PERFECTOID SPACES: AN ANNOTATED BIBLIOGRAPHY

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This annotated bibliography was prepared as part of a five-lecture series at the summer school on perfectoid spaces held at the International Centre for Theoretical Sciences (ICTS), Bengaluru, September 9–13, 2019. It is not intended to be a freestanding reference, although we do include a few short proofs and some sketches of longer proofs; instead, I have attempted to give some complements to my Arizona Winter School 2017 lecture notes [28], which provide a far more complete version of the story.

Throughout, fix a prime number p.

1. Perfectoid fields

Primary references: [26, §1], [44, §3], [32, §3.5].

Proposition 1.1 (Fontaine–Wintenberger theorem). *The Galois groups of the fields $\mathbb{Q}_p(\mu_{p^\infty})$ and $\mathbb{F}_p((t))$ are isomorphic. More precisely, this isomorphism arises from an explicit isomorphism of Galois categories.*

Proof. This is a consequence of results announced in [16, 17] and proved in detail in [51]. It is also a special case of Proposition 1.16 via Krasner's lemma (Remark 1.10).

We expand briefly on how Proposition 1.1 is embedded in the aforementioned papers of Fontaine–Wintenberger. By a theorem of Sen [43], the field $\mathbb{Q}_p(\mu_{p^\infty})$ is *strictly arithmetically profinite* in the sense of Fontaine–Wintenberger (we do not need the exact definition here). This then implies that its *norm field* is a local field of characteristic p with residue field $\mathbb{F}_p$ (see [16, Théorème 2.4], [51, Théorème 2.1.3]), and so may be identified with $\mathbb{F}_p((t))$ via the Cohen structure theorem. The norm field construction then defines a bijection between finite extensions of $\mathbb{Q}_p(\mu_{p^\infty})$ and finite separable extensions of its norm field [51, Théorème 3.2.2]. $\square$

Remark 1.2. The results of Fontaine–Wintenberger cited above also imply that for any strictly arithmetically profinite algebraic extension K of $\mathbb{Q}_p$, the Galois group of K and its norm field are isomorphic. This more general statement can also be recovered from Proposition 1.16, by showing that the completion of K is perfectoid with tilt isomorphic to the completed perfect closure of the norm field.

Before relating the Fontaine–Wintenberger theorem to perfectoid fields, we introduce some background on nonarchimedean fields.

Definition 1.3. A *nonarchimedean field* is a topological field whose topology is defined by some *nontrivial* nonarchimedean absolute value, with respect to which the field is complete. For K a nonarchimedean field, write $|K^\times|$ for the value group, $\mathfrak{o}_K$ for the valuation ring, $\mathfrak{m}_K$ for the maximal ideal, and κ_K for the residue field.

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Proposition 1.4. *Let K be a nonarchimedean field (a field complete with respect to a non-trivial nonarchimedean absolute value). Let L/K be a finite extension.*

(i) *The absolute value on K extends uniquely to a nonarchimedean absolute value on L.*
(ii) *There is a unique maximal subextension U of K which is* unramified *over K: $|U^\times| = |K^\times|$ and κ_U/κ_K is separable of degree $[U : K]$. In particular, $\mathfrak{o}_U$ can be written as $\mathfrak{o}_K[\lambda]$ where λ maps to a primitive element of the residue field extension; in particular, if we write an element x of U as $\sum_{i=0}^{d-1} a_i\lambda^i$ with $a_i \in K$, then $x \in \mathfrak{o}_U$ if and only if $a_i \in \mathfrak{o}_U$ (or equivalently $a_i \in \mathfrak{o}_K$) for all i.*
(iii) *If κ_K has characteristic p, then there is a unique maximal subextension T of K containing U which is* totally tamely ramified *over U: $\kappa_T = \kappa_U$ and $[|T^\times| : |U^\times|] = [T : U]$ is coprime to p. That is, $[T : U]$ is coprime to p, the residue fields of T and U coincide, and the value group extension of T/U has index $[T : U]$. Moreover, T can be written as $U(\lambda^{1/d})$ in such a way that $\lambda^{1/d}$ generates the quotient of the value groups; in particular, if we write an element x of T as $\sum_{i=0}^{d-1} a_i\lambda^{i/d}$ with $a_i \in U$, then $x \in \mathfrak{o}_T$ if and only if $a_i\lambda^{i/d} \in \mathfrak{o}_T$ for all i.*
(iv) *With notation as in (iii), the degree $[L : U]$ is a power of p. In particular, if L/K is Galois with group G, then $\mathrm{Gal}(L/U)$ admits a subnormal series in which each successive quotient is cyclic of order p.*

Proof. See for example [6, Chapter XIII]. □

Remark 1.5. If you are used to thinking about local fields as examples of nonarchimedean fields, a warning is in order: for L/K a finite extension of nonarchimedean fields, the inequality

$$[L : K] \geq [\left|L^\times\right| : \left|K^\times\right|][\kappa_L : \kappa_K],$$

which is always an equality when K is a local field, can be strict in general. See [36] for a detailded discussion of this phenomenon.

Definition 1.6. For $P(T) = P_nT^n + \cdots + P_0 \in K[T]$ a polynomial over a nonarchimedean field K, the *Newton polygon* of P is the open polygon which forms the lower boundary of the convex hull of the set

$$\bigcup_{i=0}^{n} \{i\} \times [-\log|P_{n-i}|, \infty) \subset \mathbb{R}^2.$$

For $i = 1, \ldots, n$, the section of this polygon with x-coordinates in the range $[i-1, i]$ is a line segment. The slopes of these n line segments form the *slope multiset* of P.

Proposition 1.7 (Properties of Newton polygons)**.** *Let $P(T)$ be a polynomial over a nonarchimedean field K.*

(i) *Choose an extension L of K over which $P(T)$ factors as $(T-\alpha_1)\cdots(T-\alpha_n)$. Then the slope multiset of P consists of $-\log|\alpha_1|, \ldots, -\log|\alpha_n|$ in some order. In particular, the slope multiset of a product of two polynomials is the union of the slope multisets of the two polynomials.*
(ii) *If P is irreducible, then the Newton polygon is a straight line segment.*

Proof. There are many references for this material; see for example [23, Chapter 2]. □

Proposition 1.8 (Krasner's lemma)**.** *Let L/K be a (not necessarily finite) extension of nonarchimedean fields. Let $P(T) \in K[T]$ be a polynomial which factors completely over L as $(T-\alpha_1)\cdots(T-\alpha_n)$. Then for any $\beta \in L$ such that*

$$|\beta - \alpha_1| < |\alpha_i - \alpha_1| \qquad (i = 2, \dots, n),$$

we have $K(\alpha_1) \subseteq K(\beta)$.

Proof. For $i = 2, \dots, n$, $|\beta - \alpha_i| = |\alpha_i - \alpha_1| > |\beta - \alpha_1|$ by the nonarchimedean triangle inequality. By Proposition 1.7, the Newton polygon of $P(T-\beta)$ includes a segment of length 1, which must correspond to an irreducible factor. Alternatively, see [42, Theorem III.1.5.1]; it is assumed therein that K is a discretely valued field, but the proof remains unchanged in the general case. □

Proposition 1.9. *Let K be a nonarchimedean field and let x be a nonzero element of K of positive valuation. Then K is algebraically closed if and only if:*

(a) *the value group of K is not discrete; and*
(b) *every polynomial over $\mathfrak{o}_K/(x)$ has a root in $\mathfrak{o}_K/(x)$.*

Proof. (The following argument is extracted from [26, Lemma 1.5.4]; see also [44, Proposition 3.8], [32, Lemma 3.5.5].) It is clear that both conditions are necessary. To check sufficiency, note first that (a) and (b) together imply that the value group of K is in fact divisible. With this in mind, we show that every polynomial $P(T) \in \mathfrak{o}_K[T]$ has a root in $\mathfrak{o}_K$, by induction on the degree n of P. (This implies the same with $\mathfrak{o}_K$ replaced by K, by rescaling in T.)

