at all!

Proposition 17.2.4 Derived Cartier isomorphism. *For $k \in \mathbf{Ring}_{\mathbb{F}_p}$ and $R \in \mathbf{Ring}_k$, there is a functorial (in R) increasing exhaustive filtration on $\mathrm{dR}_{R/k}$ in $D(R^{(1)})$, called the **conjugate filtration**, equipped with canonical identifications*

$$\mathrm{gr}^i \mathrm{dR}_{R/k} \cong \left(\bigwedge^i L_{R^{(1)}/k} \right) [-i].$$

Proof. For R a polynomial ring, we take the filtration on $\Omega^\bullet_{R/k}$ where Fil_i is the canonical truncation $\tau^{\leq i}\Omega^\bullet_{R/k}$ (Definition 10.4.1). The desired identifications in this case are just a reformulation of the Cartier isomorphism (Lemma 14.1.6). To deduce the general case, just take left derived functors. ∎

Remark 17.2.5 The conjugate filtration derives its name from the fact that it goes in the opposite direction from the usual Hodge filtration; its relationship with the Cartier isomorphism seems to have been observed first by Katz [78]. The Hodge filtration and the conjugate filtration give rise to the usual **Hodge-de Rham spectral sequence** and the **conjugate spectral sequence**, respectively; the latter is unnamed in [78], the modern terminology appearing first in [92].

Note that the following corollary is not automatic, because we don't currently have any way to control the effect of étale localization on derived de Rham cohomology; rather, we must prove this first and then deduce étale localization as a further corollary (Corollary 17.2.7).

Corollary 17.2.6 *For $k \in \mathbf{Ring}_{\mathbb{F}_p}$ and $R \in \mathbf{Ring}_k$ smooth, there is a canonical isomorphism $\mathrm{dR}_{R/k} \cong \Omega^\bullet_{R/k}$ which respects the conjugate filtrations on both sides.*

Proof. The map in question comes from the construction using the universal property of left Kan extensions. By Proposition 17.2.4, it respects the conjugate filtration on both sides.

The map on graded pieces can be written as $\left(\bigwedge^i L_{R^{(1)}/k}\right)[-i] \to \left(\Omega^i_{R^{(1)}/k}\right)[-i]$ using Proposition 17.2.4 and the usual Cartier isomorphism (Lemma 14.1.6 in the case of affine space; the general case follows by étale localization). This map is an isomorphism for $i = 1$ by Proposition 17.1.2, and hence is an isomorphism for general i as well. ∎

Corollary 17.2.7 *For $k \in \mathbf{Ring}_{\mathbb{F}_p}$ and $R \to S$ an étale morphism in $\mathbf{Ring}_k$, there is a natural isomorphism $\mathrm{dR}_{R/k} \otimes^L_R S \cong \mathrm{dR}_{S/k}$.*

Proof. This formally reduces to the case where R is a polynomial ring in finitely many variables over k, in which case S is smooth over k and Corollary 17.2.6 applies. ■

Remark 17.2.8 Note that Corollary 17.2.6 fails when k is not a ring of characteristic p. For example, if k is a $\mathbb{Q}$-algebra, then $k \cong \Omega^*_{A/k}$ by the Poincare lemma for any $A \in \mathbf{Poly}_k$, so $k \cong \mathrm{dR}_{A/k}$ for all $A \in \mathbf{Ring}_k$.

What is going on here is that the definition of derived de Rham cohomology we are using is a shortcut taking advantage of the Cartier isomorphism. The correct construction for general k requires an extra completion step that correctly accounts for the Hodge filtration, plus some care for the difference between the **direct sum totalization** and the **direct product totalization** of a double complex which is bounded above in one direction and bounded below in the other direction (see Definition 13.1.4). See [18], Lecture VII, Remark 3.8 for additional discussion and onward references.

17.3 Regular semiperfect rings

We next describe a large class of rings for which derived de Rham cohomology can be described explicitly. These can then be used in the terms of a simplicial resolution to compute derived de Rham cohomology more generally.

Definition 17.3.1 Let $k \in \mathbf{Ring}_{\mathbb{F}_p}$ be perfect. A **regular semiperfect** k-algebra is an object $S \in \mathbf{Ring}_k$ of the form R/I where $R \in \mathbf{Ring}_{\mathbb{F}_p}$ is perfect and I is an ideal of R generated by a regular sequence. Note that any such ring is semiperfect, that is, the Frobenius map is surjective; this can be used to partially recover R from S, as per Exercise 17.5.5. ◊

Example 17.3.2 A typical example of a regular semiperfect k-algebra is

$$S = k[x_1^{p^{-\infty}}, \dots, x_r^{p^{-\infty}}]/(x_1, \dots, x_r).$$

Note that there are many ways to write S as R/I; for instance, we may take $R = k[x_1^{p^{-\infty}}, \dots, x_r^{p^{-\infty}}]$ and $I = (x_1, \dots, x_r)$, but we can also replace R with its classical I-completion (or anything between). One can recover the completion from S; see Exercise 17.5.5. □

Lemma 17.3.3 *Let $k \in \mathbf{Ring}_{\mathbb{F}_p}$ be perfect and let $S \in \mathbf{Ring}_k$ be regular semiperfect. Then $\mathrm{dR}_{S/k}$ is concentrated in degree 0 (and thus can be viewed as an object in $\mathbf{Ring}_k$).*

Proof. Set notation as in Definition 17.3.1. By Exercise 17.5.1, $L_{R/k}$ vanishes in $D(R)$. From the distinguished triangle (17.1) associated to the morphisms $k \to R \to S$, we deduce that $L_{S/k} \to L_{S/R}$ is an isomorphism in $D(S)$. Since I is generated by a regular sequence, Proposition 17.1.2 asserts that $L_{S/R} \cong I/I^2[1]$ where I/I^2 is a finite projective S-module. Consequently, the derived exterior power

$$\bigwedge_S^i (L_{S/R}[-1]) = \left(\bigwedge_S^i L_{S/R} \right) [-i]$$

is also concentrated in degree 0. By Proposition 17.2.4, we may now deduce the claim. ■

Example 17.3.4 In Example 17.3.2, one may compute that

$$\mathrm{dR}_{S/k} \cong \bigoplus_{i_1,\ldots,i_r \in \mathbb{Z}[p^{-1}]_{\geq 0}} k \cdot \frac{x_1^{i_1} \cdots x_r^{i_r}}{\lfloor i_1 \rfloor ! \cdots \lfloor i_r \rfloor !}.$$

In general, we get the divided power envelope of I in R (in the sense of [15]; we cannot apply Definition 14.2.1 as we are not in the $\mathbb{Z}_p$-flat case). □

As an illustration of the technique we have in mind, let us apply this logic to ordinary de Rham cohomology in the smooth case.

Lemma 17.3.5 *Let $k \in \mathbf{Ring}_{\mathbb{F}_p}$ be perfect and let $R \in \mathbf{Ring}_k$ be smooth over k. Let S be the coperfection of R. Let $S^\bullet$ be the Čech nerve of $R \to S$.*

1. *The map $R \to S$ is flat.*

2. *For each $n \geq 0$, the ring S^n is regular semiperfect.*

Proof. Since R is a smooth k-algebra and k is perfect, the Frobenius map $\phi_R \colon R \to R$ is flat (see Remark 19.1.2). It follows that $R \to S$ is flat.

To see that S^n is regular semiperfect, we may work étale locally to reduce to the case where $R = k[x_1, \ldots, x_r]$. In this case, we may write

$$S^n = k[x_{ij}^{p^{-\infty}} : i = 1, \ldots, r; j = 0, \ldots, n]/(x_{ij} - x_{ij'} : i = 1, \ldots, r; 1 \leq j < j' \leq n)$$

to see that it is regular semiperfect. ■

Remark 17.3.6 With notation as in Lemma 17.3.5, Corollary 17.2.6 implies that we can identify $\Omega^\bullet_{R/k}$ with $\mathrm{dR}_{R/k}$ in $D(k)$. For each n, $\mathrm{dR}_{S^n/k}$ is concentrated in degree 0 by Lemma 17.3.3. It can be shown further that the functor $\mathrm{dR}_{\bullet/k}$ satisfies descent with respect to the fpqc cover $R \to S$ (see for instance [23], section 3); consequently, $\mathrm{dR}_{R/k}$ can be computed by the complex $\mathrm{dR}_{S^\bullet/k}$.

17.4 Derived crystalline cohomology

Definition 17.4.1 Derived crystalline cohomology. Let $k \in \mathbf{Ring}_{\mathbb{F}_p}$ be perfect and let $A \in \mathbf{Ring}$ be a classically p-complete ring with $A/p \cong k$. We wish to derive a **crystalline cohomology** functor $R\Gamma_{\mathrm{crys}} \colon \mathbf{Poly}_k \to D(A)$ taking $R = k[x_1, \ldots, x_r]$ to $\widehat{\Omega}^\bullet_{P/A}$ for $P = A[x_1, \ldots, x_r]$; however, we need to make sure that this construction does not depend on the choice of coordinates on R. Fortunately, we can deduce this from our proof of the Hodge-Tate comparison, which gives us a canonical isomorphism of $\widehat{\Omega}^\bullet_{P/A}$ with $\phi_A^* \mathbf{\Delta}_{R/k}$ (Corollary 14.4.10).

We define the **derived crystalline cohomology** functor $R\Gamma_{\mathrm{dcrys}} \colon \mathbf{Ring}_k \to D(A)$ by taking the left derived functor of the ordinary crystalline cohomology $R\Gamma_{\mathrm{crys}}$. From the comparison of crystalline cohomology with de Rham cohomology (Proposition 14.2.8), we obtain a natural isomorphism

$$R\Gamma_{\mathrm{dcrys}}(\bullet/A) \otimes^L_A k \cong \mathrm{dR}_{\bullet/k}\,.$$

◊

Remark 17.4.2 Using Lemma 17.3.5, we can carry out the analogue of Remark 17.3.6 to construct an explicit *functorial* complex computing the (derived) crystalline cohomology of a smooth algebra over a perfect ring. We omit the details here.

17.5 Exercises

1. Let $A \to B$ be a morphism of perfect $\mathbb{F}_p$-algebras. Show that $L_{A/B} \cong 0$.

Hint. On one hand, ϕ_B evidently induces an isomorphism on $L_{A/B}$. On the other hand, the induced map is zero when B is a polynomial ring over A.

2. Let $f\colon A \to B$ be a morphism of finite presentation between perfect $\mathbb{F}_p$-algebras. Show that f is étale.

Hint. Apply Exercise 17.5.1.

3. Let R be a perfect $\mathbb{F}_p$-algebra and let $x_1, \dots, x_r \in R$ be a regular sequence. Prove that the regular semiperfect ring $S = R/(x_1, \dots, x_r)$ is perfect if and only if S is a direct factor of R.

Hint. By Exercise 17.5.2, the map $R \to S$ is both étale and surjective, and hence a closed-open immersion.

4. Let $A \to B$ be a morphism of lenses. Using Exercise 17.5.1, show that the derived p-completion of $L_{B/A}$ vanishes.

Hint. As in Lemma 17.1.3, use derived Nakayama to reduce to Proposition 6.6.2. See also [25], Lemma 3.14.

5. With notation as in Definition 17.3.1, show that the perfection of S is canonically isomorphic to the classical I-completion of R.

18 Derived prismatic cohomology

Reference. [18], lecture VII.

In this section, we discuss how to adapt our previous statements about smooth algebras to the singular case. The idea is to use simplicial resolutions of singular algebras by smooth ones, so that all the heavy lifting gets done by the smooth case.

18.1 Derived prismatic cohomology

Definition 18.1.1 Let (A, I) be a bounded prism with slice $\overline{A}$. The **derived prismatic cohomology** functor $L\mathbf{\Delta}_{\bullet/A}\colon \mathbf{Ring}_{\overline{A}} \to D_{\mathrm{comp}}(A)$ is the left derived functor of the functor $\mathbf{Poly}_{\overline{A}} \to D_{\mathrm{comp}}(A)$ given by $R_0 \mapsto \mathbf{\Delta}_{\widehat{R_0}/A}$ (where $\widehat{R_0}$ is the derived p-completion). Note that $L\mathbf{\Delta}_{R/A}$ is a commutative algebra object in $D_{\mathrm{comp}}(A)$.

Similarly, the **derived Hodge-Tate cohomology** functor $L\overline{\mathbf{\Delta}}_{\bullet/A}\colon \mathbf{Ring}_{\overline{A}} \to D_{\mathrm{comp}}(\overline{A})$ is the left derived functor of the functor $\mathbf{Poly}_{\overline{A}} \to D_{\mathrm{comp}}(A)$ given by $R_0 \mapsto \overline{\mathbf{\Delta}}_{\widehat{R_0}/A}$. Note that $L\overline{\mathbf{\Delta}}_{R/A}$ is a commutative algebra object in $D_{\mathrm{comp}}(R)$. There is a natural isomorphism $L\mathbf{\Delta}_{R/A} \otimes^L_A \overline{A} \cong L\overline{\mathbf{\Delta}}_{R/A}$ in $D_{\mathrm{comp}}(\overline{A})$. ♢

Remark 18.1.2 The object $L\mathbf{\Delta}_{R/A}$ admits a ϕ_A-semilinear endomorphism ϕ_R. One can further show that $L\mathbf{\Delta}_{R/A}$ carries the structure of a **derived δ-ring** once one makes a precise definition of this concept (which we will not do here).

Remark 18.1.3 While ordinary prismatic and Hodge-Tate cohomology are concentrated in nonnegative degrees, the same is not true of derived prismatic and Hodge-Tate cohomology. In general, they will not even be bounded below!

Proposition 18.1.4 Derived Hodge-Tate comparison. *Let (A, I) be a bounded prism. For any $R \in \mathbf{Ring}_{\overline{A}}$, the complex $L\overline{\mathbf{\Delta}}_{R/A}$ admits a functorial (in R) multiplicative exhaustive increasing filtration* $\mathrm{Fil}^{\mathrm{HT}}_{\bullet}$ *in $D_{\mathrm{comp}}(R)$ for* *which we have canonical identifications*

$$\mathrm{gr}_i^{\mathrm{HT}}(L\overline{\mathbf{\Delta}}_{R/A}) \cong \left(\bigwedge_{\overline{A}}^{i} L_{R/\overline{A}}\{-i\}\right)[-i]^{\wedge}_{(p)}$$

where $\{-i\}$ *denotes a Breuil-Kisin twist (Definition 12.3.1).*

Proof. This follows from the same argument as in Proposition 17.2.4 upon checking that when R is the p-adic completion of a polynomial ring over A, we have $L\overline{\mathbf{\Delta}}_{R/A} \cong \overline{\mathbf{\Delta}}_{R/A}$; this amounts to an application of Lemma 17.1.3. ■

Corollary 18.1.5 Comparison with the smooth case. *Let* (A, I) *be a bounded prism. For any* p*-completely smooth* A/I*-algebra, the natural map* $L\mathbf{\Delta}_{R/A} \to \mathbf{\Delta}_{R/A}$ *is an isomorphism.*

Proof. This follows from Proposition 18.1.4 as in the proof of Corollary 17.2.6. ■

18.2 Regular semilenses

The statement that derived de Rham cohomology can be computed easily using regular semiperfect rings (Remark 17.3.6) can be adapted as follows.

Definition 18.2.1 Let (A, I) be a perfect prism. A **semilens** over $\overline{A}$ is a derived p-complete ring which can be written as the quotient of some lens over $\overline{A}$. (This corresponds to a **semiperfectoid ring** in [18] and [25].) If S is a semilens, then S/p is semiperfect and $\theta\colon W(S^\flat) \to S$ is surjective. It will follow from Remark 18.2.3 that $L\mathbf{\Delta}_{S/A} \in D^{\leq 0}(A)$, but in general it will not be concentrated in degree 0.

For (A, I) a perfect prism, a **regular semilens** over (A, I) is a ring S of the form R/J where R is a lens over $\overline{A}$ and J is an ideal of R generated by a regular sequence. ◊

Example 18.2.2 By analogy with Example 17.3.2, note that for any lens R,

$$S = R[x_1^{p^{-\infty}}, \dots, x_r^{p^{-\infty}}]^{\wedge}_{(p)}/(x_1, \dots, x_r)$$

is a regular semilens. □

Remark 18.2.3 Let (A, I) be a perfect prism and let S be a regular semilens over (A, I). For simplicity, assume also that $\overline{A}$ is p-torsion-free and S is p-completely flat over $\overline{A}$. From Proposition 18.1.4, we see that $L\overline{\mathbf{\Delta}}_{S/A}$ admits an increasing exhaustive filtration with graded pieces $\left(\bigwedge^i L_{S/\overline{A}}\{-i\}\right)[-i]$. By our assumptions on S, each of these graded pieces is a finite projective S-module (compare the proof of Lemma 17.3.3). It follows that $L\overline{\mathbf{\Delta}}_{S/A}$ is concentrated in degree 0, where it is a p-completely flat S-algebra; consequently, $L\mathbf{\Delta}_{S/A}$ is concentrated in degree 0, where it is a (p, I)-completely flat A-algebra.

It can also be shown (as in Remark 18.1.2) that the Frobenius on prismatic cohomology provides $L\mathbf{\Delta}_{S/A}$ with a δ-ring structure, so $(L\mathbf{\Delta}_{S/A}, I)$ is in fact a prism over (A, I)! This can even be made explicit: if we write $S = R/J$ with R a lens and J generated by a regular sequence, then

$$L\mathbf{\Delta}_{S/A} \cong W(R^\flat)\{J/d\}^{\wedge}_{(p)}$$

(compare Lemma 14.4.2).

18.3 Exercises

1. Let R be a lens and let $x_1, \dots, x_r$ be a regular sequence in R. Prove that the regular semilens $S = R/(x_1, \dots, x_r)$ is a lens if and only if S is a direct factor of R.

Hint. Argue by analogy with Exercise 17.5.3, using Exercise 17.5.4.

19 Coperfections in mixed characteristic

Reference. [18], lecture VIII; [25], section 7.

Given a perfect prism (A, I), we apply prismatic cohomology to construct a "canonical coperfection" of a p-complete A/I-algebra. In general this will not be a true ring but rather something derived; only in certain (important) special cases will we end up with a genuine ring. Nonetheless, this construction is quite useful in mixed-characteristic commutative algebra; for instance, it recovers the **André flatness lemma**, whose earliest proofs [5], [16] depended heavily on the theory of perfectoid spaces. We incur no such dependence here; we make the argument entirely in the world of rings and schemes, with no recourse to nonarchimedean analytic geometry.

From now on, we write $\mathbf{\Delta}_{R/A}$ and $\overline{\mathbf{\Delta}}_{R/A}$ to mean *derived* prismatic and Hodge-Tate cohomology (which were previously denoted $L\mathbf{\Delta}_{R/A}$ and $L\overline{\mathbf{\Delta}}_{R/A}$), as we will have no further use for the underived versions.

19.1 Coperfections in characteristic p revisited

To motivate the mixed-characteristic construction, we start by reconstructing the coperfection of an $\mathbb{F}_p$-algebra in a somewhat exotic-looking fashion.

Definition 19.1.1 Recall that for R an $\mathbb{F}_p$-algebra, we have defined the **coperfection** of R as the image R_{perf} of R under the left adjoint of the forgetful functor from perfect $\mathbb{F}_p$-algebras to arbitrary $\mathbb{F}_p$-algebras. Concretely,

$$R_{\mathrm{perf}} = \operatorname{colim}(R \overset{\phi}{\to} R \overset{\phi}{\to} R \to \cdots).$$

Now suppose that R is an algebra over a perfect field k of characteristic p. Let $R^{(1)} \to R$ be the relative Frobenius map (Definition 14.1.2); then the induced map $(R^{(1)})_{\mathrm{perf}} \to R_{\mathrm{perf}}$ is an isomorphism. See also Exercise 19.6.2. $\lozenge$

Remark 19.1.2 A fundamental theorem of Kunz (see [117], tag 0EC0) asserts that a noetherian $\mathbb{F}_p$-algebra is regular if and only if its Frobenius map is flat. (As a reminder, if R is a finite type k-algebra for some perfect field k of characteristic p, then R is regular if and only if it is a smooth k-algebra; see [117], tag 00TQ.) Since flatness is preserved by colimits, we see that if R is a noetherian regular $\mathbb{F}_p$-algebra, then $R \to R_{\mathrm{perf}}$ is flat; the converse is also true (see Exercise 19.6.1). For an analogue in mixed characteristic, see Remark 25.5.3.

The following can be seen as another instance of the same phenomenon that gives rise to the vanishing of the cotangent complex for a morphism of perfect rings in characteristic p (as in the proof of Lemma 3.3.5).

Proposition 19.1.3 *Let k be a perfect field of characteristic p and choose $R \in \mathbf{Ring}_k$. Then the projection $\mathrm{dR}_{R/k} \to R$ induces an isomorphism of*

$$\mathrm{dR}_{R/k,\mathrm{perf}} = \operatorname{colim}(\mathrm{dR}_{R/k} \overset{\phi_R}{\to} \mathrm{dR}_{R/k} \overset{\phi_R}{\to} \cdots)$$

with R_{perf}.

Proof. Since everything is defined using the formalism of nonabelian derived functors (Definition 17.2.1), it suffices to treat the case where R is a polynomial ring over k in finitely many variables. In this case, it will suffice to check that for each $i > 0$,

$$\operatorname{colim}(\Omega^i_{R/k} \xrightarrow{\phi_R} \Omega^i_{R/k} \xrightarrow{\phi_R} \cdots)$$

vanishes. This follows from the fact that Frobenius kills differential forms: in degree 1 we have

$$\phi_R(x\,dy) = x^p d(y^p) = px^p y^{p-1} dy = 0$$

and similarly in higher degrees. ■

To get closer to the mixed-characteristic case, let us reformulate in terms of (derived) prismatic cohomology.

Proposition 19.1.4 *Let R be a k-algebra for some perfect field k of characteristic p, and let (A, I) be the prism $(W(k), p)$. Then the colimit*

$$\overline{\mathbf{\Delta}}_{R/A,\mathrm{perf}} = \operatorname{colim}(\overline{\mathbf{\Delta}}_{R/A} \xrightarrow{\phi} \overline{\mathbf{\Delta}}_{R/A} \xrightarrow{\phi} \cdots)$$

is concentrated in degree 0*, where it coincides with* R_{perf} *(as a k-algebra). Note that there is no p-adic completion needed here because we are working modulo p.*

Proof. Again, we formally reduce to the case where R is a polynomial ring in finitely many variables. To deduce this from Proposition 19.1.3, we need to check that the map

$$\mathrm{gr}_i^{\mathrm{HT}}(\phi_R)\colon\ \mathrm{gr}_i^{\mathrm{HT}}(\overline{\mathbf{\Delta}}_{R/A}) \to \mathrm{gr}_i^{\mathrm{HT}}(\overline{\mathbf{\Delta}}_{R/A})$$

induced by the Frobenius on $\overline{\mathbf{\Delta}}_{R/A}$ conicides with the map $\Omega^i_{R/k} \to \Omega^i_{R/k}$ induced by the Frobenius on R via the identification $\mathrm{gr}_i^{\mathrm{HT}}(\overline{\mathbf{\Delta}}_{R/A}) \cong \Omega^i_{R/k}$ of Proposition 18.1.4 (note that now we have reverted from derived to ordinary prismatic cohomology). By functoriality, it suffices to treat the case $R = k[x]$; this amounts to a direct calculation in the style of Lemma 12.3.4, which we leave to the reader. (Compare [18], Lecture XIII, Proposition 1.6.) ■

We make one more change to prepare for the passage to mixed characteristic: we replace the Frobenius action on Hodge-Tate cohomology, which has no analogue in mixed characteristic, with the prismatic Frobenius.

Corollary 19.1.5 *Let R be a k-algebra for some perfect field k of characteristic p, and let (A, I) be the prism $(W(k), p)$. Then the completed colimit*

$$\mathbf{\Delta}_{R/A,\mathrm{perf}} = \operatorname{colim}(\mathbf{\Delta}_{R/A} \xrightarrow{\phi_R} \mathbf{\Delta}_{R/A} \xrightarrow{\phi_R} \cdots)^{\wedge}_{(p)} \in D(\mathbb{Z}_p)$$

is concentrated in degree 0*, where it coincides with* $W(R_{\mathrm{perf}})$ *(as a $W(k)$-algebra).*

Proof. This is a direct consequence of Proposition 19.1.4. ■

19.2 The mixed characteristic case

Definition 19.2.1 Let (A, I) be a perfect prism with slice $\overline{A}$. For $R \in \mathbf{Ring}_{\overline{A}}$ derived p-complete, define the **prismatic coperfection**

$$\mathbf{\Delta}_{R/A,\mathrm{perf}} = \operatorname{colim}(\mathbf{\Delta}_{R/A} \xrightarrow{\phi_R} \mathbf{\Delta}_{R/A} \xrightarrow{\phi_R} \cdots)^{\wedge}_{(p,I)} \in D_{\mathrm{comp}}(A)$$

using the A-linear structure on the initial term. This corresponds to the **perfection** in [18], [25].

Define the **lens coperfection** as

$$R_{\text{lens}} = \mathbf{\Delta}_{R/A,\text{perf}} \otimes^L_A \overline{A} \in D_{\text{comp}}(R)$$

(the derived completion being p-adic) using the R-linear structure coming from $R \to \overline{\mathbf{\Delta}}_{R/A} \to \mathbf{\Delta}_{R/A,\text{perf}} \otimes^L_A \overline{A}$ (the latter map coming from the identification of $\mathbf{\Delta}_{R/A}$ with the first term of the colimit defining $\mathbf{\Delta}_{R/A,\text{perf}}$). This corresponds to the **perfectoidization** in [18], [25].

By construction, $\mathbf{\Delta}_{R/A,\text{perf}}$ and R_{lens} are commutative algebra objects in $D_{\text{comp}}(A)$ and $D_{\text{comp}}(R)$, respectively. The Frobenius on $\mathbf{\Delta}_{R/A}$ induces an automorphism of $\mathbf{\Delta}_{R/A,\text{perf}}$ denoted ϕ_R. ♢

Remark 19.2.2 The notation R_{lens} suggests that the lens coperfection of R depends only on R and not on its description as an $\overline{A}$-algebra. This will be confirmed by Lemma 19.2.3.

