p-adic Differential Equations

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Based on a course given by the author at MIT, this modern treatment is accessible to graduate students and researchers. Exercises are included at the end of each chapter to help the reader review the material, and the author also provides detailed references to the literature to aid further study.

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p-adic Differential Equations

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Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore,
São Paulo, Delhi, Dubai, Tokyo, Mexico City

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521768795

First published 2010

Printed in the United Kingdom at the University Press, Cambridge

A catalogue record for this publication is available from the British Library

Library of Congress Cataloguing in Publication data
Kedlaya, Kiran Sridhara, 1974–
p-adic differential equations / Kiran S. Kedlaya.
p. cm. – (Cambridge studies in advanced mathematics ; 125)
Includes bibliographical references and index.
ISBN 978-0-521-76879-5
1. *p*-adic analysis. 2. Differential
equations. I. Title. II. Series.
QA241.K43 2010
512.7′4–dc22
2010004489

ISBN 978-0-521-76879-5 Hardback

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Preface

This book is an outgrowth of a course, taught by the author at MIT during fall 2007, on p-adic ordinary differential equations. The target audience was graduate students with some prior background in algebraic number theory, including exposure to p-adic numbers, but not necessarily with any background in p-adic analytic geometry (of either the Tate or Berkovich flavors).

Custom would dictate that ordinarily this preface would continue with an explanation of what p-adic differential equations are, and why they matter. Since we have included a whole chapter on this topic (Chapter 0), we will devote this preface instead to a discussion of the origin of the book, its general structure, and what makes it different from previous books on the subject.

The subject of p-adic differential equations has been treated in several previous books. Two that we used in preparing the MIT course, and to which we make frequent reference in the text, are those of Dwork, Gerotto, and Sullivan [**80**] and of Christol [**42**]. Another existing book is that of Dwork [**78**], but it is not a general treatise; rather, it focuses in detail on hypergeometric functions.

However, this book develops the theory of p-adic differential equations in a manner that differs significantly from most prior literature. Key differences include the following.

- We limit our use of cyclic vectors. This requires an initial investment in the study of matrix inequalities (Chapter 4) and lattice approximation arguments (especially Lemma 8.6.1), but it pays off in significantly stronger results.
- We introduce the notion of a Frobenius descendant (Chapter 10). This complements the older construction of Frobenius antecedents, particularly in dealing with certain boundary cases where the antecedent method does not apply.

As a result, we end up with some improvements of existing results, including the following. (Some of these can also be found in an upcoming book of Christol [**46**], whose development we learned about only after this book was mostly complete.)

- We refine the Frobenius antecedent theorem of Christol and Dwork (Theorem 10.4.2).
- We extend some results of Christol and Dwork, on the variation of the generic radius of convergence, to subsidiary radii (Theorem 11.3.2).
- We extend Young's geometric interpretation of subsidiary generic radii of convergence beyond the range of applicability of Newton polygons (Theorem 11.9.2).
- We give quantitative versions of the Christol–Mebkhout decomposition theorem for differential modules on an annulus that are applicable even when the modules are not solvable at a boundary (Theorems 12.2.2 and 12.3.1).
- We give a somewhat simplified treatment of the theory of p-adic exponents (Theorems 13.5.5, 13.5.6, and 13.6.1).
- We sharpen the bound in the Christol transfer theorem to a disc containing a regular singularity with exponents in $\mathbb{Z}_p$ (Theorem 13.7.1).
- We give a general version of the Dieudonné–Manin classification theorem for difference modules over a complete nonarchimedean field (Theorem 14.6.3).
- We give improvements on the Christol–Dwork–Robba effective bounds for solutions of p-adic differential equations (Theorems 18.2.1 and 18.5.1) and some related bounds that apply in the presence of a Frobenius structure (Theorem 18.3.3). The latter can be used to recover a theorem of Chiarellotto and Tsuzuki concerning the logarithmic growth of solutions of differential equations with Frobenius structure (Theorem 18.4.5).
- We state a relative version of the p-adic local monodromy theorem, formerly Crew's conjecture (Theorem 20.1.4), and describe in detail how it may be derived either from the p-adic index theory of Christol and Mebkhout, which we treat in detail in Chapter 13, or from the slope theory for Frobenius modules of Kedlaya, which we only sketch, in Chapter 16.

Some of the new results are relevant in theory (in the study of higher-dimensional p-adic differential equations, largely in the context of the *semistable reduction problem* for overconvergent F-isocrystals, for which see [**138**] and [**143**]) or in practice (in the explicit computation of solutions of p-adic differential equations, e.g., for the machine computation of zeta functions of particular varieties, for which see [**139**]). There is also some relevance, entirely outside number theory, to the study of flat connections on complex analytic varieties (see [**144**]).

Although some applications involve higher-dimensional p-adic analytic spaces, this book treats exclusively p-adic *ordinary* differential equations. In joint work with Liang Xiao [**145**], we have developed some extensions to higher-dimensional spaces.

Each individual chapter of this book exhibits the following basic structure. Before the body of the chapter, we give a brief introduction explaining what is to be discussed and often setting some running notations or hypotheses. After the body of the chapter, we typically include a section of afternotes, in which we provide detailed references for results in that chapter, fill in historical details, and add additional comments. (This practice is modeled on that in [**94**], although we do not carry it out quite as fully.) Note that we have a habit of attributing to various authors slightly stronger versions of their theorems than the ones they originally stated; to avoid complicating the discussion in the text, we resolve these misattributions in the afternotes instead. At the end of a chapter we typically include a few exercises; a fair number of these request proofs of results which are stated and used in the text but whose proofs pose no unusual difficulties.

The chapters themselves are grouped into several parts, which we now describe briefly. (Chapter 0, being introductory, does not fit into this grouping.)

Part I is preliminary, collecting some basic tools of p-adic analysis. However, it also includes some facts of matrix analysis (the study of the variation of numerical invariants attached to matrices as a function of the matrix entries) which may not be familiar to the typical reader.

Part II introduces some formalism of differential algebra, such as differential rings and modules, twisted polynomials, and cyclic vectors, and applies these to fields equipped with a nonarchimedean norm.

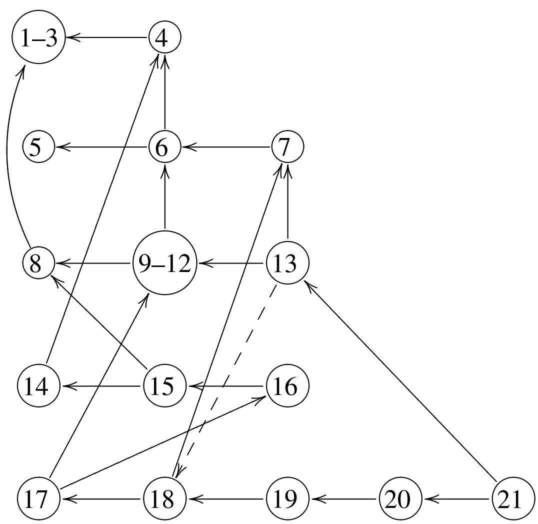
Part III begins the study of p-adic differential equations in earnest, developing some basic theory for differential modules on rings and annuli, including the Christol–Dwork theory of variation of the generic radius of convergence and the Christol–Mebkhout decomposition theory. We also include a treatment of p-adic exponents, culminating in the Christol–Mebkhout structure theorem for p-adic differential modules on an annulus satisfying the Robba condition (i.e., having intrinsic generic radius of convergence everywhere equal to 1).

Part IV introduces some formalism of difference algebra, and presents (without full proofs) the theory of slope filtrations for Frobenius modules over the Robba ring.

Part V introduces the concept of a Frobenius structure on a p-adic differential module, to the point of stating the p-adic local monodromy theorem and sketching briefly the proof techniques using either p-adic exponents or Frobenius slope filtrations. We also discuss effective convergence bounds for solutions of p-adic differential equations.

Part VI consists of a series of brief discussions of several areas of application of the theory of p-adic differential equations. These are somewhat more didactic, and much less formal, than in the other parts; they are meant primarily as suggestions for further reading.

The following diagram indicates the logical dependencies of the chapters. To keep the diagram manageable, we have grouped together some chapters (1–3 and 9–12) and omitted Chapter 0 and the chapters of Part VI. The reader should be aware that there is one forward reference, from Chapter 13 to Chapter 18, but the graph remains acyclic. (There are some additional forward references between Chapters 1 and 2, but these should not cause any difficulty.)



As noted above we have not assumed that the reader is familiar with rigid analytic geometry and so have phrased all statements more concretely in terms of rings and modules. Although we expect that the typical reader has at least a passing familiarity with p-adic numbers, for completeness we include a rapid development of the algebra of complete rings and fields in the first few chapters of the book. This development, when read on its own, may appear somewhat idiosyncratic; its design is justified by the reuse of some material in later chapters.

We would like to think that the background needed is that of a two-semester undergraduate abstract algebra course. However, some basic notions from commutative algebra do occasionally intervene, including flat modules, exact sequences, and the snake lemma. It may be helpful to have a well-indexed text on commutative algebra within arm's reach; we like Eisenbud's book [**84**], but the far slimmer Atiyah and Macdonald [**9**] should also suffice.

The author would like to thank the participants of the MIT course 18.787 ("Topics in number theory", fall 2007) for numerous comments on the lecture notes which ultimately became this book. Particular thanks are due to Ben Brubaker and David Speyer for giving guest lectures, and to Chris Davis, Hansheng Diao, David Harvey, Raju Krishnamoorthy, Ruochuan Liu, Eric Rosen, and especially Liang Xiao for providing feedback. Additional feedback was provided by Francesco Baldassarri, Laurent Berger, Bruno Chiarellotto, Gilles Christol, Ricardo García López, Tim Gowers, and Andrea Pulita.

During the preparation of the course and of this book, the author was supported by a National Science Foundation CAREER grant (DMS-0545904), a Sloan Research Fellowship, MIT's NEC Research Support Fund, and the MIT Cecil and Ida Green Career Development Professorship.

0
Introductory remarks

The theory of ordinary differential equations is a fundamental instrument of *continuous* mathematics, in which the central objects of study are functions involving real numbers. It is not immediately apparent that this theory has anything useful to say about *discrete* mathematics in general or number theory in particular.

In this book we consider ordinary differential equations in which the role of the real numbers is instead played by the field of p-adic numbers, for some prime number p. The p-adics form a number system with enough formal similarities to the real numbers to permit meaningful analogues of notions from calculus, such as continuity and differentiability. However, the p-adics incorporate data from arithmetic in a fundamental way; two numbers are p-adically close together if their difference is divisible by a large power of p.

In this chapter, we first indicate briefly some ways in which p-adic differential equations appear in number theory. We then focus on an example of Dwork, in which the p-adic behavior of Gauss's hypergeometric differential equation relates to the manifestly number-theoretic topic of the number of points on an elliptic curve over a finite field.

Since this chapter is meant only as an introduction, it is full of statements for which we give references instead of proofs. This practice is not typical of the rest of this book, except for the discussions in Part VI.

0.1 Why p-adic differential equations?

Although the very existence of a highly developed theory of p-adic ordinary differential equations is not entirely well known even within number theory, the subject is actually almost 50 years old. Here are circumstances, past and present, in which it arises; some of these will be taken up again in Part VI.

Variation of zeta functions (see Chapter 22). The original circumstance in which p-adic differential equations appeared in number theory was Dwork's work on the variation of zeta functions of algebraic varieties over finite fields. Roughly speaking, solving certain p-adic differential equations can give rise to explicit formulas for the numbers of points on varieties over finite fields.

In contrast with methods involving étale cohomology, methods for studying zeta functions based on p-adic analysis (including p-adic cohomology) lend themselves well to numerical computation. The interest in computing zeta functions for varieties where straightforward point-counting is impossible (e.g., curves over extremely large finite fields) has been driven by applications in computer science, the principal example being cryptography based on elliptic or hyperelliptic curves.

p-adic cohomology (see Chapter 23). Dwork's work suggested, but did not immediately lead to, a proper analogue of étale cohomology based on p-adic analytic techniques. Such an analogue was eventually developed by Berthelot (on the basis of some work of Monsky and Washnitzer, and also ideas of Grothendieck); it is called *rigid cohomology* (see the notes at the end of this chapter for the origin of the word "rigid"). The development of rigid cohomology has lagged somewhat behind that of étale cohomology, partly owing to the emergence of some thorny problems related to the construction of a good category of coefficients. These problems, which have only recently been resolved, are rather closely related to questions concerning p-adic differential equations; in fact, some results presented in this book have been used to address them.

p-adic Hodge theory (see Chapter 24). The subject of p-adic Hodge theory aims to do for the cohomology of varieties over p-adic fields what ordinary Hodge theory does for the cohomology of varieties over $\mathbb{C}$: that is, it aims to provide a better understanding of the cohomology of a variety in its own right, independently of the geometry of the variety. In the p-adic case, the cohomology in question is often étale cohomology, which carries the structure of a Galois representation.

The study of such representations, pioneered by Fontaine, involves a number of exotic auxiliary rings (rings of *p-adic periods*), which serve their intended purposes but are otherwise a bit mysterious. More recently, the work of Berger has connected much of the theory to the study of p-adic differential equations; notably, a key result that was originally intended for use in p-adic cohomology (the *p-adic local monodromy theorem*) turned out to imply an important conjecture about Galois representations, Fontaine's conjecture on potential semistability.

Ramification theory (see Chapter 19). There are some interesting analogies between properties of differential equations over $\mathbb{C}$ with meromorphic singularities and properties of "wildly ramified" Galois representations of p-adic fields. At some level, this is suggested by the parallel formulation of the Langlands conjectures in the number field and function field cases. One can use p-adic differential equations to interpolate between the two situations, by associating differential equations with Galois representations (as in the previous item) and then using differential invariants (for example, irregularity) to recover Galois invariants (for example, Artin and Swan conductors).

For representations of the étale fundamental group of a variety over a field of positive characteristic of dimension greater than 1, it is difficult to construct meaningful Galois-theoretic numerical invariants. Recent work of Abbes and Saito [**1, 2**] provides satisfactory definitions, but the resulting quantities are quite difficult to calculate. One can alternatively use p-adic differential equations to define invariants which can be somewhat easier to deal with; for instance, one can define a *differential Swan conductor* which is guaranteed to be an integer [**133**], whereas this is not clearly the case for the Abbes–Saito conductor. One can then equate the two conductors, deducing integrality for the Abbes–Saito conductor; this has been carried out by Chiarellotto and Pulita [**40**] for one-dimensional representations and by L. Xiao [**219**] in the general case.

0.2 Zeta functions of varieties

For the rest of this introduction, we return to Dwork's original example showing the role of p-adic differential equations and their solutions in number theory. This example refers to elliptic curves, for which see Silverman's book [**200**] for background.

Definition 0.2.1. For λ in some field K, let E_λ be the elliptic curve over K defined by the equation

$$E_\lambda : y^2 = x(x-1)(x-\lambda)$$

in the projective plane. Remember that there is one point $O = [0 : 1 : 0]$ at infinity. There is a natural commutative group law on $E_\lambda(K)$, with identity element O, characterized by the property that three points add to zero if and only if they are collinear. (It is better to say that three points add to zero if they are the three intersections of E_λ with some line, as this correctly permits degenerate cases. For instance, if two of the points coincide, the line must be the tangent to E_λ at that point.)

For elliptic curves over finite fields, one has the following result of Hasse, which generalizes some observations, made by Gauss and others, for certain special cases.

Theorem 0.2.2 (Hasse). *Suppose that λ belongs to a finite field $\mathbb{F}_q$. If we write $\#E_\lambda(\mathbb{F}_q) = q + 1 - a_q(\lambda)$, then $|a_q(\lambda)| \leq 2\sqrt{q}$.*

Proof. See [**200**, Theorem V.1.1]. □

Hasse's theorem was later vastly generalized as follows, originally as a set of conjectures by Weil. (Despite no longer being conjectural, these are still commonly referred to as the *Weil conjectures*.)

Definition 0.2.3. For X an algebraic variety over $\mathbb{F}_q$, the *zeta function* of X is defined as the formal power series

$$\zeta_X(T) = \exp\left(\sum_{n=1}^{\infty} \frac{T^n}{n} \#X(\mathbb{F}_{q^n})\right);$$

another way to write this, which makes it look more like a typical zeta function, is

$$\zeta_X(T) = \prod_x (1 - T^{\deg(x)})^{-1},$$

where x runs over the Galois orbits of $X(\overline{\mathbb{F}_q})$ and $\deg(x)$ is the size of the orbit x. (If you prefer algebro-geometric terminology, you may run x over closed points of the scheme X, in which case $\deg(x)$ denotes the degree of the residue field of x over $\mathbb{F}_q$.)

Example 0.2.4. For $X = E_\lambda$ one can verify that

$$\zeta_X(T) = \frac{1 - a_q(\lambda)T + qT^2}{(1 - T)(1 - qT)},$$

using properties of the Tate module of E_λ; see [**200**, Theorem V.2.2].

A statement of the Weil conjectures is given in the following theorem.

Theorem 0.2.5 (Dwork, Grothendieck, Deligne, *et al.*). *Let X be an algebraic variety over $\mathbb{F}_q$. Then $\zeta_X(T)$ represents a rational function of T. Moreover, if X is smooth and proper of dimension d, we can write*

$$\zeta_X(T) = \frac{P_1(T) \cdots P_{2d-1}(T)}{P_0(T) \cdots P_{2d}(T)},$$

where each $P_i(T)$ has integer coefficients, satisfies $P_i(0) = 1$, and has all roots in $\mathbb{C}$ on the circle $|T| = q^{-i/2}$.

Proof. The proof of this theorem is a sufficiently massive undertaking that even a reference is not reasonable here; instead, we give [**107**, Appendix C] as a source of references. (Another useful exposition is [**178**].) □

Remark 0.2.6. It is worth pointing out that the first complete proof of Theorem 0.2.5 used the fact that for any prime $\ell \neq p$ one has

$$\#X(\mathbb{F}_{q^n}) = \sum_i (-1)^i \operatorname{Trace}(F^n, H^i_{\text{et}}(X, \mathbb{Q}_\ell)),$$

where $H^i_{\text{et}}(X, \mathbb{Q}_\ell)$ is the i-th étale cohomology group of X (or rather, the base change of X to $\overline{\mathbb{F}_q}$) with coefficients in $\mathbb{Q}_\ell$. This is an instance of the *Lefschetz trace formula* in étale cohomology.

0.3 Zeta functions and p-adic differential equations

Remark 0.3.1. The interpretation of Theorem 0.2.5 in terms of étale cohomology (Remark 0.2.6) is all well and good, but there are several downsides. An important one is that étale cohomology is not explicitly computable; for instance, it is not straightforward to describe étale cohomology to a computer well enough that the computer can make calculations. (The main problem is that while one can write down étale cocycles, it is very hard to tell whether any given cocycle is a coboundary.)

Another important downside is that you do not get every good information about what happens to ζ_X when you vary X. This is where p-adic differential equations enter the picture. It was observed by Dwork that if one has a family of algebraic varieties defined over $\mathbb{Q}$, the same differential equations appear on the one hand when one studies the variation of complex periods and on the other hand when one studies the variation of zeta functions over $\mathbb{F}_p$.

Here is an explicit example due to Dwork.

Definition 0.3.2. Recall that the *hypergeometric series*

$$F(a, b; c; z) = \sum_{i=0}^{\infty} \frac{a(a+1)\cdots(a+i)b(b+1)\cdots(b+i)}{c(c+1)\cdots(c+i)i!} z^i \qquad (0.3.2.1)$$

satisfies the *hypergeometric differential equation*

$$z(1-z)y'' + (c - (a+b+1)z)y' - aby = 0. \qquad (0.3.2.2)$$

Set

$$\alpha(z) = F(1/2, 1/2; 1; z).$$

Over $\mathbb{C}$, α is related to an elliptic integral, for instance, by the formula

$$\alpha(\lambda) = \frac{2}{\pi} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \lambda \sin^2 \theta}} \qquad (0 < \lambda < 1).$$

(One can extend this to complex λ, but care needs to be taken with the branch cuts.) This elliptic integral can be viewed as a period integral for the curve E_λ, i.e., one is integrating some meromorphic differential form on E_λ around some loop (or more properly, around some homology class).

Let p be an odd prime. We now try to interpret $\alpha(z)$ as a function of a p-adic variable rather than a complex variable. Be aware that this means that z can take *any* value in a field with a norm extending the p-adic norm on $\mathbb{Q}$, not just in $\mathbb{Q}_p$ itself. (For the moment, you can imagine z running over a completed algebraic closure of $\mathbb{Q}_p$.)

Lemma 0.3.3. *The series $\alpha(z)$ converges p-adically for $|z| < 1$.*

Proof. Exercise. $\square$

Dwork discovered that a closely related function admits a sort of analytic continuation.

Definition 0.3.4. Define the *Igusa polynomial*

$$H(z) = \sum_{i=0}^{(p-1)/2} \binom{(p-1)/2}{i}^2 z^i.$$

Modulo p, the roots of $H(z)$ are the values of $\lambda \in \overline{\mathbb{F}_p}$ for which E_λ is a *supersingular* elliptic curve, i.e., for which $a_q(\lambda) \equiv 0 \pmod{p}$. (In fact, the roots of $H(z)$ all belong to $\mathbb{F}_{p^2}$, by a theorem of Deuring; see [**200**, Theorem V.3.1].)

Dwork's analytic continuation result is the following.

Theorem 0.3.5 (Dwork). *There exists a series $\xi(z) = \sum_{i=0}^{\infty} P_i(z)/H(z)^i$, with each $P_i(z) \in \mathbb{Q}_p[z]$, converging uniformly for those z satisfying $|z| \leq 1$ and $|H(z)| = 1$ and such that*

$$\xi(z) = (-1)^{(p-1)/2} \frac{\alpha(z)}{\alpha(z^p)} \qquad (|z| < 1).$$

Proof. See [**213**, §7]. $\square$

Remark 0.3.6. Note that ξ itself satisfies a differential equation derived from the hypergeometric equation. We will see such equations again once we introduce the notion of a Frobenius structure on a differential equation, in Chapter 17.

In terms of the function ξ, we can compute zeta functions in the Legendre family as follows.

Definition 0.3.7. Let $\mathbb{Z}_q$ be the unique unramified extension of $\mathbb{Z}_p$ with residue field $\mathbb{F}_q$. For $\lambda \in \mathbb{F}_q$, let $[\lambda]$ be the unique qth root of 1 in $\mathbb{Z}_q$ congruent to λ mod p (the *Teichmüller lift* of λ).

Theorem 0.3.8 (Dwork). *If $q = p^a$ and $\lambda \in \mathbb{F}_q$ is not a root of $H(z)$, then*

$$T^2 - a_q(\lambda)T + q = (T - u)(T - q/u),$$

where

$$u = \xi([\lambda])\xi([\lambda]^p)\cdots\xi([\lambda]^{p^{a-1}}).$$

That is, the quantity u is the "unit root" (meaning the root of valuation 0) of the polynomial $T^2 - a_q(\lambda)T + q$ occurring (up to reversal) in the zeta function.

Proof. See [**213**, §7]. □

0.4 A word of caution

Example 0.4.1. Before we embark on the study of p-adic ordinary differential equations, a cautionary note is in order concerning the rather innocuous-looking differential equation $y' = y$. Over $\mathbb{R}$ or $\mathbb{C}$, this equation is nonsingular everywhere and its solutions $y = ce^x$ are defined everywhere.

Over a p-adic field, things are quite different. As a power series around $x = 0$, we have

$$y = c\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

and the denominators hurt us rather than helping. In fact, the series only converges for $|x| < p^{-1/(p-1)}$ (assuming that we are normalizing in such a way that $|p| = p^{-1}$). For comparison, note that the logarithmic series

$$\log \frac{1}{1-x} = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

converges for $|x| < 1$.

Remark 0.4.2. The conclusion to be drawn from the previous example is that there is no fundamental theorem of ordinary differential equations over the p-adics! In fact, the hypergeometric differential equation in the previous example was somewhat special; the fact that it had a solution in a disc where it had no singularities was not a foregone conclusion. One of Dwork's discoveries is that this typically happens for differential equations that "come from geometry", such as the Picard–Fuchs equations, which arise from integrals of algebraic functions (e.g., elliptic integrals). Another of Dwork's discoveries is that, using similar techniques to those used to study obstructions to solving complex differential equations in singular discs, one can quantify the obstruction to solving a p-adic differential equation in a nonsingular disc. We will carry this out later in the book.

Notes

For detailed notes on the topics discussed in Section 0.1, see the notes for the chapters referenced.

We again mention [**107**, Appendix C] and [**178**] as starting points for further reading about the Weil conjectures.

The notion of an analytic function specified in terms of a uniform limit of rational functions with poles prescribed to certain regions is the original such notion, introduced by Krasner. For this book, we will restrict our consideration of p-adic analysis to working with complete rings in this fashion, without attempting to introduce any notion of nonarchimedean analytic geometry. However, it must be noted that it is much better in the long run to work in terms of analytic geometry; for example, it is prohibitively difficult to deal with partial differential equations without doing so.

That said, there are several ways to develop a theory of analytic spaces over a nonarchimedean field. The traditional method is Tate's theory of rigid analytic spaces, so-called because one develops everything "rigidly" by imitating the theory of schemes in algebraic geometry but using rings of convergent power series instead of polynomials. The canonical foundational reference for rigid geometry is the book of Bosch, Güntzer, and Remmert [**31**], but novices may find the text of Fresnel and van der Put [**93**] or the lecture notes of Bosch [**30**] more readable. A more recent method, which in some ways is more robust, is Berkovich's theory of nonarchimedean analytic spaces (commonly called *Berkovich spaces*), as introduced in [**19**] and further developed in [**20**]. For both points of view, see also the lecture notes of Conrad [**59**].

Dwork's original analysis of the Legendre family of elliptic curves using the associated hypergeometric equation (this analysis expands earlier work of Tate) appears in [**74**, §8]. The treatment in [**213**] is more overtly related to p-adic cohomology.

The family of hypergeometric equations with $a, b, c \in \mathbb{Q} \cap \mathbb{Z}_p$ is rich enough that one could devote an entire book to the study of its p-adic properties. Indeed, Dwork did exactly this; the result was [**78**].

