Errata for *p-adic Differential Equations* (updated 26 Oct 2021)

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Definition 0.3.2: The formula for $(a)_i$ should read $a(a+1)\cdots(a+i-1)$, not $a(a+1)\cdots(a+i)$.

Theorem 1.2.4: While the attribution of this result to Ostrowski is correct, the second reference is insufficient: it only covers the case $F = \mathbb{Q}$. For a complete reference for the second assertion, see for example J.W.S. Cassels, *Local Fields*.

Theorem 1.3.6: The last sentence of the proof is incorrect (the first inequality goes in the wrong direction); it should be replaced by the following. By the definition of the quotient norm,

$$\begin{aligned} |a_1m_1 + \cdots + a_nm_n|_V &\geq |a_2m_2 + \cdots + a_nm_n|_{V'} \\ &\geq c_2 \max\{|a_2|, \dots, |a_n|\} \\ &\geq \frac{c_2}{\max\{|m_2|_V, \dots, |m_n|_V\}} |a_2m_2 + \cdots + a_nm_n|_V . \end{aligned}$$

Put $c_1 = \min\{1, c_2/\max\{|m_2|_V, \dots, |m_n|_V\}\}$; we then have

$$\begin{aligned} c_1^{-1} |a_1m_1 + \cdots + a_nm_n|_V &= \max\{|a_1m_1 + \cdots + a_nm_n|_V, |a_2m_2 + \cdots + a_nm_n|_V\} \\ &\geq |a_1m_1|_V , \end{aligned}$$

from which it follows that $|a_1m_1 + \cdots + a_nm_n|_V \geq c_1 |a_1m_1|_V$. This proves that $|\cdot|_V$ is equivalent to the supremum norm defined by $m_1, \dots, m_n$.

Lemma 1.3.8: In the last paragraph of the proof, the equation $x_{ij} = \sum_{h=i+1}^{\infty} a_{h,j}m_j$ should read $x_{ij} = \sum_{h=i+1}^{\infty} a_{h,j}m_h$.

Theorem 1.4.9: the given proof is incorrect. A submultiplicative norm on a field need not be multiplicative; consider for instance the norm on $\mathbb{Q}$ given by taking the supremum of the p-adic norms. In fact, the given formula does not define a multiplicative norm in general; worse yet, it is possible to choose F so that there is no $\alpha \in E$ for which the given formula describes a multiplicative norm. (This is connected to the phenomenon of defect described in Chapter 3.)

Here is a correct proof of Theorem 1.4.9. For $\alpha \in E$, let $P(T)$ be the minimal polynomial of α over F, put $d = \deg(P)$, and define $|\alpha| = |P(0)|^{1/d}$. To check that this gives a multiplicative norm, choose an arbitrary $\beta \in E$ with minimal polynomial $Q(T)$ of degree f. The polynomials P and Q are irreducible, so by Theorem 2.2.1 their Newton polygons consist of single segments of some slopes r and s, respectively. Write $P(T) = \sum_i P_iT^i$ and $Q(T) = \sum_j Q_jT^j$; then $|P_i| \leq e^{-r(d-i)}$ and $|Q_j| \leq e^{-s(f-j)}$, with equality for $i = j = 0$.

Factor $P(T) = (T-\alpha_1)\cdots(T-\alpha_d)$ and $Q(T) = (T-\beta_1)\cdots(T-\beta_e)$ over some algebraic extension of F, and define

$$A(T) = \sum_k A_kT^k = \prod_{i=1}^{d}\prod_{j=1}^{e}(T - \alpha_i - \beta_j), \qquad M(T) = \sum_k M_kT^k = \prod_{i=1}^{d}\prod_{j=1}^{e}(T - \alpha_i\beta_j).$$

Then A_k is an integer polynomial in the P_i and Q_j which is homogeneous of degree $df - k$ for the weighting giving degree $d - i$ to P_i and degree $f - j$ to Q_j. This implies that $|A_k| \leq e^{-\min\{r,s\}(df-k)}$, so the Newton polygon of A has no slopes less than $\min\{r, s\}$. By the multiplicativity of Newton polygons, the same holds for the minimal polynomial of $\alpha + \beta$, so $|\alpha + \beta| \leq e^{-\min\{r,s\}} = \max\{|\alpha|, |\beta|\}$. Meanwhile, M_k is an integer polynomial in the P_i and Q_j which is homogeneous of bidegree $(df - k, df - k)$ for the weighting giving bidegree $(d - i, 0)$ to P_i and bidegree $(0, f - j)$ to Q_j. This implies that $|M_k| \leq e^{-(r+s)(df-k)}$ with equality for $k = 0$, so the Newton polygon of M has all slopes equal to $r + s$. By the multiplicativity of Newton polygons, the same holds for the minimal polynomial of $\alpha\beta$, so $|\alpha\beta| = e^{-r-s} = |\alpha||\beta|$.