Lemma 19.2.3 Base independence of lens coperfection. *Let $(A, I) \to (B, J)$ be a morphism of perfect prisms and let $S \in \mathbf{Ring}_{\overline{B}}$ be derived p-complete.*

1. *The natural map $\overline{\mathbf{\Delta}}_{S/A} \cong \overline{\mathbf{\Delta}}_{S/B}$ is an isomorphism.*
2. *If $S = \overline{B}$, then the natural map $S \to \overline{\mathbf{\Delta}}_{S/B}$ is also an isomorphism.*
3. *The natural maps*

$$\mathbf{\Delta}_{S/A} \to \mathbf{\Delta}_{S/B}, \qquad \mathbf{\Delta}_{S/A,\text{perf}} \to \mathbf{\Delta}_{S/B,\text{perf}}$$

are isomorphisms.

Proof. By derived Nakayama (Remark 6.6.6), the first statement implies the third. To check the first and second statements, we may reduce to comparing graded pieces of the Hodge-Tate filtration. Using Proposition 18.1.4 to translate the statement in terms of cotangent complexes plus (17.1), we reduce to checking that the derived p-completion of $L_{\overline{B}/\overline{A}}$ vanishes. This holds because both rings are lenses; see Exercise 17.5.4. ■

Let us consider some examples.

Example 19.2.4 Coperfection for a crystalline prism. Suppose that (A, I) is crystalline, that is, $I = (p)$ and $A = W(\overline{A})$. By Corollary 19.1.5, we have

$$R_{\text{lens}} \cong R_{\text{perf}}, \qquad \mathbf{\Delta}_{R/A,\text{perf}} \cong W(R_{\text{perf}})$$

with everything concentrated in degree 0. □

Example 19.2.5 Coperfection for a lens. Let (A, I) be a perfect prism and suppose that $R \in \mathbf{Ring}_{\overline{A}}$ is itself a lens. By Lemma 19.2.6, $\mathbf{\Delta}_{R/A} \cong W(R^\flat)$ concentrated in degree 0. Since Frobenius is already an automorphism on $W(R^\flat)$, it follows that $\mathbf{\Delta}_{R/A,\text{perf}} \cong W(R^\flat)$ and $R_{\text{lens}} \cong R$, both concentrated in degree 0. □

Lemma 19.2.6 *Let (A, I) be a perfect prism and suppose that $R \in \mathbf{Ring}_{\overline{A}}$ is itself a lens. Then $\mathbf{\Delta}_{R/A} \cong W(R^\flat)$.*

Proof. Write $R = \overline{B}$ for some perfect prism (B, J); by Theorem 7.3.5, the map $\overline{A} \to \overline{B}$ promotes uniquely to a morphism of prisms $(A, I) \to (B, J)$. Now $(R \to B/J \leftarrow B)$ is an object of $(R/A)_{\mathbf{\Delta}}$, so we have a natural map $\mathbf{\Delta}_{R/A} \to B = W(R^\flat)$. To check that this is an isomorphism, by derived Nakayama (Remark 6.6.6) it suffices to do this after applying $\bullet \otimes^L_A \overline{A}$; that is, we must check that $\overline{\mathbf{\Delta}}_{R/A} \cong R$. This follows from Lemma 19.2.3. ■

We are now ready to consider a simple example where the lens coperfection is not concentrated in degree 0, although the verification of this will come later (see Section 26). This should not necessarily be viewed as a bad thing, as the higher cohomology will carry some important geometric information.

Example 19.2.7 The q-torus. Let (A, I) be the coperfection of $(\mathbb{Z}_p[\![q-1]\!], ([p]_q))$, so that A is the classical $(p, [p]_q)$-completion of $\mathbb{Z}_p[q^{p^{-\infty}}]$. Take $R = \overline{A}[x^{\pm}]^{\wedge}_{(p)}$.

We will see later (see Section 26) that in this example $H^1(\mathbf{\Delta}_{R/A,\mathrm{perf}})$ and $H^1(R_{\mathrm{lens}})$ are both nonzero. This will follow by our later computation of $\mathbf{\Delta}_{R/A}$ using a q-de Rham complex (compare Example 12.4.3). We will eventually see that $\mathbf{\Delta}_{R/A,\mathrm{perf}}$ is given by the $(p, [p]_q)$-completion of

$$A[x^{\pm p^{p^{-\infty}}}] \stackrel{\gamma-\mathrm{id}}{\to} JA[x^{\pm p^{p^{-\infty}}}]$$

where

$$J = \left(\bigcup_n (q^{p^{-n}} - 1)\right)^{\wedge}_{(p,[p]_q)} = \ker(A \to \mathbb{Z}_p, q^{p^{-n}} \mapsto 1)$$

and γ is characterized by

$$\gamma(x^i) = q^i x^i \qquad (i \in \mathbb{Z}[p^{-1}]).$$

In particular, $(q-1) \cdot 1$ in degree 1 is not a coboundary even modulo $[p]_q$. □

19.3 More properties of coperfection

Lemma 19.3.1 Base change compatibility. *Let (A, I) be a perfect prism. The functor $R \mapsto R_{\mathrm{lens}}$ on derived p-complete $\overline{A}$-algebras commutes with faithfully flat base change on the prism (A, I).*

Proof. We treat here only the case where R has bounded p-power torsion. See [25] for a broader result that includes the general case of this assertion.

Let $(A, I) \to (B, IB)$ be a faithfully flat map of perfect prisms. Put $S = R \widehat{\otimes}^L_{\overline{A}} \overline{B}$; then S is p-completely flat over R and thus concentrated in degree 0 (because R has bounded p-power torsion). We need to show that $R_{\mathrm{lens}} \widehat{\otimes}^L_{\overline{A}} \overline{B} \cong S_{\mathrm{lens}}$; by compatibility with filtered colimits, this reduces to showing that $\overline{\mathbf{\Delta}}_{R/A} \otimes^L_{\overline{A}} \overline{B} \cong \overline{\mathbf{\Delta}}_{S/B}$. This follows by comparing the Hodge-Tate filtrations on both sides using Proposition 18.1.4, then using the analogous compatibility for the cotangent complex and its exterior powers (Proposition 17.1.2). ■

The following can be viewed as a refinement of Exercise 17.5.1.

Lemma 19.3.2 *Let $R_\bullet$ be a simplicial object of $\mathbf{Ring}_{\mathbb{F}_p}$. Then the action of Frobenius on $H^{-i}(R_\bullet)$ is zero for all $i > 0$. In particular, if ϕ_R is a homotopy equivalence, then $H^{-i}(R_\bullet) = 0$ for all $i > 0$.*

Proof. For a given $i > 0$, set $S_i = \mathrm{Sym}_{\mathbb{F}_p} \mathbb{F}_p[i]$. By construction, $H^{-i}(S_i)$ is nonzero; moreover, any class of $H^i(R_\bullet)$ is in the image of $H^{-i}(S_i)$ along some map $S_i \to R_\bullet$. Hence it suffices to check that Frobenius kills $H^{-i}(S_i)$.

For $i = 1$, we may write

$$S_1 = \mathbb{F}_p \otimes^L_{\mathbb{F}_p[x]} \mathbb{F}_p,$$

from which we read off that $H^{-1}(S_1) \cong (x)/(x^2)$, which is evidently killed by Frobenius.

For $i > 1$, we may write

$$S_{i+1} = \mathbb{F}_p \otimes^L_{S_i} \mathbb{F}_p$$

to obtain an identification

$$H^{-i-1}(S_{i+1}) \cong H^{-i-1}(\mathbb{F}_p \otimes^L_{S_i} \mathbb{F}_p) \cong H^{-i}(S_i)$$

that is compatible with Frobenius. By induction on i, we deduce the desired result. (Compare [24], Proposition 11.6.) ∎

Remark 19.3.3 It is noted in [24], Remark 11.8 that Lemma 19.3.2 admits a generalization which makes no reference to Frobenius or characteristic p: for any simplicial commutative ring $R_\bullet$, the multiplication map $R_\bullet \times R_\bullet \to R_\bullet$ induces the zero map on $H^{-i}(R_\bullet)$ for all $i > 0$.

Lemma 19.3.4 Coconnectivity of coperfection. *For any perfect prism (A, I) and any derived p-complete $R \in \mathbf{Ring}_{\overline{A}}$, $\mathbf{\Delta}_{R/A,\mathrm{perf}} \in D^{\geq 0}_{\mathrm{comp}}(A)$.*

Proof. From its construction, $\mathbf{\Delta}_{R/A,\mathrm{perf}}/p$ carries a natural Frobenius endomorphism; by Lemma 19.3.2, its negative cohomology groups must vanish. By applying derived Nakayama (Exercise 6.7.5) to the canonical truncation $\tau^{\leq -1}(\mathbf{\Delta}_{R/A,\mathrm{perf}})$, we deduce the claim. (Compare [18], Lecture VIII, Remark 2.5(1) or [25], Lemma 8.4.) ∎

Lemma 19.3.5 *For (A, I) a perfect prism and $R \in \mathbf{Ring}_{\overline{A}}$ derived p-complete, suppose that $\mathbf{\Delta}_{R/A,\mathrm{perf}} \in D^{\leq 0}_{\mathrm{comp}}(A)$. (For example, this holds whenever R is a semilens; see Corollary 19.3.6.)*

1. *The object $\mathbf{\Delta}_{R/A,\mathrm{perf}} \in D_{\mathrm{comp}}(A)$ is concentrated in degree 0, where it is a perfect (p, I)-complete δ-ring.*
2. *The pair $(\mathbf{\Delta}_{R/A,\mathrm{perf}}, I\mathbf{\Delta}_{R/A,\mathrm{perf}})$ is a perfect prism over (A, I).*
3. *The object R_{lens} is concentrated in degree 0, where it is a lens. Moreover, the map $R \to R_{\mathrm{lens}}$ is the universal map of R into a lens.*

Proof. Point (1) is a direct corollary of Lemma 19.3.4. Point (2) is a direct corollary of (1). Point (3) follows from (2) and Example 19.2.5. ∎

Corollary 19.3.6 *The embedding functor from lenses to semilenses (both viewed as full subcategories of $\mathbf{Ring}$) admits a left adjoint given by lens coperfection. (For more on this functor, see Corollary 19.4.6.)*

Proof. Let (A, I) be a perfect prism and let $R = \overline{A}/J$ be a derived p-complete quotent. Since $\overline{A} \to R$ is surjective, $\Omega^1_{R/\overline{A}} = 0$ and so $L_{R/\overline{A}}[-1] \in D^{\leq 0}_{\mathrm{comp}}(R)$. This in turn implies that $\wedge^i L_{R/\overline{A}}[-i] \in D^{\leq 0}_{\mathrm{comp}}(R)$ for all i, and similarly after derived p-completion. By the Hodge-Tate filtration (Proposition 18.1.4), we deduce that $\overline{\mathbf{\Delta}}_{R/A} \in D^{\leq 0}_{\mathrm{comp}}(R)$ and hence $\mathbf{\Delta}_{R/A} \in D^{\leq 0}_{\mathrm{comp}}(A)$. Now apply Lemma 19.3.5 to deduce that R_{lens} is concentrated in degree 0, where it is a lens. ∎

Remark 19.3.7 As indicated in [18], Lecture VIII, Remark 2.5, Lemma 19.3.4 and Lemma 19.3.5 are concrete consequences of the statement that the action of ϕ_R gives $\mathbf{\Delta}_{R/A,\mathrm{perf}}$ the structure of a "derived perfect δ-ring". We will not try to unpack this statement further here.

19.4 André flatness

We next use prismatic coperfections to construct faithfully flat morphisms of prisms; this recovers an important assertion of mixed-characteristic commutative algebra.

Definition 19.4.1 A ring R is **absolutely integrally closed** if every monic polynomial over R has a root. We often abbreviate this to **AIC**. $\lozenge$

Lemma 19.4.2 *For $R \in \mathbf{Ring}$ AIC, $f \in R$ an element, and $R[f^{-1}] \to S$ a finite étale morphism in $\mathbf{Ring}$, there exist elements $g_1, \dots, g_r \in R$ such that $(f, g_1, \dots, g_r)R = R$ and for $i = 1, \dots, r$, the morphism $R[(fg_i)^{-1}] \to S[g_i^{-1}]$ is totally split (that is, as an $R[(fg_i)^{-1}]$-algebra we can split $S[g_i^{-1}]$ as a finite product of copies of $R[(fg_i)^{-1}]$).*

Proof. By [117], tag 0DCS, the localization of R at any prime ideal is strictly henselian; this implies the claim at once. ■

Lemma 19.4.3 *Let (A, I) be a perfect prism. Let $P \in \overline{A}[x]$ be a monic polynomial. Then there exists a faithfully flat morphism $(A, I) \to (B, IB)$ of perfect prisms such that $\overline{B}$ contains a root of P.*

Proof. Define the ring

$$R = \overline{A}[x^{p^{-\infty}}]^{\wedge}_{(p)}/(P);$$

by construction, R is a regular semilens and $\overline{A} \to R$ is p-completely faithfully flat. By Remark 18.2.3, $\mathbf{\Delta}_{R/A}$ is concentrated in degree 0, where it is a (p, I)-completely flat A-algebra. By the Hodge-Tate comparison (Proposition 18.1.4), $R \to \overline{\mathbf{\Delta}}_{R/A}$ is p-completely faithfully flat.

By Corollary 19.3.6, $\mathbf{\Delta}_{R/A,\mathrm{perf}}$ is concentrated in degree 0, where it is a perfect (p, I)-complete δ-ring which we call B. By the previous paragraph, $R \to \overline{\mathbf{\Delta}}_{R/A,\mathrm{perf}}$ is p-completely faithfully flat, so $(A, I) \to (B, IB)$ is faithfully flat. By construction, $\overline{B}$ is an R-algebra, so it contains a root of P. (Compare [25], Proposition 7.11.) ■

Theorem 19.4.4 André flatness lemma. *Let R be a lens. Then there exists a p-completely faithfully flat morphism $R \to S$ of lenses such that S is AIC. In particular, every element of S admits a compatible system of p-power roots.*

Proof. This follows directly from Lemma 19.4.3 via transfinite induction. (Compare [25], Theorem 7.12.) ■

Remark 19.4.5 Theorem 19.4.4 is a key ingredient in the proof of Hochster's **direct summand conjecture**; see Theorem 25.5.1.

Corollary 19.4.6 *For any semilens R, the natural map from R to its lens coperfection (i.e., its image under the left adjoint from Corollary 19.3.6) is surjective.*

Proof. By Corollary 19.3.6, $R \to R_{\mathrm{lens}}$ is the universal map from R to a lens, which we wish to show is surjective. Since we may check this after a p-completely faithfully flat base extension, using Theorem 19.4.4 we may reduce to the case where the multiplicative map $\sharp \colon \overline{A}^{\flat} \to \overline{A}$ is surjective. Let J be the kernel of $\overline{A} \to R$. We can then choose elements $x_i \in \overline{A}^{\flat}$ for i running over some index set I such that the elements $x_i^{\sharp} \in \overline{A}$ form a set of generators of J, and check directly that the quotient R' of $\overline{A}$ by the p-completion of the ideal generated by $x_i^{p^{-j}}$ for all $i \in I, j \geq 0$ is a lens. The natural map $R \to R'$ satisfies the same universal property as $R \to R_{\mathrm{lens}}$, so $R_{\mathrm{lens}} \cong R'$ is indeed a quotient of R. (Compare [18], Corollary 3.2.) ■

Remark 19.4.7 In the theory of perfectoid spaces, the surjectivity assertion in Corollary 19.4.6 corresponds to the fact there is no difference between **Zariski closed subsets** and **strongly Zariski closed subsets** of a perfectoid space. These concepts had previously been distinguished in [108], Remark II.2.4.

19.5 Examples of lens coperfection

Remark 19.5.1 In each of the following examples, we exhibit the lens coperfection of a semilens S not from its definition (Definition 19.2.1), but from the adjunction property (Corollary 19.3.6).

We have the following analogue of Example 3.4.3.

Example 19.5.2 Let R be a lens and let S be the regular semilens $R[x^{p^{-\infty}}]^{\wedge}_{(p)}/(x)$. Then $S_{\text{lens}} \cong R$ with the kernel of $S \to S_{\text{lens}}$ being the closure of the ideal $(x^{p^{-\infty}})$. We check this from the adjunction property (Corollary 19.3.6): if $S \to T$ is a morphism with T a lens, then T is reduced (Corollary 8.4.7) and so $x^{p^{-n}} \in \ker(S \to T)$ for all n.

In this case, it is easy to see that the kernel of $S \to S_{\text{lens}}$ is strictly larger than the radical of the ideal x. For example, if R is p-torsion-free, then the element

$$\sum_{n=1}^{\infty} p^n x^{p^{-n}}$$

belongs to the kernel but no power of it is divisible by x. □

Example 19.5.3 Let R be a completed algebraic integral closure of $\mathbb{Z}_p$ and let S be the regular semilens $R[x^{p^{-\infty}}]^{\wedge}_{(p)}/(x-1)$. Fix a coherent sequence (ζ_{p^n}) of p-power roots of unity in R. Then S_{lens} can be described as the ring of continuous functions $\mathbb{Z}_p \to R$, viewed as a subring of the product $\prod_{c \in \mathbb{Z}_p} R$, via the map taking $x^{p^{-n}}$ to $(\zeta_{p^n}^c)_{c \in \mathbb{Z}_p}$. As in Example 19.5.2, this can be checked using the adjunction property of lens coperfection (Corollary 19.3.6).

The kernel of the map $S \to S_{\text{lens}}$ has been analyzed in [53]: it is the radical of the ideal $(x-1)$, but is strictly larger than $(x-1)$ itself. However, it is difficult to exhibit "explicit" elements witnessing the difference between the two ideals. □

Here is a variation of the previous example.

Example 19.5.4 Let R be a p-torsion-free lens and let S be the nonregular semilens $R[x^{p^{-\infty}}, y^{p^{-\infty}}]^{\wedge}_{(p)}/(x^{p^{-n}} - y^{p^{-n}} : n = 0, 1, \ldots)$. In this case, $S_{\text{lens}} \cong R[x^{p^{-\infty}}]^{\wedge}_{(p)}$ via the map $y^{p^{-n}} \mapsto x^{p^{-n}}$. By contrast, if we take the quotient of $R[x^{p^{-\infty}}, y^{p^{-\infty}}]^{\wedge}_{(p)}$ by the ideal $(x - y)$, we end up with something more similar to Example 19.5.3 (particularly if R contains a coherent p-power sequence of roots of unity). □

Note that in the previous examples, the complications all arise from the kernel of the map to the lens coperfection. If we exclude this by requiring the semilens to be p-torsion-free, then one can express the lens coperfection in more classical language. (One can also make some statements in the more general case, for which we defer to [70] for details.)

Definition 19.5.5 For R a p-torsion-free ring, the ***p*-root closure** (or ***p*-normalization**) of R is the minimal subring S of $R[p^{-1}]$ containing R and closed under taking p-th roots. That is, if $x \in R[p^{-1}]$ and $x^p \in S$, then also $x \in S$. ◊

Lemma 19.5.6 *For R a p-torsion-free ring, an element x of $R[p^{-1}]$ belongs to the p-root closure of R if and only if $x^{p^n} \in R$ for some nonnegative integer n.*

Proof. It is clear that every x of this form belongs to the p-root closure. It thus suffices to check that the resulting set is a ring, as then it is clear that it contains R and is closed under taking p-th roots. We leave the verification to the reader; alternatively, see [105] where the concept of the p-root closure was first considered in detail. ■

Theorem 19.5.7 Ishizuka. *Let S be a semilens. If S is p-torsion-free, then the lens coperfection of S equals the p-adic completion of the p-root closure of R.*

Proof. See [70], Main Theorem C. ■

19.6 Exercises

1. Show that for $R \in \mathbf{Ring}_{\mathbb{F}_p}$, the Frobenius map $\phi_R \colon R \to R$ is flat if and only if the canonical map from R to its coperfection R_{perf} is flat.

   Hint. Let R_1 be a copy of R viewed as an R-algebra via ϕ. The map $R_1 \to R_{\mathrm{perf}}$ induces a surjection on spectra; hence if I is a finitely generated ideal of R and

   $$0 \to K \to I \otimes_R R_1 \to IR_1 \to R_1/IR_1 \to 0$$

   is exact with $K \neq 0$, then $I \otimes R_{\mathrm{perf}} \to IR_{\mathrm{perf}}$ is not injective either. This checks a standard criterion for flatness ([117], tag 00M5).

2. Let $R \to S$ be a morphism of $\mathbb{F}_p$-algebras such that the corresponding map $\operatorname{Spec} S \to \operatorname{Spec} R$ is a universal homeomorphism. Show that the induced map $R_{\mathrm{perf}} \to S_{\mathrm{perf}}$ of coperfections is an isomorphism.

20 The arc-topology and friends

Reference. [21].

We describe an exotic Grothendieck topology on the category of schemes, the **arc-topology**, and its close relatives, the **h-topology** and **v-topology**. This will be useful in the study of the étale comparison map (Section 22).

20.1 Grothendieck topologies

In Section 11, we introduced **indiscrete Grothendieck topologies** as a shortcut to getting to the construction of prismatic cohomology. Since we will be discussing various Grothendieck topologies on the category of schemes, we must say a bit more now.

Definition 20.1.1 A **Grothendieck topology** on a category $\mathcal{C}$ consists of a collection of (set-indexed) families of morphisms $\{U_i \to U\}_{i \in I}$ with a single target U, the **coverings**, subject to the following restrictions.

- Any isomorphism, viewed as a singleton family, is a covering.
- If $\{U_i \to U\}_{i \in I}$ is a covering and, for each i, $\{V_{ij} \to U_i\}_{j \in J_i}$ is a covering, then the composition $\{V_{ij} \to U\}_{i \in I, j \in J_i}$ is a covering. (In short, a covering of the terms in a covering gives a covering.)
- If $\{U_i \to U\}_{i \in I}$ is a covering and $V \to U$ is any morphism of $\mathcal{C}$, then the fiber products $U_i \times_U V$ exist for all $i \in I$ and $\{U_i \times_U V \to V\}_{i \in I}$ is a covering. (In short, the restriction of a covering is a covering.)

A category equipped with a Grothendieck topology is called a **site**.

A **presheaf** on a site valued in **Set** is a contravariant functor $F \colon \mathcal{C} \to \mathbf{Set}$. A **sheaf** is a presheaf such that for every covering $\{U_i \to U\}_{i \in I}$, $F(U)$ is the limit of the diagram

$$\prod_{i\in I} F(U_i) \rightrightarrows \prod_{(i_0,i_1)\in I\times I} F(U_{i_0}\times_U U_{i_1}).$$

The category of sheaves of sets on the site is called the **topos** associated to the site; it is in many ways a more canonical object, in that there are usually many different ways to construct families of coverings (or even underlying categories) that give rise to equivalent topoi. In particular, one can "sheafify" the definition of a morphism of sites to obtain morphisms of topoi, some of which do not arise from morphisms of the underlying sites. We will not dwell on this too much here, but see [117], tag 00X9. ♢

20.2 Valuation rings

Definition 20.2.1 A **valuation ring** is a local integral domain V which, as a subring of its fraction field K, is maximal with respect to local inclusions of local rings. In this case, the group $\Gamma = K^\times/V^\times$ (the **value group** of A) is totally ordered with the nonnegative elements being $(V \setminus \{0\})/V^\times$. See [117], tag 00I8 for more on valuation rings.

We say that V is **eudoxian** if its value group satisfies the equivalent conditions of Lemma 20.2.2.

We define an **arc** to be a scheme of the form $\mathrm{Spec}(V)$ where V is a eudoxian valuation ring. For example, a scheme which is the spectrum of a discrete valuation ring (sometimes called a **trait** or a **dash**) is an arc. (This terminology is hinted at in [21] but not actually introduced there.) ♢

Lemma 20.2.2 *For Γ a totally ordered abelian group, the following statements are equivalent.*

1. *For any two elements $\alpha, \beta \in \Gamma$ with $\alpha, \beta > 0$, there exists a positive integer n such that $n\alpha > \beta$.*
2. *The group Γ admits an order-preserving isomorphism with a subgroup of the additive group $\mathbb{R}$.*

Proof. It is obvious that (2) implies (1). Conversely, if (1) holds and Γ is nontrivial (as otherwise there is nothing to check), we can fix a single $\alpha \in \Gamma$ and define a function $f\colon \Gamma \to \mathbb{R}$ by the formula

$$f(\beta) = \sup\left\{\frac{r}{s} \colon r, s \in \mathbb{Z}, s > 0, s\beta > r\alpha\right\}$$

(condition (1) guaranteeing that the set in question is bounded above). We leave it to the reader to verify that this indeed gives an injective order-preserving homomorphism (Exercise 20.4.1). ■

Remark 20.2.3 A typical example of a totally ordered abelian group not satisfying the conditions of Lemma 20.2.2 is the group $\mathbb{R}\times\mathbb{R}$ with the lexicographic ordering.

Remark 20.2.4 A eudoxian valuation ring is **microbial** in the sense of Huber [67], but not conversely; the latter requires that there be a "leading term" while still having intermediate specializations. An example of a totally ordered abelian group that is not microbial is the infinite direct sum $\oplus_{m\in\mathbb{Z}}\mathbb{R}$ with the lexicographic ordering.

Corollary 20.2.5 *For V a valuation ring,* $\mathrm{Spec}(V)$ *is an arc if and only if it contains at most two points (the* ***generic point*** *and the* ***special point****, which* *coincide if and only if V is a field).*

Proof. Exercise (see Exercise 20.4.2). ∎

Remark 20.2.6 Condition (1) in Lemma 20.2.2 is commonly called the **archimedean property** of a totally ordered group. We prefer the adjective **eudoxian** both for historical accuracy and to avoid creating confusion with the use of the term **nonarchimedean** in reference to an associated absolute value of a eudoxian valuation.