To this end, we construct a sequence $z_0, z_1, \dots$ as follows. Start with $z_0 = 0$. Given z_i, if $P(z_i) = 0$ there is nothing more to check. If $P(z_i) \neq 0$ but the polynomial $P(T+z_i)$ has more than one distinct slope in its slope multiset, Proposition 1.7 allows us to factor it nontrivially and proceed by induction. Otherwise, because K has divisible value group, we can find a nonzero value $u_i \in K$ for which $P(u_iT + z_i)$ has all slopes equal to 0. By hypothesis (b), there exists $y_i \in \mathfrak{o}_K$ such that $P(u_iy_i + z_i) \in x\mathfrak{o}_K$; put $z_{i+1} = z_i + u_iy_i$.

To conclude the argument, it will suffice to check that if the construction of the sequence continues infinitely, then the sequence converges to a limit z which is a root of P. Since $P(T + z_i)$ has only one slope in its slope multiset, we must have $|u_i| = |P(z_i)|^{1/n}$. Since $|P(z_{i+1})| \leq |x|\,|P(z_i)|$, it follows that $u_i \to 0$ as $i \to \infty$, so the z_i do converge to a limit z satisfying $|P(z)| = 0$. □

Remark 1.10. From the previous discussion, we deduce that an algebraic extension L of K is algebraically closed if and only if its completion is algebraically closed: the "if" assertion follows from Krasner's lemma (Proposition 1.8) while the "only if" assertion follows from Proposition 1.9.

One consequence of this observation for the Fontaine–Wintenberger theorem is that on one hand, $\mathbb{Q}_p(\mu_{p^\infty})$ and its completion have the same Galois group; on the other hand, $\mathbb{F}_p((t))$, its perfect closure, and the completion of its perfect closure all have the same Galois group.

Definition 1.11. A *perfectoid field* is a nonarchimedean field K with residue field of characteristic p and nondiscrete value group, for which the Frobenius map $x \mapsto x^p$ on $\mathfrak{o}_K/(p)$ is surjective. We allow the possibility that K is of characteristic p, in which case K is forced to be perfect.

Remark 1.12. Any algebraically closed nonarchimedean field with residue field of characteristic p is perfectoid. The completion K of $\mathbb{Q}_p(p^{p^{-\infty}})$ is perfectoid:

$$\mathfrak{o}_K/(p) \cong \mathbb{F}_p[\overline{T}_1, \overline{T}_2, \dots]/(\overline{T}_1^p, \overline{T}_2^p - \overline{T}_1, \dots).$$

The completion of $\mathbb{Q}_p(\mu_{p^\infty})$ is perfectoid: we have

$$\mathfrak{o}_K/(p) \cong \mathbb{F}_p[\overline{T}_1, \overline{T}_2, \dots]/(\overline{T}_1^{p-1} + \cdots + \overline{T}_1 + 1, \overline{T}_2^p - \overline{T}_1, \dots).$$

In both cases, the tilt is isomorphic to the completion of $\mathbb{F}_p((T))[T^{1/p^\infty}]$. In particular, one cannot recover K from $K^\flat$ alone; some extra data is needed which we describe in the next lecture.

Remark 1.13. As noted in [28, Remark 2.1.8], the definition of a perfectoid field first appeared in [39] in 1984 under the terminology *hyperperfect field* (in French, *corps hyperparfait*), but the significance of this went unnoticed at the time.

Proposition 1.14 (after Fontaine)**.** *Let K be a perfectoid field.*

(i) *The natural map*

$$\varprojlim_{x \mapsto x^p} \mathfrak{o}_K \to \varprojlim_{x \mapsto x^p} \mathfrak{o}_K/(p)$$

is an isomorphism of multiplicative monoids.

(ii) *Using (i) to upgrade $\mathfrak{o}_{K^\flat} := \varprojlim_{x \mapsto x^p} \mathfrak{o}_K$ to a ring, it becomes a perfect valuation ring of characteristic p with fraction field $K^\flat := \varprojlim_{x \mapsto x^p} K$. The valuation on $K^\flat$ is the restriction along the final projection $\sharp : K^\flat \to K$.*

(iii) *The map $\sharp$ induces an isomorphism $|K^\times| \cong |K^{\flat\times}|$; in particular, both value groups are p-divisible.*

(iv) *The fields κ_K and $\kappa_{K^\flat}$ are isomorphic; in particular, both residue fields are perfect. Moreover, for $\overline{x} \in K^\flat$ such that $\sharp(\overline{x})/p \in \mathfrak{o}_K^\times$ (which exists by (iii)), the rings $\mathfrak{o}_K/(p)$ and $\mathfrak{o}_{K^\flat}/(\overline{x})$ are isomorphic.*

We call $K^\flat$ the tilt *of K.*

Proof. See [26, Lemma 1.3.3] or [44, Lemma 3.4]. (While the basic construction described here was known to Fontaine, the term *tilt*, and the notations $\flat$ and $\sharp$, were introduced by Scholze in [44].) □

Proposition 1.15. *Let K be a perfectoid field. Then K is algebraically closed if and only if $K^\flat$ is algebraically closed.*

Proof. This follows from Proposition 1.9 and Proposition 1.14(iii, iv). □

We are not yet able to prove the following result; we state the proof modulo a key construction which we will introduce in the next lecture.

Proposition 1.16 (Generalized Fontaine–Wintenberger theorem)**.** *Let K be a perfectoid field with tilt $K^\flat$.*

(i) *Every finite extension of K is perfectoid.*

(ii) *The functor $L \mapsto L^\flat$ defines an equivalence of Galois categories between finite extensions of K and $K^\flat$, and hence an isomorphism between the absolute Galois groups of K and $K^\flat$.*

Proof of Proposition 1.16. We follow the proof of [26, Theorem 1.5.6]. See [44, Theorem 3.7] for a somewhat different approach (using almost ring theory in place of Witt vectors). We may omit the case where K is of characteristic p, as in this case $K = K^\flat$ and the claim is trivial.

We will show in the next lecture (see Proposition 2.16) that there exists a surjective homomorphism $\theta : W(\mathfrak{o}_{K^\flat}) \to \mathfrak{o}_K$ with the property that for each finite extension E of $K^\flat$, there exists a perfectoid field L with

$$W(\mathfrak{o}_E) \otimes_{W(\mathfrak{o}_{K^\flat}),\theta} \mathfrak{o}_K \cong \mathfrak{o}_L;$$

this isomorphism will induce an isomorphism $L^\flat \cong E$. By inverting p in the previous isomorphism, we will also have an identification

$$W(\mathfrak{o}_E)[p^{-1}] \otimes_{W(\mathfrak{o}_{K^\flat}),\theta} \mathfrak{o}_K \cong L;$$

in case $E/K^\flat$ is a Galois extension with group G, it will follow that G acts on L with invariant subring K, so by Artin's lemma L/K is a finite Galois extension with Galois group G. If $E/K^\flat$ is not necessarily Galois, we may first go up to a Galois closure of $E/K^\flat$ and then come back down to deduce that $[L : K] = [E : K^\flat]$.

In this way, we will obtain a functor from finite extensions of $K^\flat$ to finite perfectoid extensions of K which, when followed by the tilting functor, yields an equivalence of categories (by the degree preservation property from the previous paragraph). In particular, this functor is fully faithful, and it remains only to check that it is essentially surjective. For this, let E be a completed algebraic closure of $K^\flat$. By Proposition 2.14 again, we may realize E as the tilt of some extension L of K; by Proposition 1.15, L is algebraically closed. By Remark 1.10(ii), the union of the finite extensions of K arising from finite extensions of $K^\flat$, or equivalently finite Galois extensions of $K^\flat$, is also algebraically closed. Hence every finite extension L of K is contained in a finite Galois extension of K arising from a finite Galois extension of $K^\flat$; as in the previous paragraph, we deduce that L is itself perfectoid. This proves the claim. $\square$

Remark 1.17. Let $\mathbb{C}$ be a completed algebraic closure of $\mathbb{Q}_p$. By Proposition 1.16, we can identify $\mathbb{C}^\flat$ with a completed algebraic closure of $\mathbb{F}_p((t))$ in various ways; for example, the two calculations from Remark 1.12 give rise to two distinct isomorphisms of this sort.

Suppose now that K is an arbitrary untilt of $\mathbb{C}^\flat$ of characteristic 0. Since K is algebraically closed and contains $\mathbb{Q}_p$, the completed algebraic closure of $\mathbb{Q}_p$ within K is isomorphic to $\mathbb{C}$. However, the resulting inclusion $\mathbb{C} \subseteq K$ can be strict; see [35] for examples.