Notes for Chapter 1: Some prior references for Lemma 1.3.7 are [31, Proposition 2.6.2/3] and §3, Lemme 2 of: M. Matignon and M. Reversat, Sous-corps fermés d'un corps valué, *J. Algebra* **90** (1984), 491–515. See also Lemme 8 of: J. Fresnel and M. Matignon, Produit tensoriel topologique de corps valués, *Canad. J. Math.* **35** (1983), 218–273.

Exercises for Chapter 1: In exercise 9, it should be assumed that E is algebraic and the limit of the sequence $|\alpha_1|, |\alpha_2|, \dots$ is not in $p^{\mathbb{Q}}$.

Example 2.1.4: The last term of the polynomial should be p^3, not p^3T^3.

Theorem 2.2.1: The ring denoted R in the theorem statement is called F in the proof. The ring denoted R in the proof (to which Theorem 2.2.2 is applied) is a polynomial ring over the original ring.

Theorem 2.2.2: The hypothesis that R be a nonarchimedean ring includes the condition that the norm on R is multiplicative, which is too strong for some applications (e.g., to rings of matrices, as in Proposition 8.3.5). It should only be assumed that the norm on R is *submultiplicative*, i.e., for all $a, b \in R$, one has $|ab| \leq |a||b|$. Also, condition (d) should read $|ab - c| \leq \lambda^2|a||b|$, and the definition of B_μ should be correspondingly changed to

$$B_\mu = \{(u, v) \in U \times V : |(u, v)| \leq \mu|a||b|\}.$$

Notation 3.0.1: change "the separable closure of E" to "the separable closure of F".

Theorem 4.1.4: in the first line of the proof, replace V by $\mathbb{C}^n$.

Definition 4.1.8: change "will not necessarily form a basis for" to "will not necessarily fill out".

Definition 4.3.3: change "as in the nonarchimedean case" to "as in the archimedean case".

Corollary 4.4.8: while the statement is correct as written, later in these errata (see Theorem 6.7.4) we will need the following variant: condition (b) (but not (a)) holds for the function

$$\max\{1, \sigma_1/\delta\}^{a(n)}.$$

where $a(n)$ is a suitable function of n (e.g., $a(n) = 2^n n! - 1$). To see this, note that in the proof, we have

$$|U_1| \leq (\sigma_1/\delta)^{n-1}, \qquad |U_4| \leq (\sigma_1/\delta)\,|U_1|^2\,|U_3|^2$$

and U_3 is bounded using the induction hypothesis with n replaced by $n - i \leq n - 1$ and σ_1 replaced by $|A_2| \leq \sigma_1\,|U_1|$.

Definition 5.5.1: in the formula for multiplication, the sum on h should run from 0 to i (not j) and the binomial coefficient should be $\binom{i}{h}$ (not $\binom{j}{h}$).

Example 5.8.3: change "this makes sense only if" to "this cannot make sense except possibly if"

Remark 5.8.4: the last line in the display should read $(1 + d(r))T(r, D(m))$, not $(1 + d(r))T(r, m)$.

Definition 6.2.12: in the definition of a refined differential module, it is not necessary to assume *a priori* that V is pure, as this follows from the condition $|D|_{\mathrm{sp},V^\vee\otimes V} < |D|_{\mathrm{sp},V}$ by Lemma 6.2.8. Namely, if V is not pure, then it admits at least one subquotient V_1 with $|D|_{\mathrm{sp},V_1} = |D|_{\mathrm{sp},V}$ and at least one subquotient V_2 with $|D|_{\mathrm{sp},V_2} < |D|_{\mathrm{sp},V}$. Then $V_2^\vee \otimes V_1$ occurs as a subquotient of $V^\vee \otimes V$, so $|D|_{\mathrm{sp},V^\vee\otimes V} \geq |D|_{\mathrm{sp},V_2^\vee\otimes V_1} = |D|_{\mathrm{sp},V}$.