Remark 20.2.7 Recall (Definition 19.4.1) that a ring R is said to be **absolutely integrally closed** (or **AIC**) if every monic polynomial over R has a root in R. When $R = V$ is a valuation ring, this is equivalent to requiring that its fraction field is algebraically closed. In particular, any (eudoxian) valuation ring can be embedded in an AIC (eudoxian) valuation ring.

20.3 The arc-topology

Definition 20.3.1 As per [21] (and an as yet unavailable sequel to [106]), we say that a morphism $f : Y \to X$ of schemes is an **arc-covering** if for any morphism $\mathrm{Spec}(V) \to X$ from an arc into X, there exists a commuting diagram as in Figure 20.3.2 in which $\mathrm{Spec}(W) \to \mathrm{Spec}(V)$ is a faithfully flat morphism of arcs. (We *do not* require the map $V \to W$ to be integral.)

$$\begin{tikzcd} \mathrm{Spec}(W) \arrow[r, dashed] \arrow[d, dashed] & Y \arrow[d] \\ \mathrm{Spec}(V) \arrow[r] & X \end{tikzcd}$$

Figure 20.3.2

◊

Lemma 20.3.3 *Let $f : Y \to X$ be a morphism of schemes.*

1. *If f is faithfully flat, then it is an arc-covering.*
2. *If f is proper and surjective, then it is an arc-covering.*

Moreover, in both cases f is also a v-covering (see Remark 20.3.7).

Proof. For (1), we first lift the closed point of $\mathrm{Spec}(V)$, and then lift generizations. For (2), we first lift the generic point of $\mathrm{Spec}(V)$, and then apply the valuative criterion for properness. In both cases, the condition that V is eudoxian plays no role. (Compare [106], Remark 2.5 or [24], Example 2.3.) ∎

Example 20.3.4 Let $X = \mathrm{Spec}\, R$ where $R \in \mathbf{Ring}$ is noetherian. Then the map $R \to \prod_{\mathfrak{m}} R^{\wedge}_{\mathfrak{m}}$, where $\mathfrak{m}$ runs over the product of all maximal ideals of R, is a faithfully flat morphism and hence an arc-covering. □

Example 20.3.5 Let $X = \mathrm{Spec}(k[x, y])$ be the affine plane over a field k, let $\tilde{f} : \tilde{Y} \to X$ be the blowup at the origin, let Y be the complement in $\tilde{Y}$ of a single closed point in the exceptional locus, and let $f : Y \to X$ be the induced morphism. Then $f : Y \to X$ is surjective but not an arc-covering: we can choose an arc whose special point maps to the origin in X and whose generic point maps to the direction corresponding to the missing point in the exceptional locus, and such an arc will not lift to Y. (Again, compare [106], Remark 2.5 or

[24], Example 2.3.) $\square$

Definition 20.3.6 The **arc-topology** on the category of schemes is the Grothendieck topology in which a family $\{f_i : Y_i \to X\}_{i \in I}$ of morphisms is considered to be a covering if for any open affine $V \subseteq X$, there exists a map $t: K \to I$ of sets with K finite and some affine opens $U_k \subseteq f_{t(k)}^{-1}(V)$ for each $k \in K$ such that the induced map $\sqcup_k U_k \to V$ is an arc-covering. $\lozenge$

Remark 20.3.7 Lemma 20.3.3 shows that the arc-topology includes many more coverings than the flat topology. This leads to some potentially confusing behavior: for instance, the structure presheaf $X \mapsto \Gamma(X, \mathcal{O})$ is not a sheaf for the arc-topology, because its sheafification does not include nilpotents (the inclusion of the reduced closed subscheme is an arc-covering). This can (and generally should) be circumvented by working with derived categories.

In any case, there is plenty of precedent for considering topologies of this nature. For example, Voevodsky [123] considered the **h-topology**, generated by étale coverings and proper surjective morphisms. A more recent variant is the **universally subtrusive topology** of Rydh [106], which is defined similarly to the arc-topology except that the lifting property is required for all valuation rings, not just eudoxian ones; following [24], this is now commonly called the **v-topology**. For morphisms of finite type between noetherian schemes, the h-topology, the v-topology, and the arc-topology all coincide, but not otherwise; see Remark 20.3.11 for a minimal counterexample and [21], section 1.1 for more discussion.

We describe two fundamental examples of coverings.

Example 20.3.8 Let A be a ring. Let $(A \to V_i)_{i \in I}$ be a set of isomorphism class representatives of A-algebras which are AIC valuation rings of cardinality at most $\max\{\aleph_0, \#A\}$ and put $B = \prod_{i \in I} V_i$. The map $A \to B$ is a v-covering: any morphism $f: A \to V$ to a valuation ring factors through the intersection $\mathrm{Frac}(f(A)) \cap V$ within $\mathrm{Frac}(V)$, and hence through some V_i. (Compare [21], Proposition 3.30.) $\square$

Remark 20.3.9 In Example 20.3.8, the connected components of the ring V are indexed by the set I. However, if I is infinite, then the spectrum of V is much larger than the set of kernels of projections $V \to V_i$: it also includes maximal ideals corresponding to **ultraproducts** of the V_i.

Example 20.3.10 Let V be a valuation ring and let $\mathfrak{p}$ be a prime ideal of V. Then $V \to V_{\mathfrak{p}} \times V/\mathfrak{p}$ is an arc-covering, but not a v-covering unless $\mathfrak{p}$ is zero or the maximal ideal. (See [21], Corollary 2.9.) $\square$

Remark 20.3.11 One can modify Example 20.3.10 to obtain a finitely presented morphism, as follows. Let V be a valuation ring which is not eudoxian. Let $\mathfrak{p}$ be a prime ideal which is neither zero nor the maximal ideal (see Corollary 20.2.5). Then for any $f \in V \setminus \mathfrak{p}$, $V \to V_f \times V/f$ is an arc-covering but not a v-covering. (Compare [21], Example 1.3.)

We record some variations on Example 20.3.8.

Remark 20.3.12 In Example 20.3.8, let J be the subset of $i \in I$ for which V_i is eudoxian. Then $C = \prod_{j \in J} V_j$ is an arc-covering, but not in general a v-covering as per Example 20.3.10.

This remains true if we replace each V_j with a larger valuation ring. In particular, we can ensure that each factor is not a field, and even complete with respect to its valuation.

Remark 20.3.13 It is possible to characterize arc-coverings of qcqs schemes in purely topological terms: they are precisely the **universal spectral submer-**

sions ([21], Proposition 2.19). See Exercise 20.4.3 for a related observation.

20.4 Exercises

1. Complete the proof of Lemma 20.2.2 by proving that the map f is indeed an injective order-preserving homomorphism.
2. Prove Corollary 20.2.5.
3. Let $f\colon Y \to X$ be a v-covering of qcqs schemes. Show that f is **universally submersive**: for every morphism $X' \to X$ of qcqs schemes, the map $Y \times_X X' \to X'$ induces a quotient map on underlying topological spaces.

21 Descent for the arc-topology

Reference. [21]; [25], section 8.2.

We establish some descent properties for the arc-topology (Section 20) which will be used to establish the étale comparison theorem (Section 22).

21.1 Descent for perfect schemes

Definition 21.1.1 The functor from perfect $\mathbb{F}_p$-schemes (i.e., those on which Frobenius is an isomorphism) to arbitrary $\mathbb{F}_p$-schemes admits a right adjoint, called **perfection**; for affine schemes, this corresponds to coperfection of rings. Let X_{perf} denote the perfection of X.

Let $\mathbf{Vect}(X)$ denote the category of vector bundles on the scheme X. $\Diamond$

Lemma 21.1.2 *Consider a pullback diagram of perfect $\mathbb{F}_p$-schemes as in Figure 21.1.3. For any complex $K^\bullet$ of quasicoherent sheaves on Y, the base-change morphism*

$$Lg^* Rf_* K \to Rf'_* Lg'^* K$$

is a quasi-isomorphism.

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

Figure 21.1.3

Proof. We reduce at once to the case where all of the schemes in question are affine. In this case, the claim reduces at once to Exercise 7.4.4. (Compare [24], Lemma 3.18.) ∎

Corollary 21.1.4 *Let V be a perfect valuation ring over $\mathbb{F}_p$. Let $\mathfrak{p}$ be a prime ideal of V. Then for every perfect V-scheme X and every complex K of quasicoherent sheaves on X, the triangle*

$$R\Gamma(X, K) \to R\Gamma(X \times_V V_{\mathfrak{p}}, K) \oplus R\Gamma(X \times_V V/\mathfrak{p}, K) \to$$

$$R\Gamma(X \times_V \kappa(V), K) \to$$

is distinguished in $D(\mathbb{F}_p)$.

Proof. By Lemma 21.1.2, this reduces to the exactness of the sequence

$$0 \to V \to V_{\mathfrak{p}} \oplus V/\mathfrak{p} \to \kappa(\mathfrak{p}) \to 0,$$

which we leave as an exercise (Exercise 21.4.1). (Compare [24], Lemma 6.3.) ∎

Lemma 21.1.5 *Let X be a noetherian $\mathbb{F}_p$-scheme. Let Z be a closed subscheme of X. Let $f\colon Y \to X$ be a blowup whose center is contained in Z, and put $E = f^{-1}(Z)$.*

1. *For $\mathcal{F} \in \mathbf{Vect}(X_{\mathrm{perf}})$, the triangle*

$$R\Gamma(X_{\mathrm{perf}}, \mathcal{F}) \to R\Gamma(Y_{\mathrm{perf}}, \mathcal{F}) \oplus R\Gamma(Z_{\mathrm{perf}}, \mathcal{F}) \to R\Gamma(E_{\mathrm{perf}}, \mathcal{F}) \to$$

is distinguished in $D(\mathbb{F}_p)$ (and similarly with the Zariski topology replaced by the fppf topology).

2. *The pullback functor*

$$\mathbf{Vect}(X_{\mathrm{perf}}) \to \mathbf{Vect}(Y_{\mathrm{perf}}) \times_{\mathbf{Vect}(E_{\mathrm{perf}})} \mathbf{Vect}(Z_{\mathrm{perf}})$$

is an equivalence of categories.

Proof. For both assertions, we may assume that $X = \operatorname{Spec} A$ is affine; write $Z = \operatorname{Spec} A/I$. Write nE for the subscheme of Y cut out by I^n.

For (1), we may assume $\mathcal{F} = \mathcal{O}$. By our hypotheses, we have $\mathcal{O}(X) \cong \mathcal{O}(Y)$ and $\mathcal{O}(Z) \cong \mathcal{O}(E)$ by Stein factorization, and similarly after taking perfections. Since X and Z are both affine, it remains to check that $H^i(Y_{\mathrm{perf}}, \mathcal{O}) \to H^i(E_{\mathrm{perf}}, \mathcal{O})$ is an isomorphism for each $i > 0$.

At this point, we follow [27], Lemma 3.9 (which is written using the Nisnevich topology, but the Zariski topology works equally well). By [117], tag 02OB, point (1), there exists a constant c such that for $n \geq c$,

$$\ker(H^i(Y, \mathcal{O}) \to H^i(E_n, \mathcal{O})) \subseteq I^{n-c} H^i(Y, \mathcal{O}).$$

Note that $H^i(Y, \mathcal{O})$ is a finitely generated A-module which, since f is a blowup and $i > 0$, is supported entirely on Z. Hence for $n \gg 0$, I^{n-c} annihilates $H^i(Y, \mathcal{O})$ and so

$$H^i(Y, \mathcal{O}) \hookrightarrow H^i(E_n, \mathcal{O}) \qquad (n \gg 0). \tag{21.1}$$

On the other hand, by [117], tag 020B, point (3), for $m \gg n \gg 0$ we have

$$\operatorname{im}(H^i(E_m, \mathcal{O}) \to H^i(E_n, \mathcal{O})) = \operatorname{im}(H^i(Y, \mathcal{O}) \to H^i(E_n, \mathcal{O})). \tag{21.2}$$

Fix a value $n \gg 0$ that is large enough for both (21.1) and (21.2) to hold. Then for $e \gg 0$, the image of $\phi^e\colon H^i(E_n, \mathcal{O}) \to H^i(E_n, \mathcal{O})$ is contained in the image of $H^i(Y, \mathcal{O}) \to H^i(E_n, \mathcal{O})$: to see this, refactor the former map as

$$H^i(E_n, \mathcal{O}) \stackrel{\phi^e}{\to} H^i(E_{p^e n}, \mathcal{O}) \to H^i(E_n, \mathcal{O})$$

and then apply (21.2). By this plus (21.1), we see that

$$\operatorname{colim}_\phi H^i(Y, \mathcal{O}) = \operatorname{colim}_\phi H^i(E_n, \mathcal{O})$$

and hence

$$H^i(Y_{\mathrm{perf}}, \mathcal{O}) = \operatorname{colim}_\phi H^i(E_n, \mathcal{O})$$

$$= \operatorname{colim}_\phi H^i(E, \mathcal{O}) = H^i(E_{\text{perf}}, \mathcal{O})$$

as claimed.

For (2), we follow [24], Lemma 4.6. By the Beauville-Laszlo theorem (see Remark 21.2.7), we may assume that A is (classically) I-complete. We may also assume that we start with an object in $\mathbf{Vect}(Y) \times_{\mathbf{Vect}(E)} \mathbf{Vect}(Z)$. Let $\mathcal{I}$ be the inverse image ideal sheaf of I; by the construction of the blowup, $\mathcal{I}$ is an *ample* invertible sheaf on Y. Consequently, by Serre vanishing, we may choose some n such that

$$H^i(Y, \mathcal{I}^k/\mathcal{I}^{k+1}) = 0 \qquad (k \geq n). \tag{21.3}$$

Since X is affine and complete along Z, $\mathbf{Vect}(X) \to \mathbf{Vect}(Z)$ is an equivalence of categories (Exercise 6.7.9). We thus have objects $\mathcal{E} \in \mathbf{Vect}(X)$, $\mathcal{F} \in \mathbf{Vect}(Y)$ and an isomorphism $\psi\colon f^*\mathcal{E}|_E \cong \mathcal{F}|_E$. By pulling back by a suitable power of ϕ, we may construct another isomorphism $\psi_n\colon f^*\mathcal{E}|_{nE} \cong \mathcal{F}|_{nE}$.

We now observe that for $m \geq n$, an isomorphism $\psi_m\colon f^*\mathcal{E}|_{mE} \cong \mathcal{F}|_{mE}$ can be promoted to an isomorphism $\psi_{m+1}\colon f^*\mathcal{E}|_{(m+1)E} \cong \mathcal{F}|_{(m+1)E}$: namely, the obstruction to lifting belongs to

$$H^1(Y, \mathcal{I}^m/\mathcal{I}^{m+1} \otimes \mathcal{H}om(f^*\mathcal{E}, \mathcal{F}))$$

which vanishes by (21.3). Since

$$\mathbf{Vect}(Y) \to \lim_m \mathbf{Vect}(mE)$$

is an equivalence by the formal existence theorem ([117], tag 0885), we deduce the desired result. ∎

Remark 21.1.6 Point (1) of Lemma 21.1.5 can also be formulated as follows: for $j\colon Z \to X$ the inclusion and $g\colon E \to X$ the induced map (and reusing the names f, g, j for the images of these maps under the perfection functor), we have a distinguished triangle

$$\mathcal{F} \to Rf_*f^*\mathcal{F} \oplus Rj_*j^*\mathcal{F} \to Rg_*g^*\mathcal{F} \to$$

in the derived category of coherent sheaves on X_{perf}.

Theorem 21.1.7

1. *The structure presheaf $\mathcal{O}$ on the category of perfect $\mathbb{F}_p$-schemes is an arc-sheaf. Moreover, for any affine perfect $\mathbb{F}_p$-scheme X, the i-th cohomology group of $\mathcal{O}$ for the arc-topology on X vanishes for all $i > 0$.*

2. *The functor* **Vect** *is an arc-stack on the category of perfect $\mathbb{F}_p$-schemes.*

Proof. To begin with, both assertions hold for the flat (fpqc) topology in place of the arc-topology thanks to classical faithfully flat descent ([117], tag 0238).

We next upgrade both assertions from the flat topology to the v-topology. Every v-covering is a cofiltered limit of h-coverings, so we may reduce to considering perfections of h-coverings of finite type $\mathbb{F}_p$-schemes. Since the h-topology is generated by faithfully flat coverings and proper surjective morphisms, and we already know descent for the former. we may reduce to considering the perfection of a proper surjective morphism. Moreover, by Raynaud-Gruson flattening [103], we may further reduce to considering the case of a blowup, to which we may apply Lemma 21.1.5.

Finally, we upgrade both assertions from the v-topology to the arc-topology. By passing to affines and then pulling back along a cover as in Example 20.3.8, we may reduce to considering a covering as in Example 20.3.10 (compare [21], Theorem 4.1). For this, apply Corollary 21.1.4 and Lemma 21.2.1. ∎

Corollary 21.1.8 *Let* $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ *be an arc-covering of perfect affine schemes over* $\mathbb{F}_p$*. Then the augmented Čech-Alexander complex*

$$0 \to A \to B \to B \otimes_A B \to \cdots$$

is acyclic.

Proof. This follows at once from Theorem 21.1.7. (Compare [21], Proposition 8.9.) ∎

21.2 Additional descent arguments

We record here an argument that was used in the proof of Theorem 21.1.7 to promote a statement about acyclicity of the structure sheaf to a statement about descent for vector bundles.

Lemma 21.2.1 *Consider a commuting diagram of rings as in Figure 21.2.2, in which* $R \to R_1$ *and* $R_2 \to R_{12}$ *are localizations at the same multiplicative subset of* R *and the sequence*

$$0 \to R \to R_1 \oplus R_2 \to R_{12} \to 0$$

is exact. Then the square Figure 21.2.3 is cartesian.

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R_1 \arrow[d] \\ R_2 \arrow[r] & R_{12} \end{tikzcd}$$

Figure 21.2.2

$$\begin{tikzcd} \mathbf{Vect}(R) \arrow[r] \arrow[d] & \mathbf{Vect}(R_1) \arrow[d] \\ \mathbf{Vect}(R_2) \arrow[r] & \mathbf{Vect}(R_{12}) \end{tikzcd}$$

Figure 21.2.3

Proof. Let M_1, M_2, M_{12} be objects of $\mathbf{Vect}(R_1), \mathbf{Vect}(R_2), \mathbf{Vect}(R_{12})$ equipped with isomorphisms $M_i \otimes_{R_i} R_{12} \cong M_{12}$ and put $M = \ker(M_1 \oplus M_2 \to M_{12})$; we will show that $M \in \mathbf{Vect}(R)$ and that the induced maps $M \otimes_R R_i \to M_i$ are isomorphisms.

We first check that the maps $M \otimes_R R_i \to M_i$ are all surjective.

- Given $x \in M_1$, we can write the image of x in M_{12} as y/f for some $y \in M_2$ and some $f \in R$ which becomes a unit in R_1. Then (fx, y) is an element of M mapping to $fx \in M_1$, so $M \otimes_R R_1 \to M_1$ is surjective.

- Since $R_1 \oplus R_2 \to R_{12}$ is surjective, $M \otimes_R (R_1 \oplus R_2) \to M_{12}$ is surjective.

- Given $x \in M_2$, we may map x to M_{12} and then lift it to $(x_1, x_2) \in M_1 \oplus M_2$ in the image of $M \otimes_R (R_1 \oplus R_2)$. By construction, $(x_1, x_2 - x) \in M$, so the image of $M \otimes_{R_1} \to R_1$ contains both x_2 and $x_2 - x$. Hence $M \otimes_R R_2 \to M_2$ is also surjective.

We next check that M is a finite R-module. From the previous discussion, we see that there exist a finite free R-module F and a morphism $F \to M$ of R-modules such that, for $F_i = F \otimes_R R_*$, the induced map $F_i \to M_i$ is surjective. Put $N = \ker(F \to M)$ and $N_i = \ker(F_i \to M_i)$. We have a diagram as in Figure 21.2.4 in which all of the squares commute and all of the rows and columns are exact, except possibly for the dashed arrows. However, because the modules M_i are projective, the maps $N_i \otimes_{R_i} R_{12} \to N_{12}$ are isomorphisms, so all of the preceding logic applies to them also; this allows us to add the dashed horizontal arrow to the diagram, and hence also the dashed vertical arrow.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & N \arrow[r] \arrow[d] & N_1 \oplus N_2 \arrow[r] \arrow[d] & N_{12} \arrow[r, dashed] \arrow[d] & 0 \\
0 \arrow[r] & F \arrow[r] \arrow[d] & F_1 \oplus F_2 \arrow[r] \arrow[d] & F_{12} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d, dashed] & M_1 \oplus M_2 \arrow[r] \arrow[d] & M_{12} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Figure 21.2.4

We next check that for each i, $M \otimes_i R_i \to M_i$ is an isomorphism. Consider the commutative diagram as in Figure 21.2.5 with exact rows. By the previous logic, we know that both of the outside vertical maps are surjective. By the five lemma, the right vertical arrow is an isomorphism.

$$\begin{tikzcd}
 & N \otimes_R R_i \arrow[r] \arrow[d] & F_i \arrow[r] \arrow[d, equal] & M \otimes_R R_i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N_i \arrow[r] & F_i \arrow[r] & M_i \arrow[r] & 0
\end{tikzcd}$$

Figure 21.2.5

We finally check that M is a projective R-module. By repeating the logic used to construct Figure 21.2.4, we obtain another commutative diagram as in Figure 21.2.6 with exact rows and columns. The element of $\mathrm{Hom}_{R_1}(M_1, M_1) \oplus \mathrm{Hom}_2(M_2, M_2)$ corresponding to the identity maps has zero horizontal image, so by the snake lemma it lifts to some $\mathrm{Hom}_{R_1}(M_1, F_1) \oplus \mathrm{Hom}_{R_2}(M_2, F_2)$ which maps to zero in $\mathrm{Hom}_{R_{12}}(M_{12}, F_{12})$. This gives us maps $M_1 \to F_1, M_2 \to F_2$ which agree on M and map it into F; the resulting map $M \to F$ splits the surjection $F \to M$, showing that M is projective. (Compare [82], Lemma 1.3.8, Lemma 1.3.9.)

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \star \arrow[r] \arrow[d] & \mathrm{Hom}_{R_1}(M_1, N_1) \oplus \mathrm{Hom}_{R_1}(M_2, N_2) \arrow[r] \arrow[d] & \mathrm{Hom}_{R_{12}}(M_{12}, N_{12}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \star \arrow[r] \arrow[d] & \mathrm{Hom}_{R_1}(M_1, F_1) \oplus \mathrm{Hom}_{R_1}(M_2, F_2) \arrow[r] \arrow[d] & \mathrm{Hom}_{R_{12}}(M_{12}, F_{12}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \star \arrow[r] \arrow[d] & \mathrm{Hom}_{R_1}(M_1, M_1) \oplus \mathrm{Hom}_{R_1}(M_2, M_2) \arrow[r] \arrow[d] & \mathrm{Hom}_{R_{12}}(M_{12}, M_{12}) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

Figure 21.2.6

∎

Remark 21.2.7 A well-known instance of Lemma 21.2.1 is the Beauville-Laszlo theorem: this is the case where

$$R_1 = R_t, \quad R_2 = \lim_n R/t^n, \quad R_{12} = R_{2,t}$$

for some non-zerodivisor $t \in R$. Compare [117], tag 05E5.

Remark 21.2.8 In Lemma 21.2.1, the hypothesis that $R \to R_1$ and $R_2 \to R_{12}$ are localizations at the same multiplicative subset is only needed to ensure that $M \otimes_R R_1 \to M_1$ is surjective. In some cases one can run the same argument with a different condition; see for example [82], Theorem 2.7.7 for an application to vector bundles on adic spaces.

21.3 Arc-descent for étale cohomology

We record another form of descent for the arc-topology, this time in the realm of étale cohomology.

Theorem 21.3.1 Arc-descent for étale cohomology. *For $R \in$ **Ring**, let $\mathcal{F}$ be a torsion sheaf on $(\operatorname{Spec} R)_{\mathrm{et}}$. Then the functor*

$$(f : X \to \operatorname{Spec} R) \mapsto R\Gamma(X_{\mathrm{et}}, f^*\mathcal{F})$$

from R-schemes to $D(\mathbb{Z})^{\geq 0}$ satisfies descent for the arc-topology. That is, for $f : Y \to X$ an arc-covering, there is a natural quasi-isomorphism from $R\Gamma(X_{\mathrm{et}}, f^*\mathcal{F})$ to the totalization of the Čech-Alexander complex $R\Gamma(Y_{\bullet,\mathrm{et}}, f^*\mathcal{F})$.

Proof. We first verify descent for a v-covering $f : Y \to X$, in which we may assume both schemes are qcqs. We can then write Y as a filtered limit of some finitely presented X-schemes, each of which is itself a v-covering, with affine transition maps; we may thus reduce to dealing with a finitely presented v-covering. By arguing as in [106], Theorem 3.12, we may refine this covering by a composition of a quasicompact open covering with a proper surjective morphism. As descent for the former is immediate, we may further assume that f is proper surjective. In this case, we are in the usual setting of cohomological descent for étale cohomology. For this, we may assume that X is the spectrum of a strictly henselian local ring with closed point x. By the proper base change theorem, $\mathcal{F}(Y) \cong \mathcal{F}(Y_x)$, so we may check the claim after pulling back along $x \to X$. But the resulting map $Y_x \to x$ has a section, so it satisfies descent for purely formal reasons. See [21], Proposition 5.2 for more details.