2. Tilting, untilting, and Witt vectors

In the previous lecture, the proof of Proposition 1.16 hinged on being able to find perfectoid fields with a specified tilt using Witt vectors. In order to better understand the relationship between perfectoid fields and their tilts, we use Witt vectors to describe all possible fields with a given tilt.

Definition 2.1. A ring R of characteristic p is *perfect* if the Frobenius homomorphism $x \mapsto x^p$ is an isomorphism; note that injectivity of this map is equivalent to R being reduced. When R is a field, this is equivalent to the Galois-theoretic condition that every finite extension of R is separable.

Proposition 2.2. *Let R be a perfect ring of characteristic p.*

(a) *There exists a p-adically separated and complete ring $W(R)$ with $W(R)/(p) \cong R$ (the* ring of p-typical Witt vectors *with coefficients in R).*
(b) *The reduction map $W(R) \to R$ admits a unique multiplicative section $\overline{x} \mapsto [\overline{x}]$ (called the* Teichmüller map*).*
(c) *The construction of $W(R)$ is functorial in R. In particular, $W(R)$ itself is unique up to unique isomorphism.*

Proof. See for example [26, §1.1]. Note that the Teichmüller map can be characterized by the formula $x = \lim_{n\to\infty} x_n^{p^n}$ where $x_n \in W(R)$ is any element satisfying $x_n^{p^n} \equiv x \pmod p$; the limit exists because $x_n^{p^n} \equiv x_{n+1}^{p^{n+1}} \pmod{p^{n+1}}$. □

Remark 2.3. The Witt vector construction was first introduced in the context where R is a perfect field. In this case, $W(R)$ is a complete discrete valuation ring with maximal ideal p and residue field R. For example, $W(\mathbb{F}_p) \cong \mathbb{Z}_p$.

Remark 2.4. One may fancifully think of $W(R)$ as $R[\![p]\!]$ except with some "carries" in the arithmetic. More precisely, every element $x \in W(R)$ has a unique representation as a convergent series $\sum_{n=0}^{\infty} [\overline{x}_n] p^n$ with $\overline{x}_n \in R$, but the arithmetic operations are somewhat complicated to express in terms of these coordinates. (Note that $\overline{x}_n$ is not the n-th Witt vector coefficient, but rather its p^n-th root.)

Since $W(R)$ is functorial in R, it admits a unique lift φ of the Frobenius map on R. This map has the property that $\varphi([\overline{x}]) = [\overline{x}^p]$ for $\overline{x} \in R$; that is, the elements $[\overline{x}]$ form the kernel of the associated p-derivation

$$\delta(x) := \frac{\varphi(x) - x^p}{p}$$

which occurs prominently in the context of *prismatic cohomology* [5, 31].

Definition 2.5. For the remainder of this lecture, let F denote a perfect nonarchimedean field of characteristic p, and define the ring $\mathbf{A}_{\mathrm{inf}}(F) := W(\mathfrak{o}_F)$. This is a local ring with residue field equal to that of F.

Remark 2.6. Since $\mathbf{A}_{\mathrm{inf}}(F)$ is to be interpreted as $\mathfrak{o}_F[\![p]\!]$, one can form a tenuous analogy between $\mathbf{A}_{\mathrm{inf}}(F)$ and a two-dimensional complete local ring such as $\mathbb{F}_p[\![x, y]\!]$. On one hand, the ring $\mathbf{A}_{\mathrm{inf}}(F)$ does not have any reasonable finiteness properties. For starters, it is certainly not noetherian: for any $\overline{x} \in \mathfrak{o}_F$ of positive valuation, the ideal

$$([\overline{x}], [\overline{x}^{1/p}], [\overline{x}^{1/p^2}], \dots)$$

is not finitely generated. In fact, $\mathbf{A}_{\mathrm{inf}}(F)$ has infinite [37] and even uncountable [13] global dimension, and in general is not even coherent [30].

On the other hand, it is true that every vector bundle on the punctured spectrum of $\mathbf{A}_{\mathrm{inf}}(F)$ extends uniquely over the puncture. See [30].

Proposition 2.7 (after Fontaine)**.** *Let K be a perfectoid field.*
(i) *There is a unique homomorphism $\theta : \mathbf{A}_{\mathrm{inf}}(K^\flat) \to \mathfrak{o}_K$ whose restriction along the Teichmüller map is the map $\sharp$.*
(ii) *The map θ is surjective.*

Proof. Part (i) is a formal consequence of the basic properties of p-typical Witt vectors; see [26, §1.1]. Part (ii) follows from Proposition 1.14(iv). □

To further analyze the kernel of θ, we make a key definition.

Definition 2.8. An element $z = \sum_{n=0}^{\infty}[\overline{z}_n]p^n \in \mathbf{A}_{\text{inf}}(F)$ is *primitive* if $\overline{z}_0 \in \mathfrak{m}_F$ and $\overline{z}_1 \in \mathfrak{o}_F^\times$. An ideal of $\mathbf{A}_{\text{inf}}(F)$ is *primitive* if it is principal generated by some primitive element. (It will follow from the following remark that every generator is then primitive.)

Remark 2.9. In the definition of a primitive element, the condition that $\overline{z}_1 \in \mathfrak{o}_F^\times$ may be replaced by the condition that $(z - [\overline{z}_0])/p \in \mathbf{A}_{\text{inf}}(F)^\times$ or the condition that $\delta(z) \in \mathbf{A}_{\text{inf}}(F)^\times$ (because $\delta(z) \equiv [\overline{z}_1^p] \pmod{p}$). From the latter formulation and the identity

$$\delta(yz) = y^p\delta(z) + z^p\delta(y) + p\delta(y)\delta(z),$$

we see that the product of a primitive element with a unit is a primitive element.

Remark 2.10. In the analogy between $\mathbf{A}_{\text{inf}}(F)$ and $\mathbb{F}_p[\![x, y]\!]$, primitive elements correspond to power series $\sum_{m,n=0}^{\infty} a_{m,n}x^m y^n$ with $a_{0,0} = 0$, $a_{0,1} \neq 0$. By the Weierstrass preparation theorem, any such power series can be written as a unit of $\mathbb{F}_p[\![x, y]\!]$ times $y - cx$ for some $c \in \mathbb{F}_p$; consequently, the quotient by the ideal generated by such a power series is isomorphic to $\mathbb{F}_p[\![x]\!]$.

Proposition 2.11. *For I a primitive ideal, every class in $\mathbf{A}_{\text{inf}}(F)/I$ can be represented by some element of $\mathbf{A}_{\text{inf}}(F)$ which is a unit times a Teichmüller lift.*

Proof. See [26, Lemma 1.4.7]. To summarize, let z be a generator of I. Given $x = \sum_{n=0}^{\infty}[\overline{x}_n]p^n \in \mathbf{A}_{\text{inf}}(F)$, x generates the same class in $\mathbf{A}_{\text{inf}}(F)/(z)$ as

$$x - \frac{(x - [\overline{x}_0])/p}{(z - [\overline{z_0}])/p} z = [\overline{x}_0] + [\overline{z}_0](x - [\overline{x}_0])/p.$$

By repeating this construction, we either produce a representative of the desired form, or verify that $x \in (z)$ (in which case we take the representative 0). We will take a more detailed look at what is going on here in the third lecture. $\square$

Remark 2.12. A more "prismatic" version of the construction from Proposition 2.11 would be to replace x with

$$x - \varphi^{-1}\left(\frac{\delta(x)}{\delta(z)}\right) z.$$

However, we have not checked that this has the same convergence property as the construction given above.

Proposition 2.13. *For K a perfectoid field, the kernel of $\theta : \mathbf{A}_{\text{inf}}(K^\flat) \to \mathfrak{o}_K$ is a primitive ideal.*

Proof. We reproduce here [26, Corollary 1.4.14]. By Proposition 1.14(iii), there exists $\overline{x} \in \mathfrak{o}_{K^\flat}$ such that $y := \theta(\overline{x})/p$ is a unit in $\mathfrak{o}_K$. By Proposition 2.7, there exists $w \in \mathbf{A}_{\text{inf}}(K^\flat)$ with $\theta(w) = y$. Since w must be a unit in $\mathbf{A}_{\text{inf}}(K^\flat)$, the element $z := pw - [\overline{x}]$ is a primitive element in $\ker(\theta)$.