It is possible to resurrect in part the fundamental theorem of ordinary differential equations in the p-adic setting. The best results in that direction seem to be those of Priess-Crampe and Ribenboim [**183**]. One consequence of their work is the fact that a differential equation over $\mathbb{Q}_p$ has a solution if and only if it has a sufficiently good approximate solution; this amounts to a differential version of Hensel's lemma. We too will need noncommutative forms of Hensel's lemma; see Theorem 2.2.2.

Christol [**45**] has given an interesting retrospective on some of the key ideas of Dwork, including generic points, the transfer principle, and Frobenius structures, which resonate throughout this book.

Exercises

The reader new to p-adic numbers should postpone doing these exercises until he or she has read Part I.

(1) Prove directly from the definition that the series $F(a, b; c; z)$ converges p-adically for $|z| < 1$ whenever a, b, c are rational numbers with denominators not divisible by p. This implies Lemma 0.3.3.

(2) Using the fact that $\alpha(z)$ satisfies the hypergeometric equation, write down a nontrivial differential equation with coefficients in $\mathbb{Q}(z)$ satisfied by the function $\xi(z)$.

(3) Check that the usual formula

$$\liminf_{n\to\infty} |a_n|^{-1/n}$$

for the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ still works over a nonarchimedean field. That is, the series converges when $|z|$ is less than this radius and diverges when $|z|$ is greater than this radius.

(4) Show that in the previous exercise, just like in the archimedean case, a power series over a nonarchimedean field can either converge or diverge at a value of z for which $|z|$ equals the radius of convergence.

(5) Check that (as claimed in Example 0.4.1), under the normalization $|p| = p^{-1}$, the exponential series $\exp(z)$ over $\mathbb{Q}_p$ has radius of convergence $p^{-1/(p-1)}$, while the logarithm series $\log(1 - z)$ has radius of convergence 1.

(6) Show that, over $\mathbb{Q}_p$, while a power series in z which converges for $|z| \leq 1$ may have an antiderivative which only converges for $|z| < 1$, its derivative still converges for $|z| \leq 1$. This is the reverse of what happens over an archimedean field.

Part I

Tools of p-adic Analysis

1

Norms on algebraic structures

In this chapter, we recall some basic facts about norms (absolute values), primarily of the nonarchimedean sort, on groups, rings, fields, and modules. We also briefly discuss the phenomenon of spherical completeness, which is peculiar to the nonarchimedean setting. Our discussion is not particularly comprehensive; the reader new to nonarchimedean analysis is directed to [**191**] for a fuller treatment.

Several proofs in this chapter make forward references to Chapter 2. There should be no difficulty in verifying the absence of circular references.

Convention 1.0.1. In this book, a *ring* means a commutative ring unless commutativity is suppressed explicitly by describing the ring as "not necessarily commutative" or implicitly by its usage in certain phrases, e.g., a *ring of twisted polynomials* (Definition 5.5.1).

Notation 1.0.2. For R a ring, we denote by $R^\times$ the multiplicative group of units of R.

1.1 Norms on abelian groups

Let us start by recalling some basic definitions from analysis, before specializing to the nonarchimedean case.

Definition 1.1.1. Let G be an abelian group. A *seminorm* (or *semiabsolute value*) on G is a function $|\cdot| : G \to [0, +\infty)$ satisfying the following conditions.

(a) We have $|0| = 0$.

(b) For $f, g \in G$, $|f - g| \leq |f| + |g|$. (Equivalently, $|g| = |-g|$ and $|f + g| \leq |f| + |g|$. This condition is usually called the *triangle inequality*.)

We say that the seminorm $|\cdot|$ is a *norm* (or *absolute value*) if the following additional condition holds.

(a′) For $g \in G$, $|g| = 0$ if and only if $g = 0$.

We also express this by saying that G is *separated* under $|\cdot|$. A seminorm on an abelian group G induces a metric topology on G, in which the basic open subsets are the open balls, i.e., sets of the form $\{g \in G : |f - g| < r\}$ for some $f \in G$ and some $r > 0$.

Definition 1.1.2. Let G, G' be abelian groups equipped with seminorms $|\cdot|$, $|\cdot|'$, respectively, and let $\phi : G \to G'$ be a homomorphism. Note that ϕ is continuous for the metric topologies on G, G' if and only if there exists a function $h : (0, +\infty) \to (0, +\infty)$ such that for all $r > 0$,

$$\{g \in G : |g| < h(r)\} \subseteq \{g \in G : |\phi(g)|' < r\}.$$

We say that ϕ is *bounded* if there exists $c \geq 0$ such that $|\phi(g)|' \leq c|g|$ for all $g \in G$. We say that ϕ is *isometric* if $|\phi(g)|' = |g|$ for all $g \in G$. We say two seminorms $|\cdot|_1, |\cdot|_2$ on G are *topologically equivalent* if they induce the same metric topology, i.e., the identity morphism on G is continuous in both directions. We say that $|\cdot|_1, |\cdot|_2$ are *metrically equivalent* if there exist $c_1, c_2 > 0$ such that, for all $g \in G$,

$$|g|_1 \leq c_1|g|_2, \qquad |g|_2 \leq c_2|g|_1;$$

this implies topological equivalence but the reverse implication does not necessarily hold.

Definition 1.1.3. Let G be an abelian group equipped with a seminorm. A *Cauchy sequence* in G under $|\cdot|$ is a sequence $\{x_n\}_{n=0}^{\infty}$ in G such that, for any $\epsilon > 0$, there exists an integer N such that, for all integers $m, n \geq N$, $|x_m - x_n| < \epsilon$. We say the sequence $\{x_n\}_{n=0}^{\infty}$ is *convergent* if there exists $x \in G$ such that, for any $\epsilon > 0$ there exists an integer N such that, for all integers $n \geq N$, $|x - x_n| < \epsilon$; in this case, the sequence is automatically Cauchy, and we say that x is a *limit* of the sequence. If G is separated under $|\cdot|$, then limits are unique when they exist. We say G is *complete* under $|\cdot|$ if every Cauchy sequence has a unique limit.

Theorem 1.1.4. *Let G be an abelian group equipped with a norm $|\cdot|$. Then there exists an abelian group G', equipped with a norm $|\cdot|'$ under which it is complete, and an isometric homomorphism $\phi : G \to G'$ with dense image.*

This is standard, so we only sketch the proof.

Proof. We take the set of Cauchy sequences in G and declare two sequences $\{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty}$ to be equivalent if the sequence $x_0, y_0, x_1, y_1, \ldots$ is also Cauchy. This is easily shown to be an equivalence relation; let G' be the set of equivalence classes. It is then straightforward to construct the group operation (termwise addition) and the norm on G' (the limit of the norms of the terms of the sequence). The map ϕ takes $g \in G$ to the constant sequence $g, g, \ldots$ □

Definition 1.1.5. With the notation of Theorem 1.1.4, we call G' the *completion* of G; the group G', equipped with the norm $|\cdot|'$ and the homomorphism ϕ, is functorial in G. That is, any continuous homomorphism $G \to H$ extends uniquely to a continuous homomorphism $G' \to H'$ between the completions; in particular, G' is unique up to unique isomorphism. Note that one can also define the completion even if G is only equipped with a seminorm, but only by first quotienting by the kernel of the seminorm; in that case, the map from G to its completion need not be injective.

Definition 1.1.6. If R is a not necessarily commutative ring and $|\cdot|$ is a seminorm on its additive group, we say that $|\cdot|$ is *submultiplicative* if the following additional condition holds.

(c) For $f, g \in R$, $|fg| \leq |f||g|$.

We say that $|\cdot|$ is *multiplicative* if the following additional condition holds.

(c$'$) For $f, g \in R$, $|fg| = |f||g|$.

The completion of a ring R equipped with a submultiplicative seminorm admits a natural ring structure, because the termwise product of two Cauchy sequences is again Cauchy.

Lemma 1.1.7. *Let F be a field equipped with a multiplicative norm. Then the completion of F is also a field.*

Proof. Note that if $\{f_n\}_{n=0}^{\infty}$ is a Cauchy sequence in F then $\{|f_n|\}_{n=0}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$ by the triangle inequality, and so has a limit since $\mathbb{R}$ is complete. Since F is equipped with a true norm, if $\{f_n\}_{n=0}^{\infty}$ does not converge to 0 then $\{|f_n|\}_{n=0}^{\infty}$ also must not converge to 0. In particular, $|f_n|_{n=0}^{\infty}$ is bounded below away from 0, from which it follows easily that $\{f_n^{-1}\}_{n=0}^{\infty}$ is also a Cauchy sequence. This proves that every nonzero element of the completion of F has a multiplicative inverse, as desired. □

Proposition 1.1.8. *Two multiplicative norms $|\cdot|, |\cdot|'$ on a field F are topologically equivalent if and only if there exists $c > 0$ such that $|x|' = |x|^c$ for all $x \in F$.*

Proof. Exercise, or see [**80**, Lemma I.1.2]. □

Definition 1.1.9. Let G be an abelian group equipped with a seminorm $|\cdot|_G$, and let G' be a subgroup of G. The *quotient seminorm* on the quotient G/G' is defined by the formula

$$|g+G'|_{G/G'} = \inf_{g' \in G'} \{|g+g'|_G\}. \tag{1.1.9.1}$$

If $|\cdot|_G$ is a norm, then $|\cdot|_{G/G'}$ is a norm if and only if G' is closed in G.

1.2 Valuations and nonarchimedean norms

We now restrict our attention to nonarchimedean absolute values, which can be described additively (using valuations) as well as multiplicatively (using norms). It will be convenient to switch back and forth between these points of view throughout the book.

Definition 1.2.1. A *real semivaluation* on an abelian group G is a function $v : G \to \mathbb{R} \cup \{+\infty\}$ with the following properties.

(a) We have $v(0) = +\infty$.

(b) For $f, g \in G$, $v(f-g) \geq \min\{v(f), v(g)\}$.

We say v is a *real valuation* if the following additional condition holds.

(a′) For $g \in G$, $v(g) = +\infty$ if and only if $g = 0$.

If v is a real (semi)valuation on G, then the function $|\cdot| = e^{-v(\cdot)}$ is a (semi)norm on G which is *nonarchimedean* (or *ultrametric*), i.e., it satisfies the *strong triangle inequality*, which is given as follows.

(b′) For $f, g \in G$, $|f-g| \leq \max\{|f|, |g|\}$.

Conversely, for any nonarchimedean (semi)norm $|\cdot|$, $v(\cdot) = -\log|\cdot|$ is a real (semi)valuation. We will apply various definitions made for seminorms to semivaluations in this manner; for instance, if R is a ring and v is a real (semi)valuation on its additive group, we say that v is (sub)multiplicative if the corresponding nonarchimedean (semi)norm is.

Definition 1.2.2. We say that a group is *nonarchimedean* if it is equipped with a nonarchimedean norm; we say that a ring or field is *nonarchimedean* if it is equipped with a multiplicative nonarchimedean norm. Note that any nonarchimedean ring is an integral domain.

Definition 1.2.3. Let F be a nonarchimedean field. The *multiplicative value group* of a nonarchimedean field F is the image of $F^\times$ under $|\cdot|$, viewed as a subgroup of $\mathbb{R}^+$; we will often denote it simply as $|F^\times|$. The *additive value group* of F is the set of negative logarithms of the multiplicative value group. If these groups are discrete and nonzero (i.e., isomorphic to $\mathbb{Z}$), we say F is *discretely valued*. Define also

$$\begin{aligned}\mathfrak{o}_F &= \{f \in F : v(f) \geq 0\},\\ \mathfrak{m}_F &= \{f \in F : v(f) > 0\},\\ \kappa_F &= \mathfrak{o}_F/\mathfrak{m}_F.\end{aligned}$$

Note that $\mathfrak{o}_F$ is a local ring (the *valuation ring* of F), $\mathfrak{m}_F$ is the maximal ideal of $\mathfrak{o}_F$, and κ_F is a field (the *residue field* of F).

It is worth noting that there are comparatively few archimedean (i.e., not nonarchimedean) absolute values on fields.

Theorem 1.2.4 (Ostrowski). *Let F be a field equipped with a norm $|\cdot|$. Then $|\cdot|$ fails to be nonarchimedean if and only if the sequence $|1|, |2|, |3|, \dots$ is unbounded. In that case, F is isomorphic to a subfield of $\mathbb{C}$ equipped with the restriction of the usual absolute value.*

Proof. Exercise, or see [**191**, §2.1.6] and [**191**, §2.2.4], respectively. □

1.3 Norms on modules

When considering norms on modules, we usually require compatibility with the underlying ring.

Definition 1.3.1. Let R be a ring equipped with a multiplicative seminorm $|\cdot|$, and let M be an R-module equipped with a seminorm $|\cdot|_M$. We say that $|\cdot|_M$ is *compatible with* $|\cdot|$ (or *with* R) if the following conditions hold.

(a) For $f \in R$, $x \in M$, $|fx|_M = |f||x|_M$.

(b) If $|\cdot|$ is nonarchimedean, then so is $|\cdot|_M$.

Note that (b) is not superfluous; see the end-of-chapter exercises. If R is a field, then two norms $|\cdot|_M, |\cdot|'_M$ on M compatible with R are metrically equivalent if and only if they are topologically equivalent (exercise).

One thing to be aware of is that if M' is a quotient of M and $|\cdot|_{M'}$ is the quotient norm on M' induced by $|\cdot|_M$, then in general we cannot say that $|\cdot|_{M'}$ is compatible with R. Rather, we only have the inequality

$$|fx'|_{M'} \leq |f||x'|_{M'} \qquad (f \in R, x' \in M');$$

this implies compatibility only if R is a field.

We can generate a rich supply of norms on modules via the following construction.

Definition 1.3.2. Let R be a ring equipped with a multiplicative (semi)norm $|\cdot|$, and let M be a finite free R-module. For B a basis of M, define the *supremum (semi)norm* of M with respect to B by setting

$$\left|\sum_{b\in B} c_b b\right| = \sup_{b\in B}\{|c_b|\} \qquad (c_b \in R).$$

This (semi)norm extends canonically to $M \otimes_R S$ for any isometric inclusion $R \hookrightarrow S$. (The situation is more complicated for arbitrary (semi)norms; see Definition 1.3.10 below.)

We say that a seminorm on M is *supremum-equivalent* if it is metrically equivalent to the supremum seminorm with respect to some basis; the same is then true of any basis, by Lemma 1.3.3 below. In particular, if $|\cdot|$ is a norm then any supremum-equivalent seminorm is a norm.

Lemma 1.3.3. *Let R be a ring equipped with a multiplicative seminorm $|\cdot|$, and let M be a finite free R-module. Then for any two bases B_1, B_2 of M, the supremum seminorms of M defined by B_1 and B_2 are metrically equivalent.*

Proof. Put $B_1 = \{m_{1,1}, \dots, m_{1,n}\}$ and $B_2 = \{m_{2,1}, \dots, m_{2,n}\}$. Define the $n \times n$ matrix A over R by the formula

$$m_{2,j} = \sum_{i=1}^{n} A_{ij} m_{1,i};$$

then A is invertible. In particular, we cannot have $|A_{ij}| = 0$ for all i, j.

For $x \in M$, we can uniquely write $x = a_{1,1}m_{1,1} + \cdots + a_{1,n}m_{1,n} = a_{2,1}m_{2,1} + \cdots + a_{2,n}m_{2,n}$ with $a_{i,j} \in R$. We then have

$$a_{1,i} = \sum_{j=1}^{n} A_{ij} a_{2,j} \qquad (i = 1, \dots, n)$$

and so

$$\max_i\{|a_{1,i}|\} \leq \left(\sum_{i=1}^{n}\sum_{j=1}^{n} |A_{ij}|\right) \max_j\{|a_{2,j}|\}.$$

This inequality, together with the corresponding one with the bases reversed (involving the matrix A^{-1}), implies the claim. □

Corollary 1.3.4. *Let $R \hookrightarrow S$ be an isometric inclusion of rings equipped with multiplicative seminorms. Let M be a finite free R-module. Let $|\cdot|_M$ be a seminorm on M that is compatible with R, which is the restriction of a supremum-equivalent seminorm on $M \otimes_R S$ that is compatible with S. Then $|\cdot|_M$ is supremum-equivalent.*

Proof. Put $N = M \otimes_R S$, and let $|\cdot|_N$ be a supremum-equivalent norm on N, compatible with S, whose restriction to M equals $|\cdot|_M$. Pick any basis B

of M; then B is also a basis of N. Since $|\cdot|_N$ is equivalent to the supremum norm on N defined by some basis, by Lemma 1.3.3 it is also equivalent to the supremum norm defined by B. By restriction, we see that $|\cdot|_M$ is equivalent to the supremum norm on M defined by B. □

The notion of supremum-equivalence is well-behaved under quotients.

Lemma 1.3.5. *Let R be a ring equipped with a multiplicative seminorm $|\cdot|$, let M be a finite free R-module, and let M_1 be a finite free R-submodule of M such that M/M_1 is also free. Let $|\cdot|_M$ be a supremum-equivalent norm on M compatible with R. Then the quotient norm $|\cdot|_{M_1}$ on M_1 induced by $|\cdot|_M$ is also supremum-equivalent.*

Proof. Let $m_1, \dots, m_k$ be a basis of M_1, and choose $m_{k+1}, \dots, m_n \in M$ lifting a basis of M/M_1. Then $m_1, \dots, m_n$ is a basis of M; by Lemma 1.3.3, $|\cdot|_M$ is equivalent to the supremum norm defined by $m_1, \dots, m_n$. That is, there exist $c_1, c_2 > 0$ such that, for any $x = a_1 m_1 + \cdots + a_n m_n \in M$,

$$c_1 \max\{|a_1|, \dots, |a_n|\} \leq |x|_M \leq c_2 \max\{|a_1|, \dots, |a_n|\}.$$

Then for any $a_{k+1}, \dots, a_n \in R$, we have

$$\begin{aligned} c_1 \max\{|a_{k+1}|, \dots, |a_n|\} &= c_1 \inf_{a_1,\dots,a_k \in R} \{\max\{|a_1|, \dots, |a_n|\}\} \\ &\leq \inf_{a_1,\dots,a_k \in R} \{|a_1 m_1 + \cdots + a_n m_n|_M\} \\ &= |a_{k+1} m_{k+1} + \cdots + a_n m_n|_{M_1} \\ &\leq |a_{k+1} m_{k+1} + \cdots + a_n m_n|_M \\ &\leq c_2 \max\{|a_{k+1}|, \dots, |a_n|\}. \end{aligned}$$

Thus $|\cdot|_{M_1}$ is equivalent to the supremum norm defined by the images of $m_{k+1}, \dots, m_n$ in M_1, proving the desired result. □

In general, even over a field, not every compatible norm on a vector space need be supremum-equivalent; see the exercises. However, such supremum-equivalence is true for complete fields.

Theorem 1.3.6. *Let F be a field complete for a norm $|\cdot|$, and let V be a finite-dimensional vector space over F. Then any two norms on V compatible with F are metrically equivalent.*

Proof. In the archimedean case, apply Theorem 1.2.4 to deduce that $F = \mathbb{R}$ or $F = \mathbb{C}$, then use compactness of the unit ball. In the nonarchimedean case, we proceed as follows. (See [**80**, Theorem I.3.2] for a different proof.)

We proceed by induction on n, the case $n = 1$ being trivial. Let $m_1, \dots, m_n$ be any basis of V. It suffices to show that any given norm $|\cdot|$ on V compatible with F is equivalent to the supremum norm defined by $m_1, \dots, m_n$. One inequality is evident: for any $a_1, \dots, a_n \in F$, we have

$$|a_1 m_1 + \cdots + a_n m_n|_V \leq \max_i \{|m_i|\} \max_i \{|a_i|\}.$$

Put $V' = V/Fm_1$. Let $|\cdot|_{V'}$ denote the quotient seminorm on V' induced by $|\cdot|_V$. This seminorm is compatible with F, but we must check that it is indeed a norm. Suppose on the contrary that $a_2, \dots, a_n \in F$ are such that $|a_2 m_2 + \cdots + a_n m_n|_{V'} = 0$. Then we can choose a sequence $a_{1,1}, a_{1,2}, \dots$ of elements of F such that $|a_{1,i} m_1 + a_2 m_2 + \cdots + a_n m_n|_V \to 0$ as $i \to \infty$. But then $|a_{1,i} - a_{1,j}||m_1|_V = |(a_{1,i} - a_{1,j}) m_1|_V \to 0$ as $i, j \to \infty$, so the $a_{1,i}$ form a Cauchy sequence. Since F is complete, this Cauchy sequence has a limit a_1, and $|a_1 m_1 + \cdots + a_n m_n|_V = 0$ contrary to the hypothesis that $|\cdot|_V$ is a norm.

Hence $|\cdot|_{V'}$ is indeed a norm. By the induction hypothesis, there exists $c_2 > 0$ such that

$$|a_2 m_2 + \cdots + a_n m_n|_{V'} \geq c_2 \max_i \{|a_i|\}.$$

We then have

$$\begin{aligned}
|a_1 m_1 + \cdots + a_n m_n|_V &\geq \max\{|a_1 m_1|_V, |a_2 m_2 + \cdots + a_n m_n|_V\} \\
&\geq \max\{|a_1 m_1|_V, |a_2 m_2 + \cdots + a_n m_n|_{V'}\} \\
&\geq \min\{|m_1|, c_2\} \max_i \{|a_i|\},
\end{aligned}$$

proving that $|\cdot|_V$ is equivalent to the supremum norm defined by $m_1, \dots, m_n$. □

Even if a norm is supremum-equivalent, it need not be equal to the supremum norm defined by any basis. However, one can approximate supremum-equivalent norms using supremum norms as follows. For a stronger result in the spherically complete case, see Lemma 1.5.5.

Lemma 1.3.7 (Approximation lemma). *Let F be a nonarchimedean field, let V be a finite-dimensional vector space over F, and let $|\cdot|_V$ be a supremum-equivalent norm on V compatible with F. Assume that either:*

(a) *$c > 1$ and the value group of F is not discrete; or*

(b) *$c \geq 1$ and the value groups of F and V coincide and are discrete.*

Then there exists a basis of V defining a supremum norm $|\cdot|'_V$ for which

$$c^{-1}|x|_V \leq |x|'_V \leq c|x|_V \qquad (x \in V).$$

Proof. We induct on n, with trivial base case $n = 0$. For $n > 0$, pick any nonzero $m_1 \in V$, and put $V_1 = V/Fm_1$. Using (a) or (b), we can rescale m_1 by an element of F to force $1 \le |m_1|_V \le c^{2/3}$.

Equip V_1 with the quotient seminorm $|\cdot|_{V_1}$ induced by $|\cdot|_V$. By Lemma 1.3.5, $|\cdot|_{V_1}$ is again supremum-equivalent. Moreover, in case (b) the infimum in (1.1.9.1) is always achieved, i.e., for every $x_1 \in V_1$, there exists $x \in V$ lifting x_1 with $|x|_V = |x_1|_{V_1}$. Hence V_1 again satisfies (b).

We may now apply the induction hypothesis to V_1 to produce a basis $m_{2,1}, \dots, m_{n,1}$ of V_1 defining a supremum norm $|\cdot|'_{V_1}$ for which

$$c^{-1/3}|x_1|_{V_1} \le |x_1|'_{V_1} \le c^{1/3}|x_1|_{V_1} \qquad (x_1 \in V_1).$$

For $i = 2, \dots, n$, choose $m_i \in V$ lifting $m_{i,1}$ such that $|m_i|_V \le c^{1/3}|m_{i,1}|_{V_1}$; then

$$|m_i|_V \le c^{1/3}|m_{i,1}|_{V_1} \le c^{2/3}|m_{i,1}|'_{V_1} = c^{2/3}.$$

Let $|\cdot|'_V$ be the supremum norm defined by $m_1, \dots, m_n$. For $x \in V$, write $x = a_1m_1 + \cdots + a_nm_n$ with $a_i \in F$. On the one hand,

$$|x|_V \le \max_{1\le i\le n}\{|a_i||m_i|_V\} \le c^{2/3}|x|'_V.$$

On the other hand, if x_1 is the image of x in V_1 then

$$|x_1|'_{V_1} \le c^{1/3}|x_1|_{V_1} \le c^{1/3}|x|_V,$$

so $|a_2|, \dots, |a_n| \le c^{1/3}|x|_V$. Moreover,

$$\begin{aligned}
|a_1m_1|_V &\le \max\{|x|_V, |x - a_1m_1|_V\}\\
&\le \max\{|x|_V, c^{2/3}|x - a_1m_1|'_V\}\\
&= \max\{|x|_V, c^{2/3}\max\{|a_2|, \dots, |a_n|\}\}\\
&\le c|x|_V.
\end{aligned}$$

Since $|m_1|_V \ge 1$, we deduce $|a_1| \le c|x|_V$ and so $|x|'_V \le c|x|_V$. This proves the desired inequalities. □

We need the following infinite-dimensional analogue of Theorem 1.3.6, taken from [**195**, Proposition 10.4]. Be aware that the situation in the archimedean case is much subtler; see the notes at the end of the chapter.

Lemma 1.3.8. *Let F be a complete nonarchimedean field. Let V be an F-vector space equipped with a norm $|\cdot|_V$ compatible with F. Suppose that V contains a dense F-subspace of countable infinite dimension over F. Then there exists a sequence $m_1, m_2, \dots$ of elements of V with the following properties.*

(a) *For each $m \in V$, there is a unique sequence $a_1, a_2, \dots$ of elements of F such that the series $\sum_{i=1}^{\infty} a_i m_i$ converges to m.*
(b) *With notation as in (a), the function $|\cdot|'_V$ defined by*

$$|m|'_V = \sup_i \{|a_i m_i|_V\}$$

is a norm on V compatible with F and metrically equivalent to $|\cdot|_V$.

Proof. Choose an ascending sequence of F-subspaces $0 = V_0 \subset V_1 \subset \cdots$, with $\dim_F V_n = n$, whose union is dense in V. For each $n > 0$, pick some $m_{n,0} \in V_n \setminus V_{n-1}$.