To obtain descent for the arc-topology, as in the proof of Theorem 21.1.7

we may use v-descent to reduce to a covering as in Example 20.3.10 in which V is AIC. In this case, $V/\mathfrak{p}$ is also AIC, so both V and $V/\mathfrak{p}$ are strictly henselian with the same residue field. It follows that the functor in question takes the same values on V and $V/\mathfrak{p}$, and takes the same values on $V_\mathfrak{p}$ and $\kappa(\mathfrak{p})$. (Compare [21], Theorem 5.4.) ■

21.4 Exercises

1. Let V be a perfect valuation ring over $\mathbb{F}_p$. Let $\mathfrak{p}$ be a prime ideal of V. Prove directly that the sequence

$$0 \to V \to V_\mathfrak{p} \oplus V/\mathfrak{p} \to \kappa(\mathfrak{p}) \to 0$$

is exact.

Hint. See [24], Lemma 6.3.

22 The étale comparison theorem

Reference. [18], lecture IX; [25], section 9. We follow the latter more closely.

In this section, we establish the étale comparison theorem for prismatic cohomology (Theorem 22.6.1). The strategy is to use arc-descent to reduce to a case where everything can be calculated explicitly; this avoids any use of analytic geometry.

22.1 The Artin-Schreier-Witt exact sequence

Our entire discussion up to now has involved cohomology theores of "coherent nature", involving some sort of algebro-geometric structure sheaf. It is reasonable to wonder how then we can hope to say anything meaningful about étale cohomology. The fundamental bridge between these two words is the **Artin-Schreier exact sequence**, or in a more general form the **Artin-Schreier-Witt exact sequence**. (See [117], tag 0A3J for more discussion.)

Proposition 22.1.1 Artin-Schreier-Witt exact sequence. *For any scheme X over $\mathbb{F}_p$, the sequence*

$$0 \to \underline{\mathbb{F}_p} \to \mathbb{G}_a \stackrel{\phi-1}{\to} \mathbb{G}_a \to 0$$

of étale sheaves on X is exact. More generally, for any positive integer n, the sequence

$$0 \to \underline{\mathbb{Z}/p^n\mathbb{Z}} \to W_n \stackrel{\phi-1}{\to} W_n \to 0$$

of étale sheaves on X is exact (where W_n denotes p-typical Witt vectors of rank n, viewed as a group scheme with respect to addition).

Proof. The key point is the surjectivity of $\phi - 1$. To check this, we may assume $X = \operatorname{Spec} A$ is affine. It will suffice to check that for $x = (x_0, \dots, x_{n-1}) \in W_n(A)$, the morphism

$$A \to B = A[y_0, \dots, y_{n-1}]/((\varphi - 1)y - x)$$

is étale (and even finite étale). The defining ideal is generated by some elements of the form

$$y_i^p - y_i - P(x_0, \dots, x_i, y_0, \dots, y_{i-1}) \qquad (i = 0, \dots, n-1);$$

we may thus deduce that $A \to B$ is finite étale using the Jacobian criterion (the Jacobian matrix is triangular with units on the diagonal). ■

Remark 22.1.2 The Artin-Schreier-Witt sequence also has a nonabelian analogue; this appears in Fontaine's theory of (φ, Γ)**-modules** [49].

22.2 Frobenius fixed points and coperfections

The Artin-Schreier exact sequence leads us to the following considerations.

Definition 22.2.1 Fix an $\mathbb{F}_p$-algebra B and an element $t \in B$. Write $D_{\mathrm{comp}}(B)$ for the t-complete derived category of B.

Let $D(B[F])$ be the **derived category of Frobenius B-modules**; objects in this category are pairs (M, ϕ) where $M \in D(B)$ and $\phi \colon M \to \phi_* M$ is a morphism. Let $D_{\mathrm{comp}}(B[F])$ be the full subcategory of $D(B[F])$ spanned by pairs (M, ϕ) with $M \in D_{\mathrm{comp}}(B)$.

Given $(M, \phi) \in D_{\mathrm{comp}}(B[F])$, define $M^{\phi-1}$ as the cocone of ϕ (see Definition 10.2.2); that is,

$$M^{\phi=1} = \mathrm{Cone}(M \stackrel{\phi-1}{\to} M)[-1] = R\,\mathrm{Hom}_{D(B[F])}((B, \phi), (M, \phi))[-1].$$

We call this object the **Frobenius fixed points** of M. ♢

Lemma 22.2.2 *With notation as in Definition 22.2.1, for any $(N, \phi) \in D_{\mathrm{comp}}(B[F])$, the map $N^{\phi=1} \to (N/t)^{\phi=1}$ is an isomorphism. (Here the ϕ-structure on N/t is the one induced from N using the fact that $\phi(t) = t^p \in tB$.)*

Proof. If N is derived t-complete, then the cone F of $N \to N/t$ is complete for the t-adic filtration, and the ϕ-action is topologically nilpotent because $\phi(t) = t^p \in t^2 B$. From this we see that $F^{\phi=1} = 0$, proving the claim. ■

Proposition 22.2.3 *With notation as in Definition 22.2.1, the following statements hold.*

1. *The functors $D_{\mathrm{comp}}(B[F]) \to D(\mathbb{F}_p)$ given by $M \mapsto M^{\phi=1}$ and $M \mapsto (M[t^{-1}])^{\phi=1}$ commute with sequential colimits. (In the realm of ∞-categories, this can be upgraded to arbitrary colimits.)*

2. *For any $(M, \phi) \in D_{\mathrm{comp}}(B[F])$, for*

$$(N, \phi) = \mathrm{colim}_\phi (M, \phi) \in D_{\mathrm{comp}}(B[F]),$$

the natural maps

$$M^{\phi=1} \to N^{\phi=1}, \qquad (M[t^{-1}])^{\phi=1} \to (N[t^{-1}])^{\phi=1}$$

are isomorphisms.

Proof. For convenience, we take colimits in the ambient category $D(B)$ (in contrast to the original statement).

For (1), we first verify that the two assertions are equivalent. Consider a diagram $\{(M_i, \phi_i)\}$ in $D_{\mathrm{comp}}(B[F])$. Let F be the cone of the map from $\mathrm{colim}_i M_i$ to its derived t-completion; note that F is uniquely t-divisible, so F is also the cone of the map obtained after inverting t on both sides. Now both statements of the lemma are equivalent to the vanishing of $F^{\phi=1}$, and hence to each other.

We now prove that $M \mapsto M^{\phi=1}$ commutes with colimits. By Lemma 22.2.2, this functor refactors as

$$D_{\mathrm{comp}}(B[F]) \stackrel{N \mapsto N/t}{\to} D(B[F]) \stackrel{\bullet^{\phi=1}}{\to} D(\mathbb{F}_p)$$

and both factors commute with colimits.

For (2), using (1) it suffices to check that each map $(M, \phi) \stackrel{\phi}{\to} (M, \phi)$ induces an isomorphism upon applying either $\bullet^{\phi=1}$ or $(\bullet[t^{-1}])^{\phi=1}$, both of which are clear. (Compare [18], Lecture IX, Proposition 1.2.) ■

Corollary 22.2.4 *Let (A, I) be a perfect prism and choose a generator d of I (Theorem 7.2.2). Let R be a derived p-complete $\overline{A}$-algebra. Then the natural maps*

$$(\mathbf{\Delta}_{R/A}/p^n)^{\phi=1} \to (\mathbf{\Delta}_{R/A,\mathrm{perf}}/p^n)^{\phi=1}$$
$$(\mathbf{\Delta}_{R/A}[d^{-1}]/p^n)^{\phi=1} \to (\mathbf{\Delta}_{R/A,\mathrm{perf}}[d^{-1}]/p^n)^{\phi=1}$$

are isomorphisms.

Proof. This formally reduces to the case $n = 1$. In this case, apply Proposition 22.2.3 with $B = A/p, t = d$ where d is a generator of I (Theorem 7.2.2). ■

22.3 The arc_p-topology

We need a variant of the arc-topology that accounts for p-completion.

Definition 22.3.1 Let $f\colon R \to S$ be a morphism of derived p-complete rings. We say that f is an **arc_p-covering** if the completion property from Figure 20.3.2 (with $X = \operatorname{Spec} R$, $Y = \operatorname{Spec} S$) holds whenever the valuation ring V is p-complete (and eudoxian). Note that we can then take W to also be p-complete (and eudoxian). ♢

Remark 22.3.2 A sufficient (but not necessary) condition for a morphism $f\colon R \to S$ to be an arc_p-covering is the following: for every p-complete AIC eudoxian valuation ring V, the map

$$\mathrm{Hom}_{\mathbf{Ring}}(R, V) \to \mathrm{Hom}_{\mathbf{Ring}}(S, V)$$

is a bijection.

Lemma 22.3.3 *Let $R \to S$ be an arc_p-covering of derived p-complete rings. Then*

$$R \to S \oplus R/p \oplus R[1/p]$$

is an arc-covering.

Proof. Let $R \to V$ be a morphism in **Ring** with V a eudoxian valuation ring. The image of p in V is then one of the following.

1. A nonzero element of the maximal ideal. In this case, we can replace V with its p-completion, in which case the map factors through S because $R \to S$ is an arc_p-covering.

2. The zero element. In this case, the map factors through R/p.

3. A unit. In this case, the map factors through $R[1/p]$.

■

Theorem 22.3.4 Arc_p-descent for étale cohomology. *For $R \in$ **Ring**, let $\mathcal{F}$ be a torsion sheaf on $(\operatorname{Spec} R)_{\mathrm{et}}$. Then the functor*

$$(f\colon \operatorname{Spec} S \to \operatorname{Spec} R) \mapsto R\Gamma(\operatorname{Spec} S^{\wedge}_{(p)}[p^{-1}], f^*\mathcal{F})$$

from $\mathbf{Ring}_R^{\mathrm{op}}$ to $D(\mathbb{Z})^{\geq 0}$ satisfies descent for the arc_p-topology.

Proof. It suffices to check descent for an arc_p-covering $R \to S$ (so in particular both rings are derived p-complete). By Lemma 22.3.3, $R \to S \oplus R/p \oplus R[1/p]$ is an arc-covering. Since derived p-completion followed by inverting p kills both

(R/p)-modules and $R[1/p]$-modules, we may deduce the claim from arc-descent for étale cohomology (Theorem 22.3.4). (Compare [21], Corollary 6.17.) ■

22.4 Tilting valuation rings

In order to further relate the arc-topology with the arc_p-topology, we study the effect of tilting on valuation rings. This can be thought of as a continuation of our discussion of perfectoid fields (Subsection 8.3).

Lemma 22.4.1 *Let V be a p-complete AIC valuation ring. Then V is a lens.*

Proof. If V is a p-complete AIC valuation ring, then $\operatorname{Frac} V$ is a perfectoid field (Definition 8.3.1). We may thus deduce the claim from Lemma 8.3.3. ■

Lemma 22.4.2 *Let V be a lens. Then V is a valuation ring if and only if $V^\flat$ is. In this case, the value groups of V and $V^\flat$ are isomorphic; in particular, V is eudoxian if and only if $V^\flat$ is.*

Proof. Let $\sharp\colon V^\flat \to V$ be the multiplicative map obtained by composing the constant lift $[\bullet] : V^\flat \to W(V^\flat)$ with the quotient map $W(V^\flat) \to V$. It is customary to write the image of x under $\sharp$ as $x^\sharp$ rather than $\sharp(x)$.

Suppose that V is a valuation ring. If $\operatorname{Frac} V$ has characteristic p, then $V = V^\flat$ and there is nothing more to check; we may thus assume that $\operatorname{Frac} V$ has characteristic 0. Since V is an integral domain, the p-power map on V is injective; hence for $x \in V^\flat$, $x^\sharp = 0$ if and only if $x = 0$. This in turn implies that $V^\flat$ is an integral domain (if $xy = 0$ then $x^\sharp y^\sharp = 0$) and that the principal ideals of $V^\flat$ are totally ordered with respect to inclusion (if $x^\sharp$ is divisible by $y^\sharp$, then the ratio admits a coherent sequence of p-power roots and so is itself in the image of $\sharp$). Hence $V^\flat$ is a valuation ring.

Conversely, suppose that $V^\flat$ is a valuation ring. Again, we may assume that $p \neq 0$ in V; since $V^\flat$ is an integral domain, we may apply Exercise 8.5.3 to deduce that V is p-torsion-free. Since $V^\flat$ is a local ring, so are $W(V^\flat)$ and its quotient V. Choose $\varpi \in V$ as per Lemma 8.2.3.

We need to show that given any two nonzero elements $x, y \in V$, one is a multiple of the other. By dividing by powers of ϖ as needed, we may reduce to the case where x and y have nonzero images in V/ϖ and hence in $V/p = V^\flat/d$. Since $V^\flat$ is a valuation ring, after possibly swapping terms we can write $x = yz + pu$ for some $z, u \in V$. Similarly, we can write $\varpi \equiv yw + pv$ for some $w, v \in V$. Since V is classically ϖ-complete, $1 - (p/\varpi)v$ is a unit in V; hence ϖ is divisible by y, as then is p. Consequently, $x = yz + pu$ is also divisible by y, as desired. ■

Lemma 22.4.3 *Let V be a lens which is a valuation ring. Then V and $V^\flat$ have residue fields isomorphic to each other (and to that of $W(V^\flat)$) and $\sharp : V^\flat \to V$ induces an isomorphism between the value groups of V and $V^\flat$. In particular, V is eudoxian if and only if V' is.*

Proof. From the proof of Lemma 22.4.2, we see that $\sharp$ induces an injective map from the value group of $V^\flat$ to that of V. To prove surjectivity, we must check that every element x of V has an associate in the image of $V^\flat$. As in the proof of Lemma 22.4.2, we may prove this by first dividing by a suitable power of ϖ to ensure that $x \not\equiv 0 \pmod{\varpi}$, then showing that in this case x is an associate of $[\overline{x}]$ where $\overline{x} \in V/p$ is the image of x. ■

Lemma 22.4.4 *Let V be a lens. If V is an AIC valuation ring, then so is $V^\flat$.*

Proof. By Lemma 22.4.2, V is a valuation ring if and only if $V^\flat$ is; it thus remains to show that if $V^\flat$ is not AIC, then neither is V. We may assume that V has characteristic 0, as otherwise there is nothing to check.

Let R be the integral closure of $V^\flat$ in a nontrivial finite Galois extension of its fraction field with Galois group G. By Theorem 7.3.5, $V' = V \otimes_{W(V^\flat)} W(R)$ is a lens with $V'^\flat \cong R$ and, by Lemma 22.4.2, again a valuation ring.

By construction, G acts on $V'^\flat$ with fixed subring V. By functoriality, G also acts on V'; since V' is of characteristic 0, we can we can see by averaging over the group action that the fixed subring V'^G is equal to V.

By the Artin and Dedekind lemmas in Galois theory, we see that $\operatorname{Frac} V'$ is a finite Galois extension of $\operatorname{Frac} V$ of degree $\#G > 1$. This proves that V is not AIC. ■

Remark 22.4.5 The tilting correspondence for perfectoid fields (Theorem 8.3.4) implies the converse of Lemma 22.4.4: if V is a lens and $V^\flat$ is an AIC valuation ring, then so is V. We will recover this later as a corollary of the étale comparison theorem (Theorem 23.1.1).

22.5 Arc$_p$-descent for lenses

Lemma 22.5.1 *Let $R \to S$ be an arc$_p$-covering of lenses. Write R as the slice of a perfect prism (A, I) with $I = (d)$ (Theorem 7.2.2). Then*

$$R^\flat \to R^\flat[d^{-1}] \oplus S^\flat$$

is an arc-covering.

Proof. Let $R^\flat \to V$ be a map to a eudoxian valuation ring, which we may assume is perfect. If d maps to a unit in V, then the map extends to $R^\flat[d^{-1}]$. Otherwise, we may replace V with its d-adic completion; by Theorem 7.3.5, the map $R^\flat \to V$ corresponds to a map $R \to V^\sharp = A \otimes_R W(V)$ whose target is a lens and (by Lemma 22.4.2) a p-complete eudoxian valuation ring. Since $R \to S$ is an arc-covering, we get an extension to a map $S \to V'$ for some eudoxian valuation ring V' containing $V^\sharp$, which we may take to be p-complete and AIC and hence a lens (Lemma 22.4.1). Then $S^\flat \to V'^\flat$ gives the desired extension. (Compare [25], Proposition 8.9.) ■

Theorem 22.5.2 *Let $R \to S$ be an arc$_p$-covering of lenses. Then the augmented Čech-Alexander complex*

$$0 \to R \to S \to S \widehat{\otimes}_R S \to \cdots$$

is acyclic.

Proof. Write R as the slice of a perfect prism (A, I). By applying Corollary 21.1.8 to the arc-covering from Lemma 22.5.1, then taking derived d-completions (which kills all terms involving $R^\flat[d^{-1}]$), we deduce the stated result. (Compare [25], Proposition 8.9.) ■

22.6 The comparison theorem

We finally obtain the étale comparison theorem.

Theorem 22.6.1 Étale comparison theorem. *Let (A, I) be a perfect prism and choose a generator d of I (Theorem 7.2.2). Let R be a derived p-complete $\overline{A}$-algebra. Then for each positive integer n, there is a canonical identification*

$$R\Gamma_{\text{et}}(\operatorname{Spec} R[p^{-1}], \underline{\mathbb{Z}/p^n}) \cong (\mathbf{\Delta}_{R/A}[d^{-1}]/p^n)^{\phi=1} \tag{22.1}$$

(in the sense of Definition 22.2.1). In particular, the right-hand side of (22.1) depends only on R.

Proof. We first observe that both sides of (22.1) admits descent for the arc$p$-topology: for the left-hand side this is Theorem 22.3.4, and for the right-hand side it follows from Corollary 22.2.4 and Theorem 22.5.2. (Compare [25], Corollary 8.10.)

Using arc$_p$-descent and the Artin-Schreier-Witt exact sequence (Proposition 22.1.1), we obtain a map (from left to right in (22.1)) of the desired form. To check that it is an isomorphism, we may apply arc$_p$-descent again: using a v-covering as in Example 20.3.8, we may reduce to a case where $R = \prod_i R_i$ is a product of p-complete AIC valuation rings (and in particular a lens, by Lemma 22.4.1). Note that by Theorem 23.1.1, each ring $R_i^\flat$ is an AIC valuation ring.

In this case, the left-hand side of (22.1) equals $(\mathbb{Z}/p^n)^I$ concentrated in degree 0. As for the right-hand side, we have $\mathbf{\Delta}_{R/A,\mathrm{perf}} \cong W(R^\flat)$ concentrated in degree 0 (Example 19.2.5). By Proposition 22.1.1, for each $i \in I$, $\phi - 1$ is surjective on $W(R_i^\flat[d^{-1}])/p^n$ with kernel $\mathbb{Z}/p^n$. We thus have a canonical exact sequence

$$0 \to (\mathbb{Z}/p^n)^I \to W(R^\flat[d^{-1}])/p^n \stackrel{\phi-1}{\to} W(R^\flat[d^{-1}])/p^n \to 0$$

which is exactly what we needed. ■

Remark 22.6.2 As per [25], Remark 9.3, we point out that a similar method can be used to obtain a variant of Theorem 22.6.1 without inverting p or d: there is a canonical identification of étale sheaves on Spec $\overline{A}$:

$$\underline{\mathbb{Z}/p^n} \cong (\mathbf{\Delta}_{R/A}/p^n)^{\phi=1}$$

(so in particular the right-hand side is concentrated in degree 0). The corresponding exact sequence in the proof would be

$$0 \to (\mathbb{Z}/p^n)^I \to W(R^\flat)/p^n \stackrel{\phi-1}{\to} W(R^\flat)/p^n \to 0.$$

22.7 Exercises

1. Let R be a lens. Prove that R is a **seminormal** ring in the sense of Swan [118]: that is, the map

$$R \to \{(y, z) \in R^2 : y^3 = z^2\}, \qquad x \mapsto (x^2, x^3)$$

is a bijection.

Hint. Use Example 20.3.8 and Theorem 22.5.2 to reduce to the case where R is an AIC valuation ring.

23 Applications of étale comparison

Reference. [18], lecture IX; [25], section 9.

In this section, we describe some applications of the étale comparison theorem for prismatic cohomology (Theorem 22.6.1).

23.1 Tilting of valuation rings

We prove the converse of Lemma 22.4.4 and recover the tilting correspondence for perfectoid fields (Theorem 8.3.4). This theme will be continued in the treatment of almost purity (Section 25).

Theorem 23.1.1 *Let V be a lens. Then V is an AIC valuation ring if and only if $V^\flat$ is.*

Proof. By Lemma 22.4.2, V is a valuation ring if and only if $V^\flat$ is; it thus remains to show that if $V^\flat$ is AIC, then so is V. So suppose by way of contradiction that V is not AIC. By Lemma 22.4.3, V and $V^\flat$ have the same (algebraically closed) residue field and the same (divisible) value group. Consequently, any nontrivial finite Galois extension of $\operatorname{Frac} V$ is totally wildly ramified and so has Galois group which is a p-group. This in turn implies that $\operatorname{Frac} V$ admits a nontrivial $\mathbb{Z}/p\mathbb{Z}$-extension, and so $H^1_{\text{et}}(\operatorname{Spec} V[p^{-1}], \underline{\mathbb{F}_p}) \neq 0$. However, this contradicts Theorem 22.6.1: the right-hand side of (22.1) vanishes by Proposition 22.1.1. ■

Remark 23.1.2 Theorem 23.1.1 can be used to recover the tilting correspondence for perfectoid fields (Theorem 8.3.4) as follows. Let K be a perfectoid field and let L be a completed algebraic closure of K. Theorem 23.1.1 implies that $L^\flat$ is an algebraically closed extension of $K^\flat$, so it contains a completed algebraic closure M of $K^\flat$. Each finite subextension of M over $K^\flat$ untilts to a finite extension of K within L which is perfectoid. The completed union of these extensions is an untilt of M, so by Lemma 22.4.4 this untilt is algebraically closed. In particular it contains the integral closure of K in L, and so by completeness it equals L; in other words, $M = L^\flat$.

Now let $P \in K[x]$ be an irreducible polynomial with roots $\alpha_1, \dots, \alpha_n \in L$. By the previous paragraph, we can find a finite Galois perfectoid extension K' of K within L and an element $\beta \in L$ such that $|\beta - \alpha_1| < |\alpha_i - \alpha_1|$ for $i = 2, \dots, n$. By Krasner's lemma, we have $\beta \in K'$; it follows that every finite extension of K within L is contained in a finite Galois perfectoid extension of K within L. Using the Galois correspondence, we deduce that every finite extension of K is the untilt of some finite extension of $K^\flat$ within $L^\flat$, and so is perfectoid.

This argument is essentially the proof of Theorem 8.3.4 given in [80], Theorem 1.5.6 except that therein, Theorem 23.1.1 is proved by an explicit computation ([80], Lemma 1.5.4). The novelty here is that arc-descent allows us to deduce this from the much more basic statement that V can be extended to an AIC valuation ring, which is then automatically a lens (Lemma 22.4.1).

23.2 Torsion in étale and de Rham cohomology

Lemma 23.2.1 *Let V be a valuation ring with fraction field F and residue field k. Then for any matrix A over V, the rank of A as a matrix over F is greater than or equal to the rank of A as a matrix over k.*

Proof. Let r be the rank of A as a matrix over k. Then there exists an $r \times r$ submatrix of A whose determinant has nonzero image in k. This determinant also has nonzero image in A, and so the rank of A as a matrix over F is at least r. ■

Lemma 23.2.2 Semicontinuity for perfect complexes. *Let V be a valuation ring with fraction field F and residue field k. Let $K^\bullet$ be a perfect complex in $D(V)$. Then for each i,*

$$\dim_F H^i(K^\bullet \otimes^L_V F) \leq \dim_k H^i(K^\bullet \otimes^L_V k).$$

Remark 23.2.3 *A minimal example of strict inequality in Lemma 23.2.2 is a two-term complex $V \stackrel{\times x}{\to} V$ placed in degrees 0 and 1, where $x \in V$ is a nonzero element of the maximal ideal: over F the cohomology vanishes, but over k we have a nonzero H^1.*

Proof. We may assume that $K^\bullet$ is represented by a bounded complex of finite free V-modules. Fix bases of these modules and let A, B be the matrices representing the differentials in and out of degree i in these bases. By Lemma 23.2.1, the rank of A does not increase when passing from F to k, and the corank of B does not decrease; combining these two points yields the desired inequality. (Compare [117], tag 0BDI.) ■

Lemma 23.2.4 *Let k be an algebraically closed field of characteristic p. Let $(M, \phi) \in D(k[F])$ be a pair in which M is perfect as a complex of k-modules. Then for each integer i, the natural map*

$$H^i(M^{\phi=1}) \otimes_{\mathbb{F}_p} k \to H^i(M)$$

is injective. Moreover, for each i, the map is bijective if and only if $\phi\colon H^i(M) \to H^i(M)$ is bijective.

Proof. Exercise (Lemma 23.2.4). ■

The following statement recovers Theorem 1.2.2.

Theorem 23.2.5 *Let $\mathbb{C}$ be a complete algebraically closed extension of the field $\mathbb{Q}_p$. Let $\mathfrak{o}_\mathbb{C}$ be the valuation ring of $\mathbb{C}$ and let k be the residue field of $\mathfrak{o}_\mathbb{C}$. Let X be a smooth proper formal scheme over $\mathfrak{o}_\mathbb{C}$ with generic fiber X_η and special fiber X_k. Then for all $i \geq 0$,*

$$\dim_{\mathbb{F}_p} H^i_{\mathrm{et}}(X_\eta, \underline{\mathbb{F}_p}) \leq \dim_k H^i_{\mathrm{dR}}(X_k/k). \tag{23.1}$$

Proof. By Lemma 22.4.1, $\mathfrak{o}_\mathbb{C}$ is a lens; let (A, I) be its underlying perfect prism, choose a generator d of I (Theorem 7.2.2), and put $V = A/p = \mathfrak{o}_\mathbb{C}^\flat$. Let $(W, (p))$ be the perfect crystalline prism corresponding to k. By Theorem 7.3.5, the morphism $\mathfrak{o}_\mathbb{C} \to k$ lifts to a unique morphism $(A, I) \to (W, (p))$.