By Proposition 2.11, every nonzero class in $\mathbf{A}_{\text{inf}}(K^\flat)/(z)$ can be represented by an element of $\mathbf{A}_{\text{inf}}(K^\flat)$ which is a unit times a Teichmüller lift; any such element has nonzero image in θ. It follows that $\ker(\theta) = (z)$, as claimed. $\square$

Proposition 2.14 (Tilting correspondence)**.** *For every primitive ideal I of $\mathbf{A}_{\mathrm{inf}}(F)$, the quotient $\mathbf{A}_{\mathrm{inf}}(F)/I$ can be identified with $\mathfrak{o}_K$ for some perfectoid field K. We then have an isomorphism $K^\flat \cong F$ for which I occurs as the kernel of $\theta : \mathbf{A}_{\mathrm{inf}}(F) \cong \mathbf{A}_{\mathrm{inf}}(K^\flat) \to \mathfrak{o}_K$. (Any such K is called an* untilt *of F.)*

Proof. See [26, Theorem 1.4.13]. To summarize, by Proposition 2.11, we can represent each class in the quotient by a unit times a Teichmüller lift, and use the latter to define the valuation (modulo showing that this does not depend on the choice of representative). □

Remark 2.15. Note that $z = p$ is a primitive element, and consistently F is an untilt of itself. Any other untilt of F is of characteristic 0.

As noted earlier, the following result completes the proof of Proposition 1.16.

Proposition 2.16 (Untilting of extensions)**.** *Let K be a perfectoid field. For any nonarchimedean field E containing $K^\flat$, the ring*

$$\mathfrak{o}_L := \mathbf{A}_{\mathrm{inf}}(E) \otimes_{\mathbf{A}_{\mathrm{inf}}(K^\flat),\theta} \mathfrak{o}_K$$

is the valuation ring of a perfectoid field L with $L^\flat \cong E$.

Proof. By Proposition 2.7, the map $\theta : \mathbf{A}_{\mathrm{inf}}(K^\flat) \to \mathfrak{o}_K$ is surjective and its kernel I is a primitive ideal. The ideal $I\mathbf{A}_{\mathrm{inf}}(E)$ is again primitive, so by Proposition 2.14, there is a perfectoid field L for which $L^\flat \cong E$ and $\theta : \mathbf{A}_{\mathrm{inf}}(E) \to \mathfrak{o}_L$ has kernel $I\mathbf{A}_{\mathrm{inf}}(E)$. This field has the desired property. □

Remark 2.17. Now that we have a reasonable way to describe the untilts of F, one can try to construct a moduli space of these untilts. Before doing so, we must observe that for any primitive ideal I of $\mathbf{A}_{\mathrm{inf}}(F)$, $\varphi(I)$ is also a primitive ideal and φ induces an isomorphism $\mathbf{A}_{\mathrm{inf}}(F)/I \cong \mathbf{A}_{\mathrm{inf}}(F)/\varphi(I)$; that is, I and $\varphi(I)$ define "the same" untilt of F.

In order to construct the desired moduli space, we must therefore find a way to define a space associated to $\mathbf{A}_{\mathrm{inf}}(F)$ and then quotient by the action of φ. Since φ is of infinite order, there is no hope of doing this within the category of schemes, at least not directly. We will compare two different constructions of this form in the fourth lecture.

Remark 2.18. The ring $\mathbf{A}_{\mathrm{inf}}(K)$ plays a central role in Fontaine's construction of p-adic period rings. We recommend [3] for a development of this point in modern language.

3. Perfectoid rings and spaces

In this lecture, we describe how the tilting equivalence can be extended to certain rings and spaces. Some detailed historical remarks, including many original references for the following statements. can be found in [28, Remarks 2.1.8, 2.3.18, 2.4.11, 2.5.13]; we do not attempt to reproduce these here.

Remark 3.1. These lectures will not include any review of Huber's theory of adic spaces, as these are covered in other lectures. For the reader reading this document in isolation, some introductory sources for the theory are [11] (in the context of rigid analytic geometry), [28, Lecture 1], [47, Lectures I–V], and [50].

One caution is in order: we will only consider Huber rings A, and Huber pairs (A, A^+), in which A is Hausdorff, complete, and contains a topologically nilpotent unit (also called a *pseudouniformizer*); this last condition is usually called *Tate*. In [28, Lecture 1], the Tate condition is relaxed to the condition that the topologically nilpotent elements of A generate the unit ideal; this condition is called *analytic*.

Definition 3.2. A Huber ring A is *perfectoid* if the following conditions hold.

(a) The ring A is *uniform*: its subring A° of power-bounded elements is bounded (and hence a ring of definition).

(b) There exists a pseudouniformizer ϖ with $p \in \varpi^p A^\circ$ such that the Frobenius map $\varphi : A^\circ/(\varpi) \to A^\circ/(\varpi^p)$ is surjective.

A Huber pair (A, A^+) is *perfectoid* if A is perfectoid. This implies an analogue of (b) with A° replaced by A^+; see [28, Corollary 2.3.10].

Remark 3.3. Beware that different sources use the term *perfectoid* at different levels of generality. In [44], the only rings considered are perfectoid K-algebras where K is itself a perfectoid field. In [32], only perfectoid $\mathbb{Q}_p$-algebras are considered. The definition we give above was introduced by Fontaine [15] and adopted by Kedlaya–Liu in [33] and Scholze in [47]. Even more general definitions are also possible, as in [4].

Remark 3.4. Given a perfectoid ring A, there is not much wiggle room left in the choice of A^+; it is a subring of A° and the quotient A°/A^+ is an *almost zero* A^+-module, meaning that it is annihilated by every topologically nilpotent element of A^+.

The notion of an almost zero module is the starting point of *almost commutative algebra* as introduced by Faltings and developed by Gabber–Ramero [18], in which one systematically defines *almost* versions of various ring-theoretic and module-theoretic concepts consistent with the previous definition.

A crucial first example is given by the perfectoid analogues of Tate algebras.

Definition 3.5. For K a perfectoid field of characteristic 0, the rings

$$K\langle T^{p^{-\infty}}\rangle := (\mathfrak{o}_K[T^{p^{-\infty}}])^\wedge_p[p^{-1}], \qquad K\langle T^{\pm p^{-\infty}}\rangle := (\mathfrak{o}_K[T^{\pm p^{-\infty}}])^\wedge_p[p^{-1}].$$

are perfectoid rings for the p-adic topologies. More generally, if (A, A^+) is a perfectoid Huber pair, we may similarly define perfectoid rings $A\langle T^{p^{-\infty}}\rangle$, $A\langle T^{\pm p^{-\infty}}\rangle$.

The following is true but not straightforward to prove.

Proposition 3.6. *A perfectoid ring which is a field is a perfectoid field. That is, if the underlying ring is a field, then the topology is induced by some nonarchimedean absolute value.*

Proof. See [29, Theorem 4.2]. □

Remark 3.7. A related statement is that for A a perfectoid ring, the residue field of any maximal ideal of A is a perfectoid field. See [28, Corollary 2.9.14].

As for perfectoid fields, there is a tilting construction that plays a pivotal role in the theory.

Proposition 3.8. *Let (A, A^+) be a perfectoid Huber pair.*

(i) *The natural map*

$$\varprojlim_{x \mapsto x^p} A^+ \to \varprojlim_{x \mapsto x^p} A^+/(p)$$

is an isomorphism of multiplicative monoids.

(ii) *Using (ii) to upgrade* $\varprojlim_{x\mapsto x^p} A^+$ *to a ring* $A^{\flat +}$*, this ring occurs in a perfectoid Huber pair* $(A^\flat, A^{\flat +})$ *of characteristic* p *in which the underlying multiplicative monoid of* $A^\flat$ *is* $\varprojlim_{x\mapsto x^p} A$*. (Moreover,* $A^\flat$ *depends only on* A*, not on* A^+*.)*
(iii) *Let* $\sharp : A \to A^\flat$ *be the final projection. Then there exists a pseuoduniformer* $\overline{\varpi}$ *of* $A^\flat$ *such that* $\sharp(\overline{\varpi})/\varpi$ *is a unit in* A^+*.*
(iv) *With notation as in (iii), the rings* $A^+/(\varpi)$ *and* $A^{\flat +}/(\overline{\varpi})$ *are isomorphic.*

Proof. See [28, Theorem 2.3.9]. □

Remark 3.9. The construction of perfectoid Tate algebras (Definition 3.5) commutes with tilting.