Definition 6.4.1: delete the word "field".

Lemma 6.7.3: to apply Lemma 6.7.1, one must first reduce to the case $|\sigma_1| = 1$. This can be achieved by rescaling d after adjoining an element of suitable norm to the constant subfield of F; note that the latter step does not change the operator norm of d, and so preserves the visible spectrum by Corollary 6.5.5. One can then apply Theorem 4.3.11 to put N in the right form.

Theorem 6.7.4: while the statement is correct as written, later in these errata (see Lemma 6.8.1) we will need the following variant: the conclusion of the theorem also holds for $\theta = \max\{1, \sigma_1/\delta\}^{a(n)}$ as in the comment on Corollary 4.4.8. In particular, in this formulation the bound $\delta \geq \theta\,|d|_F$ holds for δ in some neighborhood of σ_1 depending only on n and $|d|_F$.

Lemma 6.8.1: In the last sentence, the application of Theorem 6.7.4 as published is not sufficient; one needs the variant described above.

Definition 7.1.1: there should be no minus sign in the formula for irregularity.

Remark 7.1.4: change "for $a, b \in \mathbb{C}$" to "for $a, b, c \in \mathbb{C}$".

Theorem 7.2.1: in (7.2.1.1), there should be a minus sign in front of the summation.

Definition 7.2.4: in the last line, $e^{-\alpha_1}, \dots, e^{-\alpha_n}$ should be $e^{-2\pi i\alpha_1}, \dots, e^{-2\pi i\alpha_n}$.

Theorem 7.3.8: change "informal" to "nonformal".

Proposition 7.3.12: change d to D in one place in the statement and two places in the proof.

Lemma 8.0.4: change $[\alpha, \gamma]$ to $[\alpha, \gamma)$.

Proposition 8.3.5: the matrix U should belong to $\mathrm{GL}_n(K\langle t/\beta\rangle[t^{-1}])$, not $\mathrm{GL}_n(K\langle t/\beta\rangle)$.

Lemma 8.3.6: the matrix U should belong to $\mathrm{GL}_n(K\langle t/\beta\rangle[t^{-1}])$, not $\mathrm{GL}_n(K\langle t/\beta\rangle)$.

Corollary 8.5.3: the union is over α, with β fixed.

Proposition 8.5.7: the matrix U should belong to $\mathrm{GL}_n(K\langle t/\beta\rangle[t^{-1}])$, not $\mathrm{GL}_n(K\langle t/\beta\rangle)$.

Example 9.6.2: in order to get two linearly independent solutions (without using logarithmic solutions), one must also assume that c is not a positive integer.

Example 9.9.3: the reference to M_1 after the first displayed equation should be to M_0.

Definition 10.3.1: The second displayed equation does not make sense, because pt^{p-1} is not an element of F'_ρ. It should instead read

$$D(v \otimes f) = D'(v) \otimes pt^{p-1}f + v \otimes d(f).$$

Lemma 10.3.6: The statements of (d) and (f) are not correct. To fix them, one must assume K contains the full group μ_p of p-th roots of unity. In this case, for $\zeta \in \mu_p$, define the map $\zeta : F_\rho \to F_\rho$ as the substitution $t \mapsto \zeta t$. For a finite differential module (V, D) over F_ρ, define the pullback $\zeta^*(V, D)$ as the differential module $(\zeta^* V, D')$ with

$$\zeta^* V = V \otimes_{F_\rho, \zeta} F_\rho, \qquad D'(v \otimes f) = D(v) \otimes \zeta f + v \otimes d(f).$$

Then the correct statement of (d) is that if $\mu_p \subseteq K$, then

$$\varphi^* \varphi_* V \cong \bigoplus_{\zeta \in \mu_p} \zeta^* V.$$

More precisely, the map $\zeta^* V \to \varphi^* \varphi_* V$ takes $v \otimes 1$ to $\frac{1}{p} \sum_{i=0}^{p-1} (t/\zeta)^i v \otimes t^{-i}$. Similarly, the correct statement of (f) is that if $\mu_p \subseteq K$, then

$$\varphi_* V_1 \otimes \varphi_* V_2 \cong \bigoplus_{\zeta \in \mu_p} \varphi_*(V_1 \otimes \zeta^* V_2).$$