We conflate the underlying spaces of the formal scheme X and the ordinary scheme X_k; on this space, we may define prismatic cohomology complexes of sheaves $\mathbf{\Delta}_{X/A}$ and $\mathbf{\Delta}_{X_k/W}$. By the Hodge-Tate comparison (Theorem 12.4.1) and its compatibility with base change (Lemma 15.1.3), we have

$$\mathbf{\Delta}_{X/A} \widehat{\otimes}^L_A W \cong \mathbf{\Delta}_{X_k/W} \tag{23.2}$$

(with the completion being p-adic).

Define

$$R\Gamma_A(X) = R\Gamma(X, \mathbf{\Delta}_{X/A}) \in D(A).$$

Since $R\Gamma(X, \bullet)$ preserves limits, $R\Gamma_A(X)$ is a derived (p, I)-complete object of $D(A)$. By the Hodge-Tate comparison and the usual finiteness property of coherent cohomology on a proper scheme ([117], tag 02O5), $R\Gamma_A(X) \otimes^L_A \mathfrak{o}_\mathbb{C}$ is a perfect complex of $\mathfrak{o}_\mathbb{C}$-modules. By derived Nakayama (Proposition 6.6.2) applied to suitable truncations, we deduce that $R\Gamma_A(X)$ is a perfect complex of A-modules.

In particular, $R\Gamma_A(X) \otimes^L_A V$ is a perfect complex of V-modules. By Lemma 23.2.2, we obtain

$$\dim_{\mathbb{C}^\flat} H^i(R\Gamma_A(X) \otimes^L_A \mathbb{C}^\flat) \leq \dim_k H^i(R\Gamma_A(X) \otimes^L_A k). \tag{23.3}$$

We will deduce (23.1) by comparing its terms with (23.3).

We first check that the right-hand sides of (23.1) and (23.3) coincide. From (23.2) we obtain

$$R\Gamma_A(X) \widehat{\otimes}^L_A W \cong R\Gamma_W(X_k).$$

Reduce modulo p (which gets rid of the completion) and applying the crystalline comparison theorem (Corollary 14.4.10) yields

$$R\Gamma_A(X) \otimes^L_A k \cong R\Gamma_W(X_k) \otimes^L_W k \cong \phi_* R\Gamma_{\mathrm{dR}}(X_k/k).$$

Since k is perfect, the Frobenius twist ϕ_* has no effect on k-dimensions; we thus deduce the desired equality.

We next check that the left-hand sides of (23.1) and (23.3) satisfy

$$\dim_{\mathbb{F}_p} H^i_{\mathrm{et}}(X_\eta, \mathbb{F}_p) \leq \dim_{\mathbb{C}^\flat} H^i(R\Gamma_A(X) \otimes^L_A \mathbb{C}^\flat)$$

(in fact equality will hold as perf [18], Lecture IX, Remark 5.3, but this will suffice for now). Apply the étale comparison theorem (Theorem 22.6.1) to the terms of an open affine cover of X to obtain an identification

$$R\Gamma_{\mathrm{et}}(X_\eta, \mathbb{F}_p) \cong (R\Gamma_A(X) \otimes^L_A \mathbb{C}^\flat)^{\phi=1},$$

then apply Lemma 23.2.4 to obtain for each i an injective linear map

$$H^i_{\mathrm{et}}(X_\eta, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathbb{C}^\flat \to H^i(R\Gamma_A(X) \otimes^L_A \mathbb{C}^\flat).$$

This yields the desired inequality. (Compare [18], Lecture IX, Theorem 5.1.) ■

23.3 Tate twists

Definition 23.3.1 For any scheme X and any positive integer n, let μ_n be the sheaf on X for the flat topology which is the kernel of the multiplication-by-n map on $\mathbb{G}_{m,X}$. (If n is invertible on X, we may use instead the étale topology.) Define the pro-sheaf

$$\mathbb{Z}_p(1) = \lim_m \mu_{p^m};$$

for $n \in \mathbb{Z}$, set $\mathbb{Z}_p(n) = \mathbb{Z}_p(1)^{\otimes n}$ (taking the tensor product over the constant pro-sheaf $\mathbb{Z}_p$).

For $n \geq 0$, define the presheaf $\mathbb{Z}_p(n)_{\mathrm{lens}}$ on the category of lenses, valued in $D(\mathbb{Z}_p)$, by the following formula: for (A, I) a perfect prism with slice R, let d be a generator of I and set

$$\mathbb{Z}_p(n)_{\mathrm{lens}}(R) = \left(\phi^{-1}(d)^n A \stackrel{\phi/d^n-1}{\to} A\right)$$

with the first term placed in degree 0. (Note that the resulting object does not depend on the choice of d.) By Theorem 22.5.2, this construction defines an arc_p-sheaf. ♢

Lemma 23.3.2 *Let R be a lens. Then for $n > 0$, there are natural isomorphisms*

$$\mathbb{Z}_p(n) \cong \mathbb{Z}_p(n)_{\mathrm{lens}} \cong \mathbb{Z}_p(1)^{\otimes n}_{\mathrm{lens}}$$

of arc_p-sheaves on the opposite category of lenses over R.

Proof. By arc_p-descent and Example 20.3.8, we may reduce to the case where R is a product of p-complete AIC valuation rings, and then to the case of a single such ring. In this case, the map $\phi^{-1}(d)A \stackrel{\phi/d-1}{\to} A$ is surjective modulo p (by the AIC property) and hence surjective by derived Nakayama (Proposition 6.3.1).

Suppose that R is of characteristic p. In this case, we may take $d = p$, and then $\mathbb{Z}_p(n)_{\mathrm{lens}} \cong \left(A \stackrel{\phi-p^n}{\to} A\right)$. The map $\phi - p^n$ on A is visibly injective, so both sides of the desired isomorphism are zero.

Suppose next that R is of characteristic 0. Choose a morphism $\mathbb{Z}_p[\mu_{p^\infty}] \to R$, let $\epsilon \in R^\flat$ be the element $(1, \zeta_p, \zeta_{p^2}, \dots)$, and put $q = \epsilon^\sharp$. We can then take the generator of d to be $[p]_q = (q^p - 1)/(q - 1)$; we may then identify $\mathbb{Z}_p(n)_{\mathrm{lens}}$ with $(q-1)^n \mathbb{Z}_p \subset \phi^{-1}(d^n)A$. This gives the desired natural isomorphism

$\mathbb{Z}_p(n)_{\text{lens}} \cong \mathbb{Z}_p(1)^{\otimes n}_{\text{lens}}$.

To specify a natural isomorphism $\mathbb{Z}_p(n) \cong \mathbb{Z}_p(n)_{\text{lens}}$, it now suffices to do so for $n = 1$. In this case, we must check that the action of $\text{Gal}(\mathbb{Q}_p(\mu_{p^\infty}))$ on $(q-1)\mathbb{Z}_p$ matches the action on $\lim_n \mu_{p^n}$; this follows from the fact that

$$q^m - 1 \equiv m(q-1) \pmod{d(q-1)} \qquad (m \in \mathbb{Z}).$$

∎

Remark 23.3.3 One can promote Lemma 23.3.2 to the assertion that the two definitions of Tate twists correspond to a single construction on the **quasisyntomic site**, as per [23], section 7.4. We will not spell this out further here; instead, see [25], section 14.

Theorem 23.3.4 *Let R be a lens.*

1. *We have a canonical identification*

$$\mathbb{Z}_p(0)_{\text{lens}}(R) \cong R\Gamma_{\text{et}}(\text{Spec}\, R, \mathbb{Z}_p(0)).$$

2. *For $n > 0$, we have a canonical identification*

$$\mathbb{Z}_p(n)_{\text{lens}}(R) \cong R\Gamma_{\text{et}}(\text{Spec}\, R[1/p], \mathbb{Z}_p(n)).$$

Proof. Point (1) follows from Remark 22.6.2. Point (2) follows from Theorem 22.6.1 and Lemma 23.3.2. ∎

23.4 Exercises

1. Prove Lemma 23.2.4.

2. Let R be a lens. Using Theorem 23.3.4, show that $\text{Pic}(R)$ and $\text{Pic}(R[p^{-1}])$ are both uniquely p-divisible.

Hint. Use Theorem 23.3.4 to compare $H^i_{\text{et}}(\text{Spec}\, R, \mu_p)$ with $H^i_{\text{et}}(\text{Spec}\, R[p^{-1}], \mu_p)$ for $i = 1, 2$. For more details, see [25], Corollary 9.5.

24 Almost commutative algebra

Reference. [54] (not to be confused with the much longer [55]); [25], section 10.

We introduce the framework of **almost commutative algebra** in preparation for the discussion of the **almost purity theorem** in Section 25.

24.1 A bit of motivation

We first explain the term **purity** in this context.

Theorem 24.1.1 Zariski-Nagata purity of the branch locus. *Let $X \to \overline{X}$ be an open immersion of regular noetherian schemes such that $\overline{X} \setminus X$ has codimension at least 2 in $\overline{X}$. Then every finite étale cover of X extends uniquely to a finite étale cover of $\overline{X}$.*

Proof. See [117], tag 0BMB. ∎

We next give an example where purity of the branch locus does not apply, but something "almost" as good is true.

Proposition 24.1.2 *Let L/K be a finite extension of perfectoid fields (Definition 8.3.1). Let $\mathfrak{o}_K, \mathfrak{o}_L$ be the valuation rings of K, L and let $\mathfrak{m}_K, \mathfrak{m}_L$ be the maximal ideals of $\mathfrak{o}_K, \mathfrak{o}_L$. Then* $\text{Trace}\colon L \to K$ *induces a surjection $\mathfrak{m}_L \to \mathfrak{m}_K$.*

Proof. Exercise (Exercise 24.5.2). ∎

Remark 24.1.3 A closely related phenomenon is the fact that a ramified base change can "weaken" the ramification of a covering. A classical instance of this is Abhyankar's lemma, which can be used to eliminate tame ramification; see [117], tag 0EXT.

24.2 A context for almost commutative algebra

The premise of **almost commutative algebra** is that in certain situations, one would like to treat certain types of "small" modules over a ring as if they were actually zero. For the theory of modules over a ring, this is relatively straightforward to achieve using the notion of the quotient by a thick subcategory. However, we would also like to define "almost" variants of some ring-theoretic concepts, and this is somewhat more involved; we give only the necessary details here, restricted to the minimal level of generality sufficient for our purposes. See [54] for a more comprehensive initial development.

Definition 24.2.1 By a **context** (more precisely a **context for almost commutative algebra**), we will mean a pair consisting of a base ring V and an ideal $\mathfrak{m}$ such that $\mathfrak{m}^2 = \mathfrak{m}$. ◊

Example 24.2.2 The pair $(\mathbb{Z}, (1))$ is a context for almost commutative algebra. We call this the **classical limit**, where we expect to recover concepts in ordinary commutative algebra. □

Example 24.2.3 For V a nondiscrete valuation ring with maximal ideal $\mathfrak{m}$, the pair $(V, \mathfrak{m})$ is a context for almost commutative algebra. Since $\mathfrak{m}$ is a colimit of principal ideals, the V-module $\mathfrak{m} \otimes_V \mathfrak{m}$ is flat; while adding this restriction to the definition of a context is needed for a deeper treatment (for instance, in [54] it is required starting from the end of Chapter 2), we will not need it here. □

Definition 24.2.4 Fix a context $(V, \mathfrak{m})$ for almost commutative algebra. A V-module M is **almost zero** if $\mathfrak{m}M = 0$. It is straightforward to check that the subcategory of almost zero V-modules is a thick tensor ideal in $\mathbf{Mod}_V$. It thus makes sense to say that a morphism in $\mathbf{Mod}_V$ is an **almost isomorphism** (i.e., its kernel and cokernel are almost zero). ◊

Definition 24.2.5 Fix a context $(V, \mathfrak{m})$. Choose $A \in \mathbf{Ring}_V$ and $M \in \mathbf{Mod}_A$. The **module of almost elements** of M is the object

$$M_* = \mathrm{Hom}_A(\mathfrak{m}, M) \in \mathbf{Mod}_A;$$

the natural map

$$M = \mathrm{Hom}_A(A, M) \to \mathrm{Hom}_A(\mathfrak{m}, M) = M_*$$

is an almost isomorphism. Note that for $N \in \mathbf{Mod}_A$ a second object, we have natural isomorphisms

$$\mathrm{Hom}_{A_*}(M_*, N_*) \cong \mathrm{Hom}_A(M_*, N_*) \cong \mathrm{Hom}_A(M, N)_* \tag{24.1}$$

$$M_* \otimes_{A_*} N_* \cong M_* \otimes_A N_* \cong (M \otimes N)_*. \tag{24.2}$$

To define the category of **almost A-modules**, take objects to be the objects of $\mathbf{Mod}_A$, with the morphisms from M to N being $\mathrm{Hom}_A(M, N)_*$. This makes sense because by (24.2), composition defines a morphism

$$\mathrm{Hom}_A(M, N)_* \otimes_A \mathrm{Hom}_A(N, P)_* \to \mathrm{Hom}_A(M, P)_*.$$

◊

Remark 24.2.6 The category of almost A-modules can be identified with the localization of $\mathbf{Mod}_A$ at the multiplicative system of almost isomorphisms. The easiest way to check this is not to construct the latter directly, but to check that the former satisfies the universal property that characterizes the latter: the obvious functor from $\mathbf{Mod}_A$ to the category of almost A-modules is initial for the property that every almost isomorphism becomes a genuine isomorphism in the target.

We now introduce some definitions which generalize from the classical limit in a perhaps unexpected manner.

Definition 24.2.7 Fix a context $(V, \mathfrak{m})$. Choose $A \in \mathbf{Ring}_V$ and $M \in \mathbf{Mod}_A$. We say that M is **almost finitely generated** if for every finitely generated ideal $\mathfrak{m}_0 \subseteq \mathfrak{m}$, there is a finitely generated A-submodule $M_0 \subseteq M$ with $\mathfrak{m}_0 M \subseteq M_0$.

We say that M is **almost projective** if the functor on $\mathbf{Mod}_A$ given by $N \mapsto \mathrm{Hom}_A(M, N)_*$ is exact.

We write **almost finite projective** as shorthand for **almost finitely generated and almost projective**. Note that M is almost finite projective if and only if for each $\eta \in \mathfrak{m}$, there exist a finite free A-module F and a pair of morphisms $M \to F \to M$ which compose to multiplication by η. (Compare [54], Proposition 2.3.10, Definition 2.4.4.) $\Diamond$

Remark 24.2.8 While it is true that any A-module which is almost isomorphic to a finitely generated A-module is almost finitely generated, the converse is not true. Moreover, an almost projective module is not projective in the category of almost modules; see [54], Example 2.4.5.

Definition 24.2.9 Fix a context $(V, \mathfrak{m})$. A morphism $A \to B$ in $\mathbf{Ring}_V$ is **almost finite étale** if B is an almost finite projective A-module and also an almost finite projective $(B \otimes_A B)$-module via the multiplication map. (Note that these conditions do characterize a finite étale morphism in the classical limit, by [117], tag 0CKP.) $\Diamond$

Remark 24.2.10 We will use the following limited form of "almost faithfully flat descent": if $A \to B$ is an almost injective, almost finite étale morphism of rings and $A \to C$ is another morphism of rings, then $A \to C$ is almost finite étale if and only if $B \to B \otimes_A C$ is.

24.3 Almost commutative algebra for lenses

It is convenient to make a slightly different set of definitions when working with modules over lenses.

Definition 24.3.1 For J an ideal of a lens R, Corollary 19.4.6 implies that the natural map from R to the lens coperfection $(R/J)_{\mathrm{lens}}$ is surjective. Denote its kernel by J_{lens}; this means that $R/J_{\mathrm{lens}} = (R/J)_{\mathrm{lens}}$, allowing us to omit some parentheses in what follows.

Let M be a derived p-complete R-module. We say that M is **J-almost zero** if $J_{lens} M = 0$. We say that a derived p-complete complex $K^\bullet \in D(R)$ is **J-almost zero** if $H^i(M)$ is J-almost zero for all i. (Compare [25], Definition 10.1.) $\Diamond$

Example 24.3.2 In Definition 24.3.1, in the case $J = (p)$ we have $J_{\mathrm{lens}} = \sqrt{pR}$. $\square$

Lemma 24.3.3 *Let J be an ideal of a lens R. The multiplication maps*

$$J_{\mathrm{lens}} \widehat{\otimes}^L_R J_{\mathrm{lens}} \to J_{\mathrm{lens}}$$

and

$$R/J_{\mathrm{lens}} \widehat{\otimes}^L_R R/J_{\mathrm{lens}} \to R/J_{\mathrm{lens}}$$

are quasi-isomorphisms, and moreover

$$J_{\mathrm{lens}} \widehat{\otimes}^L_R R/J_{\mathrm{lens}} = 0.$$

Proof. The second equality is a direct consequence of Proposition 8.4.8, and the others follow from this. (Compare [25], Lemma 10.3.) ■

Definition 24.3.4 Let J be an ideal of a lens R. For each positive integer n, let $J_{\mathrm{lens},n}$ be the image of J_{lens} in R/p^n. By Lemma 24.3.3, the pair $(R/p^n, J_{\mathrm{lens},n})$ is a context. ◊

Proposition 24.3.5 *Let J be an ideal of a lens R. Within the category of derived p-complete R-modules, the subcategory of J-almost zero modules is stable under kernels, cokernels, and extensions in the ambient category, and is an ideal under $\otimes$. It is also equivalent to the category of derived p-complete R/J_{lens}-modules.*

Proof. For M, N two derived p-complete $(R/J)_{\mathrm{lens}}$-modules,

$$R\operatorname{Hom}_R(M, N) = R\operatorname{Hom}_{R/J_{\mathrm{lens}}}(M \widehat{\otimes}^L_R R/J_{\mathrm{lens}}, N)$$

and by Lemma 24.3.3 we have $M \widehat{\otimes}^L_R R/J_{\mathrm{lens}} \cong M$ (reducing to the case $M = R/J_{\mathrm{lens}}$). It follows that the restriction functor from derived p-complete R/J_{lens}-modules to derived p-complete R-modules, which evidently factors through the subcategory in question, defines an equivalence to this subcategory and preserves Ext^1; this yields all of the claims. ■

Corollary 24.3.6 *Let J be an ideal of a lens R. A derived p-complete complex $K^\bullet \in D(R)$ is J-almost zero if and only if $J_{\mathrm{lens}} \widehat{\otimes}^L_R K^\bullet = 0$. Within the category of derived p-complete complexes of R-modules, the subcategory of J-almost zero complexes forms a thick tensor ideal which is equivalent to the category of derived p-complete complexes of R/J_{lens}-modules.*

Proof. This is a direct consequence of Proposition 24.3.5. (Compare [25], Proposition 10.4.) ■

Lemma 24.3.7 *Let J be an ideal of a lens R. Let $K^\bullet$ be a derived p-complete complex of R-modules. Then K is concentrated in degrees ≤ 0 if and only if $K^\bullet \widehat{\otimes}^L_R R/J_{\mathrm{lens}}$ is concentrated in degrees ≤ 0 and $H^i(K^\bullet)$ is J-almost zero for all $i > 0$.*

Proof. The "only if" is clear because tensor products are right exact. For the converse, note that in the distinguished triangle

$$J_{\mathrm{lens}} \widehat{\otimes}^L_R \tau^{\leq 0} K^\bullet \to J_{\mathrm{lens}} \widehat{\otimes}^L_R K^\bullet \to J_{\mathrm{lens}} \widehat{\otimes}^L_R \tau^{>0} K^\bullet \to$$

the first term is concentrated in degrees ≤ 0 and the last term is zero by Proposition 24.3.5. Combining this with the distinguished triangle

$$J_{\mathrm{lens}} \widehat{\otimes}^L_R K^\bullet \to K^\bullet \to R/J_{\mathrm{lens}} \widehat{\otimes}^L_R K^\bullet \to$$

yields the claim. (Compare [25], Lemma 10.5.) ■

24.4 Almost Galois extensions of rings

Just as it is sometimes useful to study field extensions using Galois theory (see Remark 23.1.2 for an example that we encountered recently), we would like to study finite étale maps of rings using Galois actions.

Definition 24.4.1 Fix a context $(V, \mathfrak{m})$. Let $A \to B$ be a morphism in $\mathbf{Ring}_V$. Let G be a finite group acting A-linearly on the ring B. We say that $A \to B$ is an **almost G-Galois extension** if the map $A \to B^G$ is an almost isomorphism and the canonical map

$$B \otimes_A B \to \prod_{g \in G} B, \qquad b \otimes b' \mapsto (\gamma(b)b')_{\gamma \in G} \tag{24.3}$$

is an almost isomorphism. Note that this property persists under base change on A. ♢

Lemma 24.4.2 *With notation as in Definition 24.4.1, the following statements hold.*

1. *The morphism $A \to B$ is almost finite étale.*

2. *Let C be the fixed subring of B under a subgroup H of G, and suppose that $C \to B$ is an almost H-Galois extension. Then $A \to C$ is almost finite étale.*

Proof. To prove (1), we only need to check that B is an almost finite projective A-module, as (24.3) already implies that B is an almost finite projective $B \otimes_A B$-module. By (24.3), the idempotent element of $\prod_{g \in G} B$ that picks out the identity component is an almost element of $B \otimes_A B$. Consequently, for each $\eta \in \mathfrak{m}$, we may multiply by η to get a genuine element $e_\eta \in B \otimes_A B$ satisfying $e_\eta^2 = e_\eta$ that kills the kernel of μ and projects to $\eta \in B$. Write $e_\eta = \sum_{i=1}^n b_i \otimes b_i'$ for some $b_i, b_i' \in B$; we then have $\sum_{i=1}^n \gamma(b_i) b_i' = 0$ for $\gamma \in G \setminus \{e\}$ and $\sum_{i=1}^n b_i b_i' = \eta$.

Define the trace map $t_{B/A} \colon B \to A$ as the sum over G-conjugates. Then

$$\sum_i t_{B/A}(bb_i) b_i' = \eta b \qquad (b \in B).$$

In other words, the composition

$$B \xrightarrow{b \mapsto (t_{B/A}(bb_i))_i} A^n \xrightarrow{(a_i) \mapsto \sum a_i b_i'} B$$

is multiplication by η; since $\eta \in \mathfrak{m}$ was arbitrary, this proves that B is an almost finite projective A-module.

To prove (2), we first apply (1) to deduce that $C \to B$ is almost finite étale. We then check that the canonical map $C \otimes_A B \to \prod_{G/H} B$ is an almost isomorphism: we can check this after tensoring over C with B, in which case we have almost isomorphisms

$$B \otimes_C (C \otimes_A B) = B \otimes_A B \to \prod_G B \to \prod_{G/H} (B \otimes_C B) = B \otimes_C \prod_{G/H} B.$$

Thus the map $A \to C$ becomes almost finite étale after tensoring over A with B, and so by Remark 24.2.10 is itself almost finite étale. (Compare [4], Proposition 9.1.) ■

Definition 24.4.3 Let J be an ideal of a lens R. We define a **J-almost G-Galois extension** of R-algebras by analogy with Definition 24.4.1. ♢

Corollary 24.4.4 *Let J be an ideal of a lens R. Let S be a derived p-complete R-algebra with an action by a finite group G, such that $R \to S$ is a J-almost G-Galois cover. Let n be a positive integer.*

1. *The morphism $R/p^n \to S/p^n$ is almost finite étale for the context $(R/p^n, J_{\mathrm{lens},n})$.*

2. *Let S' be the fixed subring of S under a subgroup H of G and suppose*

that $S' \to S$ is a J-almost H-Galois cover. Then $R/p^n \to S'/p^n$ is almost finite étale for the context $(R/p^n, J_{\mathrm{lens},n})$.

Proof. The map $R/p^n \to S/p^n$ is again a J-almost G-Galois cover, so we may apply Lemma 24.4.2 to conclude. (Compare [25], Proposition 10.8.) ∎

The construction of Galois closures of field extensions has the following analogue in this context. (One can give a version of this for almost commutative algebra, but we will only need the classical limit.)

Lemma 24.4.5 *Let $R \to S$ be a finite étale morphism of constant rank r. Then there exists an S_r-Galois extension $R \to T$ (for the classical context) factoring through an S_{r-1}-Galois extension $S \to T$.*

Proof. Let $\operatorname{Spec} T$ be the closed-open subscheme of the r-fold fiber product of $\operatorname{Spec} S$ over $\operatorname{Spec} R$ which is the complement of all of the partial diagonals; this has the desired effect. (Compare [4], Lemma 1.9.2.) ∎

24.5 Exercises

1. Prove that in the notation of Definition 24.4.1, the induced map $A \to R\Gamma(G, B)$ is also an almost isomorphism; that is, the groups $H^i(G, B)$ are almost zero for all $i > 0$.

Hint. In the classical limit, B is an induced $A[G]$-module.

2. Prove Proposition 24.1.2.

Hint. Reduce to the case of characteristic p.

25 Almost purity

Reference. [25], section 10.

We deduce a strong form of the **almost purity theorem**. The statement combines the perfectoid almost purity theorems of Scholze [107] and Kedlaya-Liu [82] (which extend the original almost purity theorem of Faltings) with André's perfectoid Abhyankar lemma [4].

25.1 Some initial remarks

To clarify a potential apparent ambiguity in the statement of Theorem 25.2.6, we issue the following reminder.

Remark 25.1.1 Let $R \to S$ be a ring homomorphism such that S is finitely generated as an R-module; such a homomorphism is usually said to be **finite**, but for added emphasis we will sometimes say that it is **module-finite**. In any case, under this condition, S is finitely presented as an R-module if and only if S is finitely presented as an R-algebra ([117], tag 0D46). That is, if we say that S is a "finitely presented, module-finite R-algebra", the two possible interpretations of this statement are equivalent.

We next give an indication of why almost purity is a highly nontrivial statement.

Remark 25.1.2 A finitely presented, module-finite algebra S over a lens R is not necessarily a lens or even a regular semilens. One rather prosaic reason is that it may not be reduced (e.g., $R[x]/x^2$). Somewhat more serious examples arise from taking quotients, as in Example 19.5.3. See Example 25.1.3 for a different sort of example.