Let $|\cdot|_n$ be the quotient seminorm on V/V_n induced by $|\cdot|_V$. As in the proof of Theorem 1.3.6, we may show by induction on n that $|\cdot|_n$ is a norm, as follows. The claim for $n = 0$ is given. Supposing that $|\cdot|_{n-1}$ is a norm, let $m \in V$ be an element with $|m|_n = 0$. There must exist a sequence a_i of elements of F with $|a_i m_{n,0} + m|_{n-1} \to 0$ as $i \to \infty$. Since $|m_{n,0}|_{n-1} \neq 0$ by the induction hypothesis, the a_i must form a Cauchy sequence in F whose limit a satisfies $|am_{n,0} + m|_{n-1} = 0$. Again by the induction hypothesis, m and $-am_{n,0}$ represent the same class in V/V_{n-1}, so m represents the zero class in V/V_n. Hence $|\cdot|_n$ is a norm.

Choose an increasing sequence of real numbers $0 < r_1 < r_2 < \cdots < 1$. Since $|\cdot|_{n-1}$ is a norm, we have $|m_{n,0}|_{n-1} \neq 0$. We can thus choose $m_n \in m_{n,0} + V_{n-1}$ with $m_1 = m_{1,0}$ and

$$|m_n|_{n-1} = |m_{n,0}|_{n-1} \geq \frac{r_n}{r_{n+1}} |m_n|_V \qquad (n > 1).$$

For $m \in V_{n-1}$ and $a \in F$, we have $|am_n + m|_V \geq |am_n|_{n-1} \geq (r_n/r_{n+1})|am_n|_V$. If $|am_n|_V = |m|_V$, this yields

$$|am_n + m|_V \geq \frac{r_n}{r_{n+1}} \max\{|am_n|_V, |m|_V\};$$

the same holds if $|am_n|_V \neq |m|_V$ since in that case $|am_n + m|_V = \max\{|am_n|_V, |m|_V\}$.

By induction on n, we deduce that, for $a_1, \dots, a_n \in F$,

$$\begin{aligned} |a_1 m_1 + \cdots + a_n m_n|_V &\geq \frac{r_1}{r_{n+1}} \max\{|a_1 m_1|_V, \dots, |a_n m_n|_V\} \\ &\geq r_1 \max\{|a_1 m_1|_V, \dots, |a_n m_n|_V\}. \end{aligned}$$

Combining this with the evident inequality

$$|a_1 m_1 + \cdots + a_n m_n|_V \leq \max\{|a_1 m_1|_V, \dots, |a_n m_n|_V\},$$

we conclude that on $\cup_n V_n$ the seminorms $|\cdot|_V$ and $|\cdot|'_V$ are metrically equivalent. Consequently, $|\cdot|'_V$ extends by continuity to a function on V which is metrically equivalent to $|\cdot|_V$ and hence is a norm (which is evidently compatible with F). This assertion will imply both (a) and (b) as soon as we have established the existence aspect of (a), which we will do now.

Given $m \in V$, by hypothesis there exists a sequence $x_1, x_2, \dots$ of elements of $\cup_n V_n$ converging to m. For $j = 1, 2, \dots$ write $x_j = \sum_{i=1}^{\infty} a_{i,j} m_i$, where only finitely many $a_{i,j}$ are nonzero. Since $\{x_j\}_{j=1}^{\infty}$ is a Cauchy sequence, for each $\epsilon > 0$ there exists N such that, for $j, j' \geq N$, $|x_j - x_{j'}|_V \leq \epsilon$. Since $|\cdot|'_V$ is metrically equivalent to $|\cdot|_V$, for each $\epsilon > 0$ there also exists N such that, for $j, j' \geq N$, $|x_j - x_{j'}|'_V \leq \epsilon$.

On the one hand this implies that, for each fixed i, the sequence $\{a_{i,j}\}_{j=1}^{\infty}$ is Cauchy. Since F is complete, this sequence has a limit a_i. On the other hand, for $j = N$ there exists some i_0 such that $a_{i,j} = 0$ for all $i \geq i_0$. If we write $x_{i,j} = \sum_{h=i+1}^{\infty} a_{h,j} m_j$, for all $j \geq N$ and all $i \geq i_0$ we have $|x_{i,j}|'_V \leq \epsilon$ and hence $|x_{i,j}|_V \leq \epsilon$. For fixed i, $x_{i,j}$ converges to $m - a_1 m_1 - \cdots - a_i m_i$ as $j \to \infty$; hence, for all $i \geq i_0$, $|m - a_1 m_1 - \cdots - a_i m_i|_V \leq \epsilon$ so the series $\sum_{i=1}^{\infty} a_i m_i$ converges to m. As noted earlier, both (a) and (b) now follow. □

Definition 1.3.9. For F a field complete for a norm $|\cdot|$, a *Banach space* over F is a vector space V over F equipped with a norm compatible with $|\cdot|$, under which it is complete. For V a Banach space and W a closed subspace, the quotient V/W is again complete. See [**214**] or [**195**] for a full development of the theory of Banach spaces and other topological vector spaces over complete nonarchimedean fields.

Definition 1.3.10. Let R be a nonarchimedean ring, and let M, N be modules over R equipped with seminorms $|\cdot|_M, |\cdot|_N$ compatible with R. The *product seminorm* on $M \otimes_R N$ is defined by the formula

$$|x|_{M \otimes_R N} = \inf \left\{ \max_{1 \leq i \leq s} \{|m_i|_M |n_i|_N\} : x = \sum_{i=1}^{s} m_i \otimes n_i \right\}.$$

As in the case of the quotient seminorm, it is clear that the product seminorm is a seminorm, but it is not clear whether it is compatible with R unless R happens to be a field. Moreover for norms $|\cdot|_M$ and $|\cdot|_N$, it is not clear whether the product seminorm is a norm. However, if M and N are finite free R-modules equipped with supremum-equivalent norms then the product seminorm will be supremum-equivalent, which forces it to be a norm. See also the following lemma.

Lemma 1.3.11. *Let F be a complete nonarchimedean field. Let V and W be (possibly infinite-dimensional) vector spaces over F equipped with norms $|\cdot|_V$ and $|\cdot|_W$ compatible with F. Then the product seminorm on $V \otimes_F W$ is a norm.*

Proof. Suppose first that V admits a dense F-subspace of at most countably infinite dimension. Then by Theorem 1.3.6 and/or Lemma 1.3.8, we can find a (finite or infinite) sequence $m_1, m_2, \dots$ of elements of V such that every element can be uniquely written as a convergent series $\sum_{i=1}^{\infty} a_i m_i$, with $a_i \in F$, and $|\cdot|_V$ is equivalent to the norm $|\cdot|'_V$ defined by

$$\left| \sum_{i=1}^{\infty} a_i m_i \right|'_V = \sup_i \{|a_i m_i|_V\} \qquad (a_i \in F).$$

More precisely, we have $c|\cdot|'_V \leq |\cdot|_V \leq |\cdot|'_V$ for some $c > 0$. Let $\pi_j : V \to F$ be the projection carrying $\sum_{i=1}^{\infty} a_i m_i$ to a_j. By tensoring with W, we obtain a projection $\pi_{j,W} : V \otimes_F W \to W$. For $x \in V \otimes_F W$, we define

$$|x|'_{V \otimes_F W} = \sup_j \{|m_j|_V |\pi_{j,W}(x)|_W\}.$$

This gives a norm by the following argument. Let $\sum_{k=1}^{s} y_k \otimes z_k$ be any representation of $x \in V \otimes_F W$ with $y_k \in V$ and $z_k \in W$, so that

$$\pi_{j,W}(x) = \sum_{k=1}^{s} \pi_j(y_k) z_k.$$

Suppose that $|x|'_{V \otimes_F W} = 0$; then choose the representation of x to minimize s. If $s > 0$ then $y_k \neq 0$ for all k and the z_k must be linearly independent over F. We can then choose j and k such that $\pi_j(y_k) \neq 0$, but then

$$0 = \pi_{j,W}(x) = \sum_{k=1}^{s} \pi_j(y_k) z_k,$$

a contradiction. Hence $s = 0$ and so $x = 0$.

For $x = \sum_{k=1}^{s} y_k \otimes z_k \in V \otimes_F W$ and any positive integer N, we can express x also as

$$m_1 \otimes \pi_{1,W}(x) + \cdots + m_N \otimes \pi_{N,W}(x) + \sum_{k=1}^{s} (y_k - \pi_1(y_k) m_1 - \cdots - \pi_N(y_k) m_N) \otimes z_k;$$

as $N \to \infty$, the product seminorm of the sum over k tends to zero. We thus conclude that, on the one hand,

$$|x|_{V \otimes_F W} \leq |x|'_{V \otimes_F W}.$$

On the other hand,

$$\begin{aligned}\max_k\{|y_k|_V|z_k|_W\} &\geq c\sup_j \max_k\{|\pi_j(y_k)m_j|_V|z_k|_W\}\\ &\geq c\sup_j\left\{|m_j|_V\left|\sum_{k=1}^s \pi_j(y_k)z_k\right|_W\right\}\\ &= c\sup_j\{|m_j|_V|\pi_{j,W}(x)|_W\},\end{aligned}$$

so that

$$|x|_{V\otimes_F W} \geq c|x|'_{V\otimes_F W}.$$

That is, the product seminorm is equivalent to $|\cdot|'_{V\otimes_F W}$ and so is a norm.

In the general case, suppose on the contrary that $x = \sum_{j=1}^s m_j \otimes n_j \in V\otimes_F W$ had product seminorm 0. This would mean that we can find a sequence $x_i \in V\otimes_F W$ in which each x_i can be represented as $\sum_{j=1}^{s_i} m_{i,j}\otimes n_{i,j}$, and so

$$\lim_{i\to\infty}\max_j\{|m_{i,j}|_V|n_{i,j}|_W\} = 0.$$

Then the same data would be available if we replaced V by the closure of the F-subspace spanned by the m_j and the $m_{i,j}$, and similarly for W. We may thus apply the previous case to obtain a contradiction. □

1.4 Examples of nonarchimedean norms

Example 1.4.1. For any field F, there is a *trivial norm* of F defined by

$$|f|_{\mathrm{triv}} = \begin{cases}1 & f\neq 0,\\ 0 & f=0.\end{cases}$$

This norm is nonarchimedean, and F is complete under it. The trivial case will always be allowed unless explicitly excluded; it is often a useful input into a highly nontrivial construction, as in the next few examples.

Example 1.4.2. Let F be any field, and let $F((t))$ denote the field of formal Laurent series. The *t-adic valuation* v_t on F is defined as follows: for $f = \sum_i c_i t^i \in F((t))$, $v_t(f)$ is the least i for which $c_i \neq 0$. This exponentiates to give a *t-adic norm* under which $F((t))$ is complete and discretely valued. (See Example 1.5.8 for a variation on this construction.)

Before introducing our next example, we make a more general definition for later use.

Definition 1.4.3. Let R be a ring equipped with a nonarchimedean submultiplicative (semi)norm $|\cdot|$. For $\rho \geq 0$, define the *ρ-Gauss (semi)norm* $|\cdot|_\rho$ on the polynomial ring $R[T]$ by

$$\left|\sum_i P_i T^i\right|_\rho = \max_i\{|P_i|\rho^i\};$$

it is clearly submultiplicative. Moreover, it is also multiplicative if $|\cdot|$ is; however, we will postpone the verification of this to the next chapter (see Proposition 2.1.2). For $r \in \mathbb{R}$, we define the *r-Gauss (semi)valuation v_r* as the (semi)valuation associated with the e^{-r}-Gauss (semi)norm.

Remark 1.4.4. The definition of the ρ-Gauss norm depends on the choice of the indeterminate T; that is, it is not equivariant for arbitrary endomorphisms of the ring $R[T]$. For clarity, we will sometimes need to specify that the Gauss norm is being defined with respect to a particular indeterminate.

Example 1.4.5. For F a nonarchimedean field and $\rho > 0$, the *ρ-Gauss norm* on $F[t]$ (with respect to t) is a multiplicative norm, so it extends to the rational function field $F(t)$. Note that $F(t)$ is discretely valued under the ρ-Gauss norm if and only if either:

(a) F carries the trivial norm and $\rho \neq 1$; or
(b) F is discretely valued and ρ belongs to the divisible closure of the multiplicative value group of F.

In case (a) the ρ-Gauss norm is equivalent to the t-adic norm if $0 < \rho < 1$, to the trivial norm if $\rho = 1$, and to the t^{-1}-adic norm if $\rho > 1$.

So far we have not mentioned the principal examples from number theory; let us do so now.

Example 1.4.6. For p a prime number, the *p-adic norm* $|\cdot|_p$ on $\mathbb{Q}$ is defined as follows. Given $f = r/s$ with $r, s \in \mathbb{Z}$, write $r = p^a m$ and $s = p^b n$ with m, n not divisible by p and then put

$$|f|_p = p^{-a+b}.$$

In particular, we have normalized in such a way that $|p|_p = p^{-1}$; this convention is usually taken in order to make the *product formula* hold. Namely, for any $f \in \mathbb{Q}$, if $|\cdot|_\infty$ denotes the usual archimedean absolute value then

$$|f|_\infty \prod_p |f|_p = 1.$$

Completing $\mathbb{Q}$ under $|\cdot|_p$ gives the *field of p-adic numbers* $\mathbb{Q}_p$; it is discretely valued. Its valuation ring is denoted $\mathbb{Z}_p$ and called the *ring of p-adic integers*.

Remark 1.4.7. When converting the p-adic norm into a valuation, it is common practice to use base-p logarithms. We have instead opted to keep the factor of $\log p$ visible when we take logarithms. One may liken this practice to using metric units rather than normalizing some dimensioned constants to 1 (e.g., the speed of light).

Just as the only archimedean norm on $\mathbb{Q}$ is the usual one, every nontrivial nonarchimedean norm on $\mathbb{Q}$ is essentially a p-adic norm, again by a theorem of Ostrowski.

Theorem 1.4.8 (Ostrowski). *Any nontrivial nonarchimedean norm on $\mathbb{Q}$ is equivalent to the p-adic norm for some prime p.*

Proof. See [**191**, §2.2.4]. □

To equip extensions of $\mathbb{Q}_p$ with norms, we use the following result. As usual, when E is a finite extension of a field F we write $[E : F]$ for the degree of the field extension, i.e., the dimension of E as an F-vector space.

Theorem 1.4.9. *Let F be a complete nonarchimedean field. Then any finite extension E of F admits a unique extension of $|\cdot|$ to a norm on E (under which E is also complete).*

Proof. We only prove uniqueness now; existence will be established in Section 2.3. Let $|\cdot|_1$ and $|\cdot|_2$ be two extensions of $|\cdot|$ to norms on E. Then these in particular give norms on E viewed as an F-vector space; by Theorem 1.3.6, these norms are metrically equivalent; that is, there exist $c_1, c_2 > 0$ such that

$$|x|_1 \leq c_1|x|_2, \qquad |x|_2 \leq c_2|x|_1 \qquad (x \in E).$$

They are also both metrically equivalent to the supremum norm for some basis of E over F, under which E is evidently complete.

We now use the extra information that $|\cdot|_1$ and $|\cdot|_2$ are multiplicative (because they are norms on E as a field in its own right). For any positive integer n, we may substitute x^n in place of x in the previous inequalities and then take nth roots, to obtain

$$|x|_1 \leq c_1^{1/n}|x|_2, \qquad |x|_2 \leq c_2^{1/n}|x|_1 \qquad (x \in E).$$

Taking limits as $n \to \infty$ gives $|x|_1 = |x|_2$, as desired. □

Remark 1.4.10. The completeness of F is crucial in Theorem 1.4.9. For instance, the 5-adic norm on $\mathbb{Q}$ extends in two different ways to the Gaussian rational numbers $\mathbb{Q}(i)$, depending on whether $|2+i| = 5^{-1}$ and $|2-i| = 1$, or vice versa.

Because of the uniqueness in Theorem 1.4.9, it also follows that any algebraic extension E of F, finite or not, inherits a unique extension of $|\cdot|$. However, if $[E : F] = \infty$ then E is not complete, so we may prefer to use its completion instead. For instance, if $F = \mathbb{Q}_p$, we define $\mathbb{C}_p$ to be the completion of an algebraic closure of $\mathbb{Q}_p$. One might worry that this may launch us into an endless cycle of completion and algebraic closure, but fortunately this does not occur.

Theorem 1.4.11. *Let F be an algebraically closed nonarchimedean field. Then the completion of F is also algebraically closed.*

For the proof, see Section 2.3.

1.5 Spherical completeness

For nonarchimedean fields there is an important distinction between two different notions of completeness, which does not appear in the archimedean case.

Definition 1.5.1. A metric space is *complete* if any decreasing sequence of closed balls with radii tending to 0 has nonempty intersection. (For an abelian group equipped with a norm this reproduces our earlier definition.) A metric space is *spherically complete* if any decreasing sequence of closed balls, regardless of radii, has nonempty intersection.

Example 1.5.2. The fields $\mathbb{R}$ and $\mathbb{C}$ with their usual absolute value are spherically complete. Any complete nonarchimedean field which is discretely valued, e.g., $\mathbb{Q}_p$ or $\mathbb{C}((t))$, is spherically complete. Any finite-dimensional vector space over a spherically complete nonarchimedean field equipped with a compatible norm is spherically complete (exercise); in particular, any finite extension of a spherically complete nonarchimedean field is again spherically complete. However, the completion of an infinite algebraic extension of $\mathbb{Q}_p$ is not spherically complete unless it is discretely valued; see the end-of-chapter exercises.

Theorem 1.5.3 (Kaplansky–Krull). *Any nonarchimedean field embeds isometrically into a spherically complete nonarchimedean field. (However, the construction is not functorial; see the notes.)*

Proof. Since completion is functorial, we may assume we are starting with a complete nonarchimedean field F. It was originally shown by Krull [**151**, Theorem 24] that F admits an extension which is *maximally complete*, in the sense of not admitting any extensions preserving both the value group and the residue field. (In fact, this is not difficult to prove using Zorn's lemma.) The equivalence of this condition with spherical completeness was then proved by Kaplansky [**118**, Theorem 4]. See also [**214**, p. 151].

One can also prove the result more directly; for instance, the case $F = \mathbb{Q}_p$ is explained in detail in [**191**, §3]. For the case $F = K((t))$, see Example 1.5.8. □

One benefit of the hypothesis of spherical completeness is that it can simplify the construction of quotient norms.

Lemma 1.5.4. *Let F be a spherically complete nonarchimedean field, let V be a finite-dimensional vector space over F, and let $|\cdot|_V$ be a norm on V compatible with F (and which must be supremum-equivalent by Theorem 1.3.6). Let V' be a quotient of V, and let $|\cdot|_{V'}$ be the quotient norm on V' induced by $|\cdot|_V$. Then, for every $x' \in V'$, there exists $x \in V$ lifting x' with $|x|_V = |x'|_{V'}$.*

Proof. We first treat the case where $\dim_F(V') = \dim_F(V) - 1$. In this case, we can choose $m_1 \in V$ so that $V' = V/Fm_1$. Given $x' \in V'$, start with any lift x_0 of x' to V. Any other lift of x' to V can be written uniquely as $x_0 + am_1$ for some $a \in F$.

For $\epsilon > 0$, let B_ϵ be the set of $a \in F$ such that $|x_0 + am_1|_V \leq |x'|_{V'} + \epsilon$. By the definition of $|x'|_{V'}$, B_ϵ is nonempty. Pick any $a \in B_\epsilon$ and define

$$r(a, \epsilon) = \sup_{b \in B_\epsilon} \{|b - a|\}.$$

Then on the one hand B_ϵ is contained in the closed ball of radius $r(a, \epsilon)$ centered at a. On the other hand, for any $r < r(a, \epsilon)$ there exists $b \in B_\epsilon$ with $r \leq |b - a|$, so

$$\begin{aligned} r|m_1|_V &\leq |b - a||m_1|_V \\ &\leq \max\{|x_0 + am_1|_V, |x_0 + bm_1|_V\} \\ &\leq |x'|_{V'} + \epsilon. \end{aligned}$$

By taking limits, we may deduce that $r(a, \epsilon)|m_1|_V \leq |x'|_{V'} + \epsilon$. Hence, for any $c \in F$ with $|c - a| \leq r(a, \epsilon)$,

$$\begin{aligned} |x_0 + cm_1|_V &\leq \max\{|x_0 + am_1|_V, |(c - a)m_1|_V\} \\ &\leq |x'|_{V'} + \epsilon \end{aligned}$$

and so $c \in B_\epsilon$.

We conclude that B_ϵ must equal the closed ball of radius $r(a, \epsilon)$ centered at a. As ϵ decreases, the B_ϵ form a decreasing family of closed balls in F. Since

F is spherically complete, the intersection of the B_ϵ is nonempty. For any a in this intersection, $x = x_0 + am_1$ is a lift of x' to V satisfying $|x|_V = |x'|_{V_1}$.

Having completed the proof in the case where $\dim_F(V') = \dim_F(V) - 1$, we may treat the general case by induction on $\dim_F(V) - \dim_F(V')$. There is nothing to check if the difference is 0, and the above argument applies if the difference is 1. Otherwise, we can choose a nontrivial quotient V'' of V which in turn has V' as a nontrivial quotient. Define the quotient norm $|\cdot|_{V''}$ on V'' induced by $|\cdot|_V$; then $|\cdot|_V$ and $|\cdot|_{V''}$ induce the same quotient norm $|\cdot|_{V'}$ on V'. To lift $x' \in V'$ to V while preserving its norm, it thus suffices to apply the induction hypothesis first to lift x' to $x'' \in V''$ while preserving its norm and then to lift x'' to $x \in V$ while preserving its norm. □

In the case of a spherically complete field, we obtain the following refinement of the approximation lemma (Lemma 1.3.7).

Lemma 1.5.5. *Let F be a spherically complete nonarchimedean field, let V be a finite-dimensional vector space over F, and let $|\cdot|_V$ be a norm on V compatible with F and having the same value group as F. Then $|\cdot|_V$ is the supremum norm defined by some basis of V.*

Proof. Note that $|\cdot|_V$ is supremum-equivalent by Theorem 1.3.6. Define V_1 as in the proof of Lemma 1.3.7. By Lemma 1.5.4, any $x' \in V_1$ lifts to some $x \in V$ with $|x|_V = |x'|_{V_1}$. We may thus imitate the proof of case (b) of Lemma 1.3.7 to prove the desired result. □

We leave as an exercise the following alternate characterizations of spherical completeness.

Definition 1.5.6. Let X be a nonarchimedean metric space with distance function d. A sequence $\{x_n\}_{n=0}^\infty$ is *pseudoconvergent* if, for some n, either $x_n = x_{n+1} = \cdots$ or $d(x_n, x_{n+1}) > d(x_{n+1}, x_{n+2}) > \cdots$. An element $x \in X$ is a *pseudolimit* if, for all sufficiently large n, $d(x, x_n) \geq d(x_n, x_{n+1})$.

Proposition 1.5.7. *Let X be a nonarchimedean metric space. Then the following conditions are equivalent.*

(a) *The space X is spherically complete.*
(b) *For any sequence $B_1, B_2, \ldots$ of balls in X, if any two of the balls have nonempty intersection then the intersection of all the balls is nonempty.*
(c) *Every pseudoconvergent sequence in X has a pseudolimit.*

Proof. Exercise. □

To conclude, we give one explicit example of a field which is spherically complete without being discretely valued. This example is used in the proof of the slope filtration theorem (Theorem 16.4.1).

Example 1.5.8. Let K be an arbitrary field. A *generalized power series*, or *Mal'cev–Neumann series*, over K is a formal sum $\sum_{i\in\mathbb{Q}} x_i t^i$ for which the set of $i \in \mathbb{Q}$ with $x_i \neq 0$ is a *well-ordered* subset of $\mathbb{Q}$ (i.e., it contains no infinite decreasing subsequence). Let $K((t^{\mathbb{Q}}))$ be the set of such series; we may equip $K((t^{\mathbb{Q}}))$ with a t-adic valuation by sending $\sum_{i\in\mathbb{Q}} x_i t^i$ to the least index i for which $x_i \neq 0$.

We can demonstrate that $K((t^{\mathbb{Q}}))$ is a spherically complete abelian group as follows. Let $B_1, B_2, \ldots$ be a decreasing sequence of balls of radii $r_1, r_2, \ldots$. Put $r = \lim_{j\to\infty} r_j$; we may assume that $r \neq r_j$ for all j (otherwise the balls are all equal after some point and their intersection is clearly nonempty). For each $i < -\log r$, choose j with $i \leq -\log r_j$; then there is a single $x_i \in K$ which occurs as the coefficient of t^i for every element of B_j. The formal sum

$$x = \sum_{i<-\log r} x_i t^i$$

is then an element of $K((t^{\mathbb{Q}}))$ belonging to the intersection of all the B_j.

It is somewhat less obvious that $K((t^{\mathbb{Q}}))$ is a field. Let $x = \sum_{i\in\mathbb{Q}} x_i t^i$ and $y = \sum_{j\in\mathbb{Q}} y_j t^j$ be two elements of $K((t^{\mathbb{Q}}))$. We would like to define their product to be

$$xy = \sum_{k\in\mathbb{Q}} \left(\sum_{i,j\in\mathbb{Q}:i+j=k} x_i y_j \right) t^k.$$

To make this definition sensible, we must check two assertions.

(a) The set of $k \in \mathbb{Q}$ admitting at least one representation as $i + j$ where $x_i y_j \neq 0$ is well-ordered.

(b) For each $k \in \mathbb{Q}$, the number of representations of k as $i + j$, where $x_i y_j \neq 0$, is finite.

We leave both these statements, plus the fact that the resulting ring $K((t^{\mathbb{Q}}))$ has multiplicative inverses, as an exercise. (Once the multiplication is known to be well-defined, such properties as associativity and distributivity over addition are fairly trivial to check.)