Definition 3.10. Let (R, R^+) be a perfectoid Huber pair of characteristic p. An element $z = \sum_{n=0}^{\infty} [\overline{z}_n] p^n \in W(R^+)$ is *primitive* if $\overline{z}_0$ is topologically nilpotent and $\overline{z}_1$ is a unit. Any associate of a primitive element is again primitive; we thus say that an ideal of $W(R^+)$ is *primitive* if it is principal and some (hence any) generator is primitive.

Proposition 3.11. *Let* (A, A^+) *be a perfectoid Huber pair.*
(i) *There is a unique homomorphism* $\theta : W(A^{\flat +}) \to A^+$ *whose restriction along the Teichmüller map is the map* $\sharp$*.*
(ii) *The map* θ *is surjective.*
(iii) *The kernel of* θ *is primitive.*

Proof. See [28, Theorem 2.3.9]. □

Proposition 3.12. *Let* (R, R^+) *be a perfectoid Huber pair of characteristic* p*. For every primitive ideal* I *of* $W(R^+)$*, there exist a perfectoid Huber pair* (A, A^+) *and an identification* $(A^\flat, A^{\flat +}) \cong (R, R^+)$ *under which* I *corresponds to the kernel of* θ*.*

Proof. See [28, Theorem 2.3.9]. □

Remark 3.13. For A a perfectoid ring, the categories of perfectoid rings over A and $A^\flat$ are equivalent via tilting, using the primitive ideal coming from A to untilt extensions of $A^\flat$. The case where A is a perfectoid field is the form of the tilting equivalence stated in [44].

The compatibility of tilting with finite extensions of fields has the following analogue for rings.

Proposition 3.14. *Let* (A, A^+) *be a perfectoid Huber pair.*
(i) *Let* $A \to B$ *be a finite étale morphism and let* B^+ *be the integral closure of* A^+ *in* B*. Then* (B, B^+) *is again a perfectoid Huber pair.*
(ii) *The categories of finite étale algebras over* A *and over* $A^\flat$ *are equivalent via tilting.*

Proof. See [28, Theorem 2.5.9]. □

A new feature in the ring case is that we also have a compatibility of tilting with localization.

Proposition 3.15. *Let* (A, A^+) *be a perfectoid Huber pair.*
(i) *Let* $(A, A^+) \to (B, B^+)$ *be a rational localization. Then* (B, B^+) *is again a perfectoid Huber pair. (In particular,* B *is again uniform.)*

(ii) *The categories of rational localizations of* (A, A^+) *and of* $(A^\flat, A^{\flat+})$ *are equivalent via tilting.*

Proof. See [28, Theorem 2.5.3]. □

This allows to construct adic spaces using the following criterion for sheafiness of Huber rings.

Definition 3.16. A Huber pair (A, A^+) is *stably uniform* if for every rational localization $(A, A^+) \to (B, B^+)$, the Huber ring B is uniform. This depends only on A, not on A^+ [28, Definition 1.2.12].

Proposition 3.17 (Buzzard–Verberkmoes, Mihara)**.** *Any stably uniform Huber pair is sheafy.*

Proof. This is due independently to Buzzard–Verberkmoes [7, Theorem 7] and Mihara [40, Theorem 4.9]. See also [32, Theorem 2.8.10] and [28, Theorem 1.2.13] (which also covers the case where A is analytic but not Tate). □

Proposition 3.18. *Every perfectoid Huber pair is sheafy.*

Proof. This follows by combining Proposition 3.15 with Proposition 3.17. □

Remark 3.19. In the previous statement, it is crucial that we have a criterion for sheafiness without a noetherian hypothesis: a perfectoid ring cannot be noetherian unless it is a finite product of perfectoid fields. See [28, Corollary 2.9.3].

Definition 3.20. For (A, A^+) a perfectoid Huber pair, by Proposition 3.18 the structure presheaf on $\mathrm{Spa}(A, A^+)$ is a sheaf. We may thus define a *perfectoid space* to be a locally v-ringed space which is locally of this form.

Remark 3.21. As a first example, one can use the perfectoid Tate algebra to define analogues of projective spaces in the category of perfectoid algebras; these play an important role in the application to the weight-monodromy conjecture given in [44], in which one exploits the fact that the conjecture is known in the equal-characteristic setting to deduce certain cases of it in mixed characteristic. One can also extend both the construction and the application to toric varieties; we leave this to the interested reader.

Proposition 3.22. *For* (A, A^+) *a sheafy Huber pair, the structure sheaf on* (A, A^+) *is acyclic. In particular, by Proposition 3.18, this holds when* (A, A^+) *is perfectoid.*

Proof. See [28, Theorem 1.4.16]. □

Remark 3.23. There are some further compatibilities of tilting with other algebraic operations or properties.

- Tilting commutes with taking completed tensor products [28, Theorem 2.4.1]. This implies the existence of fiber products in the category of perfectoid spaces.
- Certain properties of morphisms of perfectoid rings are compatible with tilting, including injectivity [28, Corollary 2.9.13], strict injectivity [28, Theorem 2.4.2], surjectivity [28, Theorem 2.4.4], or having dense image [28, Theorem 2.4.3].

4. Fargues–Fontaine curves

We now show how to construct "moduli spaces of untilts" in the spirit of Remark 2.17, leading to the schematic and adic Fargues-Fargues curves.

Throughout this lecture, let F be a perfect(oid) nonarchimedean field of characteristic p.

Definition 4.1. For any element ϖ of the maximal ideal of $\mathfrak{o}_F$, the ring $\mathbf{A}_{\mathrm{inf}}(F)$ is complete for the $(p, [\varpi])$-adic topology; we may thus view it as a Huber ring using itself as the ring of definition, and then form the adic spectrum $\mathrm{Spa}(\mathbf{A}_{\mathrm{inf}}(F), \mathbf{A}_{\mathrm{inf}}(F))$. From this space, remove the zero locus of $p[\varpi]$; we denote the resulting space by Y_F.

Proposition 4.2. *The action of φ on Y_F is without fixed points, and moreover is properly discontinuous: every point admits a neighborhood whose images under the various powers of φ are pairwise disjoint.*

Proof. For $n \in \mathbb{Z}$, define the rational subsets U_n, V_n of Y_F by the formulas

$$U_n := \{v \in Y_F : v([\varpi])^{p^n+p^{n-1}} \leq v(p) \leq v([\varpi])^{p^n}\}$$
$$V_n := \{v \in Y_F : v([\varpi])^{p^{n+1}} \leq v(p) \leq v([\varpi])^{p^n+p^{n-1}}\}.$$

Then the U_n are pairwise disjoint and $\varphi(U_n) = U_{n+1}$; the V_n are pairwise disjoint and $\varphi(V_n) = V_{n+1}$; and the union of all of the U_n and V_n is all of Y_F. □

Definition 4.3. By Proposition 4.2, we may form the quotient $X_F^{\mathrm{an}} := Y_F/\varphi$. This quotient is the *adic Fargues–Fontaine curve* associated to F. (We will define later a *schematic Fargues–Fontaine curve* which has X_F as its "analytification".)

In order to say anything more, we must analyze the rings that arise in the construction.