Nonetheless, we will see from the statements of Theorem 25.2.6 and Theorem 25.3.4 that S does inherit some good properties; for instance, its lens coperfection is concentrated in degree 0. We can thus think of a morphism of lenses as being "integral" if it arises by lens coperfection from an integral morphism from a lens to some target.

To begin with, note that if R is itself an integral domain, then we can find some nonzero $f \in R$ such that $R[f^{-1}] \to S[f^{-1}]$ is finite étale. Our strategy will be to do almost commutative algebra using the ideal $J = (f)$ to derive constructions about $R \to S$.

Example 25.1.3 Assume $p \neq 2$, and take $R = \mathbb{Z}_p[x^{p^{-\infty}}]^{\wedge}_{(p)}$ and $S = R[x^{1/2}]$. In this case S is not a lens, but the lens coperfection is easy to describe: it is $\mathbb{Z}_p[(x^{1/2})^{p^{-\infty}}]^{\wedge}_{(p)}$ concentrated in degree 0. □

25.2 Almost purity (first version)

Lemma 25.2.1 *The functor $S \mapsto S_{\mathrm{lens}}$ on derived p-complete rings satisfies descent for the arc_p-topology.*

Proof. This is a direct consequence of Theorem 22.5.2. Compare [25], Corollary 8.10. ■

Lemma 25.2.2 *Let $S \to S'$ be a integral morphism of derived p-complete rings such that for some derived p-complete ideal J of S, for every p-complete eudoxian valuation ring V, every morphism $S \to V$ which does not kill J extends uniquely to S'. (Note that we do* not *allow replacing V with a larger valuation ring!) Then Figure 25.2.3 is a pullback square in the derived category of S-modules.*

$$\begin{tikzcd} S_{\mathrm{lens}} \arrow[r] \arrow[d] & S'_{\mathrm{lens}} \arrow[d] \\ (S/J)_{\mathrm{lens}} \arrow[r] & (S'/J)_{\mathrm{lens}} \end{tikzcd}$$

Figure 25.2.3

Proof. By replacing S' with $S' \times S/J$, we may reduce to the case where $S \to S'$ is an arc_p-covering. By the hypothesis on J,

$$S' \widehat{\otimes}_S S' \to (S'/J \widehat{\otimes}_{S/J} S'/J) \times S'$$

is also an arc_p-covering. We may then deduce the claim from Lemma 25.2.1 and the universal property of lens coperfection. (Compare [25], Corollary 8.11.) ■

Remark 25.2.4 While Lemma 25.2.2 must be stated in the derived category $D(S)$ because that is the best we can prove right now, once we finish the proof of almost purity (Theorem 25.3.4) we will know that all of the objects in Figure 25.2.3 will be concentrated in degree 0. Hence we will also end up with a pullback square in **Ring**.

Corollary 25.2.5 *Let $S \to S'$ be a integral morphism of derived p-complete rings such that for some derived p-complete ideal J of S, $\operatorname{Spec} S' \to \operatorname{Spec} S$ is an isomorphism outside $V(J)$. Then $S_{\mathrm{lens}} \to S'_{\mathrm{lens}}$ is a J-almost isomorphism in $D(S)$.*

Proof. The hypothesis on $S \to S'$ implies the hypothesis of Lemma 25.2.2, so Figure 25.2.3 is a pullback square in $D(S)$. In particular, the cones of the two rows are isomorphic in $D(S)$. The bottom row consists of two objects which by construction are J-almost zero (Corollary 24.3.6), so its cone is also J-almost zero; hence the top row is a J-almost isomorphism. ■

Theorem 25.2.6 *Let J be a finitely generated ideal of a lens R. Let S be a finitely presented and module-finite R-algebra such that $\operatorname{Spec} S \to \operatorname{Spec} R$ is finite étale away from $V(J)$.*

1. *We have $J_{\text{lens}} H^i(S_{\text{lens}}) = 0$ for all $i > 0$.*

2. *The map $S \to S_{\text{lens}}$ is an isomorphism away from $V(J)$.*

3. *For every $n > 0$, the morphism $R/p^n \to H^0(S_{\text{lens}})/p^n$ is almost finite étale with respect to the context $(R/p^n, J_{\text{lens},n})$.*

4. *Suppose that S admits an action of a finite group G such that $\operatorname{Spec} S \to \operatorname{Spec} R$ is a G-Galois cover outside $V(J)$. Then $R \to H^0(S_{\text{lens}})$ is a J-almost G-Galois extension.*

Proof. We may assume from the outset that $p \in J$. Suppose first that S admits an action by a finite group G such that $R \to S$ is a J-almost G-Galois cover. Note that this hypothesis is preserved by a p-completely flat base extension, as it can be checked modulo p thanks to derived Nakayama (Remark 6.6.6); moreover, all of the conclusions can also be checked after such a base extension. By Theorem 19.4.4, we may thus assume that R is absolutely integrally closed. By Lemma 19.4.2, we can then find generators $f_1, \dots, f_r$ of J such that $R \to S$ splits outside $V(f_i)$ for $i = 1, \dots, r$. Since being a J-almost isomorphism is equivalent to being an (f_i)-almost isomorphism for $i = 1, \dots, r$, we may reduce to the case where $J = (f)$ and we have an R-algebra isomorphism $S[f^{-1}] \cong \prod_{i \in I} R[f^{-1}]$ for some finite index set I. Put $S' = \prod_{i \in I} R$ and let S' be the integral closure of R in $S[f^{-1}]$; we then have maps $S \to S'', S' \to S''$ to which we may apply Corollary 25.2.5. This allows us to equate all of the desired assertions about S to the corresponding statements about S', which are self-evident.

Assume next that $R \to S$ has constant degree r outside $V(J)$. By Lemma 24.4.5, we can find an S_r-Galois covering of $\operatorname{Spec}(R) \setminus V(J)$ which is an S_{r-1}-Galois covering of $\operatorname{Spec}(S) \setminus V(J)$. Using Corollary 24.4.4, we may reduce to the previous case.

Now consider the general case. In this case, $\operatorname{Spec}(R) \setminus V(J)$ can be partitioned as a finite union $\bigsqcup_i U_i$ of closed-open subsets, on each of which the degree of $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$ is constant. For each i, let R_i be the image of R in $H^0(U_i, \mathcal{O})$ and let $R_{i,\text{lens}}$ be the lens coperfection of R_i. The map $R \to \prod_i R_{i,\text{lens}}$ satisfies the condition of Lemma 25.2.2, so we may reduce to the previous case. ■

25.3 Almost purity (second version)

It turns out that Theorem 25.2.6 can be formally upgraded by first deducing a statement about lens coperfections of integral extensions of lenses, which amounts to a major upgrade of Corollary 19.3.6. We turn to this next.

Lemma 25.3.1 *Let $R \to S$ be a module-finite and finitely presented morphism in $\mathbf{Ring}_{\mathbb{Z}_{(p)}}$. Then there exist elements $g_1, \dots, g_n \in R$ such that for*

$$R_i = R/(g_1, \dots, g_{i-1})_{\text{red}}[g_i^{-1}], S_i = S/(g_1, \dots, g_{i-1})_{\text{red}}[g_i^{-1}],$$

the following statements hold.

1. *The ideal $(g_1, \dots, g_n)$ of R is the unit ideal. That is, $\bigcup_i \operatorname{Spec} R_i = \operatorname{Spec} R$.*

2. *The map $R_i \to S_i$ factors as the composition of a finite étale morphism $R_i \to T_i$ and a universal homeomorphism $T_i \to S_i$. Moreover, $T_i[p^{-1}] \to S_i[p^{-1}]$ is an isomorphism.*

3. *In each R_i, either $p = 0$ or $p \in R_i^\times$.*

Proof. Exercise (Exercise 25.6.1), or see [25], Lemma 10.12. ■

Remark 25.3.2 In Lemma 25.3.1, condition (2) implies that $R[g_1^{-1}] \to S_{\mathrm{red}}[g_1^{-1}]$ factors as $R[g_1^{-1}] \to T_1 \to S_{\mathrm{red}}[g_1^{-1}]$ where the first map is finite étale and the second map is again a universal homeomorphism. The latter is forced to be an isomorphism if either $p \in R_1^\times$ or R is a lens.

Theorem 25.3.3 *Let (A, I) be a perfect prism with associated lens R. Let $R \to S$ be the derived p-completion of an integral map. Then $\mathbf{\Delta}_{S/A,\mathrm{perf}}$ is concentrated in degree 0, where it is a derived p-complete perfect δ-ring over A. Consequently, S_{lens} is concentrated in degree 0, where it is a lens.*

Proof. By passage to filtered colimits, we may assume that $R \to S$ is module-finite and finitely presented. By Lemma 19.3.4 and Lemma 19.3.5, we know that $\mathbf{\Delta}_{S/A,\mathrm{perf}}$ is concentrated in degrees ≥ 0, and everything will follow once we show that it is also concentrated in degrees ≤ 0. For this, we fix a sequence $g_1, \dots, g_n$ as in Lemma 25.3.1 and induct on n.

For the base case $n = 1$, the map $R \to S_{\mathrm{red}}$ is finite étale, and so S_{red} is a lens. By arc-descent for lenses (Theorem 22.5.2), $S_{\mathrm{lens}} \to S_{\mathrm{red,lens}}$ is an isomorphism.

For the induction step $n > 1$, the induction hypothesis (and arc-descent) implies that $(S/g_1)_{\mathrm{lens}}$ is concentrated in degrees ≤ 0; by Lemma 24.3.7, it is enough to check that S_{lens} is g_1-almost concentrated in degrees ≤ 0. By Remark 25.3.2, $R[g_1^{-1}] \to S_{\mathrm{red}}[g_1^{-1}]$ is a finite étale covering.

Let S' be the integral closure of R in $S_{\mathrm{red}}[g_1^{-1}]$. By Lemma 25.2.2, the map $S_{\mathrm{lens}} \to S'_{\mathrm{lens}}$ is a g_1-almost isomorphism, so it will be enough to check that S'_{lens} is g_1-almost concentrated in degrees ≤ 0. But this may be deduced from Theorem 25.2.6 by approximating S' with module-finite, finitely presented R-algebras. (Compare [25], Theorem 10.11.) ■

Theorem 25.3.4 *Let J be a finitely generated ideal of a lens R. Let S be a finitely presented and module-finite R-algebra such that $\operatorname{Spec} S \to \operatorname{Spec} R$ is finite étale away from $V(J)$.*

1. *The lens coperfection S_{lens} is concentrated in degree 0, where it is a lens.*
2. *The map $S \to S_{\mathrm{lens}}$ is an isomorphism away from $V(J)$.*
3. *For every $n > 0$, the morphism $R/p^n \to S_{\mathrm{lens}}/p^n$ is almost finite étale with respect to the context $(R/p^n, J_{\mathrm{lens},n})$*
4. *Suppose that S admits an action of a finite group G such that $\operatorname{Spec} S \to \operatorname{Spec} R$ is a G-Galois cover outside $V(J)$. Then $R \to S_{\mathrm{lens}}$ is a J-almost G-Galois extension.*

Proof. Combine Theorem 25.2.6 with Theorem 25.3.3. ■

Remark 25.3.5 In the case where R is a p-torsion-free lens and $J = (p)$, Theorem 25.3.4 recovers the almost purity theorem for perfectoid spaces, as in [107], [82]; the conclusion in this case includes the statement that $S[p^{-1}] \cong S_{\mathrm{lens}}[p^{-1}]$. The case where $J \neq (p)$ incorporates the perfectoid Abhyankar lemma of [4].

25.4 An application to cohomological dimension

Lemma 25.4.1 *Let R be a p-torsion-free lens. Let $R \to S$ be the derived p-completion of a finitely presented morphism. Let J be a finitely generated ideal of S. Then $\operatorname{Cone}(S_{\mathrm{lens}} \to (S/J)_{\mathrm{lens}})$ is concentrated in degrees ≤ 0.*

Proof. By Theorem 25.3.3, S_{lens} and $(S/J)_{\text{lens}}$ are both lenses concentrated in degree 0. By Corollary 19.4.6, the map $S_{\text{lens}} \to (S_{\text{lens}}/JS_{\text{lens}})_{\text{lens}} = (S/J)_{\text{lens}}$ is surjective. This proves the claim: the cone is actually concentrated in degrees -1 and 0. ∎

Corollary 25.4.2 *With notation as in Lemma 25.4.1, write R as the slice of the perfect prism (A, I). Then $\operatorname{Cone}(\mathbf{\Delta}_{S/A,\text{perf}} \to \mathbf{\Delta}_{(S/J)/A,\text{perf}})$ is concentrated in degrees ≤ 0.*

Proof. By derived Nakayama (Proposition 6.6.2), this reduces to the corresponding statement after reduction modulo I, which is Lemma 25.4.1. ∎

Corollary 25.4.3 *Let R be a p-torsion-free lens. Let $R \to S$ be a finitely presented morphism of rings. Let J be a finitely generated ideal of S. Put $Y = \operatorname{Spec} S[p^{-1}]$, let $U \subseteq Y$ be the complement of $V(J)$, and let $j \colon U \to Y$ be the canonical open immersion. Then*

$$R\Gamma_{\text{et}}(Y, j_! \underline{\mathbb{F}_p}) \in D^{\leq 1}(\mathbb{F}_p).$$

Proof. Combine Corollary 25.4.2 with the étale comparison theorem (Theorem 22.6.1). ∎

Theorem 25.4.4 *Let R be a p-torsion-free lens and put $X = \operatorname{Spec} R[p^{-1}]$. Then for every etale $\mathbb{F}_p$-sheaf $\mathcal{F}$ on X, we have $H^i(X, \mathcal{F}) = 0$ for all $i > 0$. That is, the $\mathbb{F}_p$-étale cohomological dimension of X is at most 1.*

Proof. This reduces to Corollary 25.4.3 using the "method of the trace" ([117], tag 03SH) as in [25], Theorem 11.1. ∎

Remark 25.4.5 Echoing a remark from [25], we point out that Theorem 25.4.4 fails completely if we replace the scheme X with the Huber adic spectrum of the ring $R[p^{-1}]$; for example, the homotopy type of this space can contribute to cohomology in higher degrees.

Remark 25.4.6 In connection with Theorem 25.4.4, we should mention some results of Achinger ([1]). First, every connected affine scheme over $\mathbb{F}_p$ is a $K(\pi, 1)$ space for the étale topology. Second, every noetherian adic affinoid space over $\mathbb{Q}_p$, and every perfectoid space over $\mathbb{Q}_p$, is a $K(\pi, 1)$ space. Both of these results can be interpreted as saying that the fundamental groups of these space are so large as to "absorb" all higher homotopy groups.

25.5 The direct summand conjecture

The following application of almost purity to Hochster's **direct summand conjecture** is given in [5], [16]. This has various consequences in commutative algebra which we do not discuss here; see instead [66].

Theorem 25.5.1 *Put $R = \mathbb{Z}_p[\![x_1, \dots, x_r]\!]$ and let $R \to S$ be an injective, module-finite ring homomorphism. Then this map splits in $\mathbf{Mod}_R$.*

Proof. It suffices to check that $R/p^n \to S/p^n$ splits in $\mathbf{Mod}_{R/p^n}$ for every n, as then an application of the Artin-Rees lemma shows that $R \to S$ splits (see [16], Lemma 5.3). That is, we must show that the boundary class $\alpha \in \operatorname{Ext}^1_R(S/R, R)$ vanishes modulo p^n for all $n \geq 2$.

Define the lens

$$R_1 = \mathbb{Z}_p[p^{p^{-\infty}}][\![x_1^{p^{-\infty}}, \dots, x_r^{p^{-\infty}}]\!]^\wedge_{(p)}.$$

The apparent map $R \to R_1$ is faithfully flat because R_1 is the p-completion of a free R-module. By Theorem 19.4.4, there exists a p-completely faithfully flat morphism $R_1 \to R_2$ of lenses such that R_2 is AIC. Put $S_i = S \otimes_R R_i$ and let $\alpha_i \in \operatorname{Ext}^1_{R_i}(S_i/R_i, R_i)$ be the image of α; by faithfully flat descent, it is enough to check that α_2 vanishes modulo p^n for all $n \geq 3$.

Choose a nonzero element $f \in R$ such that $R[f^{-1}] \to S[f^{-1}]$ is finite étale, and define the ideal $J = (p, f)R_2$. By Theorem 25.3.4, $R_2/p^n \to S_{2,\mathrm{lens}}/p^n$ is almost finite étale for the context $(R_2/p^n, J_{\mathrm{lens}}/p^n)$. Consequently, α_2/p^n is (J_{lens}/p^n)-almost zero; in particular, it is killed by $(pf)^{p^{-m}} \in R_2$ for all $m \geq 0$. (Note that $f^{p^{-m}}$ makes sense in R_2 because the latter is AIC; this is why we didn't stop at R_1. Also, we are using that S_2 maps to $S_{2,\mathrm{lens}}$ but not any closer relationship between these two objects.)

Now suppose that $\alpha/p^n \neq 0$ for some $n \geq 2$. By Krull's intersection theorem ([117], tag 00IP), $pf \notin (\mathrm{Ann}_{R/p^n}(\alpha/p^n))^{p^m}$ for $m \gg 0$. Since $R \to R_2$ is p-completely faithfully flat, we also have $pf \notin (\mathrm{Ann}_{R_2/p^n}(\alpha/p^n))^{p^m}$; but this contradicts the previous paragraph. This conclusion yields the desired result. (Compare [16], Theorem 5.4.) ■

Remark 25.5.2 A similar argument (see [16], Theorem 6.1) yields the **derived direct summand conjecture**: if $X \to \operatorname{Spec} R$ is a proper surjective morphism, then $R \to R\Gamma(X, \mathcal{O})$ splits in $D(A_0)$.

Remark 25.5.3 Another result that can be deduced from almost purity is a mixed-characteristic analogue of the Kunz criterion of regularity in positive characteristic (Remark 19.1.2): a classically p-complete noetherian ring is regular if and only if it admits a faithfully flat morphism to some lens. See [20].

Remark 25.5.4 Yet another result in this context (but which requires methods beyond the scope of these notes) is the following. Let A be an excellent noetherian integral domain. Let A^+ be an absolute integral closure of A (that is, take the integral closure of A in some algebraic closure of $\operatorname{Frac} A$). Then for every positive integer n, the A/p^n-module A^+/p^n is Cohen-Macaulay ([17], Theorem 1.1).

Remark 25.5.5 See [25], Remark 10.13 for an indication of how to apply Theorem 25.3.4 to recover some additional results in commutative algebra, such as the results of [63].

25.6 Exercises

1. Prove Lemma 25.3.1.

Hint. We can ignore condition (3), as we may enforce it at the end by refining the stratification. To handle (1) and (2), by noetherian approximation we may reduce to the case where R is a finitely generated $\mathbb{Z}_{(p)}$-algebra; in that case, see [25], Lemma 10.12.

26 q-de Rham cohomology

Reference. [18], lecture IX; [25], section 16. See also [7] and [109].

We use the following notation frequently: for n a nonnegative integer,

$$[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + \cdots + q^{n-1}$$
$$[n]_q! = [1]_q \cdots [n]_q.$$

26.1 A brief history of q

Definition 26.1.1 For parameters $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n$, the **hypergeometric series**

$$ {}_mF_n(\alpha_1, \dots, \alpha_m; \beta_1, \dots, \beta_n; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_m)_k}{(\beta_1)_k \cdots (\beta_n)_k} \frac{z^k}{k!} $$

where $(x)_k$ denotes the **Pochhammer symbol**

$$ (x)_k = x(x+1)\cdots(x+k-1). $$

These were first considered in the case $m = 2, n = 1$ by Wallis in 1655, and later by Euler and more systematically by Gauss. The case $m = 3, n = 3$ was considered by Clausen in 1828; the general case was introduced by Thomae [121] in 1870. $\diamondsuit$

Definition 26.1.2 For parameters $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n$, the **basic hypergeometric series** (or q**-hypergeometric series)**

$$ {}_m\phi_n(\alpha_1, \dots, \alpha_m; \beta_1, \dots, \beta_n; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1; q)_k \cdots (\alpha_m; q)_k}{(\beta_1; q)_k \cdots (\beta_n; q)_k (q; q)_k} \left((-1)^k q^{k(k-1)/2}\right)^{1+n-m} z^k $$

where $(x; q)_k$ denotes the q**-Pochhammer symbol**

$$ (x; q)_k = \prod_{i=0}^{k-1} (1 - xq^k). $$

Note that the term $(q; q)_k$ corresponds (up to signs and factors of q) to the factor $k!$ in the ordinary hypergeometric series. For the Gaussian case (i.e., $m = 2, n = 1$) this was first introduced by Heine [64] in 1846. $\diamondsuit$

Remark 26.1.3 The process of q-deformation has a long rich history, of which we give a scandalously brief summary here.

1. Products of q-Pochhammer symbols appear naturally in the study of generating functions connected to partitions, which first appear in the work of Euler, were systematically studied by Jacobi, occur prominently in the notebooks of Ramanujan, and have a continuing history far beyond the scope of this remark. See [48] for a comprehensive development (circa 1988).

2. A **quantum group** is a certain noncommutative algebra which can be viewed as a q-deformation of the universal enveloping algebra of a Lie algebra (or an affine Lie algebra). These were introduced by Drinfeld and Jimbo in the 1980s with a view towards quantum statistical mechanics.

3. One way to make sense of (some) statements about the putative **field with one element** is to consider statements about the finite field of q elements and then specialize at $q = 1$. As an elementary example, taking the formula

$$ \#\,\mathrm{GL}_n(\mathbb{F}_q) = (q^n - 1)(q^n - q)\cdots(q^n - q^{n-1}) = (-1)^n q^{n(n-1)/2}(1; q)_n $$

and setting $q = 1$ gives $n!$, the order of the group S_n, which by chance happens to be the Weyl group of GL_n. One is thus led to treat Weyl groups as "algebraic groups over the field with one element".

4. Going in the opposite direction, it is quite common in combinatorics to look for statements about finite sets that can be promoted to statements about finite-dimensional vector spaces. One recent development in this direction is the q-analogue of a **matroid** ([76]).

Remark 26.1.4 Note that Heine uses the letter q as we do nowadays for the deformation parameter. However, this was presumably following the model of Jacobi, who used the letter q in the notation for the **Jacobi theta function**. This comes from the usage where q stands for the value $e^{2\pi i\tau}$ where τ is a value in the upper half-plane, as in the theory of elliptic functions.

The point of all this is that q does not stand for "quantum", as it was entrenched in the notation a full 50 years before the first inklings of quantum mechanics!

26.2 Jackson's q-calculus

We focus here on a specific instance of q-deformation introduced by Jackson [71], [72] in 1908.

Definition 26.2.1 Given a function $f(x)$ and a parameter q, the **q-derivative** of $f(x)$ is defined as

$$D_q f(x) = \frac{f(qx) - f(x)}{qx - x}.$$

It is obviously additive and satisfies a modified product rule

$$D_q(f(x)g(x)) = f(x)D_q(g(x)) + g(qx)D_q(f(x)). \tag{26.1}$$

◊

Definition 26.2.2 Given a function $f(x)$ and a parameter q, the **q-integral** (or **Jackson integral**) of $f(x)$ is defined as

$$\int f(x) d_q x = (1-q)x \sum_{k=0}^{\infty} q^k f(q^k x)$$

provided that we are working in some context where the infinite sum makes sense. For example, if this is taking place in a q-adically complete ring and f is itself q-adically continuous, then one may check that performing this operation and then taking the q-derivative recovers $f(x)$. ◊

Remark 26.2.3 Without the q-deformation, hypergeometric series have long been viewed as solutions of the **hypergeometric differential equation**

$$P(\underline{\alpha}; \underline{\beta})(y) = 0, \qquad P(\underline{\alpha}; \underline{\beta}) = z \prod_i (z\frac{d}{dz} + \alpha_i) - \prod_j (z\frac{d}{dz} + \beta_j - 1)$$

starting with the work of Gauss in the case $m = 2, n = 1$. In modern language, hypergeometric differential equations (particularly with $n = m - 1$) provide important examples of **Picard-Fuchs equations** which describe the variation of algebraic periods on some family of algebraic varieties; in particular, there is a natural construction via which hypergeometric equations emerge from de Rham cohomology.

The q-analogue of de Rham cohomology was first considered by Aomoto [7], [8] in 1990 in order to provide a similar geometric description of the Jackson integrals that appear when one tries to transport the previous construction to the setting of q-hypergeometric series.

26.3 The q-de Rham complex of Aomoto

Definition 26.3.1 For $R \in \mathbf{Ring}$, define the **framed q-de Rham complex** associated to $R[x]$ as

$$q\Omega^\bullet_{R[x]/R,\square} = \left(R[x][\![q-1]\!] \xrightarrow{\nabla_q} R[x][\![q-1]\!]\, dx \right)$$

where ∇_q denotes the q-differential

$$\nabla_q(f(x)) = D_q(f(x))\, dx = \frac{f(qx) - f(x)}{qx - x}\, dx.$$

We refer to this as a "framed" construction because it depends implicitly on the choice of the polynomial generator x; see Remark 26.3.4.

We similarly define

$$q\Omega^\bullet_{R[x_1,\ldots,x_r],\square} = q\Omega^\bullet_{R[x_1]/R} \otimes_{R[\![q-1]\!]} \cdots \otimes_{R[\![q-1]\!]} q\Omega^\bullet_{R[x_r]/R}.$$

There is an evident isomorphism

$$q\Omega^\bullet_{R[x_1,\ldots,x_r],\square}/(q-1) \cong \Omega^\bullet_{R[x_1,\ldots,x_r]/R}.$$

$\lozenge$

Remark 26.3.2 For $R \in \mathbf{Ring}_{\mathbb{Q}}$, we can identify $q\Omega^\bullet_{R[x_1,\ldots,x_r]/R,\square}$ with the usual de Rham complex by a Taylor series construction. In particular, in this case the construction is independent of the choice of coordinates. (Compare [109], Lemma 4.1.)