Notes

The concept of a real valuation is a special case of Krull's notion of a *valuation* (sometimes called a *Krull valuation* for emphasis), in which the role of the real numbers is replaced by an arbitrary totally ordered group. For instance, on the polynomial ring $k[x, y]$ one can define a degree function taking monomials to elements of $\mathbb{Z} \times \mathbb{Z}$. If we then equip the latter with the lexicographic ordering (i.e., we compare pairs in their first component, using the second component only to break a tie in the first component), we may then define a valuation taking each polynomial to the lowest degree of any of its monomials.

While not all concepts from archimedean analysis generalize as nicely to Krull valuations as to real valuations, Krull valuations are important in algebraic geometry; indeed, they were originally advocated by Zariski as a key component of a rigorous foundation for the study of algebraic varieties. They were later ousted from this role by the more flexible theory of schemes, but they continue to play an important role in the study of birational properties of varieties and schemes, particularly in questions about the resolution of singularities by blowups. See [**187**] for a full account of the theory of valuations and [**113**] for discussion of some points concerning Zariski's use of valuations in algebraic geometry.

A sequence $m_1, m_2, \dots$ as in the statement of Lemma 1.3.8(a) is called a *Schauder basis* for V. For more discussion, see [**31**, §2.7.2]. Over a complete archimedean field (i.e., $\mathbb{R}$ or $\mathbb{C}$), the statement of Lemma 1.3.8 becomes false, as there are Banach spaces admitting countable dense subsets (i.e., Banach spaces which are *separable*) but not admitting Schauder bases. The first example is due to Enflo [**85**]. For a general discussion of the problem of constructing various sorts of Banach-space bases in the archimedean setting, see [**159**, Part I] or [**37**].

For a direct proof of Theorem 1.4.11 in the case of the completed algebraic closure of $\mathbb{Q}_p$, see [**191**, §3.3.3].

The fact that the completion of an infinite extension of $\mathbb{Q}_p$ fails to be spherically complete if it is not discretely valued (see Example 1.5.2 and the exercises) is a special case of a more general fact, namely, that any nonarchimedean metric space which admits a countable dense subset and whose metric takes values which are dense in $\mathbb{R}^+$ fails to be spherically complete [**193**, Theorem 20.5].

The condition of spherical completeness is quite important in nonarchimedean functional analysis, as it is needed for the Hahn–Banach theorem to hold. (By contrast, the nonarchimedean version of the open mapping theorem requires only completeness of the field.) For an expansion of this remark we recommend [**195**]; an older reference is [**214**].

Condition (c) of Proposition 1.5.7 is due to Ostrowski [**179**]; it is the definition used by Kaplansky in [**118**].

Example 1.5.8 was originally due to Hahn [**100**]. It was later generalized independently by Mal'cev and by Neumann to the case where the ordered abelian group $\mathbb{Q}$ is replaced by a possibly nonabelian group. Then it was used by Poonen [**182**] to describe the spherical completion of an arbitrary complete nonarchimedean field, even in the mixed-characteristic case. For instance, one obtains a description of a spherical completion of $\mathbb{C}_p$ in terms of a generalized power series in p.

Exercises

(1) Prove Proposition 1.1.8. (Hint: first note that $|x| < 1$ if and only if the sequence $x, x^2, \ldots$ converges to 0.)

(2) Prove Ostrowski's theorem (Theorem 1.2.4).

(3) Give an example to show that, even for a finite-dimensional vector space V over a complete nonarchimedean field F, the requirement that a norm on $|\cdot|_V$ compatible with F must satisfy the strong triangle inequality is not superfluous; that is, condition (b) in Definition 1.3.1 is not a consequence of the rest of the definition. (Hint: use a modification of a supremum norm.)

(4) Let M be a module over a nonarchimedean field F. Prove that any two norms on M compatible with F are topologically equivalent if and only if they are metrically equivalent.

(5) Use Remark 1.4.10 to give an example showing that the statement of Theorem 1.3.6 may fail if F is not complete.

(6) Prove that the valuation ring $\mathfrak{o}_F$ of a nonarchimedean field is noetherian if and only if F is trivially or discretely valued.

(7) Use Theorem 1.4.9 to prove that, for any field F, any nonarchimedean norm $|\cdot|$ on F, and any extension of E, there exists *at least one* extension of $|\cdot|$ to a norm on E. (Hint: reduce to the cases where E is a finite extension or a purely transcendental extension.)

(8) Here is a more exotic variation of the t-adic valuation. Let F be a field, and choose $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$.
   (a) Prove that on the rational function field $F(t_1, \ldots, t_n)$ there is a valuation v_α such that $v(f) = 0$ for all $f \in F^\times$ and $v(t_i) = \alpha_i$ for $i = 1, \ldots, n$. (Hint: you may construct it by iterating the definition of Gauss valuations.)
   (b) Prove that if $\alpha_1, \ldots, \alpha_n$ are linearly independent over $\mathbb{Q}$, the valuation v_α is uniquely determined by (a).
   (c) Prove that if $\alpha_1, \ldots, \alpha_n$ are not linearly independent over $\mathbb{Q}$, the valuation v_α is not uniquely determined by (a). (Hint: try for a starter the case $n = 2, \alpha_1 = \alpha_2 = 1$.)

(9) Let E be the completion of an infinite extension of $\mathbb{Q}_p$ which is not discretely valued. Let $\alpha_1, \alpha_2, \ldots \in E$ be any sequence of elements such that $|\alpha_1|, |\alpha_2|, \ldots$ form a strictly decreasing sequence with positive limit.

(Since E is not discretely valued, such a sequence must exist.) Prove that the sequence of discs

$$\{z \in E : |z - \alpha_1 - \cdots - \alpha_i| \leq |\alpha_i|\}$$

is decreasing, but its intersection is empty. (Hint: note that if the intersection were nonempty then it would contain an element algebraic over $\mathbb{Q}_p$, since such elements are dense in E.) Deduce as a corollary that E is not spherically complete.

(10) Prove that any finite-dimensional vector space over a spherically complete nonarchimedean field equipped with a compatible norm is also spherically complete. (Hint: use Lemma 1.5.5.)

(11) Prove Proposition 1.5.7.

(12) Prove that a subset S of $\mathbb{R}$ is well-ordered (contains no infinite decreasing sequence) if and only if every nonempty subset of S has a least element.

(13) Check the unproved assertions at the end of Example 1.5.8. (Hint: reduce both (a) and (b) to the fact that, given any sequence of pairs $(i, j) \in \mathbb{Q}^2$ for which $i + j$ is nonincreasing, one can pass to a subsequence in which one component is nonincreasing. Reduce (c) to checking that $x \in K((t^{\mathbb{Q}}))$ has an inverse whenever $x - 1$ has positive t-adic valuation, then use a geometric series.)

2

Newton polygons

In this chapter, we recall the traditional theory of Newton polygons for polynomials over a nonarchimedean field. In the process, we introduce a general framework which will allow us to consider Newton polygons in a wider range of circumstances; it is based on a version of Hensel's lemma that applies in not necessarily commutative rings. As a first application, we fill in a few missing proofs from Chapter 1.

2.1 Introduction to Newton polygons

We start with the possibly familiar notion of the Newton polygon associated with a polynomial over a nonarchimedean ring.

Definition 2.1.1. Let R be a ring equipped with a nonarchimedean submultiplicative (semi)norm $|\cdot|$. For $\rho > 0$ and $P = \sum_i P_i T^i \in R[T]$, define the *width* of P under the ρ-Gauss norm $|\cdot|_\rho$ as the difference between the maximum and minimum values of i for which $\max_i\{|P_i|\rho^i\}$ is achieved.

Proposition 2.1.2. *Let R be a ring equipped with a nonarchimedean multiplicative seminorm $|\cdot|$. For $\rho > 0$ and $P, Q \in R[T]$ the following results hold.*

(a) *We have $|PQ|_\rho = |P|_\rho |Q|_\rho$. That is, $|\cdot|_\rho$ is multiplicative.*
(b) *The width of PQ under $|\cdot|_\rho$ equals the sum of the widths of P and Q under $|\cdot|_\rho$.*

Proof. For $* \in \{P, Q\}$, let j_*, k_* be the minimum and maximum values of i for which $\max_i\{|*_i|\,\rho^i\}$ is achieved. Write

$$PQ = \sum_i (PQ)_i T^i = \sum_i \left(\sum_{g+h=i} P_g Q_h \right) T^i.$$

In the sum $(PQ)_i = \sum_{g+h=i} P_g Q_h$, each summand has norm at most $|P|_\rho |Q|_\rho \rho^{-i}$, with equality if and only if $|P_g| = |P|_\rho \rho^{-g}$ and $|Q_h| = |Q|_\rho \rho^{-h}$. This cannot occur for $i < j_P + j_Q$, and for $i = j_P + j_Q$ it can only occur for $g = j_P, h = j_Q$. Hence

$$\begin{aligned} |(PQ)_i| &< |P|_\rho |Q|_\rho \rho^{-i} && (i < j_P + j_Q), \\ |(PQ)_i| &= |P|_\rho |Q|_\rho \rho^{-i} && (i = j_P + j_Q), \\ |(PQ)_i| &\le |P|_\rho |Q|_\rho \rho^{-i} && (i > j_P + j_Q). \end{aligned}$$

Similarly, we also have

$$\begin{aligned} |(PQ)_i| &\le |P|_\rho |Q|_\rho \rho^{-i} && (i < k_P + k_Q), \\ |(PQ)_i| &= |P|_\rho |Q|_\rho \rho^{-i} && (i = k_P + k_Q), \\ |(PQ)_i| &< |P|_\rho |Q|_\rho \rho^{-i} && (i > k_P + k_Q). \end{aligned}$$

This proves both claims. □

Definition 2.1.3. Let R be a ring equipped with a nonarchimedean submultiplicative seminorm $|\cdot|$. Let $v = -\log|\cdot|$ denote the corresponding valuation. Given a polynomial $P(T) = \sum_{i=0}^n P_i T^i \in R[T]$, draw the set of points

$$\{(-i, v(P_i)) : i = 0, \dots, n;\ v(P_i) < +\infty\} \subset \mathbb{R}^2,$$

then form the lower convex hull of these points. (That is, take the intersection of every closed half-plane which contains all the points and lies above some nonvertical line.) The boundary of this region is called the *Newton polygon* of P. The *slopes* of P are the slopes of this open polygon, viewed as a multiset in which each slope r counts with multiplicity equal to the horizontal width of the segment of the Newton polygon of slope r (or equal to zero if there is no such segment); the latter can also be interpreted as the width of P under $|\cdot|_{e^{-r}}$. (In the case where this multiset has cardinality less than $\deg(P)$, we include $+\infty$ with sufficient multiplicity to make up the shortfall.)

Example 2.1.4. For $R = \mathbb{Q}_p$ equipped with the p-adic norm $|\cdot|_p$, the Newton polygon of the polynomial $T^3 + pT^2 + pT + p^3T^3$ has vertices $(-3, 0)$, $(-1, \log p)$, $(0, 3\log p)$. Its slopes are $\frac{1}{2}\log p$ with multiplicity 2 and $2\log p$ with multiplicity 1.

Proposition 2.1.5. *Let R be a nonarchimedean ring, and suppose that $P(T) = (T - \lambda_1) \cdots (T - \lambda_n)$. Then the slope multiset of P consists of $-\log|\lambda_1|, \ldots, -\log|\lambda_n|$.*

Proof. This is immediate from the multiplicativity of $|\cdot|_{e^{-r}}$. □

Here is an explicit example of Proposition 2.1.5 which we will use repeatedly.

Example 2.1.6. For p a prime and m a positive integer, let ζ_m be a primitive mth root of unity in an algebraic closure of $\mathbb{Q}_p$. According to Theorem 1.4.9 (which we will finish proving later in this chapter), $\mathbb{Q}_p(\zeta_m)$ admits a unique extension of the p-adic norm $|\cdot|_p$ on $\mathbb{Q}_p$. Assuming this, let us calculate the norm of the element $1 - \zeta_m$.

The easiest case is when m is not divisible by p. In this case $\zeta_m - 1$ is a root of the polynomial $((T+1)^m - 1)/T$, which has integer coefficients and constant term $m \not\equiv 0 \pmod{p}$. Hence the Newton polygon has all slopes equal to 0, so $|1 - \zeta_m|_p = 1$.

Suppose next that $m = p^h$ for some positive integer h. In this case ζ_{p^h} is a root of the polynomial

$$P(T) = \frac{T^{p^h} - 1}{T^{p^{h-1}} - 1} = \sum_{i=0}^{p-1} T^{ip^{h-1}},$$

so $\zeta_{p^h} - 1$ is a root of $Q(T) = P(T+1)$. The image $\overline{Q}(T)$ of $Q(T)$ in $\mathbb{F}_p[T]$ satisfies

$$\overline{Q}(T) = \frac{(T+1)^{p^h} - 1}{(T+1)^{p^{h-1}} - 1} = \frac{T^{p^h} + 1^{p^h} - 1}{T^{p^{h-1}} + 1^{p^{h-1}} - 1} = T^{(p-1)p^{h-1}},$$

so all the coefficients of Q except for its leading coefficient are divisible by p. Moreover $Q(0) = p$, so the Newton polygon of Q is a straight line segment with endpoints $(-(p-1)p^{h-1}, 0)$ and $(0, \log p)$. We conclude that

$$|1 - \zeta_{p^h}|_p = p^{-p^{-h+1}/(p-1)} \tag{2.1.6.1}$$

(see the exercises for an alternate derivation of (2.1.6.1)).

Finally, suppose that $m = p^h j$, where j is not divisible by p. If $j = 1$, we have $|1 - \zeta_m|_p = p^{-p^{-h+1}/(p-1)}$ by (2.1.6.1). Otherwise, we can write $\zeta_m = \zeta_{p^h}\zeta_j$ and

$$1 - \zeta_m = (1 - \zeta_{p^h}) + \zeta_{p^h}(1 - \zeta_j).$$

On the right-hand side, the first term has norm less than 1 whereas the second term has norm equal to 1. We thus have $|1 - \zeta_m|_p = 1$.

Remark 2.1.7. The data of the Newton polygon is equivalent to the data of the ρ-Gauss norms for all ρ. This amounts to the statement that the graph of a convex continuous function is determined by the positions of its supporting lines of all slopes; this holds because the graph is the lower boundary of the intersection of the closed halfspaces bounded below by the supporting lines. For an example of the use of this observation, see the proof of Theorem 11.2.1.

2.2 Slope factorizations and a master factorization theorem

A key property of the operation of completing a nonarchimedean ring, first noticed by Hensel, is that it vastly enlarges the collection of polynomials over the ring which can be factored. Here is a sample statement of this form, which gives us a factorization that separates slopes in the Newton polygon.

Theorem 2.2.1. *Let R be a complete nonarchimedean ring. Suppose that $S \in R[T]$, $r \in \mathbb{R}$, and $m \in \mathbb{Z}_{\geq 0}$ satisfy*

$$v_r(S - T^m) > v_r(T^m).$$

Then there exists a unique factorization $S = PQ$ satisfying the following conditions.

(a) *The polynomial $P \in R[T]$ has degree* $\deg(S) - m$*, and its slopes are all less than r.*

(b) *The polynomial $Q \in R[T]$ is monic of degree m, and its slopes are all greater than r.*

(c) *We have $v_r(P - 1) > 0$ and $v_r(Q - T^m) > v_r(T^m)$.*

Moreover, for this factorization,

$$\min\{v_r(P-1), v_r(Q-T^m) - v_r(T^m)\} \geq v_r(S-T^m) - v_r(T^m). \quad (2.2.1.1)$$

(In fact, this statement turns out to be an equality; see the exercises.)

Let us translate this statement into plainer language. The hypothesis $v_r(S - T^m) > v_r(T^m)$ is equivalent to the following two conditions.

(a) The coefficient S_m of T^m in S satisfies $|S_m - 1| < 1$; in particular, S_m is a unit.

(b) The supporting line of slope r of the Newton polygon of S touches the polygon at the point $(-m, 0)$ and nowhere else.

In particular, r does not occur as a slope of the Newton polygon of S. Note that if (b) holds and S_m is a unit, we can apply the theorem to $S_m^{-1}S$ instead.

It is not difficult to prove Theorem 2.2.1 directly. However, we will be stating a number of similar results as we go along, with similar proofs. To save some effort, we state a master factorization theorem applicable to not necessarily commutative rings, from which we can deduce Theorem 2.2.1 and all the variants we will use later. The theorem, and the proof given here, are due to Christol [**42**, Proposition 1.5.1].

Theorem 2.2.2 (Master factorization theorem). *Let R be a nonarchimedean, not necessarily commutative, ring. Suppose that the nonzero elements $a, b, c \in R$ and the additive subgroups $U, V, W \subseteq R$ satisfy the following conditions.*

(a) *The spaces U, V are complete under the norm, and $UV \subseteq W$.*
(b) *The map $f(u, v) = av + ub$ is a surjection of $U \times V$ onto W.*
(c) *There exists $\lambda > 0$ such that*

$$|f(u, v)| \geq \lambda \max\{|a||v|, |b||u|\} \qquad (u \in U, v \in V).$$

(d) *We have $ab - c \in W$ and*

$$|ab - c| < \lambda^2 |c|.$$

Then there exists a unique pair $(x, y) \in U \times V$ such that

$$c = (a + x)(b + y), \qquad |x| < \lambda|a|, \qquad |y| < \lambda|b|.$$

For this x, y we also have

$$|x| \leq \lambda^{-1}|ab - c||b|^{-1}, \qquad |y| \leq \lambda^{-1}|ab - c||a|^{-1}.$$

Before proving Theorem 2.2.2, let us see how it implies Theorem 2.2.1.

Proof of Theorem 2.2.1. We apply Theorem 2.2.2 with the following parameters:

$$\begin{aligned}
R &= F[T],\\
|\cdot| &= |\cdot|_{e^{-r}},\\
U &= \{P \in F[T] : \deg(P) \leq \deg(S) - m\},\\
V &= \{P \in F[T] : \deg(P) \leq m - 1\},\\
W &= \{P \in F[T] : \deg(P) \leq \deg(S)\},\\
a &= 1,\\
b &= T^m,\\
c &= S,\\
\lambda &= 1
\end{aligned}$$

and then put $P = a + x$ and $Q = b + y$.

To see that this works, let us verify explicitly that condition (c) is satisfied in this setup (the other conditions are more obvious). If $u \in U, v \in V$ are such that $\max\{|a||v|, |b||u|\}$ is achieved by some term of av then that term appears unchanged in $av + ub$ because ub is divisible by T^m. Hence $|f(u, v)| \geq |av|$ in that case. Otherwise we have $|av| < |bu|$ and so $|f(u, v)| = |bu|$. □

With this motivation in mind, we now prove Theorem 2.2.2.

Proof of Theorem 2.2.2. We define a norm on $U \times V$ by setting

$$|(u, v)| = \max\{|a||v|, |b||u|\},$$

so that (c) implies

$$\lambda|(u, v)| \leq |f(u, v)| \leq |(u, v)|.$$

In particular $\lambda \leq 1$, so $|ab - c| < |ab| = |c|$.

Since a, b are nonzero, (c) implies that f is injective. By (b), f is in fact a bijective group homomorphism between $U \times V$ and W. It follows that, for all $w \in W$,

$$|f^{-1}(w)| \leq \lambda^{-1}|w|.$$

By (d) we may choose $\mu \in (0, \lambda)$ with $|ab - c| \leq \lambda\mu|c|$. Define

$$B_\mu = \{(u, v) \in U \times V : |(u, v)| \leq \mu|c|\}.$$

For $(u, v) \in B_\mu$ we have

$$|a||v| \leq |(u, v)| \leq \mu|c| = \mu|a||b|,$$

so $|v| \leq \mu|b|$. Similarly $|u| \leq \mu|a|$. As a result,

$$\begin{aligned}
|f^{-1}(c - ab - uv)| &\leq \lambda^{-1}|c - ab - uv| \\
&\leq \lambda^{-1}\max\{|c - ab|, |uv|\} \\
&\leq \lambda^{-1}\max\{\lambda\mu|c|, \mu^2|a||b|\} \\
&= \mu|c|.
\end{aligned}$$

Consequently, the map $g(u, v) = f^{-1}(c - ab - uv)$ carries B_μ into itself.

We next show that g is contractive. For $(u, v), (t, s) \in B_\mu$,

$$\begin{aligned}
|g(u, v) - g(t, s)| &\leq |f^{-1}(ts - uv)| \\
&\leq \lambda^{-1}|ts - uv| \\
&= \lambda^{-1}|t(s - v) + (t - u)v| \\
&\leq \lambda^{-1}\max\{\mu|a||s - v|, \mu|t - u||b|\} \\
&= \lambda^{-1}\mu|(u - t, v - s)| \\
&= \lambda^{-1}\mu|(u, v) - (t, s)|,
\end{aligned}$$

which has the desired effect because $\lambda^{-1}\mu < 1$.

Since g is contractive on B_μ and $U \times V$ is complete, by the Banach contraction mapping theorem (exercise) there is a unique $(x, y) \in U \times V$ fixed by g. That is,

$$ay + xb = f(x, y) = f(g(x, y)) = c - ab - xy$$

and so

$$c = (a + x)(b + y).$$

Moreover, there is a unique such (x, y) in the union of all the B_μ, and that element belongs to the intersection of all the B_μ. □

Remark 2.2.3. One can also use Theorem 2.2.2 to recover other instances of Hensel's lemma. For instance, if F is a complete nonarchimedean field, $P(x) \in \mathfrak{o}_F[x]$, and the reduction of $P(x)$ into $\kappa_F[x]$ factors as $\overline{QR}$ with $\overline{Q}, \overline{R}$ coprime, then there exists a unique factorization $P = QR$ in $\mathfrak{o}_F[x]$ with Q, R lifting $\overline{Q}, \overline{R}$ (exercise).

2.3 Applications to nonarchimedean field theory

We now go back and apply Theorem 2.2.1 to prove some facts about extensions of nonarchimedean fields which were omitted from Chapter 1. We first complete the proof of Theorem 1.4.9; for this we need the following lemma.

Lemma 2.3.1. *Let F be a complete nonarchimedean field. Let $P(T) \in F[T]$ be a polynomial whose slopes are all greater than or equal to r. Let $S(T) \in F[T]$ be any polynomial, and write $S = PQ + R$ with* $\deg(R) < \deg(P)$ *using the division algorithm. Then*

$$v_r(S) = \min\{v_r(P) + v_r(Q), v_r(R)\}.$$

Proof. Exercise. □

Proof of Theorem 1.4.9 (continued). It remains to show that if F is a complete nonarchimedean field then any finite extension E of F admits an extension of $|\cdot|$ to a norm on E. If E' is a field intermediate between F and E, we may first extend the norm to E' and then to E. Consequently, it suffices to check the case where $E = F(\alpha)$ for some $\alpha \in E$, that is, $E \cong F[T]/(P(T))$ for some monic irreducible polynomial $P \in F[T]$ (the minimal polynomial of α). Apply Theorem 2.2.1; since $P(T)$ cannot factor nontrivially, we deduce that P must have a single slope r.

We now define a norm on E as follows: for $\beta = c_0 + c_1\alpha + \cdots + c_{n-1}\alpha^{n-1}$, with $n = \deg(P) = [E : F]$, put

$$|\beta|_E = \max_i \{|c_i| e^{-ri}\}.$$

That is, take $|\beta|_E$ to be the e^{-r}-Gauss norm of the polynomial $c_0 + c_1 T + \cdots + c_{n-1}T^{n-1}$. The submultiplicativity of $|\cdot|_E$ is then a consequence of Lemma 2.3.1; however, since E is a field, submultiplicativity implies multiplicativity. $\square$

We next give the proof of Theorem 1.4.11. For this, we need a crude version of the principle that the roots of a polynomial over a complete algebraically closed nonarchimedean field should vary continuously in the coefficients.

Lemma 2.3.2. *Let F be an algebraically closed nonarchimedean field with completion E, and suppose that $P \in E[T]$ is monic of degree d. Then, for any $\epsilon > 0$, there exists $z \in F$ such that $|z| \leq |P(0)|^{1/d}$ and $|P(z)| < \epsilon$.*

Proof. If $P(0) = 0$ we may pick $z = 0$, so assume that $P(0) \neq 0$. Put $P = T^d + \sum_{i=0}^{d-1} P_i T^i$. For any $\delta > 0$, we can pick a polynomial $Q = T^d + \sum_{i=0}^{d-1} Q_i d^i \in F[T]$ with $|Q_i - P_i| < \delta$ for $i = 0, \dots, d-1$.

Now assume that $\delta < \min\{|P_0|, \epsilon, \epsilon/|P_0|\}$, so that $|Q_0| = |P_0|$. By Proposition 2.1.5 there exists $z \in F$ with $Q(z) = 0$ and $|z| \leq |Q_0|^{1/d} = |P_0|^{1/d}$. We now have

$$|P(z)| = |(P - Q)(z)| \leq \delta \max\{1, |z|\}^d \leq \delta \max\{1, |P(0)|\} < \epsilon,$$

as desired. $\square$

Proof of Theorem 1.4.11. We must check that the completion E of an algebraically closed nonarchimedean field F is itself algebraically closed. Let $P(T) \in E[T]$ be a monic polynomial of degree d. Define a sequence of polynomials $P_0, P_1, \dots$ as follows. Put $P_0 = P$. Given P_i, apply Lemma 2.3.2 to construct z_i with $|z_i| \leq |P_i(0)|^{1/d}$ and $|P_i(z_i)| < 2^{-i}$; then set $P_{i+1}(T) = P_i(T + z_i)$, so that $P_{i+1}(0) = P_i(z_i)$. If some P_i satisfies $P_i(0) = 0$ then $z_0 + \cdots + z_{i-1}$ is a root of P. Otherwise, we get an infinite sequence $z_0, z_1, \dots$ such that $z_0 + z_1 + \cdots$ converges to a root of P. $\square$

Notes

There is a good reason for Newton's name to be attached to the polygons considered in this chapter. He considered them in the context of finding a series expansion at the origin for a function $y = f(x)$ implicitly defined by a polynomial relation $P(x, y) = 0$ over $\mathbb{C}$. This was later reinterpreted by Puiseux, in terms of computing roots of polynomials over the subfield of $\mathbb{C}((x))$ consisting of series which represent meromorphic functions in a neighborhood of $x = 0$.