Theorem 10.4.2: In the proof, in order to conclude that $V'' \otimes W_0$ is contained in a factor of $V' \otimes W_0$, it must be shown not only that $IR(V' \otimes W_m) = p^{-p/(p-1)}$, but also that $IR(X') = p^{-p/(p-1)}$ for every Jordan-Hölder constituent X' of $V' \otimes W_m$. Since $W_m \otimes W_{-m} \cong W_0$, we can write $X' = X \otimes W_m$ for $X = X' \otimes W_{-m}$. Then $IR(X) \geq IR(V') > p^{-p/(p-1)}$ by Lemma 9.4.6(a), so $IR(X') = p^{-p/(p-1)}$ by Lemma 9.4.6(c).

Theorem 10.4.4: The equivalence between checking the condition for $\rho = \alpha$ and $\rho = \beta$ with checking the condition for all $\rho \in [\alpha, \beta]$ is not yet known at this point in the text; it will follow later from Theorem 11.3.2(e).

Theorem 10.5.1: The proof needs to be corrected to avoid the use of the incorrect formulation of Lemma 10.3.6(d) in the last paragraph (see above). This may be accomplished as follows.

Assume first that K contains the full group μ_p of p-th roots of unity. In the first sentence of the last paragraph of the proof, it is noted that $\varphi^* W'$ is a subquotient of $\varphi^* \varphi_* V$. By the corrected formulation of Lemma 10.3.6(d), the latter is isomorphic to $\oplus_{\zeta \in \mu_p} \zeta^* V$. Note that $IR(\zeta^* V) = IR(V)$ for each $\zeta \in \mu_p$ by Corollary 6.2.7. Since each $\zeta^* V$ is irreducible, each Jordan-Hölder constituent of $\varphi^* W'$ must be isomorphic to $\zeta^* V$ for some $\zeta \in \mu_p$, yielding $IR(\varphi^* W') = IR(V)$. We may then continue as in the original proof.

Still assuming that K contains μ_p, we may now deduce Proposition 10.6.1 and Theorem 10.6.2. To obtain Theorem 10.5.1 for general K, it is sufficient to verify that the subsidiary radii of V and $V \otimes_K K(\mu_p)$ coincide. For this, we may again assume V is irreducible. From the definition of the spectral radius, we see that $IR(V) = IR(V \otimes_K K(\mu_p))$. This is not enough to deduce the desired result because $V \otimes_K K(\mu_p)$ may fail to be irreducible. However, by Theorem 10.6.2 applied over $K(\mu_p)$, $V \otimes_K K(\mu_p)$ admits a strong decomposition, which by Corollary 6.2.7 again is $\mathrm{Gal}(K(\mu_p)/K)$-invariant. The strong decomposition of $V \otimes_K K(\mu_p)$ must therefore contain only a single summand, from which the claim follows.

Section 10.6: Change $IR(V) = p^{-1/(p-1)}\rho$ to $IR(V) = p^{-1/(p-1)}$.

Proposition 10.6.1, Theorem 10.6.2: Note that for a given K, these results depend on Theorem 10.5.1 for that particular K. As noted above (see the correction to Theorem 10.5.1), we must first prove Theorem 10.5.1 assuming that K contains the full group of p-th roots of unity, then deduce Proposition 10.6.1 and Theorem 10.6.2 under this assumption, then deduce Theorem 10.5.1 in full, then deduce Proposition 10.6.1 and Theorem 10.6.2 in full.

Proposition 10.6.6: The application of Corollary 6.2.7 in the proof is invalid, because it is not assumed that s lies in the visible spectrum. In fact, it is only necessary to execute the first paragraph for a Galois extension of constant subfields, in which case the preservation of the spectral radius is apparent from its definition (as in the corrected proof of Theorem 10.5.1 given above).

Theorem 10.6.7: Since we may include μ_p in E, we may assume from the outset that $\mu_p \subset K$. In the proof that X is refined, the displayed equation should read

$$(\varphi_* X^\vee) \otimes (\varphi_* X) \cong \bigoplus_{\zeta \in \mu_p} \varphi_*(X^\vee \otimes \zeta^* X).$$

The rest of the proof continues unchanged.