Definition 26.3.3 To promote $q\Omega^\bullet_{R[x_1,\ldots,x_r]/R,\square}$ to a $\mathbb{Z}[\![q-1]\!]$-dga, we must account for the asymmetry in the product rule (26.1). To this end, we equip $q\Omega^1_{R[x_1,\ldots,x_r]/R,\square}$ with a $q\Omega^0_{R[x_1,\ldots,x_r]/R,\square}$-bimodule structure using the standard action on the left and the action on the right via $f(x) \mapsto f(qx)$.

With this, we may view $q\Omega^\bullet_{R[x_1,\ldots,x_r]/R,\square}$ as a $\mathbb{Z}[\![q-1]\!]$-dga; however, the asymmetry we just introduced means that this dga is not commutative. $\lozenge$

Remark 26.3.4 We wish to emphasize that by contrast with the ordinary de Rham complex, the definition of the q-de Rham complex exhibits a genuine dependence on the choice of coordinates (so in particular it is not functorial enough to admit a left Kan extension). For example, there is no way to promote the automorphism $x \mapsto x+1$ on $R[x]$ to an $R[\![q-1]\!]$-linear morphism of complexes as illustrated in Figure 26.3.5.

$$\begin{tikzcd}
R[\![q-1]\!][x] \arrow[r, "\nabla_q"] \arrow[d, "x \mapsto x+1"] & R[\![q-1]\!][x]\, dx \arrow[d, dashed, "??"] \\
R[\![q-1]\!][x] \arrow[r, "\nabla_q"] & R[\![q-1]\!][x]\, dx
\end{tikzcd}$$

Figure 26.3.5

Namely, such a map would have to send x^{n-1} to

$$\sum_{i=1}^{n} \binom{n}{i} \frac{[i]_q}{[n]_q} x^{i-1} = \sum_{i=1}^{n} \binom{n-1}{i-1} \frac{[i]_q}{i} \frac{n}{[n]_q} x^{i-1},$$

but the coefficients in the latter expression are not necessarily contained in $R[\![q-1]\!]$ unless R is a $\mathbb{Q}$-algebra.

However, we can instead hope to prove that $q\Omega^\bullet_{R[x_1,\dots,x_r]/R,\square}$ is indeed functorial in $R[x_1,\dots,x_r]$ as an object (and even a commutative algebra object) in $D(R[\![q-1]\!])$; this was conjectured by Scholze in [109]. To make this more explicit in the previous example, consider the isomorphisms

$$R[x,y]/(x+y-1) \to R[x], \qquad R[x,y]/(x+y-1) \to R[y];$$

the claim then is that the map

$$q\Omega^\bullet_{R[x,y]/R,\square} \to q\Omega^\bullet_{R[y]/R,\square}$$

becomes a quasi-isomorphism once we take the universal quotient that kills $x-y+1$ in degree 0 (and similarly with the variables reversed).

Using Remark 26.3.2, one can reduce the claim to the corresponding statement after derived p-completion. In this case, one can provide a coordinate-free interpretation of q-de Rham cohomology using prismatic cohomology; see Theorem 27.3.8.

27 q-crystalline cohomology

Reference. [18], lecture X; [25], section 16. Some of this material was developed independently in the PhD thesis of Masullo [95]. However, we diverge significantly in form from these references; see below.

In this section, we introduce a q-analogue of crystalline cohomology, derive a comparison with prismatic cohomology, and use this to establish a statement about the functoriality of q-de Rham cohomology after p-completion. We follow closely our analysis of the Hodge-Tate comparison for crystalline prisms (Section 14).

We will only present the affine part of the story, but one can globalize to obtain the "Wach module cohomology" of a smooth proper $\mathbb{Z}_p$-scheme. This is a primary motivation for seeking a global analogue (Section 29).

To simplify the presentation, we only consider q-crystalline cohomology relative to $\mathbb{Z}_p$. A more general relative setup is described in [25].

27.1 q-divided powers

In order to discuss q-crystalline cohomology, we first need to define a q-analogue of divided powers. It is not at all clear how to do this in general, but fortunately for our purposes it is sufficient to do this for δ-rings. In that case, we can use the fact that divided powers can be accounted for using Frobenius (Remark 14.3.1) to come up with a suitable analogue.

Definition 27.1.1 Throughout this section, view $A = \mathbb{Z}_p[\![q-1]\!]$ as a δ-ring in which q is constant, and identify $A/([p]_q)$ with $\mathbb{Z}_p[\zeta_p]$ via the map taking q to ζ_p. We will use frequently the fact that the ideals $(p, q-1)$ and $(p, [p]_q)$ of A, although distinct, do define the same topology on A; keep in mind that $(A, ([p]_q))$ is a prism but $(A, (q-1))$ is not.

We will also use on several occasions the congruence

$$\phi([p]_q) = q^{p(p-1)} + \cdots + q^p + 1 \equiv p \pmod{[p]_q}. \tag{27.1}$$

$\diamondsuit$

Remark 27.1.2 To see the difficulty at work here, imagine trying to define q-divided power operations $\gamma_{n,q}$ using the formula

$$\gamma_{n,q}(x) = \frac{x^n}{[n]_q!}.$$

We would then have the rather awkward formula

$$\gamma_{n,q}(x+y) = \sum_{i=0}^{n} \frac{n![i]_q![n-i]_q!}{[n]_q!i!(n-i)!} \gamma_{i,q}(x)\gamma_{n-i,q}(y),$$

from which it is no longer apparent that being able to take q-divided powers of x and y implies the same for $x+y$.

Lemma 27.1.3 *The map*

$$A \to A\{x_1, \ldots, x_r, \phi(x_1)/[p]_q, \ldots, \phi(x_r)/[p]_q\}^{\wedge}_{(p,[p]_q)}$$

is $(p, [p]_q)$-completely flat.

Proof. For ease of notation we treat only the case $r = 1$, identifying x_1 with x. Consider the diagram as in Figure 27.1.4, in which the first row is given and the squares below are pushouts.

$$\begin{tikzcd}
A = \mathbb{Z}_p[[q-1]] \arrow[r] \arrow[d] & B = A\{x\}^{\wedge}_{(p,[p]_q)} \arrow[r] \arrow[d] & C = B\{\phi(x)/[p]_q\}^{\wedge}_{(p,[p]_q)} \arrow[d] \\
A' = \mathbb{Z}_p[[q^{1/p}-1]] \arrow[r] & B' = A'\{x\}^{\wedge}_{(p,[p]_q)} \arrow[r] & C' = A'\{x, \phi(x/[p]_q)\}^{\wedge}_{(p,[p]_q)}
\end{tikzcd}$$

Figure 27.1.4

By inspection, the arrow $A \to A'$ is faithfully flat; we are thus reduced to checking that $A' \to C'$ is $(p, [p]_q)$-completely flat. This can be checked by inspection: it is clear that $A' \to A'\{\phi(x/[p]_q)\} \cong A'\{y\}$ is faithfully flat, and the quotient of C' by the completion of A'-submodule is itself the completion of a free module on the basis consisting of products of the form $x^{e_0}\delta(x)^{e_1}\delta^2(x)^{e_2}\cdots$, where $e_0, e_1, \ldots \in \{0, \ldots, p-1\}$ are almost all zero. (Compare [25], Proposition 3.13, which gives a more general result.) ■

Definition 27.1.5 Recall that in the ordinary divided power setting, a δ-ring flat over $\mathbb{Z}_{(p)}$ admits divided powers on an ideal if and only if $\gamma_p(x) = x^p/p!$ sends the ideal into the ring (Remark 14.3.1).

With this in mind, for D a $[p]_q$-torsion-free δ-ring over A, for any $x \in D$ with $\phi(x) \in [p]_q D$, write

$$\gamma(x) = \frac{\phi(x)}{[p]_q} - \delta(x) \in D.$$

◊

Remark 27.1.6 With notation as in Definition 27.1.5, for $x, y \in \phi^{-1}([p]_q D)$ we have

$$\gamma(x+y) = \gamma(x) + \gamma(y) - \sum_{i=1}^{p-1} \frac{(p-1)!}{i!(p-i)!} x^i y^{p-i};$$

for $x \in \phi^{-1}([p]_q D)$ and $y \in D$, we have

$$\gamma(xy) = \phi(y)\gamma(x) - x^p\delta(y).$$

Consequently, for any ideal I of D, the set

$$J = \{x \in I \colon \phi(x) \in [p]_q D, \gamma(x) \in I\}$$

is itself an ideal of D; hence to check that $J = I$, it suffices to check that J contains a generating set of I. (Compare [25], Remark 16.6.)

We have the following analogue of Exercise 2.5.10.

Lemma 27.1.7 *Let D be a $[p]_q$-torsion-free δ-ring over A. Then the ideal $\phi^{-1}([p]_q D)$ is stable under γ.*

Proof. We need to show that if $x \in D$ and $\phi(x) \in [p]_q D$, then $\phi(\gamma(x)) \in [p]_q D$. It will suffice to check this in the universal case $D = A\{x, \phi(x)/[p]_q\}^{\wedge}_{(p,[p]_q)}$.

Since $D/[p]_q$ is p-torsion-free (by Lemma 27.1.3), to show that $\phi(\gamma(x)) \in [p]_q D$, it will suffice to show that $p\phi(\gamma(x)) \in [p]_q D$. Moreover, by (27.1) we may replace p with $\phi([p]_q)$ on the left-hand side. At this point we may proceed by direct computation:

$$\begin{aligned}
\phi([p]_q)\phi(\gamma(x)) &= \phi^2(x) - \phi([p]_q \delta(x)) \\
&= \phi(x)^p + p\phi(\delta(x)) - \phi([p]_q)\phi(\delta(x)) \\
&= [p]_q^p(\phi(x)/[p]_q)^p + (p - \phi([p]_q))\phi(\delta(x)) \\
&\equiv 0 \pmod{[p]_q D}
\end{aligned}$$

where we use (27.1) again in the last line. (Compare [25], Lemma 16.7.) ■

27.2 q-divided power pairs and envelopes

We now define the q-analogue of divided power envelopes.

Definition 27.2.1 A **q-pd pair** is a pair (D, I) in which D is a δ-ring over A and I is an ideal of D satisfying the following conditions.

1. The rings D and D/I are derived $(p, [p]_q)$-complete.
2. The ideal I contains $q - 1$ and satisfies $\phi(I) \subseteq [p]_q D$ (so that γ is defined on I) and $\gamma(I) \subseteq I$.
3. The ring D is $[p]_q$-torsion-free and the quotient $D/[p]_q$ has bounded p-power torsion. Consequently, $(D, [p]_q))$ is a bounded prism over $(A, [p]_q)$.
4. The ring $D/(q - 1)$ is p-torsion-free with finite $(p, [p]_q)$-complete Tor amplitude over D.

These form a category in which a morphism $(D, I) \to (D', I')$ is a morphism $D \to D'$ of δ-rings carrying I into I'. ♢

Example 27.2.2 The pair $(A, (q - 1))$ is the initial object in the category of q-pd pairs. More generally, if D is a δ-ring over A which is $(p, [p]_q)$-completely flat over A, then $(D, (q - 1))$ is a q-pd pair. □

Example 27.2.3 Let B be a perfect δ-ring over A which is derived $(p, [p])_q)$-complete. Since $[p]_q$ is distinguished in B, it is a non-zerodivisor (Theorem 7.2.2). Then $(B, (\phi^{-1}([p]_q))$ is a q-pd pair. □

Example 27.2.4 Let D be a p-torsion-free, p-complete δ-ring over A in which $q = 1$. Let I be an arbitrary ideal of D. Then by Remark 14.3.1, (D, I) is a δ-pd pair if and only if D admits divided powers on I in the (strong) classical sense, that is, the divided power operations carry I into $I \subset D[p^{-1}]$. □

Proposition 27.2.5 *Let P be a δ-ring over A equipped with a surjection $P \to R = \mathbb{Z}_p[x_1, \dots, x_r]^\wedge_{(p)}$ with kernel $J = (q-1, y_1, y_2, \dots)$ where each initial segment $y_1, \dots, y_s$ is a regular sequence on $P/(p, q-1)$. Define the δ-ring*

$$D = P\{\phi(y_1)/[p]_q, \dots, \phi(y_s)/[p]_q\}^\wedge_{(p,[p]_q)}.$$

Let I be the kernel of $D \to D/(q-1) \to R$.

1. *The ring D is $(p, [p]_q)$-completely flat over A.*
2. *The map $P \to D$ induces an isomorphism $P/J \cong D/I$.*
3. *The ring $D/(q-1)$ identifies with the p-completed divided power envelope of the map $P/(q-1) \to R$.*
4. *The map $(P, J) \to (D, I)$ of δ-pairs is universal for the target being a q-pd pair.*

Proof. Point (1) is contained in Lemma 27.1.3. Point (2) follows from Remark 27.1.6 (applied to the indicated generators of J). Point (3) comes from Corollary 14.3.4. Point (4) is straightforward. For the rest, see [18], Lecture XI, Proposition 1.6. ■

27.3 Comparison with prismatic cohomology

We now reprise the comparison of prismatic and Hodge-Tate cohomology in the crystalline case (Section 14).

Lemma 27.3.1 *For $P = R[x_1, \dots, x_r]$, let P^i be the $(i+1)$-fold tensor product of P over R, viewed as a polynomial ring whose generators are the various images of $x_1, \dots, x_r$. For every $i > 0$, the complex*

$$q\Omega^i_{P^0/R,\square} \to q\Omega^i_{P^1/R,\square} \to q\Omega^i_{P^2/R,\square} \to \cdots$$

vanishes in the homotopy category $K(R)$, and similarly after p-adic completion. (More precisely, this is witnessed by a homotopy at the level of $P^\bullet$-cosimplicial modules.)

Proof. The proof of Lemma 14.4.1 carries over. ■

Definition 27.3.2 Put $R = \mathbb{Z}_p[x_1, \dots, x_r]^\wedge_{(p)}$. Put $P = A\{x_1, \dots, x_r\}^\wedge_{(p,[p]_q)}$, and form the map $P \to R$ taking q to 1 and taking $\delta^m(x_i)$ to 0 for $m \geq 0$. Let $P^\bullet$ be the $(p, [p]_q)$-completed Cech nerve of $A \to P$; let J^n be the kernel of $P^n \to P \to R$ where the first map is multiplication. We view J^n as being generated by p, the differences between copies of x_i in different factors, and the images of copies of x_i under δ^m for all $m > 0$.

Let $D_{J^n,q}(P^n)$ be the q-divided power envelope of the δ-pair (P^n, J^n) as provided by Proposition 27.2.5. We refer to the Čech-Alexander complex $D_{J^\bullet,q}(P^\bullet)$ as the **q-crystalline cohomology** of R.

By viewing P^n as the completion of a polynomial ring with generators being the various images of $\delta^m(x_1), \dots, \delta^m(x_r)$, we may define the framed completed q-de Rham complex $q\widehat{\Omega}^\bullet_{P^n/R,\square}$. ♢

Remark 27.3.3 To motivate the terminology in Definition 27.3.2, we give the definition of the **q-crystalline site** of R (relative to A) following [25], Definition 16.12. We take the opposite category to the category of q-pd pairs (D, I) over $(A, [p]_q)$ equipped with isomorphisms $D/I \cong R$, in which the morphisms are morphisms of q-pd pairs which respect the isomorphisms with R. By Proposition 27.2.5, $D_{J^n,q}(P^n)$ is a weakly final object in this category. We use the indiscrete Grothendieck topology; then by Lemma 11.1.7 the cohomology of the sheaf $(D, I) \mapsto D$ is computed by $D_{J^\bullet,q}(P^\bullet)$.

Lemma 27.3.4 ***q*-Poincaré lemma.** *With notation as in Definition 27.3.2, for any n, each of the maps*

$$D_{J^n,q}(P^n)\widehat{\otimes}_{P^n} q\widehat{\Omega}^\bullet_{P^n/\mathbb{Z}_p,\square} \to D_{J^{n+1},q}(P^{n+1})\widehat{\otimes}_{P^{n+1}} q\widehat{\Omega}^\bullet_{P^{n+1}/\mathbb{Z}_p,\square}$$

is a quasi-isomorphism. Moreover, the natural map

$$q\widehat{\Omega}^\bullet_{R/\mathbb{Z}_p,\square} \to D_{J^0,q}(P^0)\widehat{\otimes}_{P^0} q\widehat{\Omega}^\bullet_{P^0/\mathbb{Z}_p,\square}$$

is a quasi-isomorphism.

Proof. By derived Nakayama (Remark 6.6.6), we may check this modulo $q - 1$. By Proposition 27.2.5, the claim in this case becomes the usual Poincaré lemma (Proposition 14.2.6). ∎

Lemma 27.3.5 *With notation as in Definition 27.3.2, the totalization of the double complex displayed in Figure 14.4.9 is quasi-isomorphic to both its first row and its first column via the inclusion maps.*

$$\begin{tikzcd}
D_{J^0,q}(P^0) \arrow[r] \arrow[d] & D_{J^1,q}(P^1) \arrow[r] \arrow[d] & D_{J^2,q}(P^2) \arrow[r] \arrow[d] & \cdots \\
D_{J^0,q}(P^0)\widehat{\otimes}_{P^0} q\widehat{\Omega}^1_{P^0/\mathbb{Z}_p,\square} \arrow[r] \arrow[d] & D_{J^1,q}(P^1)\widehat{\otimes}_{P^1} q\widehat{\Omega}^1_{P^1/\mathbb{Z}_p,\square} \arrow[r] \arrow[d] & D_{J^2,q}(P^2)\widehat{\otimes}_{P^2} q\widehat{\Omega}^1_{P^2/\mathbb{Z}_p,\square} \arrow[r] \arrow[d] & \cdots \\
D_{J^0,q}(P^0)\widehat{\otimes}_{P^0} q\widehat{\Omega}^2_{P^0/\mathbb{Z}_p,\square} \arrow[r] \arrow[d] & D_{J^1,q}(P^1)\widehat{\otimes}_{P^1} q\widehat{\Omega}^2_{P^1/\mathbb{Z}_p,\square} \arrow[r] \arrow[d] & D_{J^2,q}(P^2)\widehat{\otimes}_{P^2} q\widehat{\Omega}^2_{P^2/\mathbb{Z}_p,\square} \arrow[r] \arrow[d] & \cdots \\
\vdots & \vdots & \vdots &
\end{tikzcd}$$

Figure 27.3.6

Proof. We make the same argument as in Lemma 14.4.8: each row is homotopic to zero by Lemma 27.3.1, and all of the simplicial maps induce quasi-isomorphisms of columns by Lemma 27.3.4, so Corollary 13.3.8 yields the desired quasi-isomorphism. (Compare [18], Theorem 2.9.) ∎

To link up with prismatic cohomology, we need a q-analogue of the Cartier isomorphism.

Definition 27.3.7 For R an $\mathbb{Z}_p$-algebra, define $R^{(1)} = R \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\zeta_p]$; this will play the role of the "Frobenius twist" in this setting.

Let (D, I) be a q-pd pair with $D/I \cong R$. By assumption, $\phi(I) \subseteq [p]_q D$ and so we get an induced map $R \cong D/I \to D/[p]_q$ which is linear over the Frobenius on A. Linearizing yields an A-algebra map $A \otimes_{\phi,A} R \to D/[p]_q$, which then factors through $(A \otimes_{\phi,A} R)/[p]_q \cong R^{(1)}$.

The upshot of this is that for $R = \mathbb{Z}_p[x_1, \dots, x_r]^\wedge_{(p)}$, there is a morphism from the q-crystalline site of R to the prismatic site of $R^{(1)}$ over A, and hence a morphism of cohomology in the other direction. ◊

Theorem 27.3.8 *Put $R = \mathbb{Z}_p[x_1, \dots, x_r]^\wedge_{(p)}$ and $R^{(1)} = R \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\zeta_p]$. We then have an isomorphism*

$$\mathbf{\Delta}_{R^{(1)}/A} \cong q\Omega^\bullet_{R/\mathbb{Z}_p,\square}$$

of commutative algebra objects in $D_{\mathrm{comp}}(A)$.

Proof. Using Definition 27.3.7, we obtain a morphism from $\mathbf{\Delta}_{R^{(1)}/A}$ to the q-crystalline cohomology (the top row of Figure 27.3.6. To check that it is an isomorphism, we may invoke derived Nakayama (Proposition 6.6.2) to reduce modulo $q-1$, at which point we get back to the corresponding statement in the case of a crystalline prism (Remark 14.4.6).

Using Lemma 27.3.5, we obtain a quasi-isomorphism between the top row of Figure 27.3.6 and the left column of the same diagram. Using Lemma 27.3.4, we get a quasi-isomorphism of the left column with $q\Omega^\bullet_{R/\mathbb{Z}_p,\square}$. (Compare [18], Lecture XI, Theorem 2.5 or [25], Theorem 16.22.) ■

Corollary 27.3.9 **q-Hodge-Tate comparison.** *Put $R = \mathbb{Z}_p[x_1,\dots,x_r]^\wedge_{(p)}$. Then there is a natural identification*

$$H^\bullet(q\Omega_{R/\mathbb{Z}_p} \otimes^L_A A/([p]_q)) \cong \Omega^\bullet_{R/\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\zeta_p]$$

of graded algebras over $A/([p]_q) \cong \mathbb{Z}_p[\zeta_p]$.

Proof. This can be read off from the proof of Theorem 27.3.8, or by combining that result with Proposition 14.4.12. (Compare [18], Lecture XI, Corollary 2.6.) ■

Definition 27.3.10 Theorem 27.3.8 gives us a way to regard $q\Omega^\bullet_{R/\mathbb{Z}_p,\square}$ as a commutative ring object in $D_{\mathrm{comp}}(A)$ functorially associated to $\mathbb{Z}_p[x_1,\dots,x_r]$. We can thus use left Kan extension (Proposition 16.4.6) to extend the definition of $q\Omega^\bullet_{R/\mathbb{Z}_p,\square}$ to any derived p-complete $\mathbb{Z}_p$-algebra R. ◊

27.4 Frobenius is an isogeny

Remark 27.4.1 Put $R = \mathbb{Z}_p[x]^\wedge_{(p)}$ and view $R[\![q-1]\!]$ as a δ-ring in which q and x are constant. Then $q\Omega^\bullet_{R/\mathbb{Z}_p,\square}$ carries an action of ϕ given in degree 0 by $f \mapsto \phi(f)$ and in degree 1 by $g\,dx \mapsto \phi(g)x^{p-1}[p]_q\,dx$. A similar statement applies to $R = \mathbb{Z}_p[x_1,\dots,x_r]^\wedge_{(p)}$.

It follows that for any derived p-complete $\mathbb{Z}_p$-algebra R, the linearized Frobenius $\phi_R \colon \phi_A^*\mathbf{\Delta}_{R^{(1)}/A} \to \mathbf{\Delta}_{R^{(1)}/A}$ is an isogeny, in that it becomes an isomorphism after inverting $[p]_q$. More precisely, because the action on cohomology in degree i factors through multiplication by $[p]_q^i$, one can apply the Berthelot-Ogus functor $\eta_{[p]_q}$ ([117], tag 0F7N); this is related to the discussion of the **Nygaard filtration** in [25].

27.5 Étale localization

Remark 27.5.1 One can establish a form of étale localization (Lemma 15.1.2) in order to extend the preceding discussion to the case where R is a p-completely smooth $\mathbb{Z}_p$-algebra. In particular, for such rings the left Kan extension of Definition 27.3.10 can be computed by "naive" q-de Rham complexes using local coordinate choices.

28 Some further developments: a whirlwind tour

Reference. See the various sections below.

In this section, we survey some further developments. This is primarily meant to serve as a point of departure for further reading; as we have come nearly to the end of the course, we will not be able to provide much detail on any individual aspect.

28.1 Topological Hochschild homology

Definition 28.1.1 Let $A \to B$ be a morphism in **Ring**. The **Hochschild homology** of B over A is the complex of A-modules associated to the simplicial object $K_\bullet$ of $\mathbf{Ring}_A$ in which K_n is the $(n+1)$-fold tensor product of B over A and the various maps $K_n \to K_{n-1}$ act by taking the product of some pair of consecutive factors:

$$b_0 \otimes \cdots \otimes b_i \otimes b_{i+1} \otimes \cdots \otimes b_n \mapsto b_0 \otimes \cdots \otimes b_i b_{i+1} \otimes \cdots \otimes b_n.$$

$\Diamond$

Remark 28.1.2 It has been anticipated for some time that there should be deep links between structures arising in p-adic Hodge theory and parallel structures arising in algebraic topology, particularly with regard to **topological Hochschild homology** (THH), the analogue of Hochschild homology with rings replaced by ring spectra; working with THH means that one takes tensor products over the sphere spectrum, which lies "below" the ordinary ring of integers and thus provides a base more closely resembling the "field of one element". Much of the early work in this direction is due to Hesselholt; see for example [65].

A systematic link between THH and p-adic Hodge theory was developed more systematically in [23], in which the A_{inf}-cohomology of [22] is reconstructed using THH. This link is revisited in [25] using prismatic techniques.

28.2 The absolute prismatic site

This material comes from announcements by Bhatt and Scholze. There is not yet a primary reference; in the interim, the recorded lecture [111] of Scholze will have to suffice.

Definition 28.2.1 For R a derived p-complete ring, the **absolute prismatic oppo-site** of R, denoted $(\operatorname{Spec} R)^{\mathrm{op}}_{\mathbf{\Delta}}$, is the category in which an object is a prism (B, J) equipped with a ring homomorphism $R \to B/J$, and a morphism is a morphism of underlying prisms $(B, J) \to (B', J')$ for which the induced morphism $B/J \to B'/J'$ is R-linear. Taking the opposite category yields the **absolute prismatic site** of R, denoted $(\operatorname{Spec} R)_{\mathbf{\Delta}}$, which we equip with the indiscrete topology. Note that there is *no base prism* in the definition. $\Diamond$

Definition 28.2.2 Let $\mathcal{C}$ be a site equipped with a sheaf of rings $\mathcal{O}$ (or more generally a **ringed topos**). A **crystal** on $(\mathcal{C}, \mathcal{O})$ is a sheaf of $\mathcal{O}$-modules locally obtained by tensoring $\mathcal{O}$ with a finite projective module over the ring of global sections.