Definition 4.4. Fix a normalization of the absolute value on F. For $\rho \in (0,1)$, we define the *ρ-Gauss norm* on $\mathbf{A}_{\mathrm{inf}}(F)$ as the function $|\bullet|_\rho : \mathbf{A}_{\mathrm{inf}}(F) \to [0, +\infty)$ defined by

$$x = \sum_{n=0}^{\infty} [\overline{x}_n] p^n \mapsto \max_n \{\rho^n |\overline{x}_n|\}.$$

Remark 4.5. Recalling that we think of $W(\mathfrak{o}_F)$ as an interpretation of the nonsensical expression $\mathfrak{o}_F[\![p]\!]$, we keep in mind that the following facts about the ρ-Gauss norm on $\mathbf{A}_{\mathrm{inf}}(F)$ parallel more elementary facts about the ρ-Gauss norm on $\mathfrak{o}_F[\![T]\!]$:

$$x = \sum_{n=0}^{\infty} \overline{x}_n T^n \mapsto \max_n \{\rho^n |\overline{x}_n|\}.$$

For any closed interval $I \subset (0,1)$, define also

$$|x|_I = \sup\{|x|_\rho : \rho \in I\}.$$

Proposition 4.6 (Hadamard three circles property)**.** *For any fixed $x \in \mathbf{A}_{\mathrm{inf}}(F)$, the function $\rho \mapsto |x|_\rho$ is continuous and log-convex. The latter means that for $\rho_1, \rho_2 \in (0,1)$ and $t \in [0,1]$, for $\rho := \rho_1^t \rho_2^{1-t}$ we have*

$$|x|_\rho \leq |x|_{\rho_1}^t |x|_{\rho_2}^{1-t}.$$

In particular, for any closed interval $I = [\rho_1, \rho_2] \subset (0,1)$, we have

$$|x|_I = \max\{|x|_{\rho_1}, |x|_{\rho_2}\}.$$

Proof. The log-convexity inequality is an equality in case $x = [\overline{x}_n]p^n$, and hence a valid inequality in general. This in turn implies continuity. $\square$

Proposition 4.7. *For $\rho \in (0,1)$, the function $|\bullet|_\rho$ is a nonarchimedean absolute value on $\mathbf{A}_{\mathrm{inf}}(F)$.*

Proof. Modulo changes of notation, this can be found in any of [14, Lemme 1.4.2], [22, Lemma 2.1.7], [24, Lemma 2.2], [25, Lemma 4.1], [32, Proposition 5.1.2]. To summarize, the strong triangle inequality follows from the homogeneity of Witt vector arithmetic: we have

$$\sum_{n=0}^{\infty}[\overline{x}_n]p^n + \sum_{n=0}^{\infty}[\overline{y}_n]p^n = \sum_{n=0}^{\infty}[\overline{z}_n]p^n, \qquad \overline{z}_n = \overline{x}_n + \overline{y}_n + P(\overline{x}_0, \dots, \overline{x}_{n-1}, \overline{y}_0, \dots, \overline{y}_{n-1})$$

where P is homogeneous of degree 1 with coefficients in $\mathbb{Z}$. The multiplicative property is easiest to derive in an indirect way. For any given x and y, the multiplicativity is clear for those values of ρ for which both maxima are achieved by a unique index; this omits a discrete set of values of ρ, which we can fill in by continuity (Proposition 4.6). $\square$

Definition 4.8. The *Newton polygon* associated to an arbitrary element $x = \sum_{n=0}^{\infty}[\overline{x}_n]p^n$ of $\mathbf{A}_{\mathrm{inf}}(F)$ is the lower boundary of the convex hull of the set

$$\bigcup_{n=0}^{\infty}[n,\infty) \times [-\log|\overline{x}_n|, \infty) \subset \mathbb{R}^2.$$

The multiplicativity of the Gauss norms implies that this Newton polygon has the usual property: the slope multiset of a product xy is the multiset union of the slope multisets of x and y. (See [14, Définition 1.6.18].)

Definition 4.9. For $I \subseteq (0,1)$ a closed interval, let B_I be the completion of $\mathbf{A}_{\mathrm{inf}}(F)[p^{-1}, [\varpi]^{-1}]$ with respect to the norm $|\bullet|_I = \sup\{|\bullet|_\rho : \rho \in I\}$ (extending $|\bullet|_\rho$ to $\mathbf{A}_{\mathrm{inf}}(F)[p^{-1}, [\varpi]^{-1}]$ by multiplicativity). This norm is power-multiplicative (for all x, $|x|_I^2 = |x^2|_I$); consequently, B_I is a uniform Huber ring.

In case $I = [\rho_1, \rho_2]$ where $\rho_i = |\varpi|^{s_i}$ for some $s_i \in \mathbb{Q}$, the ring B_I is the ring associated to the rational subspace

$$\{v \in Y_F : v([\varpi])^{s_2} \leq v(p) \leq v([\varpi])^{s_1}\}$$

of Y_F. In the analogy between $\mathbf{A}_{\mathrm{inf}}(F)$ and $\mathfrak{o}_F[\![p]\!]$, B_I corresponds to the expression

$$F\left\langle \frac{p}{\varpi^{s_1}}, \frac{\varpi^{s_2}}{p} \right\rangle.$$

Remark 4.10. Beware that one cannot express an arbitrary element of B_I as a sum $\sum_{n\in\mathbb{Z}}[\overline{x}_n]p^n$ (see the published erratum to [22]). However, for any $x \in B_I$ and any $\epsilon > 0$, one can find a finite sum $y = \sum_{n\in\mathbb{Z}}[\overline{y}_n]p^n$ such that $|x-y|_I < \epsilon$.

Proposition 4.11. *For $I \subseteq (0,1)$ a closed interval, the ring B_I is a principal ideal domain.*

Proof. See [14, Théorème 2.5.1], [22, Proposition 2.6.8]. The key point is that the Banach ring B_I has the property that its associated graded ring is a Laurent polynomial ring (generated by the image of p) over the associated graded ring of F. $\square$

Proposition 4.12. *The ring B_I is* strongly noetherian *(every Tate algebra over it is noetherian) and sheafy. Consequently, the structure presheaf on Y_F is a sheaf, so we may view Y_F and X_F^{an} as "honest" noetherian adic spaces, and consider coherent sheaves on them.*

Proof. The strongly noetherian property is proved in [27], using similar ideas as in the proof of Proposition 4.11. This implies the sheafy property by a result of Huber [21, Theorem 2]. □

Remark 4.13. The rings B_I share other properties with the usual affinoid algebras appearing in rigid analytic geometry; in particular, they are known to be excellent [49].

Remark 4.14. One can also define the ring B_I when I is a half-open or open interval, but not as a Banach ring. Rather, one takes the *Fréchet completion* of $\mathbf{A}_{\mathrm{inf}}(F)[p^{-1}, [\varpi]^{-1}]$ with respect to the family of norms $|\bullet|_\rho$ for $\rho \in I$; that is, one declares a sequence to be Cauchy (and thus to have a limit) if it is Cauchy for each Gauss norm individually, but with no uniform control on the rate of convergence. (One can also use this definition when I is closed; by the last part of Proposition 4.6, it gives the same definition as before.)

The rings B_I correspond to the *extended Robba rings* of [32].

Definition 4.15. Since $Y_F \to X_F^{\mathrm{an}}$ is a free quotient by the action of φ, we can specify sheaves on X_F^{an} by specifying φ-equivariant sheaves on Y_F. For example, for $n \in \mathbb{Z}$, we can define a line bundle $\mathcal{O}(n)$ on X_F^{an} by taking the trivial line bundle on Y_F on a generator $\mathbf{v}$, then specifying that the action of φ takes $\mathbf{v}$ to $p^{-n}\mathbf{v}$.

Define the graded ring

$$P_F := \bigoplus_{n=0}^{\infty} P_{F,n}, \qquad P_{F,n} = \Gamma(X_F^{\mathrm{an}}, \mathcal{O}(n)) = \Gamma(Y_F, \mathcal{O})^{\varphi = p^n}.$$

The scheme $X_F := \operatorname{Proj} P_F$ is the *schematic Fargues–Fontaine curve* associated to F. It is a scheme over $\operatorname{Spec} \mathbb{Q}_p$ but not over $\operatorname{Spec} F$ (because P_F is not an F-algebra).

Proposition 4.16. *The scheme X_F has the following properties.*

(a) *It is connected, separated, noetherian, and regular of dimension* 1.
(b) *For each closed point $x \in X_F$, the residue field of x is a perfectoid field whose tilt may be naturally viewed as a finite extension of F; we write* $\deg(x)$ *for the degree of this extension. (In particular, if F is algebraically closed, then* $\deg(x) = 1$ *always.)*
(c) *The degree map on divisors induces a morphism* $\deg : \operatorname{Pic}(X_F) \to \mathbb{Z}$ *taking $\mathcal{O}(n)$ to n. Moreover, if F is algebraically closed, then* $\operatorname{Pic}(X_F) \cong \mathbb{Z}$.