Remark 2.1.7 is perhaps best viewed within the broader context of duality for convex functions, which is a central theme in the study of linear optimization problems. One reference for the rigorous mathematical theory of convex functions is [**192**].

Although the formulation of Theorem 2.2.2 is due to Christol, the basic observation that Hensel's lemma does not rely on commutativity is rather old. One early instance is due to Zassenhaus [**223**].

The Banach contraction mapping theorem (also known as the Banach fixed point theorem), used in the proof of Theorem 2.2.2, is probably familiar to most readers except possibly in name. For instance, it appears in the most common proofs of the implicit function theorem, the inverse function theorem, and the fundamental theorem of ordinary differential equations.

The notion of a nonarchimedean metric space can be generalized to the case where the metric takes values in an arbitrary partially ordered set. One can then give a number of extensions of the Banach contraction mapping theorem, at least in the case of spherically complete metric spaces. For instance, Priess-Crampe has proved that if X is a spherically complete metric space with distance function d, and $f : X \to X$ is a function such that $d(f(x), f(y)) < d(x, y)$ for all $x, y \in X$ with $x \neq y$, then f has a fixed point. See [**197**] for an exposition of a number of results of this type.

Exercises

(1) With notation as in Example 2.1.6, rederive (2.1.6.1) as follows. First note that for j coprime to p, $1 - \zeta_{p^h}$ and $1 - \zeta_{p^h}^j$ are multiples of each other in the ring $\mathbb{Z}[\zeta_{p^h}]$, so that $|1 - \zeta_{p^h}|_p \geq |1 - \zeta_{p^h}^j|_p$ and vice versa. Then note that the product of $1 - \zeta_{p^h}^j$ over those $j \in \{0, \dots, p^h - 1\}$ not divisible by p equals $Q(0) = p$.

(2) Prove that equality holds in (2.2.1.1). (Hint: split $S - T^m$ into $(P - 1)Q + (Q - T^m)$.)

(3) Prove the claim in Remark 2.2.3. (Hint: since $\overline{Q}$ and $\overline{R}$ are coprime, we can choose $\overline{S}, \overline{T} \in \kappa_F[x]$ such that $\overline{QS} + \overline{RT} = 1$. We can also ensure that $\deg(\overline{S}) < \deg(\overline{R})$ and $\deg(\overline{T}) < \deg(\overline{Q})$. Use lifts of these to set up the conditions of Theorem 2.2.2.)

(4) Prove Lemma 2.3.1. (Hint: separate P as a sum $P_1 + P_2$ in which $v_r(P)$ is achieved by the leading coefficient of P_1, while $v_r(P_2) > v_r(P)$.)

(5) Prove the Banach contraction mapping theorem (used in the proof of Theorem 2.2.2). That is, suppose that X is a nonempty complete metric space (not necessarily nonarchimedean) with distance function d, and let $g : X \to X$ be a map for which there exists $\mu \in (0, 1)$ such that $d(g(x), g(y)) \leq \mu d(x, y)$ for all $x, y \in X$. Prove that g has a unique fixed point. (Hint: show that, for any $x \in X$, the sequence $x, g(x), g(g(x)), \ldots$ is Cauchy and that its limit is the desired fixed point.)

3

Ramification theory

Recall (Theorem 1.4.9) that any finite extension of a complete nonarchimedean field carries a unique extension of the norm and is also complete. In this chapter, we study the relationship between a complete nonarchimedean field and its finite extensions; this relationship involves the residue fields, value groups, and Galois groups of the fields in question. We distinguish some important types of extensions, the *unramified* and *tamely ramified* extensions. See [**187**] (especially Chapter 6) for a more thorough treatment.

We also briefly discuss the special case of a discretely valued field with a perfect residue field, in which one can say much more. We introduce the standard ramification filtrations on the Galois groups of extensions of local fields; these will not reappear again until Part IV, at which point they will relate to the study of the convergence of solutions of p-adic differential equations made in Part III. We make no attempt to be thorough in our treatment; instead we refer the reader to the standard text [**198**] for more details.

Notation 3.0.1. For E/F a Galois extension of fields, write $G_{E/F}$ for $\mathrm{Gal}(E/F)$. If $E = F^{\mathrm{sep}}$, the separable closure of E, write G_F for the absolute Galois group $G_{F^{\mathrm{sep}}/F}$. Of course, if F has characteristic 0 (or is perfect, e.g., if F is a finite field) then F^{sep} coincides with the algebraic closure F^{alg}. (We avoid the usual notation $\overline{F}$ for the latter because we prefer to reserve the overbar to denote reduction modulo the maximal ideal of a local ring, e.g., from $\mathbb{Z}_p$ to $\mathbb{F}_p$.)

Remark 3.0.2. Throughout this chapter, when a nonarchimedean field F is assumed to be complete it will be sufficient to assume it is only *henselian* instead. One of many equivalent formulations of this condition (for which see [**175**, 43.2]) is as follows: for any monic polynomial $P(x) \in \mathfrak{o}_F[x]$ and any simple root $\overline{r} \in \kappa_F$ of $\overline{P} \in \kappa_F[x]$ there exists a unique root $r \in \mathfrak{o}_F$ of P lifting $\overline{r}$.

3.1 Defect

Let F be a complete nonarchimedean field, and let E be a finite extension of F. Then the value group $|E^\times|$ contains $|F^\times|$, while the residue field κ_E may naturally be viewed as an extension of κ_F. The first fundamental fact about the extension E/F is that both these containments are finite in appropriate senses closely related to the degree of E over F.

Lemma 3.1.1. *Let F be a complete nonarchimedean field, and let E be a finite extension of F. Then*

$$[E : F] \geq [\kappa_E : \kappa_F]\#(|E^\times|/|F^\times|), \tag{3.1.1.1}$$

with equality at least when F is discretely valued.

Proof. Choose $\alpha_1, \dots, \alpha_m \in \mathfrak{o}_E$ lifting a basis of κ_E over κ_F, and choose $\beta_1, \dots, \beta_n \in E$ so that $|\beta_1|, \dots, |\beta_n|$ form a set of coset representatives of $|F^\times|$ in $|E^\times|$. Then the $\alpha_i\beta_j$ are linearly independent over F, proving (3.1.1.1).

In the case where F is discretely valued, there exists a unique $\rho \in (0, 1)$ for which $|F^\times| = \rho^{\mathbb{Z}}$ and $|E^\times| = (\rho^{1/n})^{\mathbb{Z}}$. If we choose β_j to have norm ρ^{j-1} then it is not hard to show that the $\alpha_i\beta_j$ form a basis of $\mathfrak{o}_E$ over $\mathfrak{o}_F$ (exercise). This proves the desired equality. □

If F is not discretely valued, the situation can be more complicated. One does however have the following refinement of (3.1.1.1), which we will not prove here.

Theorem 3.1.2 (Ostrowski). *Let F be a complete nonarchimedean field, and let E be a finite extension of F. Then the quantity*

$$\operatorname{defect}(E/F) = \frac{[E : F]}{[\kappa_E : \kappa_F]\#(|E^\times|/|F^\times|)}$$

is a positive integer. Moreover, if F has characteristic 0 *then* $\operatorname{defect}(E/F) = 1$*; otherwise* $\operatorname{defect}(E/F)$ *is a power of the characteristic of F.*

Proof. See [**187**, Theorem 6.2]. □

Definition 3.1.3. The quantity $\operatorname{defect}(E/F)$ in Theorem 3.1.2 is called the *defect* of E over F. (Note that some sources instead define the defect as $\log_p \operatorname{defect}(E/F)$, where p is the characteristic of F.) An extension for which $\operatorname{defect}(E/F) = 1$ is said to be *defectless*. For example, any finite extension of a spherically complete field is defectless (see Remark 3.3.11

below). For an example of an extension with nontrivial defect, see [**187**, §6.3].

3.2 Unramified extensions

The easiest finite extensions of a complete nonarchimedean field to describe are the unramified extensions, which have a particularly simple effect on the residue field and the value group.

Definition 3.2.1. Let F be a complete nonarchimedean field. A finite extension E of F is *unramified* if κ_E is separable over κ_F and $[E : F] = [\kappa_E : \kappa_F]$. This forces E itself to be separable over F; see Proposition 3.2.3.

Lemma 3.2.2. *Let F be a complete nonarchimedean field, and let U be a finite extension of F. Then, for any subextension E of U over F, U is unramified over F if and only if U is unramified over E and E is unramified over F.*

Proof. Since κ_E sits between κ_U and κ_F, having κ_U separable over κ_F is equivalent to having both κ_U separable over κ_E and κ_E separable over κ_F. By Lemma 3.1.1

$$[U : F] \geq [\kappa_U : \kappa_F], \qquad [U : E] \geq [\kappa_U : \kappa_E], \qquad [E : F] \geq [\kappa_E : \kappa_F];$$

since $[U : F] = [U : E][E : F]$ and $[\kappa_U : \kappa_F] = [\kappa_U : \kappa_E][\kappa_E : \kappa_F]$, the first of the three inequalities is an equality if and only if the other two are. $\square$

What makes unramified extensions so simple to describe is that they are uniquely determined by their residue field extensions. Here is a precise statement to this effect.

Proposition 3.2.3. *Let F be a complete nonarchimedean field, and let E be a finite extension of F. Then, for any separable subextension λ of κ_F over κ_E, there exists a unique unramified extension U of F contained in E with $\kappa_U \cong \lambda$; moreover, U is separable over F.*

Proof. By the primitive element theorem, one can always write $\lambda \cong \kappa_F[x]/(\overline{P}(x))$ for some monic irreducible separable polynomial $\overline{P} \in \kappa_F[x]$. Lift $\overline{P}$ to a monic polynomial $P \in \mathfrak{o}_F[x]$. Choose $t \in \mathfrak{o}_E$ such that the image of t in κ_E corresponds to x in $\kappa_F[x]/(\overline{P}(x))$; then the reduction of $P(x + t)$ into $\kappa_E[x]$ is divisible by x but not by x^2. We may thus apply the slope factorization theorem (Theorem 2.2.1) to deduce that $P(x + t)$ has a root in $\mathfrak{o}_E$. This proves the existence and separability of U over F.

To prove uniqueness, let U' be another such extension. Then the previous argument applied to U' in place of E shows that $\mathfrak{o}_{U'}$ contains a root of $P(x+t)$

congruent to 0 modulo $\mathfrak{m}_{U'}$. However, there can only be one such root in U' because the Newton polygon of $P(x+t)$ has only one positive slope (since $\overline{P}$ is a separable polynomial), so in fact $U \subseteq U'$. Again, by comparing degrees we have $U = U'$. □

Corollary 3.2.4. *For each finite separable extension λ of κ_F, there exists a unique unramified extension E of F with $\kappa_E \cong \lambda$.*

Proof. Choose $P(x)$ as in the proof of Proposition 3.2.3. Then $E = F[x]/(P(x))$ is an unramified extension of F with residue field λ. The proof of Proposition 3.2.3 shows that any other unramified extension with residue field λ must contain E; by once again comparing degrees we see that this containment must be an equality. □

Lemma 3.2.5. *Let F be a complete nonarchimedean field, and let E be a finite extension of F. Let U_1, U_2 be unramified subextensions of E over F. Then the compositum $U = U_1U_2$ is also unramified over F, and $\kappa_U = \kappa_{U_1}\kappa_{U_2}$ inside κ_E.*

Proof. Put $U_3 = U_1 \cap U_2$ inside E; by Lemma 3.2.2, U_3 is unramified over F and U_1 is unramified over U_3. By Proposition 3.2.3, inside κ_E we have $\kappa_{U_1} \cap \kappa_{U_2} = \kappa_{U_3}$. Consequently

$$\begin{aligned}
[\kappa_U : \kappa_{U_2}] &\leq [U : U_2] \qquad \text{(by Lemma 3.1.1)} \\
&= [U_1 : U_3] \\
&= [\kappa_{U_1} : \kappa_{U_3}] \qquad \text{(because } U_1 \text{ is unramified over } U_3\text{)} \\
&= [\kappa_{U_1}\kappa_{U_2} : \kappa_{U_2}] \\
&\leq [\kappa_U : \kappa_{U_2}] \qquad \text{(because } \kappa_{U_1}\kappa_{U_2} \subseteq \kappa_U\text{)}.
\end{aligned}$$

We deduce first that $\kappa_U = \kappa_{U_1}\kappa_{U_2}$ and second that $[U : U_2] = [\kappa_U : \kappa_{U_2}]$. Hence U is unramified over U_2, hence also over F by Lemma 3.2.2 again. □

Definition 3.2.6. Let F be a complete nonarchimedean field, and let E be a finite extension of F. By Lemma 3.2.5 there is a maximal unramified subextension U of E over F; by Proposition 3.2.3, κ_U is the maximal separable subextension of κ_E over κ_F. (We will also say that $\mathfrak{o}_U$ is the "maximal unramified subextension" of $\mathfrak{o}_E$ over $\mathfrak{o}_F$.) We say E is *totally ramified* over F if $U = F$.

3.3 Tamely ramified extensions

For Galois extensions of complete nonarchimedean fields, we can refine the ramification theory introduced in the previous section. In the case of a discretely valued base field, this material may be familiar from [**198**]; we will continue the review of that case in Section 3.4.

Hypothesis 3.3.1. In this section, let F be a complete nonarchimedean field, and let E be a finite Galois extension of F.

Definition 3.3.2. The *inertia subgroup* $I_{E/F}$ of $G_{E/F}$ is the kernel of the map $G_{E/F} \to \mathrm{Aut}(\kappa_E)$; it may be interpreted as $G_{E/U}$, where U is the maximal unramified subextension of E over F. In particular, E is unramified over F if and only if $I_{E/F}$ is trivial, whereas E is totally ramified over F if and only if $I_{E/F} = G_{E/F}$.

Definition 3.3.3. Given $g \in I_{E/F}$ and $x \in E^\times$, let $\langle g, x \rangle$ denote the image of $g(x)/x$ in $\kappa_E^\times$. For fixed g, this is a homomorphism from $E^\times \to \kappa_E^\times$; moreover, it is trivial on $\mathfrak{o}_E^\times$ because $g \in I_{E/F}$. We thus obtain a homomorphism $I_{E/F} \to \mathrm{Hom}(|E^\times|, \kappa_E^\times)$; let $W_{E/F}$ denote the kernel of this map, called the *wild inertia subgroup* of $G_{E/F}$. Note that $\kappa_E^\times$ has no p-torsion, so neither does $\mathrm{Hom}(|E^\times|, \kappa_E^\times)$; hence $I_{E/F}/W_{E/F}$ is abelian and of order not divisible by p.

Remark 3.3.4. Some readers may recall that if F is discretely valued then $I_{E/F}/W_{E/F}$ is *cyclic* of order not divisible by p. This can be seen as follows. First, let μ denote the group of roots of unity in $\kappa_E^\times$ of order dividing $[E : F]$, and note that the image of $I_{E/F}/W_{E/F}$ in $\mathrm{Hom}(|E^\times|, \kappa_E^\times)$ includes only maps from $|E^\times|$ into μ. Second, note that μ is a finite cyclic group. Finally, note that $|F^\times| \cong \mathbb{Z}$ implies $|E^\times| \cong \mathbb{Z}$. (This last step does not apply if F is not discretely valued, and indeed $I_{E/F}/W_{E/F}$ need not be cyclic in general.)

Definition 3.3.5. We say E is *tamely ramified* over F if $W_{E/F}$ is trivial; in this case, the degree of E over its maximal unramified subextension is not divisible by p, and we call this the *tame degree* of E over F. Otherwise, we say E is *wildly ramified* over F; if $W_{E/F} = G_{E/F}$, we say E is *totally wildly ramified* over F. Note that if $p = 0$ then *every* finite extension of F is tamely ramified.

The structure of tamely ramified extensions is almost as simple as that of unramified extensions, by an observation due at this level of generality to Abhyankar (although the special case of discrete F was known long previously).

Proposition 3.3.6 (Abhyankar). *Suppose that E/F is tamely ramified (so that its inertia group $I_{E/F}$ is abelian of order not divisible by p). Let m be an integer not divisible by p and annihilating $I_{E/F}$ (e.g., its order). Suppose that $t_1, \dots, t_h \in F^\times$ have images in $|F^\times|$ which generate $|F^\times|/|F^\times|^m$. Then $E(t_1^{1/m}, \dots, t_h^{1/m})$ is unramified over $F(t_1^{1/m}, \dots, t_h^{1/m})$.*

Proof. Since m is not divisible by p, $F(\zeta_m)$ is unramified over F; here ζ_m denotes a primitive mth root of unity. It is thus harmless to assume $\zeta_m \in F$ (by Lemma 3.2.2). We may also assume that E is totally ramified over F.

In this case, by Kummer theory, E is contained in an extension of F of the form $F(x_1^{1/m}, \dots, x_n^{1/m})$ for some $x_1, \dots, x_n \in F^\times$. To prove the claim, it suffices to check that $F(x_1^{1/m}, t_1^{1/m}, \dots, t_h^{1/m})$ is unramified over $F(t_1^{1/m}, \dots, t_h^{1/m})$. Since $|t_1|, \dots, |t_h|$ generate $|F^\times|/|F^\times|^m$, we can choose integers $\ell_1, \dots, \ell_h$ and an element $z \in F^\times$ such that $x_1 t_1^{\ell_1} \cdots t_h^{\ell_h} z^m \in \mathfrak{o}_F^\times$, and we can write

$$F(x_1^{1/m}, t_1^{1/m}, \dots, t_h^{1/m}) = F(t_1^{1/m}, \dots, t_h^{1/m})((x_1 t_1^{\ell_1} \cdots t_h^{\ell_h} z^m)^{1/m}).$$

It now suffices to check that if m is an integer not divisible by p and $y \in \mathfrak{o}_F^\times$ then $F(y^{1/m})$ is unramified over F. Again, it is safe to replace F by an unramified extension before checking this, so we may assume that y reduces to an mth power in κ_F. In this case, we will show that y already has an mth root in F. Namely, we may now assume that $y \equiv 1 \pmod{\mathfrak{m}_F}$; then the binomial series

$$(1 + (y-1))^{1/m} = \sum_{i=0}^{\infty} \binom{1/m}{i} (y-1)^i$$

converges (since its coefficients are p-adically integral, given that m is not divisible by p) to an mth root of y in F. $\square$

Our next argument may be viewed as a preview of the filtration construction of the next section.

Proposition 3.3.7. *The group $W_{E/F}$ is a p-group.*

Proof. We proceed by induction on $[E : F]$; we may assume that E is totally wildly ramified over F (so $W_{E/F} = G_{E/F}$) and that $E \neq F$. Let v_E denote the valuation on E. Pick any $x \in \mathfrak{o}_E \setminus \mathfrak{o}_F$, and set

$$j = \min\{v_E(1 - g(x)/x) : g \in G_{E/F}\}.$$

Note that $j < +\infty$ because $x \notin F$, and $j > 0$ because E is totally wildly ramified over F.

The map $g \mapsto g(x)/x$ from $G_{E/F}$ to $\mathfrak{o}_E^\times$ is not a homomorphism; however, it does induce a nonzero homomorphism

$$G_{E/F} \to \frac{\{y \in \mathfrak{o}_E^\times : v_E(y-1) \geq j\}}{\{y \in \mathfrak{o}_E^\times : v_E(y-1) > j\}}.$$

Since $j > 0$, the group on the right is isomorphic to the additive group of κ_E and so is a p-torsion group. Hence $G_{E/F}$ surjects onto a nontrivial p-group; let E' be the fixed field of the kernel of this surjection. It is clear that E is again totally wildly ramified over E', so that $G_{E/E'}$ is also a p-group; hence $G_{E/F}$ is an extension of two p-groups and is thus a p-group itself. □

The following are easily verified using Proposition 3.3.6.

Lemma 3.3.8. *Let E' be a subextension of E over F. Then E is tamely ramified over F if and only if E is tamely ramified over E' and E' is tamely ramified over F.*

Lemma 3.3.9. *Let F be a complete nonarchimedean field, let E be a finite extension of F, and let T_1, T_2 be tamely ramified subextensions of E over F. Then $T = T_1T_2$ is also tamely ramified over F.*

Remark 3.3.10. Let T be the maximal tamely ramified subextension of E over F, so that $G_{E/T} = W_{E/F}$ is a p-group by Proposition 3.3.7. A fact from elementary group theory (exercise) allows us to construct a tower $E_0 = T \subset E_1 \subset \cdots \subset E_m = E$ such that each E_i is Galois over T, and each group $G_{E_i/E_{i-1}}$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. This is particularly important if F is of characteristic p, because every $\mathbb{Z}/p\mathbb{Z}$-extension of a field L of characteristic $p > 0$ is isomorphic to $L[z]/(z^p - z - x)$ for some $x \in L$. (Such an extension is called an *Artin–Schreier extension*.)

Remark 3.3.11. Using the result of Kaplansky mentioned in the proof of Theorem 1.5.3 (i.e., the fact that maximal completeness is equivalent to spherical completeness), it is possible to show that any finite extension E of a spherically complete field F is defectless. It suffices to check this first for E separable over F and then for E purely inseparable over F (since any E can be obtained by first making a separable extension and then a purely inseparable extension on top of that). As we go along, keep in mind that any finite extension of F is also spherically complete (Example 1.5.2).

The key initial observation is that if $[E : F] = p$ then by Theorem 3.1.2 either E is defectless over F or $[\kappa_E : \kappa_F] = [|E^\times| : |F^\times|] = 1$. If F is spherically complete, the latter case is ruled out by Kaplansky's theorem.

We now consider the separable case. Since it suffices to check defectlessness for an extension containing E (by Theorem 3.1.2), we may replace E by its Galois closure. Let T be the maximal tamely ramified subextension, so that T is defectless over F. By Remark 3.3.10 we can form E from T by a sequence of $\mathbb{Z}/p\mathbb{Z}$-extensions; by the previous paragraph, each of these is defectless. Hence E is defectless over F.

We next consider the purely inseparable case (which of course only applies if F has characteristic p). This case is even easier: we can obtain E from F by a sequence of extensions, each of which simply involves adjoining a pth root; as above, each such extension is defectless.

3.4 The case of local fields

We now specialize the discussion of ramification theory to local fields, following [**198**, Chapter IV]. This material will not be used until Chapter 19, where we will relate ramification theory to the convergence of solutions of p-adic differential equations.

Hypothesis 3.4.1. In this section, let F be a complete discretely valued nonarchimedean field whose residue field κ_F is perfect. (For more on what happens when the perfectness hypothesis is lifted, see the notes.) Let E be a finite Galois extension of F.

Definition 3.4.2. The *lower numbering filtration* of $G_{E/F}$ is defined as follows. For $i \geq -1$ an integer, put

$$G_{E/F,i} = \ker(G_{E/F} \to \mathrm{Aut}(\mathfrak{o}_E/\mathfrak{m}_E^{i+1})).$$

In particular,

$$\begin{aligned} G_{E/F,-1} &= G_{E/F}, \\ G_{E/F,0} &= I_{E/F}, \\ G_{E/F,1} &= W_{E/F}. \end{aligned}$$

For $i \geq -1$ real, we define $G_{E/F,i} = G_{E/F,\lceil i \rceil}$. The lower numbering filtration behaves nicely with respect to subgroups of $G_{E/F}$ but not with respect to quotients; thus it cannot be defined on the absolute Galois group G_F.

Definition 3.4.3. The *upper numbering filtration* of $G_{E/F}$ is defined by the relation $G_{E/F}^{\phi_{E/F}(i)} = G_{E/F,i}$, where

$$\phi_{E/F}(i) = \int_0^i \frac{1}{[G_{E/F,0} : G_{E/F,t}]}\, dt.$$

Note that the indices where the filtration jumps are now rational numbers but not necessarily integers. In any case, Proposition 3.4.4 below implies that there is a unique filtration G_F^i on G_F which induces the upper numbering filtration on each $G_{E/F}$; that is, $G_{E/F}^i$ is the image of G_F^i under the surjection $G_F \to G_{E/F}$. It is this filtration which plays the more important role in, e.g., class field theory.

Proposition 3.4.4 (Herbrand). *Let E' be a Galois subextension of E over F and put $H = G_{E/E'}$, so that H is normal in $G_{E/F}$ and $G_{E/F}/H = G_{E'/F}$. Then $G_{E'/F}^i = (G_{E/F}^i H)/H$; that is, the upper numbering filtration is compatible with the formation of quotients of $G_{E/F}$.*

Proof. The proof is elementary but slightly involved, so we will not give it here. See [**198**, § IV.3]. □

Notes

Ramification theory originally emerged in the study of algebraic number fields; the formalism we see nowadays is due largely to Hilbert. The upper and lower numbering filtrations were originally introduced by Hilbert as part of class field theory (the study of abelian extensions of number fields and their Galois groups), but not in the form we see today; the modern definitions were introduced slightly later by Herbrand. See [**198**] for more discussion.

Ramification theory for a complete discrete nonarchimedean field becomes substantially more complicated when one drops the requirement of a perfect residue field. However, the case of an imperfect residue field is of great interest in the study of finite covers of schemes of dimension greater than 1. A satisfactory theory for abelian extensions was introduced by Kato [**119**]. A generalization to nonabelian extensions was later introduced by Abbes and Saito [**1, 2**]. However, a number of alternate approaches exist, the relationships among which are not fully understood. These include Borger's theory of residual perfection [**29**], Kedlaya's differential Swan conductor [**133**], and methods from higher local class field theory [**224, 225**]. See also the notes for Chapter 19.

A henselian-valued field is called *stable* if every finite extension of it is defectless. A deep theorem of Kuhlmann [**152**, Theorem 1] (generalizing earlier theorems of Grauert-Remmert and Gruson) states that if F is a stable henselian field, and E is a henselian extension of F of finite transcendence degree with transcendence defect equal to 0, then E is also stable. In this statement, the *transcendence defect* of E/F is defined as

$$\operatorname{trdeg}(E/F) - \dim_{\mathbb{Q}}(|E^\times|^{\mathbb{Q}}/|F^\times|^{\mathbb{Q}}) - \operatorname{trdeg}(\kappa_E/\kappa_F),$$

where trdeg denotes transcendence degree. A theorem of Abhyankar [**215**, Théorème 9.2] implies that the transcendence defect is always nonnegative; moreover, when the transcendence defect is 0, the group $|E^\times|/|F^\times|$ and the extension κ_E/κ_F are both finite. By contrast, both of these can fail when the transcendence degree is positive.