Definition 10.8.1: The quantity $t - 1$ should be $t + 1$ in all three places where it appears. Similarly, the quantity $t - \mu$ should be $t + \mu$. The inclusion $du/dt \in K\langle t \rangle^\times$ should read $du/dt \in K\langle t \rangle \cap K[\![t]\!]_{\mathrm{an}}^\times$.

Remark 11.6.4: change "differential" to "different".

Question 11.8.3: This question has a negative answer. See Exercises for Chapter 11.

Notes for Chapter 11: Footnote 1 is not quite accurate: Baldassarri's paper (F. Baldassarri, Continuity of the radius of convergence of differential equations on p-adic analytic curves, *Invent. Math.* **182** (2010), 513–584) only treats the radius of convergence, not all of the radii of optimal convergence.

Exercises for Chapter 11: Exercise (4) is false as stated. For example, if $p = 0$ and M is free of rank 1 with a single generator $\mathbf{v}$ satisfying $D(\mathbf{v}) = \lambda t^n \mathbf{v}$ for some nonnegative integer n, then $R(M) = \max\{\beta, |\lambda|^{-1/(n+1)}\}$ need not belong to $|K^\times|$. A similar counterexample can be used against Question 11.8.3 by taking n coprime to p.

Theorem 12.2.2: There is a subtlety in the induction step because $\varphi_* M$ is not a differential module: the Frobenius pushforward introduces a pole at $t = 0$. Note that this issue does not arise in the corresponding arguments for Theorem 12.3.1, Theorem 12.4.2, or Theorem 12.5.2; for Theorem 12.4.1 and Theorem 12.5.1, one may replace $\varphi_* M$ with the off-center Frobenius pushforward $\psi_* M$ to remedy the original argument. To address Theorem 12.2.2 itself, apply Theorem 12.3.1 to $M \otimes K\langle \beta/t, t/\beta \rangle$ (noting that condition (b) is vacuous when $\alpha = \beta$) and Theorem 12.5.1 to $M \otimes K[\![t/\beta]\!]_{\mathrm{an}}$, then glue the resulting decompositions.

Lemma 12.2.7: change $f_i(-\log \beta)$ to $f_i(M, -\log \beta)$.

Theorem 12.7.2: In the proof (page 213, line 10), ζ should be a primitive p^{h+1}-st root of unity.

Theorem 13.2.3: Above the theorem statement, "which may viewed" should be "which may be viewed". In the theorem statement, it should be assumed that the exponents of the regular singularity have p-adic non-Liouville differences, not that they are p-adic non-Liouville numbers; and the final inclusion should read $y \in M \otimes_{K\langle t/\beta \rangle} K\langle t/\rho \rangle$.

Remark 13.2.4: the displayed equation should read

$$F(a,b,c;z) = \sum_{i=0}^{\infty} \frac{a(a+1)\cdots(a+i-1)b(b+1)\cdots(b+i-1)}{c(c+1)\cdots(c+i-1)i!} z^i.$$

Proposition 13.4.5: From what is written, it is unclear how Proposition 13.1.4 implies that $B_{\sigma_m(i)} - B_{\sigma_m(i+1)}$ is forced to be zero for m large, since it varies with m. The point is that we may apply Proposition 13.1.4 to each element of the finite set $\{B_j - B_k : 1 \leq j, k \leq n\}$ to obtain a uniform lower bound $m_0 > 0$ such that for $1 \leq j, k \leq n$ and $m \geq m_0$, if $B_j \neq B_k$, then $|B_j - B_k|^{(m)} > 3cm \geq 2cm + c$. We then apply this with $j = \sigma_m(i), k = \sigma_{m+1}(i)$.

Definition 13.5.2: The group μ_{p^m} is the group of p^m-th roots of unity (not p-th roots of unity). The displayed equation should read

$$\zeta(x) = \sum_{i=0}^{\infty} (\zeta - 1)^i \binom{D}{i}(x)$$

because D is a derivation with respect to $t\frac{d}{dt}$ rather than $\frac{d}{dt}$. Delete "w" at the end of page 227, line 10. For the proofs of Theorem 13.5.6 and Theorem 13.6.1, we must also note that for any α', β' with $\alpha < \alpha' < \beta' < \beta$, the matrices $S_{m,A}$ are invertible over $K\langle \alpha'/t, t/\beta' \rangle$ for m sufficiently large (depending on α' and β); this follows from (b) and (c) as described in [79, Lemma 1.2].