Proof. See [14, Théorème 6.5.2] for the case where F is algebraically closed, and [14, Théorème 7.3.3] for the general case. □

Remark 4.17. Proposition 4.16 states that X_F, together with the degree function on closed points, constitute an *abstract complete curve* in the sense of [14, §5].

Definition 4.18. By construction, there is a morphism $X_F^{\mathrm{an}} \to X_F$ of locally ringed spaces, along which the canonical ample line bundle $\mathcal{O}(1)$ on X_F pulls back to the prescribed $\mathcal{O}(1)$ on X_F^{an}. This morphism should be thought of as a form of "analytification", analogous to the morphism $X^{\mathrm{an}} \to X$ where X is a scheme locally of finite type over $\mathbb{C}$ and X^{an} is its associated complex analytic space [20, Expose XII], or similarly with $\mathbb{C}$ replaced by a nonarchimedean field, using rigid analytic geometry in place of complex analytic geometry [10, Appendix].

Proposition 4.19 (GAGA for X_F). (a) *The line bundle $\mathcal{O}(1)$ on X_F^{an} is ample. More precisely, for every coherent sheaf $\mathcal{F}$ on X_F^{an}, there exists a positive integer N such that for each integer $n \geq N$, $\mathcal{F}(n)$ is generated by global sections and $H^i(X_F^{\mathrm{an}}, \mathcal{F}(n)) = 0$ for all $i > 0$. (Note that this vanishing only has content for $i = 1$, because X_F^{an} admits a covering by two affinoids.)*

(b) *Pullback from X_F to X_F^{an} defines an equivalence of categories between coherent sheaves on the two spaces. Moreover, the sheaf cohomology of a coherent sheaf is preserved by pullback from X_F to X_F^{an}.*

Proof. See [14, Théoréme 11.3.1]. □

Remark 4.20. In general, the cohomology groups of a coherent sheaf on X_F are Banach spaces over $\mathbb{Q}_p$ which are typically not finite dimensional. However, they do have a somewhat weaker finiteness property: they are *Banach–Colmez spaces* [9]. In fact, the derived categories of coherent sheaves on X_F and Banach–Colmez spaces are equivalent [38].

Proposition 4.21. *One consequence of Proposition 4.19 is that the category of vector bundles on X_F is equivalent to the category of φ-equivariant vector bundles on Y_F. These can themselves be described algebraically: the space Y_F is a quasi-Stein space, so vector bundles correspond to finite projective modules over $\Gamma(Y_F, \mathcal{O}) = B_{(0,\infty)}$; and moreover the ring $B_{(0,\infty)}$ is a* Bézout domain *(every finitely generated ideal is principal), which implies that finite projective modules are free. Consequently, vector bundles on X_F can be equated with φ-modules over $B_{(0,\infty)}$; this is the basis for the description of (φ, Γ)-modules in the sense of Berger using vector bundles on a Fargues–Fontaine curve.*

Another description of vector bundles can be given using the Beauville–Laszlo theorem [1] *to glue them from their restriction to the completed local ring at some point and to the complement of that point. In the case where have a specified untilt K of F in mind, that defines a degree-*1 *point of X_F and the completion of the local ring is Fontaine's period ring $\mathbf{B}_{\mathrm{dR}}^+$ associated to K. This then leads to the description of Berger's (φ, Γ)-modules in terms of B-pairs* [2].

5. Vector bundles on Fargues–Fontaine curves

We give the classification of vector bundles on Fargues–Fontaine curves, then briefly introduce the relative version of the construction. See [28, Lecture 3] for a more detailed discussion.

As in the previous lecture, let F be a perfect nonarchimedean field of characteristic p.

Definition 5.1. Let V be a vector bundle on either X_F or X_F^{an} (by Proposition 4.19 these are interchangeable). Since X_F is connected, the *rank* of V is a well-defined nonnegative integer. The *degree* of V is the degree of the top exterior power $\wedge^{\mathrm{rank}(V)} V$ via the map $\deg : \mathrm{Pic}(X_F) \to \mathbb{Z}$. For V nonzero, the *slope* of V is the ratio $\mu(V) := \deg(V)/\operatorname{rank}(V)$. We say that V is *semistable* (resp. *stable*) if every proper nonzero subbundle W of V satisfies $\mu(W) \leq \mu(V)$ (resp. $\mu(W) < \mu(V)$).

Remark 5.2. The definitions in Definition 5.1 are copied verbatim from the theory of vector bundles on curves in algebraic geometry. In particular, the term *semistable*, having its origins in geometric invariant theory, is quite entrenched within that subject. This creates a terminological issue in p-adic Hodge theory, where we also consider *semistable* Galois representations. This may be unfortunate but is in no way an accident; this second use of the word can be traced back to the notion of *semistable reduction* of families of curves, which is named as such again because it relates to the same phenomenon in geometric invariant theory.

Proposition 5.3. *Let V, V' be semistable vector bundles on X_F. If $\mu(V) > \mu(V')$, then* $\mathrm{Hom}(V, V') = 0$.

Proof. As per [28, Lemma 3.4.5], this reduces to the fact that rank-1 bundles are stable, which in turn reduces to the case of $\mathcal{O}$. This case follows by calculating that $H^0(X_F, \mathcal{O}) = \mathbb{Q}_p$. $\square$

Proposition 5.4. *Every vector bundle V on X_F admits a unique filtration*

$$0 = V_0 \subset \cdots \subset V_l = V$$

in which each quotient V_i/V_{i-1} is a vector bundle which is semistable of some slope μ_i, and $\mu_1 > \cdots > \mu_l$. This is called the Harder–Narasimhan filtration *of V.*

Proof. This is essentially a formal consequence of Proposition 4.16 and Proposition 5.3. See [14, Théorème 5.5.2] or [28, Lemma 3.4.9]. $\square$

Definition 5.5. For V a vector bundle on X_F, the *Harder–Narasimhan polygon* (or *HN polygon*) of V is the Newton polygon associated to the Harder–Narasimhan filtration. It has length equal to the rank of V, and for $i = 1, \dots, l$, the slope μ_i occurs with multiplicity $\mathrm{rank}(V_i/V_{i-1})$.

When F is algebraically closed, one can give a complete classification of vector bundles on X_F.

Definition 5.6. Let r/s be a rational number written in lowest terms; that is, r and s are integers with $\gcd(r, s) = 1$ and $s > 0$. Let $\mathcal{O}(r/s)$ be the vector bundle of rank s on X_F corresponding (via Proposition 4.19) to the trivial vector bundle generated by $\mathbf{v}_1, \dots, \mathbf{v}_s$ on Y_F equipped with the φ-action defined by

$$\varphi(\mathbf{v}_1) = \mathbf{v}_2, \quad \cdots \quad \varphi(\mathbf{v}_{s-1}) = \mathbf{v}_s, \quad \varphi(\mathbf{v}_s) = p^{-r}\mathbf{v}_1.$$

In case $s = 1$, this reproduces the definition of $\mathcal{O}(r)$.

Proposition 5.7 (Classification of vector bundles)**.** *Suppose that F is algebraically closed.*

(ii) *A vector bundle V on F of slope μ is stable if and only if it is isomorphic to $\mathcal{O}(\mu)$.*
(ii) *Every vector bundle V on F can be expressed (nonuniquely) as a direct sum of stable subbundles (of various slopes). In particular, the HN filtration of V splits (nonuniquely).*

Proof. This result has a slightly complicated history. As formulated, it is due to Fargues–Fontaine [14, Théorème 8.2.10], who give two distinct proofs: one using periods of p-divisible groups, and another using the theory of Banach–Colmez spaces (see Remark 4.20). However, using Proposition 4.19 it can also be deduced from earlier results of Kedlaya; see [28, Theorem 3.6.13] for more discussion of this point (and a high-level sketch of the proof). The key point is to show that any V which sits in a nonsplit short exact sequence

$$0 \to \mathcal{O}(-1/n) \to V \to \mathcal{O}(1) \to 0$$

is trivial; the space of such extensions is essentially the Scholze–Weinstein moduli space of p-divisible groups [46]. $\square$

Remark 5.8. Proposition 5.7 is formally similar to the classification of vector bundles on the projective line over a field, in which every vector bundle splits as a direct sum of various $\mathcal{O}(n)$. A more apt analogy is the classification of rational Dieudonné modules over an algebraically closed field (Dieudonné–Manin classsification) in which some higher-rank objects with fractional slopes also appear; indeed, some of the precursor statements to Proposition 5.7 mentioned above are formulated as Dieudonné–Manin classifications for φ-modules over the ring $B_{(0,1)}$ or other related rings.