Exercises

(1) Complete the proof of Lemma 3.1.1. (Hint: construct a sequence of elements, in the span of the purported basis, converging to a given element of $\mathfrak{o}_E$.)

(2) Let G be a group of order p^n. Prove that there exists a chain of subgroups $G_0 = \{e\} \subset G_1 \subset \cdots \subset G_n = G$ such that each inclusion is proper and each G_i is normal in G. Deduce that, for any finite Galois extension L/K of fields of characteristic p of pth degree, there exists a sequence $K = K_0 \subset K_1 \subset \cdots \subset K_n = L$ of subextensions such that each inclusion is an extension of degree p and each K_i is Galois over K. (A somewhat easier exercise is to produce such a sequence in which each K_i is only Galois over K_{i-1}.)

4

Matrix analysis

We come now to the subject of metric properties of matrices over a field complete for a given norm. While this topic is central to our study of differential modules over nonarchimedean fields, it is based on ideas which have their origins largely outside number theory. We have thus opted to present the main points first in the archimedean setting and then to repeat the presentation for nonarchimedean fields.

The main theme is the relationship between the norms of the eigenvalues of a matrix, which are core invariants but depend on the entries of the matrix in a somewhat complicated fashion, and some less structured but more readily visible invariants. The latter are the *singular values* of a matrix, which play a key role in numerical linear algebra in controlling the numerical stability of certain matrix operations (including the extraction of eigenvalues). Their role in our work is similar.

Before proceeding, we set some basic notation and terminology for matrices.

Notation 4.0.1. Let $\mathrm{Diag}(\sigma_1, \dots, \sigma_n)$ denote the $n \times n$ diagonal matrix D with $D_{ii} = \sigma_i$ for $i = 1, \dots, n$.

Notation 4.0.2. For A a matrix, let A^T denote the transpose of A. For A an invertible square matrix, let A^{-T} denote the inverse transpose of A.

Definition 4.0.3. An $n \times n$ *elementary matrix* over a ring R is an $n \times n$ matrix obtained from the identity matrix by performing one of the following operations:

(a) exchanging two rows;
(b) adding c times one row to another row, for some $c \in R$;
(c) multiplying one row by some $c \in R^{\times}$.

If B is an elementary matrix, multiplying another $n \times n$ matrix C on the left by B effects the corresponding operation on C; such an operation is called an *elementary row operation*. Multiplying on the right by B instead effects an *elementary column operation*. (It is possible to omit type (a); see the exercises.)

4.1 Singular values and eigenvalues (archimedean case)

Hypothesis 4.1.1. In this section and the next, let A be an $n \times n$ matrix over $\mathbb{C}$. Identify $\mathbb{C}^n$ with the space of column vectors equipped with the L^2 norm, i.e.,

$$|(z_1, \ldots, z_n)| = (|z_1|^2 + \cdots + |z_n|^2)^{1/2}.$$

View A as a linear transformation on $\mathbb{C}^n$, and write

$$|A| = \sup_{v \in \mathbb{C}^n \setminus \{0\}} \{|Av|/|v|\};$$

that is, $|A|$ is the *operator norm* of A, as will be defined in Definition 6.1.2.

As mentioned above, we are interested in two sets of numerical invariants of A. One of these is the familiar set of eigenvalues.

Definition 4.1.2. Let $\lambda_1, \ldots, \lambda_n$ be the list of eigenvalues of A, arranged so that $|\lambda_1| \geq \cdots \geq |\lambda_n|$.

A second set of numerical invariants of A, which is better behaved from the point of view of numerical analysis, is the set of singular values.

Definition 4.1.3. Let A^* denote the conjugate transpose (or *Hermitian transpose*) of A. The matrix A^*A is Hermitian and nonnegative definite and so has nonnegative real eigenvalues. The (nonnegative) square roots of these eigenvalues comprise the *singular values* of A; we denote them $\sigma_1, \ldots, \sigma_n$ with $\sigma_1 \geq \cdots \geq \sigma_n$. These are not invariant under conjugation, but they are invariant under the multiplication of A on either side by a unitary matrix.

Theorem 4.1.4 (Singular value decomposition). *There exist unitary $n \times n$ matrices U, V such that $UAV = \mathrm{Diag}(\sigma_1, \ldots, \sigma_n)$.*

Proof. This is equivalent to showing that there is an orthonormal basis of $\mathbb{C}^n$ which remains orthogonal upon applying A. To construct it, start with a vector $v \in \mathbb{C}^n$ maximizing $|Av|/|v|$ and then show that, for any $w \in \mathbb{C}^n$ orthogonal to v, Aw is also orthogonal to Av. For further details, see the references in the notes. $\square$

Corollary 4.1.5. *The singular values of* A^{-1} *are* $\sigma_n^{-1}, \ldots, \sigma_1^{-1}$.

From the singular value decomposition, we may infer a convenient interpretation of σ_i.

Corollary 4.1.6. *The number* σ_i *is the smallest value of* λ *for which the following holds: for any* i*-dimensional subspace* V *of* $\mathbb{C}^n$*, there exists* $v \in V$ *which is nonzero and such that* $|Av| \leq \lambda |v|$.

Proof. Theorem 4.1.4 provides an orthonormal basis $v_1, \ldots, v_n$ of V such that $Av_1, \ldots, Av_n$ is again orthogonal and $|Av_i| = \sigma_i |v_i|$ for $i = 1, \ldots, n$. Let W be the span of $v_i, \ldots, v_n$; then, on the one hand, for any i-dimensional subspace V of $\mathbb{C}^n$, $V \cap W$ is nonzero, and any nonzero $v \in V \cap W$ satisfies $|Av| \leq \sigma_i |v|$. On the other hand, if we take V to be the span of $v_1, \ldots, v_i$, then we have $|Av| \geq \sigma_i |v|$ for all $v \in V$. This proves the claim. $\square$

The relationship between the singular values and the eigenvalues is controlled by the following inequality of Weyl [**218**].

Theorem 4.1.7 (Weyl). *We have*

$$\sigma_1 \cdots \sigma_i \geq |\lambda_1 \cdots \lambda_i| \qquad (i = 1, \ldots, n),$$

with equality for $i = n$.

For the proof, we need a construction that will recur frequently in what follows.

Definition 4.1.8. Let M be a module over a ring R. The ith *exterior power* or *wedge power* $\wedge_R^i M$ (or $\wedge^i M$ if there is no ambiguity about R) of M is the R-module generated by the symbols $m_1 \wedge \cdots \wedge m_i$ for $m_1, \ldots, m_i \in M$, modulo those relations that force the map $(m_1, \ldots, m_i) \mapsto m_1 \wedge \cdots \wedge m_i$ to be R-linear in each variable (while the others are held fixed) and alternating. The latter means that $m_1 \wedge \cdots \wedge m_i = 0$ if $m_1, \ldots, m_i$ are not all distinct.

Any element of $\wedge^i M$ of the form $m_1 \wedge \cdots \wedge m_i$ is said to be *decomposable*; it is also called the *exterior product* of $m_1, \ldots, m_i$. The set of decomposable elements is not closed under addition and consequently will not necessarily form a basis for $\wedge^i M$ (see the exercises). However, if M is freely generated by $e_1, \ldots, e_n$ then it is true that the decomposable elements of the form $e_{j_1} \wedge \cdots \wedge e_{j_i}$ with $1 \leq j_1 < \cdots < j_i \leq n$ do form a basis of $\wedge^i M$.

Note also that the exterior power is a functor on the category of R-modules; in particular, any linear transformation $T : M \to N$ induces a linear transformation $\wedge^i T : \wedge^i M \to \wedge^i N$. In the special case where $M = N$ is free and of rank i, $\wedge^i M$ is a one-dimensional space and $\wedge^i T$ turns out to be multiplication by $\det(T)$.

One easily checks that the formula

$$(m_1 \wedge \cdots \wedge m_i, m'_1 \wedge \cdots \wedge m'_j) \mapsto m_1 \wedge \cdots \wedge m_i \wedge m'_1 \wedge \cdots \wedge m'_j$$

gives a well-defined bilinear map $\wedge^i M \times \wedge^j M \to \wedge^{i+j} M$. This map is usually denoted by $\wedge$, since it corresponds to adding an extra wedge between an i-fold and a j-fold wedge product.

Lemma 4.1.9. *The singular values (resp. eigenvalues) of $\wedge^i A$ are the i-fold products of the singular values (resp. eigenvalues) of A.*

Proof. Remember that, starting with any basis of $\mathbb{C}^n$, we can obtain a basis of $\wedge^i A$ by taking i-fold exterior products of basis elements. In particular, if we choose $U \in \mathrm{GL}_n(\mathbb{C})$, so that $U^{-1}AU$ is upper triangular with its eigenvalues on the diagonal, then

$$\wedge^i(U^{-1}AU) = (\wedge^i U)^{-1}(\wedge^i A)(\wedge^i U)$$

will be upper triangular with the i-fold products of the eigenvalues on the diagonal. Similarly, if we apply Theorem 4.1.4 to construct unitary matrices U, V such that UAV is diagonal with its singular values on the diagonal then

$$\wedge^i(UAV) = (\wedge^i U)(\wedge^i A)(\wedge^i V)$$

will be diagonal with the i-fold products of its singular values on the diagonal. $\square$

We now return to Theorem 4.1.7.

Proof of Theorem 4.1.7. The equality for $i = n$ holds because $\det(A^*A) = |\det(A)|^2$. We check the inequality first for $i = 1$. Note that $\sigma_1 = |A|$ is the operator norm. Since there exists $v \in \mathbb{C}^n \setminus \{0\}$ with $Av = \lambda_1 v$, we deduce that $\sigma_1 \geq |\lambda_1|$.

To handle the general case, we consider an exterior power $\wedge^i \mathbb{C}^n$ having the action of $\wedge^i A$. By Lemma 4.1.9, the largest singular value (resp. eigenvalue) of $\wedge^i A$ is equal to the product of the i largest singular values (resp. eigenvalues) of A. Consequently, the previous inequality applied to $\wedge^i A$ gives exactly the desired result. $\square$

We mention in passing the following converse of Theorem 4.1.7, due to Horn [**114**, Theorem 4].

Theorem 4.1.10. *For $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $\sigma_1, \ldots, \sigma_n \in \mathbb{R}_{\geq 0}$ satisfying*

$$\sigma_1 \cdots \sigma_i \geq |\lambda_1 \cdots \lambda_i| \qquad (i = 1, \ldots, n),$$

with equality for $i = n$, there exists an $n \times n$ matrix A over $\mathbb{C}$ with singular values $\sigma_1, \ldots, \sigma_n$ and eigenvalues $\lambda_1, \ldots, \lambda_n$.

Equality in Weyl's theorem at an intermediate stage has a structural meaning.

Lemma 4.1.11. *Let F be any field. If $v_1, \ldots, v_i, w_1, \ldots, w_i \in F^n$ are such that $v_1 \wedge \cdots \wedge v_i$ and $w_1 \wedge \cdots \wedge w_i$ are nonzero and equal, then the F-span of $v_1, \ldots, v_i$ equals the F-span of $w_1, \ldots, w_i$.*

Proof. If $v_1 \wedge \cdots \wedge v_i$ is nonzero then $v_1, \ldots, v_i$ must be linearly independent. We may thus extend $v_1, \ldots, v_i$ to a basis $v_1, \ldots, v_n$ of F^n.

Consider the bilinear map

$$\wedge : \wedge^i F^n \times F^n \to \wedge^{i+1} F^n.$$

Suppose that $w \in F^n$ pairs to zero with $v_1 \wedge \cdots \wedge v_i$. If we write $w = c_1 v_1 + \cdots + c_n v_n$, we must then have $c_{i+1} = 0$ or else the coefficient of $v_1 \wedge \cdots \wedge v_{i+1}$ in $v_1 \wedge \cdots \wedge v_i \wedge w$ will be nonzero. Similarly $c_{i+2} = \cdots = c_n = 0$, so w belongs to the F-span of $v_1, \ldots, v_i$.

Since $(v_1 \wedge \cdots \wedge v_i) \wedge w_1 = (w_1 \wedge \cdots \wedge w_i) \wedge w_1 = 0$, w_1 must belong to the F-span of $v_1, \ldots, v_i$ and likewise for $w_2, \ldots, w_i$. Consequently, one of the two spans is contained in the other, and vice versa by the same argument. □

Theorem 4.1.12. *Suppose that for some $i \in \{1, \ldots, n-1\}$ we have*

$$\sigma_i > \sigma_{i+1}, \qquad |\lambda_i| > |\lambda_{i+1}|,$$
$$\sigma_1 \cdots \sigma_i = |\lambda_1 \cdots \lambda_i|.$$

Then there exists a unitary matrix U such that $U^{-1}AU$ is block diagonal, the first block accounting for the first i singular values and eigenvalues and the second block accounting for the others.

Proof. Let $v_1, \ldots, v_n$ be a basis of $\mathbb{C}^n$ such that $v_1, \ldots, v_i$ span the generalized eigenspaces having eigenvalues $\lambda_1, \ldots, \lambda_i$ and $v_{i+1}, \ldots, v_n$ span the generalized eigenspaces having eigenvalues $\lambda_{i+1}, \ldots, \lambda_n$. Apply the singular value decomposition (Theorem 4.1.4) to construct an orthonormal basis $w_1, \ldots, w_n$ such that $Aw_1, \ldots, Aw_n$ are also orthogonal and $|Aw_i| = \sigma_i |w_i|$.

Since $\sigma_i > \sigma_{i+1}$, the only nonzero vectors $v \in \wedge^i \mathbb{C}^n$ for which $|Av|/|v|$ achieves its maximum value $\sigma_1 \cdots \sigma_i$ are the nonzero multiples of $w_1 \wedge \cdots \wedge w_i$.

However, this is also true for $v_1 \wedge \cdots \wedge v_i$. By Lemma 4.1.11, $w_1, \dots, w_i$ span V; this implies that the orthogonal complement of V is spanned by $w_{i+1}, \dots, w_n$, and so it is also preserved by A. This yields the desired result. $\square$

Theorem 4.1.13. *The following are equivalent.*

(a) *There exists a unitary matrix U such that $U^{-1}AU$ is diagonal.*
(b) *The matrix A is* normal*, i.e., $A^*A = AA^*$.*
(c) *The eigenvalues $\lambda_1, \dots, \lambda_n$ and singular values $\sigma_1, \dots, \sigma_n$ of A satisfy $|\lambda_i| = \sigma_i$ for $i = 1, \dots, n$.*

Proof. It is clear that (a) implies both (b) and (c), because U, being unitary, is equivalent to $U^* = U^{-1}$. Given (b), we can perform a joint eigenspace decomposition for the commuting matrices A and A^*. On any common generalized eigenspace, A has some eigenvalue λ, A^* has eigenvalue $\overline{\lambda}$, and so A^*A has eigenvalue $|\lambda|^2$. This implies (c).

Given (c), Theorem 4.1.12 implies that A can be conjugated by a unitary matrix into a block diagonal matrix in which each block has a single eigenvalue λ and a single singular value σ, such that $|\lambda| = \sigma$. Let B be such a block, corresponding to a subspace V of $\mathbb{C}^n$. If $\sigma = 0$ then $B = 0$. Otherwise, $\lambda \neq 0$ and $\lambda^{-1}B$ is unitary. Hence given orthogonal eigenvectors $v_1, \dots, v_i \in V$ of B, the orthogonal complement in V of their span is preserved by B, so is either zero or contains another eigenvector v_{i+1}. This shows that B is diagonalizable with a single eigenvalue and thus is itself a scalar matrix. (One can also argue this last step using the compactness of the unitary group.) $\square$

In general, we can conjugate any matrix into an almost normal matrix; the "almost" only applies when the matrix is not semisimple.

Lemma 4.1.14. *For any $\eta > 1$, we can choose $U \in \mathrm{GL}_n(\mathbb{C})$ such that, for $i = 1, \dots, n$, the ith singular value of $U^{-1}AU$ is at most $\eta|\lambda_i|$. If A is semisimple (i.e., diagonalizable), we can also take $\eta = 1$.*

Proof. Put A in Jordan normal form, then rescale so that, for each eigenvalue λ, the superdiagonal terms have absolute value at most $(\eta^2 - 1)^{1/2}|\lambda|$ and all other terms are zero. $\square$

4.2 Perturbations (archimedean case)

Another inequality of Weyl [**217**] shows that the singular values do not change much under a small (additive) perturbation.

Theorem 4.2.1 (Weyl). *Let B be an $n \times n$ matrix, and let $\sigma'_1, \ldots, \sigma'_n$ be the singular values of $A + B$. Then*

$$|\sigma'_i - \sigma_i| \leq |B| \qquad (i = 1, \ldots, n).$$

It is more complicated to describe what happens to the eigenvalues under a small additive perturbation. The best we can do here is to quantify the effect of an additive perturbation on the characteristic polynomial.

Theorem 4.2.2. *Let B be an $n \times n$ matrix. Let $P(T) = T^n + \sum_{i=0}^{n-1} P_i T^i$ and $Q(T) = T^n + \sum_{i=0}^{n-1} Q_i T^i$ be the characteristic polynomials of A and $A + B$. Then*

$$|P_{n-i} - Q_{n-i}| \leq \left| 2^i \binom{n}{i} \right| |B| \prod_{j=1}^{i-1} \max\{\sigma_j, |B|\} \qquad (i = 1, \ldots, n).$$

The superfluous enclosure of the integer $2^i \binom{n}{i}$ in absolute value signs is quite deliberate; it will become relevant in the nonarchimedean setting.

Proof. Note that Q_{n-i} is the sum of the $\binom{n}{i}$ principal $i \times i$ minors of $A + B$. (A *minor* is the determinant of the $i \times i$ submatrix obtained by choosing a set of i rows and i columns. A *principal minor* is a minor in which the rows and columns correspond; for instance, if the first row is included, then the first column must also be included.) By multilinearity of the determinant, each principal minor can be written as a sum of 2^i terms, each of which is the product of a sign, a $k \times k$ minor of A, and an $(i - k) \times (i - k)$ minor of B. The terms with $k = i$ sum to P_{n-i} itself; the others all have $k < i$ and so the norm of each is bounded by $\sigma_1 \cdots \sigma_k |B|^{i-k}$. This proves the claim. □

We also need to consider multiplicative perturbations. For a considerable generalization of the following inequality, see Theorem 4.5.2.

Proposition 4.2.3. *Let $B \in \mathrm{GL}_n(\mathbb{C})$ satisfy $|B| \leq \eta$. Let $\sigma'_1, \ldots, \sigma'_n$ be the singular values of BA. Then*

$$\sigma'_i \leq \eta \sigma_i \qquad (i = 1, \ldots, n).$$

(The analogous result with BA replaced by AB follows from this, since transposal does not change singular values.)

Proof. We use the interpretation of singular values given by Corollary 4.1.6. Choose an i-dimensional subspace V of $\mathbb{C}^n$ such that $|BAv| \geq \sigma'_i |v|$ for all $v \in V$. Then choose a nonzero $v \in V$ such that $|Av| \leq \sigma_i |v|$. We have

$$\sigma_i'|v| \leq |BAv| \leq |B||Av| \leq \sigma_i|B||v|,$$

proving the claim. □

This implies that the norms of the eigenvalues can be recovered from the singular values, provided that we consider not just the matrix A but also its powers.

Proposition 4.2.4. *Let $\sigma_{k,1}, \dots, \sigma_{k,n}$ be the singular values of A^k. Then*

$$\lim_{k\to\infty} \sigma_{k,i}^{1/k} = |\lambda_i| \qquad (i = 1, \dots, n).$$

Proof. Pick $\eta > 1$, and choose U as in Lemma 4.1.14; that is, $U^{-1}AU$ is upper triangular and each block of eigenvalue λ differs from the scalar matrix $\mathrm{Diag}(\lambda, \dots, \lambda)$ by a matrix of norm at most $(\eta^2 - 1)^{1/2}|\lambda|$. In a block with eigenvalue λ, the singular values of the kth power are bounded below by $|\lambda|^k$ and above by $\eta^k|\lambda|^k$. Consequently, we may apply Proposition 4.2.3 to deduce that

$$|\lambda_i|^k|U|^{-1}|U^{-1}|^{-1} \leq \sigma_{k,i} \leq \eta^k|\lambda_i|^k|U||U^{-1}|.$$

Taking kth roots and then taking $k \to \infty$, we deduce

$$|\lambda_i| \leq \liminf_{k\to\infty} \sigma_{k,i}^{1/k}, \qquad \limsup_{k\to\infty} \sigma_{k,i}^{1/k} \leq \eta|\lambda_i|.$$

Since $\eta > 1$ was arbitrary, we obtain the desired result. □

Remark 4.2.5. The case $i = 1$ of Proposition 4.2.4, and the corresponding case for its nonarchimedean analogue (Proposition 4.4.10), are instances of a general fact about the spectra of bounded operators on Banach spaces. See Remark 6.1.7.

4.3 Singular values and eigenvalues (nonarchimedean case)

We now pass to nonarchimedean analogues.

Hypothesis 4.3.1. Throughout this section and the next, let F be a complete nonarchimedean field, and let A be an $n \times n$ matrix over F. View A as a linear transformation on F^n, equip F^n with the supremum norm

$$|(z_1, \dots, z_n)| = \max\{|z_1|, \dots, |z_n|\},$$

and again define the operator norm

$$|A| = \sup_{v\in F^n\setminus\{0\}} \{|Av|/|v|\}.$$

Note, however, that we also have the simpler expression

$$|A| = \max_{i,j}\{|A_{ij}|\}.$$

Definition 4.3.2. Given a sequence $s_1, \dots, s_n$, we define the *associated polygon* for this sequence to be the polygonal line joining the points

$$(-n + i, s_1 + \cdots + s_i) \qquad (i = 0, \dots, n).$$

This polygon is the graph of a convex function on $[-n, 0]$ if and only if $s_1 \leq \cdots \leq s_n$.

Definition 4.3.3. Let $s_1, \dots, s_n$ be the sequence with the property that, for $i = 1, \dots, n$, $s_1 + \cdots + s_i$ is the minimum valuation of an $i \times i$ minor of A; that is, the s_i are the *elementary divisors* (or *invariant factors*) of A. The associated polygon is called the *Hodge polygon* of A (see the notes for an explanation of the terminology). Define the *singular values* of A as $\sigma_1, \dots, \sigma_n = e^{-s_1}, \dots, e^{-s_n}$; these are invariant under multiplication on either side by a matrix in $\mathrm{GL}_n(\mathfrak{o}_F)$. Note that, as in the nonarchimedean case,

$$\sigma_1 = |A|.$$

We also have an analogue of the singular value decomposition.

Theorem 4.3.4 (Smith normal form). *There exist $U, V \in \mathrm{GL}_n(\mathfrak{o}_F)$ such that UAV is a diagonal matrix whose entries have norms $\sigma_1, \dots, \sigma_n$.*

Proof. It is equivalent to prove that, starting with A, one can perform elementary row and column operations defined over $\mathfrak{o}_F$ so as to produce a diagonal matrix (this amounts to a limited Gauss–Jordan elimination over A). To do this, find the largest entry of A, permute rows and columns to put this entry at the top left, and then use it to clear the remainder of the first row and column. Repeat with the matrix obtained by removing the first row and column, and so on. □

Corollary 4.3.5. *The slopes $s_1, \dots, s_n$ of the Hodge polygon satisfy $s_1 \leq \cdots \leq s_n$.*

Proof. The ith slope s_i is evidently the ith smallest valuation of a diagonal entry of the Smith normal form. □

We again have a characterization as in Corollary 4.1.6.

Corollary 4.3.6. *The number σ_i is the smallest value of λ for which the following holds: for any i-dimensional subspace V of F^n, there exists $v \in V$ nonzero and such that $|Av| \leq \lambda |v|$.*

Definition 4.3.7. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of A in some algebraic extension of F equipped with the unique extension of $|\cdot|$, arranged so that $|\lambda_1| \geq \cdots \geq |\lambda_n|$. The associated polygon is the *Newton polygon* of A; this is invariant under conjugation by any element of $\mathrm{GL}_n(F)$.

The nonarchimedean analogue of Weyl's inequality is the following.

Theorem 4.3.8 (Newton polygon above Hodge polygon). *We have*

$$\sigma_1 \cdots \sigma_i \geq |\lambda_1 \cdots \lambda_i| \qquad (i = 1, \dots, n),$$

with equality for $i = n$. In other words, the Hodge and Newton polygons have the same endpoints and the Newton polygon is everywhere on or above the Hodge polygon.

Proof. Again, the case $i = 1$ is clear because σ_1 is the operator norm of A, and the general case follows by considering exterior powers (using the obvious analogue of Lemma 4.1.9). □

Like its archimedean analogue, Theorem 4.3.8 also has a converse, but in this case we can write the construction down quite explicitly.

Definition 4.3.9. For $P = T^n + \sum_{i=0}^{n-1} P_i T^i$ a monic polynomial of degree n over a ring R, the *companion matrix* of P is defined as the matrix

$$\begin{pmatrix} 0 & \cdots & 0 & -P_0 \\ 1 & \cdots & 0 & -P_1 \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & 1 & -P_{n-1} \end{pmatrix}$$

with 1's on the subdiagonal, the negated coefficients of P in the far right-hand column, and 0's elsewhere. The companion matrix is constructed to have its characteristic polynomial equal to P (exercise).