We will typically apply this definition in a situation where descent of finite projective modules is effective. In this case, a crystal can be specified by assigning to each $X \in \mathcal{C}$ to a finite projective $\mathcal{O}(X)$-module $M(X)$ and to each morphism $Y \to X$ in $\mathcal{C}$ an isomorphism $M(X) \otimes_{\mathcal{O}(X)} \mathcal{O}(Y) \cong M(Y)$ in a manner compatible with composition. $\Diamond$

Definition 28.2.3 For R a derived p-complete ring, a **prismatic F-crystal** on R is a crystal M on the absolute prismatic site of R equipped with an isomorphism

$$F \colon \phi^* M[I^{-1}] \to M[I^{-1}].$$

That is, for each object $(B, J) \in (\operatorname{Spec} R)_{\mathbf{\Delta}}$, we specify an isomorphism $\phi^* M(B)[J^{-1}] \to M(B)[J^{-1}]$ compatible with the morphisms in the site (where $M(B)[J^{-1}] = \operatorname{colim}_n M(B) \otimes_B J^{-n}$; this makes sense because J is an invertible ideal). $\Diamond$

Theorem 28.2.4 *Let K be a completely valued field of mixed characteristics $(0,p)$ with perfect residue field k. Then the category of prismatic F-crystals on $\mathcal{O}_K$ is equivalent to the category of* ***crystalline lattices****, i.e., finite free $\mathbb{Z}_p$-modules equipped with continuous G_K-action whose base extensions from $\mathbb{Z}_p$ to $\mathbb{Q}_p$ are crystalline Galois representations (see Remark 28.2.5).*

Proof. See [26]. The key ingredients are the étale comparison theorem (Theorem 22.6.1), Kisin's description of crystalline Galois representations via Breuil-Kisin modules ([85]), and Beauville-Laszlo glueing (Remark 21.2.7). ■

Remark 28.2.5 It would take us well beyond the scope of these notes to explain enough of Fontaine's theory of p-adic representations and p-adic periods to define the notion of a **crystalline** Galois representation. The motivating example is the étale cohomology $H^i_{\mathrm{et}}(X_{\overline{K}}, \mathbb{Q}_p)$ where X is a smooth proper $\mathcal{O}_K$-scheme. For $i > 0$ such an extension cannot be unramified, as it would be if $\mathbb{Q}_p$ were replaced by $\mathbb{Q}_\ell$ for some prime $\ell \neq p$, because the kernel field of the Galois representation contains the p-cyclotomic tower (the determinant of cohomology is a nonzero power of the cyclotomic character); the crystalline condition is a replacement. For approachable treatments of this subject, see [33] or [51].

28.3 Prismatization

The primary reference for this topic is to be a preprint of Bhatt and Lurie which is not yet available; however, in the meantime Drinfeld has produced an independent writeup [42].

Definition 28.3.1 Let W be the ring scheme of p-typical Witt vectors. That is, for any ring A, there is a natural (in A) bijection between the underlying set of the ring $W(A)$ and the set of morphisms $\operatorname{Spec} A \to W$.

Write W as $\operatorname{Spec} \mathbb{Z}[x_0, x_1, \dots]$ in terms of the Witt coordinates, and let $W_n = \operatorname{Spec} \mathbb{Z}[x_0, \dots, x_{n-1}]$ be the n-th truncation of W. For $n \geq 1$, let $W_{\mathrm{prim},n}$ be the completion of W_n along the locally closed subscheme defined by the conditions

$$p = x_0 = 0, \qquad x_1 \neq 0.$$

Since $W_n^\times$ acts on $W_{\mathrm{prim},n}$ by multiplication, we can form the quotient

$$\mathrm{CW}_n = W_{\mathrm{prim},n}/W_n^\times$$

in the category of sheaves on the category of p-adic formal schemes. Similarly, we may form the sheaf $\mathrm{CW} = \lim_n \mathrm{CW}_n$, called the **Cartier-Witt stack**.

By definition, for any oriented prism $(A,(d))$ we get a morphism $\operatorname{Spec} A \to W_{\mathrm{prim}}$ under which the distinguished element $(x_0, x_1, \dots)$ of $\mathcal{O}(W_{\mathrm{prim}})$ pulls back to d. Consequently, for any prism (A, I), we get a morphism $\operatorname{Spec} A \to \mathrm{CW}$. ♢

Remark 28.3.2 Some caution is in order because the objects CW and CW_n are not algebraic stacks but rather **formal stacks**. We will not elaborate on what this means; see [42].

28.4 Prismatic Dieudonné theory

The reference for this topic is [6].

Definition 28.4.1 We say that $R \in \mathbf{Ring}$ is **quasi-syntomic** if R is p-complete with bounded p-power torsion and the cotangent complex $L_{R/\mathbb{Z}_p}$ has p-complete Tor amplitude in $[-1, 0]$. For example, a noetherian lci ring is quasi-syntomic, as in a regular semilens (Definition 18.2.1) with bounded p-power torsion. $\lozenge$

Theorem 28.4.2 *For any quasi-syntomic ring R, there is an anti-equivalence betewen the category of p-divisible groups over R and the category of* ***prismatic Dieudonné crystals*** *over R.*

Proof. See [6]. ■

Remark 28.4.3 Theorem 28.4.2 builds upon a long history of describing p-divisible groups in terms of objects of semilinear algebra (e.g., see [90]), as well as more recent work classifying p-divisible groups over perfectoid spaces ([112], [89]; see also [113], Appendix to Lecture 17).

28.5 Logarithmic prismatic cohomology

The reference for this topic is [88].

Definition 28.5.1 A **prelog structure** on a ring A consists of a monoid M and a morphism $\alpha\colon M \to A$ of monoids. In general, one prefers to "sheafify" this definition to define a **log structure**, as in [77]. $\lozenge$

Example 28.5.2 Suppose that Z is an effective Cartier divisor on Spec A. If the components of Z are cut out by some elements $x_1, \dots, x_r$ of A, we can use the monoid generated by these and its inclusion into A as a prelog structure. The resulting log structure will then depend only on Z and not on the components, and also makes sense even when the components of Z are not globally cut out by regular functions (as this is always true locally).

Note that there is a difference between sheafifying with respect to the Zariski topology versus the étale topology, and we generally prefer the latter. For example, if Z is a nodal cubic curve in the plane, we would like the monoid to have two independent generators corresponding to the two branches at the node, and this is true étale-locally but not Zariski-locally. □

Definition 28.5.3 As per [88], Definition 2.2, a $\delta_{\log}$**-ring** is a tuple $(A, \delta, \alpha, \delta_{\log})$ in which (A, δ) is a δ-ring, $\alpha\colon M \to A$ defines a prelog structure on A, and $\delta_{\log}\colon M \to A$ is a function satisfying the following conditions.

1. For $e \in M$ the identity element, $\delta_{\log}(e) = 0$.

2. For $m \in M$,
$$\delta(\alpha(m)) = \alpha(m)^p \delta_{\log}(m).$$

3. For $m, m' \in M$,
$$\delta_{\log}(mm') = \delta_{\log}(m) + \delta_{\log}(m') + p\delta_{\log}(m)\delta_{\log}(m').$$

An important special case is when $\delta_{\log} = 0$ identically. In this case, we say that the $\delta_{\log}$-ring in question is **constant** (or **of rank 1** in Koshikawa's terminology). $\lozenge$

We report some examples from [88], Example 2.4.

Example 28.5.4 For $(A, \delta) \in \mathbf{Ring}_\delta$, consider the canonical log structure where $M = A^\times$ and $\alpha\colon A^\times \to A$ is the canonical inclusion. There is then a unique $\delta_{\log}$-ring structure given by

$$\delta_{\log}(x) = \frac{\delta(x)}{x^p} \qquad (x \in A^\times).$$

□

Example 28.5.5 Let $R \in \mathbf{Ring}_{\mathbb{F}_p}$ be perfect and view $W(R)$ as a δ-ring via the Witt vector Frobenius. The prelog structure given by the constant lift $[\bullet]: R \to W(R)$ then admits a constant $\delta_{\log}$-structure. $\square$

Example 28.5.6 For any monoid M, we may view the monoid ring $\mathbb{Z}_{(p)}[M]$ as a δ-ring in such a way that the elements of M are all constant. The prelog structure given by the natural map $M \to \mathbb{Z}_{(p)}[M]$ then admits a constant $\delta_{\log}$-structure. $\square$

Example 28.5.7 Given a $\delta_{\log}$-ring A and a morphism $A \to B$ of δ-rings, we may upgrade to a morphism of $\delta_{\log}$-rings by equipping B with the prelog structure $M \to A \to B$ and the $\delta_{\log}$-structure $M \stackrel{\delta_{\log}}{\to} A \to B$. $\square$

Remark 28.5.8 One can continue in this manner to extend much of the formalism of δ-rings; define logarithmic prisms and logarthmic prismatic sites; establish crystalline and Hodge-Tate comparison theorems; and obtain q-analogues. The purpose of this (not yet fully realized) is to develop a prismatic theory that provides a geometric construction of **semistable** Breuil-Kisin modules associated to the cohomology of smooth proper schemes over p-adic fields that do not have good reduction, building on the adaptation of [22] carried out in [38].

However, it may be possible to give an alternate development using the formalism of prismatization (Subsection 28.3) and the fact that logarithmic structures on a given scheme X can be described locally in terms of morphisms from X to the quotient stack $\mathbb{A}^1/\mathbb{G}_m$.

29 Some global speculation

Reference. [100] for the basic setup of q-de Rham cohomology in the λ-ring context. Beyond that, we are into *terra incognita*.

We conclude with some wild speculation about a potential link back to λ-rings (Section 4), particularly with regard to q-de Rham cohomology in the context of λ-rings.

Notational warning: in this lecture, we use the notation $R\{J\}$ to denote a free object in the category of λ-rings, not δ-rings.

29.1 Divided power envelopes of λ-rings

We revisit the discussion of divided power envelopes of δ-rings from Section 14, starting by formulating a variant of Lemma 14.3.2, in which we start with a larger ideal but do not require the base ring to be p-local.

Lemma 29.1.1 *Let $R_0 \in \mathbf{Ring}$ be $\mathbb{Z}$-torsion-free. Let R be the free δ-ring over R_0 on a single generator x, and let J be the ideal of R generated by $\delta^m(x)$ for all $m \geq 0$. Then the map from R to the divided power envelope of (R, J) promotes to a morphism of δ-rings.*

Proof. Let D be the divided power envelope; it is the smallest subring of $R \otimes_{\mathbb{Z}} \mathbb{Q}$ containing R and $\gamma_n(\delta^m(x))$ for all $m \geq 0$, $n \geq 1$. The maximal ideal on which D admits divided powers includes $\delta^m(x)$ for all $m \geq 0$ by construction, and hence also $\phi(\delta^m(x)) = \delta^m(x)^p + p\delta^{m+1}(x)$ for all $m \geq 0$; consequently, for all $m \geq 0$, $n \geq 1$,

$$\phi(\gamma_n(\delta^m(x))) = \gamma_n(\phi(\delta^m(x))) \in D.$$

Hence ϕ induces an endomorphism of D.

We next check that ϕ induces a Frobenius lift on D; this amounts to checking that for all $m \geq 0$, $n \geq 1$,

$$\phi(\gamma_n(\delta^m(x))) \equiv \gamma_n(\delta^m(x))^p \pmod{pD}.$$

We will see that in fact both sides are divisible by p. For $\phi(\gamma_n(\delta^m(x))) = \gamma_n(\phi(\delta^m(x)))$, this holds by writing

$$\phi(\delta^m(x))) = p(\delta^m(x)^p/p) + p\delta^{m+1}(x) \in pD$$

and

$$\phi(\gamma_n(\delta^m(x))) = \gamma_n(\phi(\delta^m(x))) = p^n \gamma_n(\phi(\delta^m(x))/p).$$

For $\gamma_n(\delta^m(x))^p$, this holds by writing $\gamma_n(\delta^m(x))^p = p!\gamma_p(\gamma_n(\delta^m(x)))$ and applying (14.4).

Since D is p-torsion-free, by Lemma 2.1.3 we obtain a δ-structure compatible with R, as desired. ■

This statement and its proof transpose naturally to the λ-ring setting.

Lemma 29.1.2 *Let $R_0 \in$ **Ring** be $\mathbb{Z}$-torsion-free. Put $R = R\{x\}$ (the free λ-ring over R on a single generator x), and let J be the ideal of R generated by $\lambda^m(x)$ for all $m \geq 1$. Then the map from R to the divided power envelope of (R, J) promotes to a morphism of λ-rings.*

Proof. Let D be the divided power envelope; it is the smallest subring of $R \otimes_{\mathbb{Z}} \mathbb{Q}$ containing R and $\gamma_n(\lambda^m(x))$ for all $m, n \geq 1$. The maximal ideal on which D admits divided powers includes $\delta^m(x)$ for all $m \geq 0$ by construction, and hence also $\psi^p(\delta^m(x)) = \delta^m(x)^p + p\delta^{m+1}(x)$ for each prime p and all $m \geq 0$; consequently, for each prime p and all $m, n \geq 1$,

$$\psi^p(\gamma_n(\delta^m(x))) = \gamma_n(\psi^p(\delta^m(x))) \in D.$$

We thus obtain a commuting family of endomorphisms ψ^j for all $j \geq 1$ satisfying $\psi^{j_1 j_2} = \psi^{j_1} \circ \psi^{j_2}$.

As in the proof of Lemma 29.1.1, we verify that for each prime p, ψ^p induces a p-Frobenius lift on D. Since D is $\mathbb{Z}$-torsion-free, by Wilkerson's criterion (Remark 4.2.3) we obtain a λ-structure compatible with R, as desired. ■

This in turn leads to a λ-analogue of Corollary 14.3.3.

Corollary 29.1.3 *In Lemma 29.1.2, the divided power envelope is generated as a λ-ring over R_0 by the elements $\psi^p(x)/p$ for p prime.*

Proof. Let D be the divided power envelope and let D' be the subring of $D \otimes_{\mathbb{Z}} \mathbb{Q} = R \otimes_{\mathbb{Z}} \mathbb{Q}$ which is the λ-subring over R_0 generated by the elements $\psi^p(x)/p$ for p prime. By Lemma 29.1.2, we have $D' \subseteq D$; it thus remains to check the converse.

We check that $\gamma_n(\delta^m(x)) \in D'$ for all $m \geq 0$, $n \geq 1$ by induction on n, with trivial base case $n = 1$. The induction step is also trivial if n is not a prime power, as in this case $1/n!$ belongs to the fractional ideal of $\mathbb{Z}$ generated by $1/i!$ for $i \in \{1, \dots, n-1\}$. If on the other hand $n = p^\ell$, then the difference between the fractional ideal of $\mathbb{Z}$ generated by $1/n!$ and the one generated by $1/i!$ for $i \in \{1, \dots, n-1\}$ is a factor of n; consequently, we may reduce to the case where R_0 is p-local, in which case Corollary 14.3.3 implies the claim. ■

Remark 29.1.4 At this point, we can formulate a striking but not immediately meaningful analogue of Lemma 14.4.8 as follows. Let $R_0 \in$ **Ring** be arbitrary and put $R = R_0[x_1, \dots, x_r]$, viewed as a λ-ring with $x_1, \dots, x_r$ constant. For $n \geq 0$, put

$$P^n = R_0\{x_{ij} \colon i = 1, \dots, r; j = 0, \dots, n\}$$

(remembering that now we mean the free λ-ring) and let J^n be the kernel of the morphism $P^n \to R$ in $\mathbf{Ring}_\lambda$ taking x_{ij} to x_i and $\lambda^m(x_{ij})$ to 0 for all $m > 1$. Now consider the double complex displayed in Figure 29.1.5.

$$\begin{tikzcd}
D_{J^0}(P^0) \arrow[r] \arrow[d] & D_{J^1}(P^1) \arrow[r] \arrow[d] & D_{J^2}(P^2) \arrow[r] \arrow[d] & \cdots \\
D_{J^0}(P^0) \otimes_{P^0} \Omega^1_{P^0/R_0} \arrow[r] \arrow[d] & D_{J^1}(P^1) \otimes_{P^1} \Omega^1_{P^1/R_0} \arrow[r] \arrow[d] & D_{J^2}(P^2) \otimes_{P^2} \Omega^1_{P^2/R_0} \arrow[r] \arrow[d] & \cdots \\
D_{J^0}(P^0) \otimes_{P^0} \Omega^2_{P^0/R_0} \arrow[r] \arrow[d] & D_{J^1}(P^1) \otimes_{P^1} \Omega^2_{P^1/R_0} \arrow[r] \arrow[d] & D_{J^2}(P^2) \otimes_{P^2} \Omega^2_{P^2/R_0} \arrow[r] \arrow[d] & \cdots \\
\vdots & \vdots & \vdots &
\end{tikzcd}$$

Figure 29.1.5

In the diagram, we may argue as in Lemma 14.4.1 to see that each row except the first is homotopic to zero. We may then apply the Poincaré lemma (Proposition 14.2.6) to deduce that each column is quasi-isomorphic to $\Omega^\bullet_{R/R_0}$, and in particular each of the $n+1$ morphisms between the n-th column and the $(n+1)$-st column induces the same isomorphism on cohomology groups. By Corollary 13.3.8, we deduce that the top row of Figure 29.1.5 is quasi-isomorphic to $\Omega^\bullet_{R/R_0}$.

Moreover, by Corollary 29.1.3, we may identify $D_{J^n}(P^n)$ with $P^n\{\psi^p(J^n)/p\}$ (running over all primes p). Here is where we get stuck: the latter object does not have any evident site-theoretic interpretation. However, in the q-analogue of this setup we will be able to provide such an interpretation.

29.2 q-divided powers for λ-rings

Definition 29.2.1 In the following discussion, we view $A = \mathbb{Z}[\![q-1]\!]$ as a λ-ring (Definition 4.2.2) with q constant; that is, $\lambda^i(q) = 0$ for all $i > 1$ (and so $\psi^i(q) = q^i$ for all $i > 0$). ♢

We introduce the following analogue of Definition 27.1.5.

Definition 29.2.2 For D an A-torsion-free λ-ring over A, for p prime and $x \in D$ with $\psi^p(x) \in [p]_q D$, write

$$\gamma_{p,q}(x) = \frac{\psi^p(x)}{[p]_q} - \delta_p(x) \in D.$$

♢

Lemma 29.2.3 *Let D be an A-torsion-free λ-ring over A. Then for each prime p, the ideal $(\psi^p)^{-1}([p]_q D)$ is stable under $\gamma_{p,q}$.*

Proof. View D as a δ-ring for the prime p and apply Lemma 27.1.7. ■

Remark 29.2.4 As in Remark 27.1.6, for any ideal I of D, the set

$$J = \bigcap J_p, \qquad J_p = \{x \in I \colon \psi^p(x) \in [p]_q D, \gamma_{p,q}(x) \in I\}$$

is itself an ideal of D; hence to check that $J = I$, it suffices to check that J contains a generating set of I.

Definition 29.2.5 A **λ-pair** over A is a pair (D, I) in which D is a λ-ring over A and I is an ideal of D. A morphism $(D, I) \to (E, J)$ of λ-pairs is a morphism $D \to E$ of λ-rings carrying I into J. ♢

Definition 29.2.6 A **global q-pd pair** is a λ-pair (D, I) in which D is derived $(q-1)$-complete, $q - 1 \in I$, and for each prime p, $\psi^p(I) \subseteq [p]_q D$ (so that $\gamma_{p,q}$ is defined on I) and $\gamma_{p,q}(I) \subseteq I$. $\lozenge$

Example 29.2.7 The λ-pair $(A, (q-1))$ is the initial object in the category of global q-pd pairs. $\square$

Proposition 29.2.8 *Let P be a λ-ring over A equipped with a surjection $P \to R = \mathbb{Z}[x_1, \dots, x_r]$ with kernel $J = (q-1, y_1, y_2, \dots)$ where each initial segment $y_1, \dots, y_s$ is a regular sequence in $P/(q-1)$. Define the λ-ring*

$$D = P\{\psi_p(y_i)/[p]_q \colon i = 1, 2, \dots; p \text{ prime}\}^\wedge_{(q-1)}.$$

Let I be the kernel of $D \to D/(q-1) \to R$.

1. *The map $P \to D$ induces an isomorphism $P/J \cong D/I$.*
2. *The ring $D/(q-1)$ identifies with the divided power envelope of the map $P/(q-1) \to R$.*
3. *The map $(P, J) \to (D, I)$ of λ-pairs is universal for the target being a global q-pd pair.*

Proof. Analogous to Proposition 27.2.5, using Corollary 29.1.3. ■

29.3 A global site

We now globalize the earlier discussion of q-crystalline cohomology.

Definition 29.3.1 Put $R = \mathbb{Z}[x_1, \dots, x_r][\![q-1]\!]$, viewed as a λ-ring over A with $q, x_1, \dots, x_r$ all constant. For $n \geq 0$, put

$$P^n = \mathbb{Z}\{x_{ij} \colon i = 1, \dots, r; j = 0, \dots, n\}[\![q-1]\!]$$

and let J^n be the kernel of the morphism $P^n \to R$ in $\mathbf{Ring}_\lambda$ taking x_{ij} to x_i and $\lambda^m(x_{ij})$ to 0 for all $m > 1$. Let $D_{J^n,q}(P^n)$ be the ring D obtained by applying Proposition 29.2.8 to the pair (P^n, J^n). $\lozenge$

Lemma 29.3.2 *In Figure 29.3.3, the first row of the diagram is quasi-isomorphic to the left column, which is in turn quasi-isomorphic to $q\widehat{\Omega}^\bullet_{R/\mathbb{Z},\square}$. (Here all derived completions are with respect to $q-1$.)*

$$\begin{tikzcd}
D_{J^0,q}(P^0) \arrow[r] \arrow[d] & D_{J^1,q}(P^1) \arrow[r] \arrow[d] & D_{J^2,q}(P^2) \arrow[r] \arrow[d] & \cdots \\
D_{J^0,q}(P^0)\widehat{\otimes}_{P^0} q\widehat{\Omega}^1_{P^0/\mathbb{Z},\square} \arrow[r] \arrow[d] & D_{J^1,q}(P^1)\widehat{\otimes}_{P^1} q\widehat{\Omega}^1_{P^1/\mathbb{Z},\square} \arrow[r] \arrow[d] & D_{J^2,q}(P^2)\widehat{\otimes}_{P^2} q\widehat{\Omega}^1_{P^2/ZZ,\square} \arrow[r] \arrow[d] & \cdots \\
D_{J^0,q}(P^0)\widehat{\otimes}_{P^0} q\widehat{\Omega}^2_{P^0/\mathbb{Z},\square} \arrow[r] \arrow[d] & D_{J^1,q}(P^1)\widehat{\otimes}_{P^1} q\widehat{\Omega}^2_{P^1/\mathbb{Z},\square} \arrow[r] \arrow[d] & D_{J^2,q}(P^2)\widehat{\otimes}_{P^2} q\widehat{\Omega}^2_{P^2/ZZ,\square} \arrow[r] \arrow[d] & \cdots \\
\vdots & \vdots & \vdots &
\end{tikzcd}$$

Figure 29.3.3

Proof. In the following argument, all applications of derived Nakayama (Remark 6.6.6) will be modulo $q-1$.

In the diagram, by derived Nakayama plus Lemma 14.4.1 (or a direct argument), each row except the first is homotopic to zero. By derived Nakayama plus the Poincaré lemma (Proposition 14.2.6, each column is quasi-isomorphic to $q\Omega^\bullet_{R/\mathbb{Z},\square}$, and in particular each of the $n+1$ morphisms between the n-th column and the $(n+1)$-st column induces the same isomorphism on cohomology groups. By Corollary 13.3.8, we deduce that the top row of Figure 29.3.3 is quasi-isomorphic to $q\widehat{\Omega}^{\bullet}_{R/\mathbb{Z},\square}$. ■

Definition 29.3.4 Put $R = \mathbb{Z}[x_1, \dots, x_r]$. Define the **global q-crystalline site** to be the opposite category to the category of global q-pd-pairs (P, J) equipped with isomorphisms $P/J \cong R$, equipped with the indiscrete Grothendieck topology. By Proposition 29.2.8, the ring $D_{J^0,q}(P^0)$ from Definition 29.3.1 yields a weakly final object in this category, so by Lemma 11.1.7 we can compute the cohomology of this site using the associated Čech-Alexander complex. This is precisely the top row of Figure 29.3.3, so Lemma 29.3.2 gives us a quasi-isomorphism with $q\widehat{\Omega}^{\bullet}_{R/\mathbb{Z},\square}$. ◊

29.4 Okay, now what?

Remark 29.4.1 One would like to pursue the analogy with the p-local situation further, e.g., by comparing étale localization with the left Kan extension. One dangerous point is that the Hodge-Tate isomorphism is going to be more subtle due to the lack of a conjugate filtration when not working in characteristic p; compare Remark 17.2.8.

Nonetheless, one could try to define the analogue of a **prism** in this context, at least relative to the pair $(A, (q-1))$, and see where this leads. We shall see...

29.5 Exercises

1. View $\mathbb{Z}[x, y, q]$ as a Λ-ring in such a way that x, y, q are all constant. Show that in the Λ-localization $\mathbb{Z}[x, y, q]\{(q-1)^{-1}\}$, we have

$$\begin{aligned}\lambda^k\left(\frac{y-x}{q-1}\right) &= \frac{(y-x)(y-qx)\cdots(y-q^{k-1}x)}{(q-1)^k[k]_q!} \\ &= \sum_{j=0}^{k} \frac{q^{j(j-1)/2}(-x)^j y^{k-j}}{[j]_q![k-j]_q!}.\end{aligned}$$

Hint. Observe that it is enough to check the claim for $y = q^n x$. For omre details, see [100], Lemma 1.3.

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