Theorem 13.5.5: The notation $j(i)$ should be read as $j(i,m)$. The first sentence after (13.5.5.2) should read "Write $\det(S_{m,A}) = \sum_{i \in \mathbb{Z}} s_{m,A,i} t^i$." Thereafter, every expression of the form $\det(S_{m,A,0})$ (for various values of m and A) should be replaced with $s_{m,A,0}$.

Theorem 13.5.6: The definition of T_m should read $T_m = S_{m,A}^{-1} S_{m,B}$. The quantity $T_{i,\sigma_m(i)}$ is never defined: it is shorthand for the matrix entry $(T_m)_{i,\sigma_m(i)}$. In the last displayed equation, $A_1, \dots, A_n, B_1, \dots, B_n$ should be $A_1^{(m)}, \dots, A_n^{(m)}, B_1^{(m)}, \dots, B_n^{(m)}$, respectively.

Theorem 13.6.1: The given proof is incorrect starting from the third paragraph: the upper bound on $|T_{m',m}|_\rho, |T_{m',m}^{-1}|_\rho$ should be $p^{nkm'}$ rather than p^{nkm}. The argument from this point should instead be carried out as follows.

As in the proof of Theorem 13.5.6, for all m we have

$$|S_{m,A}|_\rho \leq p^{km}, \quad |S_{m,A}^{-1}|_\rho \leq p^{(n-1)km} \qquad (\rho \in [\alpha'', \beta'']).$$

Choose $\lambda \in (0,1)$ and $c > 0$ so that $p^{8nk}\eta^{-c} \leq \lambda$. Since A has p-adic non-Liouville differences, there exists $m_0 > 0$ such that for $m \geq m_0$, the congruence $h \equiv A_i - A_j \pmod{p^m}$ forces either $h = A_i - A_j = 0$ or $|h| \geq cm$. By enlarging m_0 if needed, we may also ensure that $A_i \equiv A_j \pmod{p^{m_0}}$ if and only if $A_i = A_j$.

The strategy is to renormalize the matrices $S_{m,A}$ to obtain a convergent sequence. To this end, we will construct invertible matrices R_m over K for $m \geq m_0$ such that $R_{m_0} = I_n$, $(R_m)_{ij} = 0$ whenever $A_i \neq A_j$ (or equivalently $A_i \not\equiv A_j \pmod{p^m}$), and

$$|I_n - R_m S_{m,A}^{-1} S_{m+1,A} R_{m+1}^{-1}|_\rho \leq \lambda^m \qquad (\rho \in [\alpha', \beta'], m \geq m_0).$$

This will imply that for $m \geq m_0$ and $\rho \in [\alpha', \beta']$, $|I_n - S_{m_0,A}^{-1} S_{m,A} R_m^{-1}|_\rho < 1$ and $|S_{m_0,A}^{-1} S_{m,A} R_m^{-1} - S_{m_0,A}^{-1} S_{m+1,A} R_{m+1}^{-1}|_\rho \leq \lambda^m$. Consequently, the sequence $S_{m_0,A}^{-1} S_{m,A} R_m^{-1}$ for $m = m_0, m_0+1, \dots$ will converge to an invertible matrix U over $K\langle \alpha'/t, t/\beta' \rangle$ which will have the desired effect (more on this below).

The construction of the R_m proceeds recursively as follows. Given $R_{m_0}, \dots, R_m$, we first verify that

$$|R_m|, |R_m^{-1}| \leq p^{nkm}.$$

This is clear for $m = m_0$, so we may assume $m > m_0$. Choose any $\rho \in [\alpha', \beta']$. As noted above, we have $|I_n - S_{m_0,A}^{-1} S_{m,A} R_m^{-1}|_\rho < 1$, so $|S_{m_0,A}^{-1} S_{m,A} R_m^{-1}|_\rho = |R_m S_{m,A}^{-1} S_{m_0,A}|_\rho = 1$. We then deduce the claim using the bounds on $|S_{m,A}|_\rho, |S_{m,A}^{-1}|_\rho$ from above.