Proposition 4.3.10. *Choose $\lambda_1, \dots, \lambda_n \in F^{\mathrm{alg}}$ such that $|\lambda_1| \geq \cdots \geq |\lambda_n|$, and such that the polynomial $P(T) = (T - \lambda_1) \cdots (T - \lambda_n) = T^n + \sum_{i=0}^{n-1} P_i T^i$ has coefficients in F. Choose $c_1, \dots, c_n \in F$ with $\sigma_i = |c_i|$ such that $\sigma_1 \geq \cdots \geq \sigma_n$ and*

$$\sigma_1 \cdots \sigma_i \geq |\lambda_1 \cdots \lambda_i| \qquad (i = 1, \dots, n),$$

with equality for $i = n$. Then the matrix

$$\begin{pmatrix} 0 & \cdots & 0 & -c_1^{-1} \cdots c_{n-1}^{-1} P_0 \\ c_{n-1} & \cdots & 0 & -c_1^{-1} \cdots c_{n-2}^{-1} P_1 \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & c_1 & -P_{n-1} \end{pmatrix}$$

has singular values $\sigma_1, \dots, \sigma_n$ and eigenvalues $\lambda_1, \dots, \lambda_n$.

Proof. The given matrix is conjugate to the companion matrix of P, so its eigenvalues are also $\lambda_1, \dots, \lambda_n$. To compute the singular values, we note that, for $i = 1, \dots, n-1$,

$$\begin{aligned} \left| -c_1^{-1} \cdots c_{n-i-1}^{-1} P_i \right| &= \sigma_1^{-1} \cdots \sigma_{n-i-1}^{-1} |P_i| \\ &\leq \sigma_1^{-1} \cdots \sigma_{n-i-1}^{-1} |\lambda_1 \cdots \lambda_{n-i}| \\ &\leq \sigma_{n-i}. \end{aligned}$$

Thus we can perform column operations over $\mathfrak{o}_F$ to clear everything in the far right-hand column except $-c_1^{-1} \cdots c_{n-1}^{-1} P_0$, which has norm $\sigma_1^{-1} \cdots \sigma_{n-1}^{-1} |\lambda_1 \cdots \lambda_n| = \sigma_n$. By permuting the rows and columns we obtain a diagonal matrix with entries that are the norms $\sigma_1, \dots, \sigma_n$. This proves the claim. □

Again equality has a structural meaning, but the proof requires a bit more work than in the archimedean case since we no longer have access to orthogonality. However, this extra work is rewarded by a slightly stronger result.

Theorem 4.3.11 (Hodge–Newton decomposition). *Suppose that for some $i \in \{1, \dots, n-1\}$ we have*

$$|\lambda_i| > |\lambda_{i+1}|, \qquad \sigma_1 \cdots \sigma_i = |\lambda_1 \cdots \lambda_i|.$$

(That is, the Newton polygon has a vertex with x-coordinate $-n + i$ and this vertex also lies on the Hodge polygon.) Then there exists $U \in \mathrm{GL}_n(\mathfrak{o}_F)$ such that $U^{-1}AU$ is block upper triangular, with the top left block accounting for the first i singular values and eigenvalues and the bottom right block accounting for the others. Moreover, if $\sigma_i > \sigma_{i+1}$ then we can ensure that $U^{-1}AU$ is block diagonal.

Proof. We first note that, by Theorem 2.2.1 applied to the characteristic polynomial of A, $P(T) = (T - \lambda_1) \cdots (T - \lambda_i)$ and $Q(T) = (T - \lambda_{i+1}) \cdots (T - \lambda_n)$ have coefficients in F. Since P and Q have no common roots, we can write

$1 = PB + QC$ for some $B, C \in F[T]$, and then the products $P(A)B(A)$ and $Q(A)C(A)$ give the projectors for a direct sum decomposition separating the first i generalized eigenspaces from the others.

In other words, we can find a basis $v_1, \dots, v_n$ of F^n such that $v_1, \dots, v_i$ span the generalized eigenspaces with eigenvalues $\lambda_1, \dots, \lambda_i$ and $v_{i+1}, \dots, v_n$ span the generalized eigenspaces with eigenvalues $\lambda_{i+1}, \dots, \lambda_n$. Choose a basis $w_1, \dots, w_n$ of $\mathfrak{o}_F^n$ such that $w_1, \dots, w_i$ is a basis of $\mathfrak{o}_F^n \cap (Fv_1 + \cdots + Fv_i)$. Let $e_1, \dots, e_n$ be the standard basis of F^n, and define $U \in \mathrm{GL}_n(\mathfrak{o}_F)$ by $w_j = \sum_i U_{ij} e_i$. Then

$$U^{-1}AU = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$$

is block upper triangular. By Cramer's rule, each entry of $B^{-1}C$ is an $i \times i$ minor of A divided by the determinant of B. Since $|\det(B)| = \sigma_1 \cdots \sigma_i$, $B^{-1}C$ must thus have entries in $\mathfrak{o}_F$. Writing

$$U^{-1}AU = \begin{pmatrix} B & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I_i & B^{-1}C \\ 0 & I_{n-i} \end{pmatrix},$$

we see that the singular values of B and D together must comprise $\sigma_1, \dots, \sigma_n$. The only way for this to happen, given the constraint that the product of the singular values of B equals $\sigma_1 \cdots \sigma_i$, is for B to account for $\sigma_1, \dots, \sigma_i$ and for D to account for $\sigma_{i+1}, \dots, \sigma_n$.

This proves the first claim; we may thus assume now that $\sigma_i > \sigma_{i+1}$. In that case, conjugating by the matrix

$$\begin{pmatrix} I_i & -B^{-1}C \\ 0 & I_{n-i} \end{pmatrix}$$

gives a new matrix,

$$\begin{pmatrix} B & C_1 \\ 0 & D \end{pmatrix},$$

with $C_1 = B^{-1}CD$. Since

$$|C_1| \leq |B^{-1}||C||D| = \sigma_i^{-1}|C|\sigma_{i+1} < |C|,$$

this process converges. More explicitly, we obtain a sequence of matrices $U_i \in \mathrm{GL}_n(\mathfrak{o}_F)$ converging to the identity, such that the convergent product $U = U_1 U_2 \cdots$ satisfies

$$U^{-1}AU = \begin{pmatrix} B & 0 \\ 0 & D \end{pmatrix},$$

as desired. $\square$

Note that the slopes of the Hodge polygon are forced to be in the additive value group of F, whereas the slopes of the Newton polygon need only lie in the divisible closure of the additive value group. Consequently, it is possible for a matrix to have no conjugates over $\mathrm{GL}_n(F)$ for which the Hodge and Newton polygons coincide. However, the following is true; see also Corollary 4.4.8 below.

Lemma 4.3.12. *Suppose that one of the following holds.*

(a) *The value group of $|F^\times|$ is dense in $\mathbb{R}_{>0}$, and $\eta > 1$.*

(b) *We have $|\lambda_i| \in |F^\times|$ for $i = 1, \dots, n$ (so in particular $\lambda_i \neq 0$), and $\eta \geq 1$.*

Then there exists $U \in \mathrm{GL}_n(F)$ such that the ith singular value of $U^{-1}AU$ is at most $\eta|\lambda_i|$.

Proof. Case (a) will follow from Corollary 4.4.8 below. Case (b) is directly analogous to Lemma 4.1.14. □

One also has the following variant.

Lemma 4.3.13. *Suppose that $|F^\times|$ is discrete. Then there exists $U \in \mathrm{GL}_n(F)$ such that, for each positive integer m, $|U^{-1}A^mU|$ is the least element of $|F^\times|$ greater than or equal to $|\lambda_1^m|$.*

Proof. We may normalize the valuation on F for convenience so that $\log|F^\times| = \mathbb{Z}$. As in the proof of Theorem 4.3.11, we may also reduce to the case where all the eigenvalues of A have the same norm.

Let E be a finite extension of F containing an element λ with $|\lambda| = |\lambda_1|$. (For instance, we could take $E = F(\lambda_1)$ and $\lambda = \lambda_1$, but any other choice would also work.) By Lemma 4.3.12, there exists $U_0 \in \mathrm{GL}_n(E)$ such that $\lambda^{-1}U_0^{-1}AU_0 \in \mathrm{GL}_n(\mathfrak{o}_E)$; in other words, there exists a supremum norm $|\cdot|_0$ on E^n such that $|\lambda^{-1}Av|_0 = |v|_0$ for all $v \in E^n$.

Put $V = \{v \in F^n : |v|_0 \leq 1\}$. Let $v_1, \dots, v_n$ be a basis of V over $\mathfrak{o}_F$, and let $|\cdot|_1$ be the supremum norm on F^n defined by $v_1, \dots, v_n$. Given a nonzero $v \in F^n$, choose $\mu \in F^\times$ with $-\log|\mu v|_0 \in [0, 1)$; then μv is an element of V which is not divisible by $\mathfrak{m}_F$, so $|\mu v|_1 = 1$. We conclude that

$$e^{-1}|v|_1 < |v|_0 \leq |v|_1 \qquad (v \in F^n). \tag{4.3.13.1}$$

Let $e_1, \dots, e_n$ be the standard basis of F^n, and define $U \in \mathrm{GL}_n(F)$ by $v_j = \sum_i U_{ij}e_i$. Then, for each positive integer m, $-\log|U^{-1}A^mU| \in \mathbb{Z}$ is at most $-m\log|\lambda_1|$ by Theorem 4.3.8 but is strictly greater than $-m\log|\lambda_1| - 1$ by (4.3.13.1). We thus obtain the desired equality. □

4.4 Perturbations (nonarchimedean case)

Again, we can ask about the effect of perturbations. The analogue of Weyl's second inequality is more or less trivial.

Proposition 4.4.1. *If B is a matrix with $|B| < \sigma_i$ then the first i singular values of $A + B$ are $\sigma_1, \dots, \sigma_i$.*

Proof. Exercise. □

We next consider the effect on the characteristic polynomial.

Theorem 4.4.2. *Let B be an $n \times n$ matrix. Let $P(T) = T^n + \sum_{i=0}^{n-1} P_i T^i$ and $Q(T) = T^n + \sum_{i=0}^{n-1} Q_i T^i$ be the characteristic polynomials of A and $A + B$. Then*

$$|P_{n-i} - Q_{n-i}| \leq |B| \prod_{j=1}^{i-1} \max\{\sigma_j, |B|\} \qquad (i = 1, \dots, n).$$

Proof. The proof is as for Theorem 4.2.2, except that now the factor $|2^i \binom{n}{i}|$ is dominated by 1. □

Question 4.4.3. *Is the inequality in Theorem 4.4.2 the best possible?*

We may also consider multiplicative perturbations.

Proposition 4.4.4. *Let $B \in \mathrm{GL}_n(F)$ satisfy $|B| \leq \eta$. Let $\sigma_1', \dots, \sigma_n'$ be the singular values of AB. Then*

$$\sigma_i' \leq \eta \sigma_i \qquad (i = 1, \dots, n).$$

Proof. As for Proposition 4.2.3 but using the Smith normal form (Theorem 4.3.4) instead of the singular value decomposition. □

Corollary 4.4.5. *Suppose that the Newton and Hodge slopes of A coincide and that $U \in \mathrm{GL}_n(F)$ satisfies $|U|\,|U^{-1}| \leq \eta$. Then each Newton slope of $U^{-1}AU$ differs by at most* $\log \eta$ *from the corresponding Hodge slope.*

Here is a weak converse to Corollary 4.4.5. (We leave the archimedean analogue to the reader's imagination.)

Proposition 4.4.6. *Suppose that the Newton slopes of A are nonnegative and that $\sigma_1 \geq 1$. Then there exists $U \in \mathrm{GL}_n(F)$ such that*

$$|U^{-1}AU| \leq 1, \quad |U^{-1}| \leq 1, \quad |U| \leq \sigma_1^{n-1}.$$

Proof. Let $e_1, \dots, e_n$ denote the standard basis vectors of F^n. Let M be the smallest $\mathfrak{o}_F$-submodule of F^n containing $e_1, \dots, e_n$ and stable under A. For each i, if $j = j(i)$ is the least integer such that $e_i, Ae_i, \dots, A^j e_i$ are linearly dependent then we have $A^j e_i = \sum_{h=0}^{j-1} c_h A^h e_i$ for some $c_h \in F$. The polynomial $T^j - \sum_{h=0}^{j-1} c_h T^h$ has roots which are eigenvalues of A, so the nonnegativity of the Newton slopes forces $|c_h| \leq 1$. Hence M is finitely generated, and thus free, over $\mathfrak{o}_F$.

Let $v_1, \dots, v_n$ be a basis of M, and let U be the change-of-basis matrix $v_j = \sum_i U_{ij} e_i$; then $|U^{-1}AU| \leq 1$ because M is stable under A and $|U^{-1}| \leq 1$ because M contains $e_1, \dots, e_n$. The desired bound on $|U|$ follows from the fact that, for any $x = c_1 e_1 + \cdots + c_n e_n \in M$, we have

$$\max_i \{|c_i|\} \leq \sigma_1^{n-1}. \tag{4.4.6.1}$$

It suffices to check (4.4.6.1) for $x = A^h e_i$ for $i = 1, \dots, n$ and $h = 0, \dots, j(i) - 1$, as these generate M over $\mathfrak{o}_F$. But it is evident that $|A^h e_i| \leq \sigma_1^h |e_i| = \sigma_1^h$; since $j(i) \leq n$, we are done. □

Example 4.4.7. The example

$$A = \begin{pmatrix} 1 & c & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

with $|c| > 1$ shows that the bound of Proposition 4.4.6 is sharp; in particular, the bound $|U| \leq \sigma_1^{n-1}$ cannot be improved to $|U| \leq \sigma_1$, as one might initially have expected. However, one should be able to get a more precise bound (which agrees with the bound given in this example) by accounting for the other singular values; see the exercises.

Corollary 4.4.8. *There exists a continuous function*

$$f_n(\sigma_1, \dots, \sigma_n, \sigma'_1, \dots, \sigma'_n, \delta) : (0, +\infty)^{2n} \times [0, +\infty) \to (0, +\infty)$$

(independent of F) with the following properties.

(a) *Suppose that, for each $i = 1, \dots, n$, either $\sigma_i = \sigma'_i$ or $\delta \geq \max\{\sigma_i, \sigma'_i\}$. Then*

$$f_n(\sigma_1, \dots, \sigma_n, \sigma'_1, \dots, \sigma'_n, \delta) = 1.$$

(b) *If A has singular values $\sigma_1, \dots, \sigma_n$ and eigenvalues $\lambda_1, \dots, \lambda_n$, if $\sigma'_i = |\lambda_i|$ for $i = 1, \dots, n$, and if $\sigma'_i \in |F^\times|$ whenever $\sigma'_i > \delta$ then there exists $U \in \mathrm{GL}_n(F)$ such that*

$$|U^{-1}| \leq 1, \quad |U| \leq f_n(\sigma_1, \dots, \sigma_n, \sigma'_1, \dots, \sigma'_n, \delta),$$

for which the multiset of singular values of $U^{-1}AU$ matches $\sigma'_1, \dots, \sigma'_n$ in its values greater than δ.

Proof. This follows by induction on n as follows. If $\sigma'_1 \leq \delta$, we can deduce the whole claim by Proposition 4.4.6 (after rescaling in the case $\delta \neq 1$). Otherwise, again by Proposition 4.4.6 (and again after rescaling), we can find $U_1 \in \mathrm{GL}_n(F)$ such that

$$|U_1^{-1} A_1 U_1| \leq \sigma'_1, \qquad |U_1^{-1}| \leq 1, \qquad |U_1| \leq (\sigma_1/\sigma'_1)^{n-1}.$$

Let i be the largest index such that $\sigma'_i = \sigma'_1$. Then the first i singular values of $U_1^{-1} A U_1$ are all at most σ'_1 but at least σ'_i. Hence they are all equal, and $A_1 = U_1^{-1} A U_1$ satisfies the hypothesis of Theorem 4.3.11. We may thus choose $U_2 \in \mathrm{GL}_n(\mathfrak{o}_F)$ such that $A_2 = U_2^{-1} A_1 U_2$ is block upper triangular, the top left block accounting for the first i singular values and eigenvalues and the bottom right block accounting for the others.

If $i = n$ then we may take $U = U_1 U_2$ and be done. Otherwise, note that by applying Proposition 4.4.4 we may bound the singular values of A_2 by a continuous function of $\sigma_1, \dots, \sigma_n, \sigma'_1, \dots, \sigma'_n, \delta$. We may then apply the induction hypothesis to construct a block diagonal matrix U_3, where the top left block of U_3 is the identity, $|U_3^{-1}| \leq 1$, $|U_3|$ is bounded by a continuous function of $\sigma_1, \dots, \sigma_n, \sigma'_1, \dots, \sigma'_n, \delta$, and the multiset of singular values of the bottom right block of $A_3 = U_3^{-1} A_2 U_3$ agrees with $\sigma'_{i+1}, \dots, \sigma'_n$ in its values greater than δ.

We may bound the norm of the top right block of A_3 by a continuous function of $\sigma_1, \dots, \sigma_n, \sigma'_1, \dots, \sigma'_n, \delta$. We can then conjugate by a suitable block diagonal matrix U_4, with scalar matrices in the diagonal blocks, to ensure that the multiset of singular values of $A_4 = U_4^{-1} A_3 U_4$ agrees with $\sigma'_1, \dots, \sigma'_n$ in its values greater than δ. We then take $U = U_1 \cdots U_4$. $\square$

For the purposes of this book, it is immaterial what the function f_n is, as long as it is continuous. However, for numerical applications it may be quite helpful to identify a good function f_n; here is a conjectural best possible result in the case $\delta = 0$, phrased in a somewhat stronger form. (One can formulate an archimedean analogue. It should also be possible to prove, using the Horn inequalities, that this conjecture cannot be improved; see the next section.)

Conjecture 4.4.9. *If A has singular values $\sigma_1, \dots, \sigma_n$ and eigenvalues $\lambda_1, \dots, \lambda_n$, none of which is equal to* 0, *and if $\sigma'_1, \dots, \sigma'_n \in |F^\times|$ satisfy*

$$\sigma_1 \cdots \sigma_i \geq \sigma'_1 \cdots \sigma'_i \geq |\lambda_1 \cdots \lambda_i| \qquad (i = 1, \dots, n),$$

then there exists $U \in \mathrm{GL}_n(F)$ such that

$$|U^{-1}| \leq 1, \quad |U| \leq \max_i \{(\sigma_1 \cdots \sigma_i)/(\sigma'_1 \cdots \sigma'_i)\},$$

for which $U^{-1} A U$ has singular values $\sigma'_1, \dots, \sigma'_n$.

By imitating the proof of Proposition 4.2.4 (after enlarging F to contain the eigenvalues of A), we obtain the following.

Proposition 4.4.10. *Let $\sigma_{k,1}, \dots, \sigma_{k,n}$ be the singular values of A^k. Then*

$$\lim_{k\to\infty} \sigma_{k,i}^{1/k} = |\lambda_i| \qquad (i = 1, \dots, n).$$

4.5 Horn's inequalities

Although they will not be needed in this book, it is quite natural to mention here some stronger versions of the perturbation inequalities in the archimedean and nonarchimedean cases, introduced conjecturally by Horn [**115**] in the archimedean case and resolved in the work of Klyachko, Knutson, Speyer, Tao, Woodward, and others. See the beautiful survey article of Fulton [**95**] for more information.

Definition 4.5.1. To introduce these stronger inequalities, we must set up some notation. Put

$$U_r^n = \left\{ (I, J, K) : I, J, K \subseteq \{1, \dots, n\}, \#I = \#J = \#K = r, \right.$$
$$\left. \sum_{i\in I} i + \sum_{j\in J} j = \sum_{k\in K} k + \frac{r(r+1)}{2} \right\}.$$

For $(I, J, K) \in U_r^n$, write $I = \{i_1 < \cdots < i_r\}$ and similarly for J, K. For $r = 1$, put $T_1^n = U_1^n$. For $r > 1$, put

$$T_r^n = \left\{ (I, J, K) \in U_r^n : \text{for all } p < r \text{ and } (F, G, H) \in T_p^r, \right.$$
$$\left. \sum_{f\in F} i_f + \sum_{g\in G} j_g \le \sum_{h\in H} k_h + \frac{p(p+1)}{2} \right\}.$$

For multiplicative perturbations, we obtain the following results, which include Propositions 4.2.3 and 4.4.4. It is important for the proofs that one can rephrase the Horn inequalities in terms of Littlewood–Richardson numbers; see [**95**, §3].

Theorem 4.5.2. *For $* \in \{A, B, C\}$, let $\sigma_{*,1}, \ldots, \sigma_{*,n}$ be a nonincreasing sequence of nonnegative real numbers. Then the following are equivalent.*

(a) *There exist $n \times n$ matrices A, B, C over $\mathbb{C}$ with $AB = C$ such that, for $* \in \{A, B, C\}$, $*$ has singular values $\sigma_{*,1}, \ldots, \sigma_{*,n}$.*

(b) *We have $\prod_{i=1}^n \sigma_{A,i} \prod_{j=1}^n \sigma_{B,j} = \prod_{k=1}^n \sigma_{C,k}$ and, for all $r < n$ and $(I, J, K) \in T_r^n$,*

$$\prod_{k \in K} \sigma_{C,k} \leq \prod_{i \in I} \sigma_{A,i} \prod_{j \in J} \sigma_{B,j}.$$

Proof. See [**95**, Theorem 16]. Note that the first condition in (b) is omitted in the statement given in [**95**], but this was a typographical error. □

Theorem 4.5.3. *Let F be a complete nonarchimedean field. For $* \in \{A, B, C\}$, let $\sigma_{*,1}, \ldots, \sigma_{*,n}$ be a nonincreasing sequence of elements of $|F|$. Then the following are equivalent.*

(a) *There exist $n \times n$ matrices A, B, C over F with $AB = C$ such that, for $* \in \{A, B, C\}$, $*$ has singular values $\sigma_{*,1}, \ldots, \sigma_{*,n}$.*

(b) *We have $\prod_{i=1}^n \sigma_{A,i} \prod_{j=1}^n \sigma_{B,j} = \prod_{k=1}^n \sigma_{C,k}$ and, for all $r < n$ and $(I, J, K) \in T_r^n$,*

$$\prod_{k \in K} \sigma_{C,k} \leq \prod_{i \in I} \sigma_{A,i} \prod_{j \in J} \sigma_{B,j}.$$

Proof. See [**95**, Theorem 7]. □

Example 4.5.4. Let us see explicitly how Theorem 4.5.2 implies Proposition 4.2.3. Since $T_1^n = U_1^n$, condition (b) of Theorem 4.5.2 includes all cases with $(I, J, K) \in U_1^n$. In particular, we may take

$$I = \{i\}, \qquad J = \{1\}, \qquad K = \{i\}$$

to obtain the inequality $\sigma_{C,i} \leq \sigma_{A,i}\sigma_{B,1}$; this is precisely Proposition 4.2.3.

Remark 4.5.5. For additive perturbations, one has an analogous result in the archimedean case; see [**95**, Theorem 15]. We are not aware of an additive result in the nonarchimedean case. Also, in the archimedean case one has analogous results (with slightly different statements) in which one restricts to Hermitian matrices.

Notes

The subject of archimedean matrix inequalities is an old one, with many important applications. A good reference for this is [**28**]; for instance, see [**28**, §I.2] for the singular value decomposition, [**28**, Theorem II.3.6] for the Weyl inequalities, in a much stronger form known as Weyl's majorant theorem, [**28**, Theorem III.4.5] for a strong form of Proposition 4.2.3 (also a consequence of the Horn inequalities), and so on. (A variant of our Theorem 4.2.2 appears as [**28**, Problem I.6.11].)

The strong analogy between archimedean and p-adic matrix inequalities appears to be a little-known piece of folklore. As a result, we have been unable to locate a suitable reference.

It should be pointed out that most of what we have done here is the special case for GL_n of a more general theory encompassing the other reductive algebraic groups. This point of view can be seen in [**95**], where GL_n makes some explicit appearances for which other groups can be substituted.

In Theorem 4.1.13, the equivalence of (a) and (b) is standard. We do not have a reference for the equivalence with (c), although it is implicit in most proofs of the equivalence of (a) and (b).

The reader familiar with the notions of elementary divisors or invariant factors may be wondering why the terminology "Hodge polygon" is necessary or reasonable. The answer is that the Hodge numbers of a variety over a p-adic field are reflected by the elementary divisors of the action of Frobenius groups on crystalline cohomology. The fact that the Newton polygon lies above the Hodge polygon then implies a relation between the characteristic polynomial of Frobenius and the Hodge numbers of the original variety; this relationship was originally conjectured by Katz and proved by Mazur. See [**27**] for further discussion of this point and of crystalline cohomology as a whole.

Much of the work in this chapter can be carried over to the case of a transformation which is only semilinear for some isometric endomorphism of F. We will adopt that point of view in Chapter 14; for instance, this will lead to a generalization (in Theorem 14.5.5) of the Hodge–Newton decomposition theorem (Theorem 4.3.11). In this case, the carrying over is really in the other direction: it is the latter result (due to Katz; see the notes for Chapter 14) which inspired our presentation of Theorem 4.3.11 and its archimedean analogue (Theorem 4.1.12). Similarly, our treatment of Proposition 4.4.10 and its archimedean analogue (Proposition 4.2.4) are modeled on [**120**, Corollary 1.4.4].

The question of quantifying the sensitivity to perturbation of the characteristic polynomial of a square matrix arises in numerical applications. The question is familiar in the archimedean case but perhaps less so in the nonarchimedean case; numerical applications of the latter include using p-adic cohomology to compute zeta functions of varieties over finite fields. See for instance [**3**, §1.6] and [**96**, §3].

Exercises

(1) Prove that any elementary matrix of type (a) (swapping two rows) can be factored as a product of elementary matrices of types (b) and (c); see Definition 4.0.3.

(2) Let $e_1, \dots, e_4$ be a basis of $\mathbb{C}^4$. Prove that in $\wedge^2\mathbb{C}^4$ the element $e_1 \wedge e_2 + e_3 \wedge e_4$ is not decomposable.

(3) Check that the characteristic polynomial of the companion matrix of a polynomial P (Definition 4.3.9) is equal to P.

(4) Prove Proposition 4.4.1. (Hint: use Corollary 4.3.6.)