Proposition 5.9 (Analogue of Narasimhan–Seshadri)**.** *The functor*

$$V \mapsto \Gamma(X_{\widehat{\overline{F}}}, V)$$

defines an equivalence of categories between semistable vector bundles of slope 0 *on* X_F *and continuous representations of the absolute Galois group* G_F *of* F *on finite-dimensional* $\mathbb{Q}_p$*-vector spaces.*

Proof. This follows from Proposition 5.7 and the equality $\Gamma(X_F, \mathcal{O}) = \mathbb{Q}_p$. $\square$

Remark 5.10. For line bundles, Proposition 5.9 gives rise to a canonical isomorphism

$$\mathrm{Pic}(X_F) \cong \mathbb{Z} \oplus \mathrm{Hom}_{\mathrm{cont}}(G_F, \mathbb{Q}_p^\times).$$

Remark 5.11. Proposition 5.9 is meant to evoke the Narasimhan–Seshadri theorem [41]: for X a compact Riemann surface, there is a canonical equivalence of categories between stable vector bundles of slope 0 on X and irreducible finite-dimensional unitary representations of the fundamental group of X.

In the theory of vector bundles on curves in algebraic geometry, the Narasimhan–Seshadri theorem implies that the tensor product of two semistable vector bundles on a curve is semistable *provided* that the base field is of characteristic 0. The fact that this is a highly nonformal statement can be seen by its failure to carry over to positive characteristic, which was first observed by Gieseker [19]. Correspondingly, Proposition 5.9 implies that the tensor product of two semistable vector bundles on X_F is semistable.

Many applications of the theory of vector bundles on curves involve moduli spaces of these bundles. In order to study these for Fargues–Fontaine curves, we need to introduce the relative form of the construction, in which the base field is replaced by a perfectoid ring (or a space, or...).

Definition 5.12. Let (R, R^+) be a perfectoid Huber pair of characteristic p. Let Y_R be the complement of the zero locus of $p[\varpi]$ in $\mathrm{Spa}(W(R^+), W(R^+))$, where $\varpi \in R$ is any pseudouniformizer (the answer does not depend on the choice).

Now fix a power-multiplicative Banach norm on R. For $\rho \in (0,1)$, we may define the ρ-Gauss norm on $W(R^+)$ by the same formula as before. For I a closed interval in $(0,1)$, we may then define a ring $B_{I,R}$ by completing $W(R^+)[p^{-1}, [\varpi]^{-1}]$ for the supremum of the ρ-Gauss norms for all $\rho \in I$, and again use the spectra of these rings to cover Y_R.

One cannot hope for the ring $B_{I,R}$ to be noetherian in general, nor is it perfectoid (because this was already not true when R was a field). However, it is close enough to being perfectoid to inherit the sheafy property.

Proposition 5.13. *The Huber ring* $B_{I,R}$ *is stably uniform, and hence sheafy.*

Proof. While $B_{I,R}$ is not a perfectoid ring, it turns out that it becomes perfectoid after taking the completed tensor product over $\mathbb{Q}_p$ with any perfectoid field. This can be used to recover the stably uniform property by a splitting construction; see [28, Lemma 3.1.3]. $\square$

Definition 5.14. By Proposition 5.13, Y_R is an adic space. Following the previous model, we form the quotient $X_R^{\mathrm{an}} := Y_R/\varphi$ by the totally discontinuous action of φ; we define the line bundles $\mathcal{O}(n)$ on X_R^{an} in terms of φ-equivariant line bundles on Y_R; we define the graded ring $P_R := \bigoplus_{n=0}^{\infty} P_{R,n}$ by taking $P_{R,n}$ to be the sections of $\mathcal{O}(n)$; we define the scheme $X_R := \operatorname{Proj} P_R$; and we obtain a morphism $X_R^{\mathrm{an}} \to X_R$ of locally ringed spaces.

Remark 5.15. There is a natural continuous map $X_R^{\mathrm{an}} \to \operatorname{Spa}(R, R^+)$ of topological spaces; however, this morphism does not promote to a morphism of locally ringed spaces due to the mismatch of characteristics (namely, p is invertible on the source and zero on the target). That said, any untilt (A, A^+) of (R, R^+) over $\mathbb{Q}_p$ gives rise to a section of this map which does promote to a morphism of adic spaces.

Remark 5.16. Since neither X_R nor X_R^{an} is noetherian, we cannot easily handle coherent sheaves on these spaces. In [33] and [28, Lecture 1] one finds a theory of *pseudocoherent* sheaves, which obey a stronger finiteness condition; we omit this here and instead restrict attention to vector bundles in what follows. Before doing so, we point out that the following discussion implicitly uses the analogue of Kiehl's theorem for vector bundles on affinoid adic spaces: for (A, A^+) a sheafy Huber pair, the global sections functor defines an equivalence of categories between vector bundles on $\operatorname{Spa}(A, A^+)$ and finite projective A-modules [28, Theorem 1.4.2].

Proposition 5.17 (GAGA revisited)**.**
(a) *For every vector bundle V on X_R^{an}, there exists a positive integer N such that for each integer $n \geq N$, $V(n)$ is generated by global sections and $H^i(X_R^{\mathrm{an}}, V(n)) = 0$ for all $i > 0$.*
(b) *Pullback from X_R to X_R^{an} defines an equivalence of categories between vector bundles on the two spaces. Moreover, the sheaf cohomology of a vector bundle is preserved by pullback from X_R to X_R^{an}.*

Proof. See [32, Theorem 8.7.7]. $\square$

The following is analogue of the usual semicontinuity for families of vector bundles on a curve, or more generally on a family of varieties [48].

Proposition 5.18 (Kedlaya–Liu semicontinuity theorem)**.** *Let V be a vector bundle on X_R.*
(i) *The Harder–Narasimhan polygons of the fibers of V form a lower semicontinuous function on* $\operatorname{Spa}(R, R^+)$.
(ii) *If this function is constant, then the Harder–Narasimhan filtrations of the fibers of V arise by specialization from a filtration of V.*

Proof. See [32, Theorem 4.7.5, Corollary 7.4.10]. Additional discussion found in [28, Theorem 3.7.2]. $\square$

There is also a relative form of the Narasimhan–Seshadri theorem.

Proposition 5.19. *There is an equivalence of categories between étale $\mathbb{Q}_p$-local systems on* $\operatorname{Spa}(R, R^+)$ *(see below) and vector bundles on X_R which are fiberwise semistable of degree* 0.

Proof. See [32, Theorem 9.3.13]. Additional discussion found in [28, Theorem 3.7.5] □

Remark 5.20. In Proposition 5.19, one must be careful about the meaning of the phrase "étale $\mathbb{Q}_p$-local system". One way to interpret this correctly is via de Jong's theory of étale fundamental groups [12]; this amounts to saying that an étale $\mathbb{Q}_p$-local system is étale-locally the isogeny object associated to a $\mathbb{Z}_p$-local system. Another correct interpretation can be obtained by replacing the étale topology with a certain *pro-étale topology*; this is the approach taken in [32] based on a construction of Scholze [45].

Remark 5.21. The preceding discussion lies at the heart of the construction of moduli spaces of vector bundles on Fargues–Fontaine curves. This of course requires a globalization of the definition of the relative Fargues–Fontaine curve, first to perfectoid spaces, and second to certain stacks on the category of perfectoid spaces (in particular to what Scholze calls *diamonds*). See [47] for further discussion of these stacks and their role in the study of moduli spaces of vector bundles.

Another application of relative Fargues–Fontaine curves is to the study of cohomology of $\mathbb{Q}_p$-local systems on rigid analytic spaces over p-adic fields. See [34].

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