Next, put $T_m = R_m S_{m,A}^{-1} S_{m+1,A}$; we then have $|T_m|_\rho, |T_m^{-1}|_\rho \leq p^{2nk(m+1)}$ for all $\rho \in [\alpha'', \beta'']$. If we write $T_m = \sum_{h \in \mathbb{Z}} T_{m,h} t^h$, then $(T_{m,h})_{ij}$ can only be nonzero if $h \equiv A_i - A_j \pmod{p^m}$, which forces either $h = 0$ or $|h| \geq cm$. If $h > 0$, we have

$$|(T_{m,h})_{ij} t^h|_{\alpha'} \leq |(T_{m,h})_{ij} t^h|_{\beta'} \leq |(T_{m,h})_{ij} t^h|_{\beta''} \eta^{-cm} \leq p^{2nk(m+1)} \eta^{-cm},$$

while if $h < 0$, we have

$$|(T_{m,h})_{ij} t^h|_{\beta'} \leq |(T_{m,h})_{ij} t^h|_{\alpha'} \leq |(T_{m,h})_{ij} t^h|_{\alpha''} \eta^{-cm} \leq p^{2nk(m+1)} \eta^{-cm}.$$

We may now take $R_{m+1} = T_{m,0}$, because

$$|I_n - R_{m+1} T_m^{-1}|_\rho \leq |T_m^{-1}|_\rho |T_m - T_{m,0}|_\rho \leq p^{2nk(m+1)} p^{2nk(m+1)} \eta^{-cm} \leq \lambda^m < 1 \qquad (\rho \in [\alpha', \beta'])$$

and so $|I_n - T_m R_{m+1}^{-1}|_\rho \leq \lambda^m$. (Note that indeed $(R_m)_{ij} = 0$ whenever $A_i \not\equiv A_j \pmod{p^m}$, and that the latter condition is equivalent to $A_i \neq A_j$ by our choice of m_0.) This completes the construction of the R_m.

At this point, we have exhibited a invertible matrix U over $K\langle \alpha'/t, t/\beta' \rangle$ such that $S_{m_0,A} U$ is the change-of-basis matrix to a basis $e_1, \dots, e_n$ of $M \otimes K\langle \alpha'/t, t/\beta' \rangle$ with the property that

$$\zeta(e_i) = \zeta^{A_i} e_i \qquad (\zeta \in \mu_{p^\infty}; i = 1, \dots, n).$$

We next check that the matrix of action N of D on $e_1, \dots, e_n$ has entries in K. To this end, note that the actions of ζ and D commute; writing $N_{ij} = \sum_{n \in \mathbb{Z}} N_{ijn} t^n$, we have

$$\begin{aligned}
\sum_i \left(\zeta^{A_j} \sum_{n \in \mathbb{Z}} N_{ijn} t^n \right) e_i &= D(\zeta^{A_j} e_j) \\
&= (D \circ \zeta)(e_j) = (\zeta \circ D)(e_j) \\
&= \sum_i \left(\sum_{n \in \mathbb{Z}} \zeta^{n+A_i} N_{ijn} t^n \right) e_i.
\end{aligned}$$

Since we previously enforced the assumption that A has no nonzero integer differences, it follows that $N_{ijn} = 0$ whenever $n \neq 0$ or $A_i \neq A_j$.

From the previous paragraph, we see that M decomposes into summands, each corresponding to a single value in A. It remains to check that the multiset of eigenvalues of N equals A; for this, we first reduce to the case where A consists of a single element, and then to the case where that single element equals 0. In this case, if b occurs as an eigenvalue of N, then by Example 9.5.2 we must have $b \in \mathbb{Z}_p$, and there must exist a nonzero eigenvector v of N with eigenvalue b of the form $c_1e_1 + \cdots + c_ne_n$ for some $c_1, \dots, c_n \in K$. However, we then have

$$\zeta^b v = \zeta(v) = \zeta(c_1e_1 + \cdots + c_ne_n) = c_1e_1 + \cdots + c_ne_n = v,$$

a contradiction if $b \neq 0$.

Exercises for Chapter 13: in exercise 10, it should be assumed that M has p-adic non-Liouville exponent differences.

Hypothesis 14.0.1: change "Part V" to "Part IV".

Definition 14.1.1: change $\varphi^*(M)$ to $\varphi^* M$.

Definition 14.5.1: change $s_{H,i}$ to $s_{H,1}$.

Notes for Chapter 14: change "C_k is a complete discretely valued field" to "C_k is a complete discrete valuation ring".

Theorem 15.3.4: the isomorphism is one of difference modules, not differential modules.

Theorem 16.1.1: in the sentence containing (16.1.1.1), the difference module V is the space of $n \times n$ matrices with the action $X \mapsto c^i A_0 \phi(X) A_0^{-1}$.

Lemma 17.3.4: in the proof, change $\Phi(A)$ to $\varphi(A)$.

Section 18.2: at the end of the first sentence, the ring should be $K\{t/\beta\}$.

Proposition 18.4.3: it should also be assumed that the fundamental solution matrix of M is an invertible matrix over $K\{t\}$, so that Theorem 18.2.1 applies.

Theorem 18.5.1: The bound as stated is not correct. This can be seen concretely by considering examples of the form

$$N = \begin{pmatrix} 0 & t \\ 0 & \alpha + \sum_{n=1}^\infty t^n \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & \sum_{n=1}^\infty \frac{t^n}{\alpha - n} \\ 0 & \sum_{n=0}^\infty t^n \end{pmatrix}$$

where $\alpha \in \mathbb{Z}_p$ has type 1 in the sense of Definition 13.1.1 (to enforce condition (e)). Such examples can have arbitrarily large log-growth: for k a positive integer and $\alpha = -\sum_{i=1}^\infty p^{k^i}$, U has order of log-growth k.

The error in the original proof occurs in the step "calculate explicitly as in Proposition 18.1.1." To illustrate this in more detail, and to give a corrected statement of Theorem 18.5.1, we state and prove a generalization of Proposition 18.1.1 under the hypothesis that N_0 is not nilpotent with nilpotency index m, but has prepared em $\alpha_1, \dots, \alpha_l \in \mathbb{Z}_p$ and the minimal polynomial of N_0 equals $\prod_{j=1}^l (T - \alpha_j)^{m_j}$. Under this condition, the map $f(X) = N_0 X - X N_0$ has eigenvalues $\alpha_j - \alpha_k$ for $1 \leq j, k \leq l$. By the hypothesis on prepared eigenvalues, the map $X \mapsto iX + f(X)$ remains invertible, but we cannot compute the inverse using the same formula as before.

Instead, note that by Lemma 7.3.5, the minimal polynomial of f is then $P(T) =$

$\prod_{j,k=1}^{l}(T-\alpha_j+\alpha_k)^{e_j+e_k-1}$. We may thus invert $X \mapsto iX + f(X)$ using the formula

$$X \mapsto \sum_{j=1}^{\deg(P)} -\frac{P^{(j)}(i)}{j!P(i)} X^j,$$

keeping in mind that $P^{(j)}(i)/j! \in \mathbb{Z}_p$. For the generalization of Proposition 18.1.1, we obtain the bound

$$|U_i|\,\beta^i \leq \prod_{j,k=1}^{l} \prod_{h=1}^{i} |h-\alpha_j+\alpha_k|^{-(m_j+m_k-1)}.$$

In the proof of Theorem 18.5.1, we must apply this bound for $i \leq p$. In this case, for each pair (j,k), at most one value of $h \in \{1,\dots,i\}$ contributes nontrivially to the product, so the

Using this bound, we may now follow the original proof of Theorem 18.5.1 to obtain the corrected bound

$$|U_i|\,\beta^i \leq cp^{n+(n-1)\lceil \log_p i \rceil} \max\{1, |N|_\beta^{n-1}\}$$

where

$$c = \max\left\{ \prod_{1 \leq j < k \leq l} \left| (\alpha_j - \alpha_j^{(s)} - \alpha_k + \alpha_k^{(s)})/p^s \right|^{-(m_j+m_k-1)} : 0 \leq s \leq \lfloor \log_p i \rfloor \right\}.$$

(Condition (d) ensures that $c < \infty$.)

Corollary 19.4.2: change $G_{\kappa_K((t)),i}$ to $G^i_{\kappa_K((t))}$.

Lemma 21.4.1: change $p^{p^{-h}/(p-1)} < IR(N \otimes F_\rho) < p^{p^{-h+1}/(p-1)}$ to $p^{-p^{-h+1}/(p-1)} < IR(N \otimes F_\rho) < p^{-p^{-h}/(p-1)}$.

Definition 24.4.1: change $D(V)$ to $D^\dagger_{\mathrm{rig}}(V)$.

Definition 24.4.3: change $D(V)$ to $D^\dagger_{\mathrm{rig}}(V)$ in three places.