(5) With notation as in Theorem 4.3.11, suppose that $U, V \in \mathrm{GL}_n(\mathfrak{o}_F)$ are congruent to the identity matrix modulo $\mathfrak{m}_F$. Prove that the product of the i largest eigenvalues of UAV again has norm $|\lambda_1 \cdots \lambda_i|$. (Hint: use exterior powers to reduce to the case $i = 1$.) This yields as a corollary [**34**, Lemma 5]: if $D \in \mathrm{GL}_n(F)$ is diagonal and $U, V \in \mathrm{GL}_n(\mathfrak{o}_F)$ are congruent to the identity matrix modulo $\mathfrak{m}_F$ then the Newton polygons of D and UDV coincide.

(6) State and prove an archimedean analogue of the previous problem.

(7) Prove the following improved version of Proposition 4.4.6. Suppose that the Newton slopes of A are nonnegative. Then there exists $U \in \mathrm{GL}_n(F)$ such that

$$|U^{-1}AU| \leq 1, \quad |U^{-1}| \leq 1, \quad |U| \leq \prod_{i=1}^{n-1} \max\{1, \sigma_i\}.$$

We do not know of an appropriate archimedean analogue.

Part II

Differential Algebra

5

Formalism of differential algebra

In this chapter, we introduce some basic formalism of differential algebra. This may be viewed as a mild perturbation of commutative algebra in which we consider commutative rings equipped with the additional noncommutative structure of a derivation. This allows us to manipulate differential equations and differential systems in a manner that keeps the more useful structure visible, though we will need to convert back and forth from this point of view. (One thing we will not do is generalize further, to the realm of differential schemes and sheaves; we leave this as a thought exercise for the curious reader.)

A particularly important result that we introduce is the *cyclic vector theorem*, which gives a compact but highly noncanonical way to represent a finite differential module over a field. While the cyclic vector theorem will prove indispensable at a few key points in our treatment of p-adic differential equations, we will ultimately make more progress by limiting its use. See Remark 5.7.1 for further discussion.

5.1 Differential rings and differential modules

Definition 5.1.1. A *differential ring* is a commutative ring R equipped with a derivation $d : R \to R$; the latter is an additive map satisfying the Leibniz rule

$$d(ab) = ad(b) + bd(a) \qquad (a, b \in R).$$

We expressly allow $d = 0$ unless otherwise specified; this will be useful in some situations. A differential ring which is also a domain or field, etc., will be called a *differential domain, field*, etc. Note that there is a unique extension of d to any localization of R, using the quotient rule; in particular, if R is a domain then there is a unique extension of d to $\mathrm{Frac}(R)$.

Definition 5.1.2. A *differential module* over a differential ring (R, d) is a module M equipped with an additive map $D : M \to M$ satisfying

$$D(am) = aD(m) + d(a)m;$$

such a D will also be called a *differential operator* on M relative to d. For example, (R, d) is a differential module over itself; any differential module isomorphic to a direct sum of copies of (R, d) is said to be *trivial*. (If we refer to "the trivial differential module", though, we mean (R, d) itself.) A differential module which is a successive extension of trivial modules is said to be *unipotent* (see Proposition 7.2.5 for the reason why). A *differential ideal* of R is a differential submodule of R itself, i.e., an ideal stable under d.

Definition 5.1.3. For (M, D) a differential module, define

$$H^0(M) = \ker(D), \qquad H^1(M) = \operatorname{coker}(D) = M/D(M).$$

The latter computes Yoneda extensions; see Lemma 5.3.3 below. Elements of $H^0(M)$ are said to be *horizontal* (see the notes). Note that $H^0(R) = \ker(d)$ is a subring of R; if R is a field then $\ker(d)$ is a subfield. We call them the *constant subring* and *constant subfield* of R.

We now make an observation about base changes of the constant subring. For the definition of a more general base change, see Definition 5.3.2. (See also Proposition 6.9.1.)

Lemma 5.1.4. *Let R_0 be the constant subring of the differential ring (R, d), and let R_0' be an R_0-algebra. Let (M, D) be a differential module over (R, d). View $R' = R \otimes_{R_0} R_0'$ as a differential ring by defining the derivation d' by*

$$d'\left(\sum_i a_i \otimes r_i\right) = \sum_i d(a_i) \otimes r_i \qquad (a_i \in R, r_i \in R_0').$$

Similarly, view $M' = M \otimes_{R_0} R_0'$ as a differential module over (R', d') by defining the differential operator

$$D'\left(\sum_i m_i \otimes r_i\right) = \sum_i D(m_i) \otimes r_i \qquad (m_i \in M, r_i \in R_0').$$

(a) *There are natural maps $H^i(M) \otimes_{R_0} R_0' \to H^i(M')$ for $i = 0, 1$.*
(b) *The map in (a) is always an isomorphism for $i = 1$.*
(c) *If R_0' is flat over R_0 then the map in (a) is an isomorphism for $i = 0$.*

Proof

(a) Tensoring the structure morphism $R_0 \to R_0'$ with M induces a map $M \to M'$. This in turn induces maps $H^i(M) \to H^i(M')$ of R_0-modules; using the R'-module structure on $H^i(M')$, we also obtain maps $H^i(M) \otimes_{R_0} R_0' \to H^i(M')$.

(b) Tensoring with R_0' is always a right exact functor on R_0-modules. Since

$$M \xrightarrow{D} M \to H^1(M) \to 0$$

is an exact sequence in the category of R_0-modules,

$$M' \xrightarrow{D'} M' \to H^1(M) \otimes_{R_0} R_0' \to 0$$

is also exact. Hence the induced map $H^1(M) \otimes_{R_0} R_0' \to H^1(M')$ is an isomorphism.

(c) Recall that, by definition, R_0' is flat over R_0 if and only if tensoring with R_0' is an exact functor on R_0-modules. Since

$$0 \to H^0(M) \to M \xrightarrow{D} M$$

is an exact sequence in the category of R_0-modules,

$$0 \to H^0(M) \otimes_{R_0} R_0' \to M' \xrightarrow{D'} M'$$

is also exact. Hence the induced map $H^0(M) \otimes_{R_0} R_0' \to H^0(M')$ is an isomorphism.

□

Another frequently used observation is the following.

Lemma 5.1.5. *Let (R, d) be a differential field with constant subfield R_0. Then, for any differential module (M, D) over (R, d), the natural map $H^0(M) \otimes_{R_0} R \to M$ is injective. In particular,* $\dim_{R_0} H^0(M) \leq \dim_R M$.

Proof. An equivalent statement is that if $m_1, \ldots, m_n \in H^0(M)$ are linearly dependent over R then they are also linearly dependent over R_0. Suppose on the contrary that, for some positive integer n, there exist $m_1, \ldots, m_n \in H^0(M)$ which are linearly dependent over R but linearly independent over R_0. Choose n as small as possible with this property; then there exist $c_1, \ldots, c_n \in R$ all nonzero such that $c_1 m_1 + \cdots + c_n m_n = 0$. We may rescale the c_i so that $c_1 = 1$.

Since $m_1, \ldots, m_n \in H^0(M)$, we also have

$$d(c_1)m_1 + \cdots + d(c_n)m_n = 0.$$

That is, $d(c_1), \dots, d(c_n)$ form another linear dependence relation between $m_1, \dots, m_n$. But $d(c_1) = d(1) = 0$, so to avoid contradicting the choice of n we must have $d(c_2) = \cdots = d(c_n) = 0$. That is, $c_1, \dots, c_n \in R_0$, contradicting the hypothesis that $m_1, \dots, m_n$ are linearly independent over R_0. This contradiction proves the claim. $\square$

Definition 5.1.6. Let M be a differential module over a differential ring R admitting a finite exhaustive filtration with irreducible successive quotients. (For instance, R could be a differential field and M could be a finitely generated differential module over R.) Then the multiset of these quotients is independent of the choice of the filtration; we call them the *(Jordan–Hölder) constituents* of M.

5.2 Differential modules and differential systems

We now describe the link between differential modules and linear differential systems.

Definition 5.2.1. Let R be a differential ring, and let M be a finite free differential module of rank n over R. Let $e_1, \dots, e_n$ be a basis of M. Then, for any $v \in M$, we can write $v = v_1 e_1 + \cdots + v_n e_n$ for some $v_1, \dots, v_n \in R$ and compute

$$D(v) = v_1 D(e_1) + \cdots + v_n D(e_n) + d(v_1)e_1 + \cdots + d(v_n)e_n.$$

Define the *matrix of action* of D (i.e., the matrix representing the action of D) on the basis $e_1, \dots, e_n$ to be the $n \times n$ matrix N over R given by the formula

$$D(e_j) = \sum_{i=1}^{n} N_{ij} e_i.$$

We then have

$$D(v) = \sum_{i=1}^{n} \left(d(v_i) + \sum_{j=1}^{n} N_{ij} v_j \right) e_i.$$

That is, if we identify v with the column vector $[v_1 \quad \cdots \quad v_n]$ then

$$D(v) = Nv + d(v).$$

Conversely, it is clear that, given the underlying finite free R-module, any differential module structure is given by such an equation.

Definition 5.2.2. With notation as in Definition 5.2.1, let $v_1, \dots, v_n$ be a second basis of M. The *change-of-basis matrix* from $e_1, \dots, e_n$ to $v_1, \dots, v_n$ is the $n \times n$ matrix U defined by

$$v_j = \sum_i U_{ij} e_i.$$

The effect of changing basis is that the matrix of action of D on $v_1, \dots, v_n$ is

$$U^{-1} N U + U^{-1} d(U).$$

Remark 5.2.3. In other words, differential modules are a coordinate-free version of differential systems. If you are a geometer, you may wish to go further and think of *differential bundles*, i.e., vector bundles equipped with a differential operator. A differential operator on a vector bundle is usually called a *connection*.

5.3 Operations on differential modules

We now describe the basic operations in the category of differential modules over a differential ring.

Definition 5.3.1. For R a differential ring, we regard the differential modules over R as a category in which the morphisms (or *homomorphisms*) from M_1 to M_2 are R-module homomorphisms $f : M_1 \to M_2$ satisfying $D(f(m)) = f(D(m))$ (we sometimes say these maps are *horizontal*).

Definition 5.3.2. The category of differential modules over a differential ring admits certain functors corresponding to familiar functors on the category of modules over an ordinary ring, such as the following. (Be aware that in the following notation the subscripted R on such symbols as the tensor product will often be suppressed when it is unambiguous. Our habit tends to be to drop the subscript when tensoring modules over a single ring but not when performing a base change.)

Given two differential modules M_1, M_2, the tensor product $M_1 \otimes_R M_2$ in the category of rings may be viewed as a differential module via the formula

$$D(m_1 \otimes m_2) = D(m_1) \otimes m_2 + m_1 \otimes D(m_2).$$

This in particular gives meaning to the *base change* $M \otimes_R R'$ of a differential R-module M to a differential R-algebra R'; we also denote this by $f^* M$ if f

is the map from R to R'. Similarly, the exterior power $\wedge_R^n M$ may be viewed as a differential module via the formula

$$D(m_1 \wedge \cdots \wedge m_n) = \sum_{i=1}^n m_1 \wedge \cdots \wedge m_{i-1} \wedge D(m_i) \wedge m_{i+1} \wedge \cdots \wedge m_n.$$

(A similar fact is true for the symmetric power $\mathrm{Sym}_R^n M$, but we will have no need of this.) The module of R-homomorphisms $\mathrm{Hom}_R(M_1, M_2)$ may be viewed as a differential module via the formula

$$D(f)(m) = D(f(m)) - f(D(m));$$

the homomorphisms from M_1 to M_2 as differential modules are precisely the horizontal elements of $\mathrm{Hom}_R(M_1, M_2)$. (In this case, the subscript is quite crucial: we have $\mathrm{Hom}(M_1, M_2) = H^0(\mathrm{Hom}_R(M_1, M_2))$.)

We write $M_1^\vee$ for $\mathrm{Hom}_R(M_1, R)$ and call it the *dual* of M_1; if M_1 is finite free then $\mathrm{Hom}_R(M_1, M_2) \cong M_1^\vee \otimes M_2$ and the natural map $M_1 \to (M_1^\vee)^\vee$ is an isomorphism. In particular, $M_1^\vee \otimes M_1$ contains a horizontal element corresponding to the identity map $M_1 \to M_1$; we call this the *trace (element)* of $M_1^\vee \otimes M_1$, and we call the trivial submodule generated by the trace element the *trace component* of $M_1^\vee \otimes M_1$. If R is a $\mathbb{Q}$-algebra then $M_1^\vee \otimes M_1$ splits as the direct sum of the trace component with the set of elements of $\mathrm{Hom}_R(M_1, M_2)$ of trace zero; we call the latter the *trace-zero component* of $M_1^\vee \otimes M_1$. Even if R is not a $\mathbb{Q}$-algebra, we can still view the trace component as a quotient of $M_1^\vee \otimes M_1$ by duality, but the map given by embedding the trace component into $M_1^\vee \otimes M_1$ and then projecting onto the trace component need not be an isomorphism.

Lemma 5.3.3. *Let M, N be differential modules with M finite free. Then the group $H^1(M^\vee \otimes N)$ is canonically isomorphic to the Yoneda extension group* $\mathrm{Ext}(M, N)$.

Proof. The group $\mathrm{Ext}(M, N)$ consists of equivalence classes of exact sequences $0 \to N \to P \to M \to 0$ under the relation that this sequence is equivalent to a second sequence $0 \to N \to P' \to M \to 0$ if there is an isomorphism $P \cong P'$ that induces the identity maps on M and N. Addition consists of taking two such sequences and returning the *Baer sum* $0 \to N \to Q/\Delta \to M \to 0$, where Q is the set of elements in $P \oplus P'$ whose images in M coincide and $\Delta = \{(n, -n) \in Q : n \in N\}$. The identity element is the split sequence $0 \to N \to M \oplus N \to M \to 0$. The inverse of a sequence $0 \to N \to P \to M \to 0$ is the same sequence with the map $N \to P$ negated. (See [**216**, §3.4] for the proof that this indeed gives a group.)

We first construct a canonical isomorphism $\mathrm{Ext}(M, N) \to \mathrm{Ext}(R, M^\vee \otimes N)$. Given an extension $0 \to N \to P \to M \to 0$, tensor with $M^\vee$ to get

$$0 \to M^\vee \otimes N \to M^\vee \otimes P \to M^\vee \otimes M \to 0.$$

Let Q be the inverse image of the trace component of $M^\vee \otimes M$; we then get an extension

$$0 \to M^\vee \otimes N \to Q \to R \to 0,$$

yielding a map $\mathrm{Ext}(M, N) \to \mathrm{Ext}(R, M^\vee \otimes N)$, which is easily shown to be a homomorphism. In the other direction, given an extension

$$0 \to M^\vee \otimes N \to Q \to R \to 0,$$

tensor with M to get

$$0 \to M \otimes M^\vee \otimes N \to M \otimes Q \to M \to 0,$$

then quotient $M \otimes Q$ by the kernel of the projection $M \otimes M^\vee \to R$ tensored with N. These are seen to be inverses by a diagram-chasing argument, which we omit.

By the previous paragraph, we may reduce the statement of the lemma to the case $M = R$. (One can also describe the construction without first making this reduction, but it is a bit harder to follow.) Given an extension $0 \to N \to P \to R \to 0$, compute H^0 and H^1 and apply the snake lemma to obtain a connecting homomorphism $H^0(R) \to H^1(N)$. The image of $1 \in R$ under this homomorphism determines an element of $H^1(N)$, thus giving a map $\mathrm{Ext}(R, N) \to H^1(N)$. This map is easily shown to be a homomorphism (exercise).

It remains to construct an inverse map. Given an element of $H^1(N)$ represented by $x \in N$, we equip $N \oplus R$ with the structure of a differential module by setting

$$D(n, r) = (D(n) + rx, d(r)).$$

This module is indeed an extension of the desired form, and the image of $1 \in R$ under the resulting connecting homomorphism is precisely the class of x in $H^1(N)$. In the other direction, any extension splits at the level of modules and so must have this form for some $x \in N$. This yields the claim. □

Remark 5.3.4. Although we will not need this, we note that the isomorphisms $\mathrm{Hom}_R(M_1, M_2) \cong M_1^\vee \otimes M_2$ and $M_1 \to (M_1^\vee)^\vee$, and the assertion of Lemma 5.3.3, also carry over to the case where M_1 is a finite *projective* R-module, i.e., a direct summand of a finite free R-module. Such a module is always flat. If R is noetherian then a finitely generated R-module M is projective if and only if it is *locally free*, i.e., if there exists a finite subset $f_1, \dots, f_m$ of R, generating the unit ideal, such that $M[f_i^{-1}]$ is free over $R[f_i^{-1}]$ for each i [**84**, Exercise 4.12].

5.4 Cyclic vectors

Definition 5.4.1. Let R be a differential ring, and let M be a finite free differential module of rank n over R. A *cyclic vector* for M is an element $m \in M$ such that $m, D(m), \dots, D^{n-1}(m)$ form a basis of M.

Theorem 5.4.2 (Cyclic vector theorem). *Let R be a differential field of characteristic* 0 *with nonzero derivation. Then every finite differential module over R has a cyclic vector.*

Many proofs are possible; we give here the proof from [**80**, Theorem III.4.2]. For another proof that applies over some rings other than fields, see Theorem 5.7.3 below. See also the notes for further discussion. (For a comment on characteristic p, see the exercises.)

Proof. We start by normalizing the derivation. For $u \in R^{\times}$, given one differential module (M, D) over (R, d), we get another differential module (M, uD) over (R, ud), and m is a cyclic vector for one of these modules if and only if it is a cyclic vector for the other (because the image of m under $(uD)^j$ is in the span of $m, D(m), \dots, D^j(m)$). We may thus assume (thanks to the assumption that the derivation is nontrivial) that there exists a nonzero element $x \in R$ such that $d(x) = x$.

Let M be a differential module of dimension n, and choose $m \in M$ such that the dimension μ of the span of $m, D(m), \dots$ is as large as possible. We derive a contradiction under the hypothesis $\mu < n$.

For $z \in M$ and $\lambda \in \mathbb{Q}$, we have

$$(m + \lambda z) \wedge D(m + \lambda z) \wedge \cdots \wedge D^{\mu}(m + \lambda z) = 0$$

in the exterior power $\wedge^{\mu+1} M$. If we write this expression as a polynomial in λ, it vanishes for infinitely many values so must be identically zero. Hence each coefficient must vanish separately, including the coefficient of λ^1, which is

$$\sum_{i=0}^{\mu} m \wedge \cdots \wedge D^{i-1}(m) \wedge D^i(z) \wedge D^{i+1}(m) \wedge \cdots \wedge D^{\mu}(m). \qquad (5.4.2.1)$$

Pick $s \in \mathbb{Z}$, substitute $x^s z$ for z in (5.4.2.1), divide by x^s, and set the result equal to zero. We get

$$\sum_{i=0}^{\mu} s^i \Lambda_i(m, z) = 0 \qquad (s \in \mathbb{Z}) \tag{5.4.2.2}$$

for

$$\Lambda_i(m, z) = \sum_{j=0}^{\mu-i} \binom{i+j}{i} m \wedge \cdots \wedge D^{i+j-1}(m) \wedge D^j(z) \wedge D^{i+j+1}(m) \wedge \cdots \wedge D^{\mu}(m).$$

Again because we are in characteristic zero, we may conclude that (5.4.2.2), viewed as a polynomial in s, has all coefficients equal to zero; that is, $\Lambda_i(m, z) = 0$ for all $m, z \in M$.

We now take $i = \mu$ to obtain

$$(m \wedge \cdots \wedge D^{\mu-1}(m)) \wedge z = 0 \qquad (m, z \in M);$$

since $\mu < n$, we may use this to deduce that

$$m \wedge \cdots \wedge D^{\mu-1}(m) = 0 \qquad (m \in M).$$

But this means that the dimension of the span of $m, D(m), \ldots$ is at most $\mu - 1$, contradicting the definition of μ. $\square$

5.5 Differential polynomials

We now give the interpretation of differential modules as modules over a mildly noncommutative ring.

Definition 5.5.1. Let (R, d) be a differential ring. The *ring of twisted polynomials* $R\{T\}$ over R in the variable T is the additive group

$$R \oplus (R \cdot T) \oplus (R \cdot T^2) \oplus \cdots,$$

with noncommuting multiplication given by the formula

$$\left(\sum_{i=0}^{\infty} a_i T^i\right)\left(\sum_{j=0}^{\infty} b_j T^j\right) = \sum_{i,j=0}^{\infty} \sum_{h=0}^{j} \binom{j}{h} a_i d^h(b_j) T^{i+j-h}.$$

In other words, we impose the relation

$$Ta = aT + d(a) \qquad (a \in R)$$

and then check that the result is a not necessarily commutative ring (see the exercises). We define the *degree* of a twisted polynomial, in the usual way, as the exponent of the largest power of T with a nonzero coefficient; the degree of the zero polynomial may be taken to be any particular negative value.

Proposition 5.5.2 (Ore). *For R a differential field, the ring $R\{T\}$ admits a left division algorithm. That is, if $f, g \in R\{T\}$ and $g \neq 0$ then there exist unique $q, r \in R\{T\}$ with $\deg(r) < \deg(g)$ and $f = gq + r$. (There is also a right division algorithm.)*

Proof. Exercise. □

Using the Euclidean algorithm, this yields the following consequence as in the untwisted case.

Theorem 5.5.3 (Ore). *Let R be a differential field. Then $R\{T\}$ is both left principal and right principal; that is, any left ideal (resp. right ideal) has the form $R\{T\}f$ (resp. $fR\{T\}$) for some $f \in R\{T\}$.*

Definition 5.5.4. Note that the ring opposite to $R\{T\}$, i.e., the ring in which multiplication is performed by first switching the order of the factors, is again a twisted polynomial ring except that the derivation is $-d$ instead of d. Given $f \in R\{T\}$, we define the *formal adjoint* of f as the element f in the opposite ring. This operation looks a bit less formal if we also take the coefficients over to the other side, giving what we will call the *adjoint form* of f. For instance, the adjoint form of $T^3 + aT^2 + bT + c$ is

$$T^3 + T^2 a + T(b - 2d(a)) + d^2(a) - d(b) + c.$$

Remark 5.5.5. The twisted polynomial ring is engineered precisely so that, for any differential module M over R, we obtain an action of $R\{T\}$ on M under which T acts like D. In particular, $R\{T\}$ acts on R itself with T acting like d. In fact, the category of differential modules over R is equivalent to the category of left $R\{T\}$-modules. Moreover, if M is a finite differential module over R, any cyclic vector $m \in M$ corresponds to an isomorphism $M \cong R\{T\}/R\{T\}P$ for some monic twisted polynomial P, where the isomorphism carries m to the class of 1. (We can think of f as a sort of "characteristic polynomial" for M, except that it depends strongly on the choice of the cyclic vector.) Under such an isomorphism, a factorization $P = P_1 P_2$ corresponds to a short exact sequence $0 \to M_1 \to M \to M_2 \to 0$ with

$$M_1 \cong R\{T\}P_2/R\{T\}P \cong R\{T\}/R\{T\}P_1, \qquad M_2 \cong R\{T\}/R\{T\}P_2.$$

5.6 Differential equations

You may have been wondering when differential equations will appear, as these purport to be the objects of study of this book. If so, your wait is over.

Definition 5.6.1. A *differential equation of order n* over the differential ring (R, d) is an equation of the form

$$(a_n d^n + \cdots + a_1 d + a_0)y = b,$$

with $a_0, \ldots, a_n, b \in R$, and y indeterminate. We say the equation is *homogeneous* if $b = 0$ and *inhomogeneous* otherwise.

Remark 5.6.2. Using our setup, we may write this equation as $f(d)y = b$ for some $f \in R\{T\}$. Similarly, we may view systems of differential equations as being equations of the form $f(D)y = b$, where b lives in some differential module (M, D). By the usual method (i.e. introducing extra variables corresponding to derivatives of y), we can convert any differential system into a first-order system $Dy = b$. We can also convert an inhomogeneous system into a homogeneous one by adding an extra variable, with the understanding that we would like the value of that last variable to be 1 in order to get back a solution of the original equation.

Remark 5.6.3. Here is a more explicit relationship between adjoint polynomials and the process of solving differential equations. Suppose that you start with the cyclic differential module $M \cong R\{T\}/R\{T\}f$ and you want to find a horizontal element. That means that you want to find some $g \in R\{T\}$ such that $Tg \in R\{T\}f$; we may as well assume that $\deg(g) < \deg(f)$. Then, by comparing degrees, we see that in fact $Tg = rf$ for some $r \in R$. Write f in adjoint form as $f_0 + Tf_1 + \cdots + T^n$; then

$$rf \equiv rf_0 - d(r)f_1 + d^2(r)f_2 - \cdots + (-1)^n d^n(r) \pmod{TR\{T\}}.$$

In this manner, finding a horizontal element becomes equivalent to solving a differential equation.

5.7 Cyclic vectors: a mixed blessing

The reader may at this point be wondering why so many points of view are necessary, since the cyclic vector theorem can be used to transform any differential module into a differential equation, and ultimately differential equations are the things one writes down and wants to solve. Permit me to interject here a countervailing opinion.

Remark 5.7.1. In ordinary linear algebra (or in other words, when considering differential modules for the trivial derivation), one can pass freely between linear transformations on a vector space and square matrices if one is willing to choose a basis. The merits of doing this depend on the situation, so it is valuable to have both the matricial and coordinate-free viewpoints well in hand. One can then pass to the characteristic polynomial, but not all information is retained (one loses information about nilpotency) and even information that in principle is retained is sometimes not so conveniently accessed. In short, no one would seriously argue that one can dispense with studying matrices because of the existence of the characteristic polynomial.

The situation is not so different in the differential case. The difference between a differential module and a differential system is merely the choice of a basis, and again it is valuable to have both points of view in mind. However, the cyclic vector theorem may seduce one into thinking that collapsing a differential system into a differential polynomial is an operation without drawbacks, but this is far from the case. For instance, determining whether two differential polynomials correspond to the same differential system is not straightforward.

More seriously for our purposes, the cyclic vector theorem only applies over a differential field. Many differential modules are more naturally defined over some ring which is not a field. For instance, differential modules arising from geometry (such as Picard–Fuchs modules) are usually defined over a ring of functions on some geometric space. While there are forms of the cyclic vector theorem available over nonfields (see for instance Theorem 5.7.3 below), these do not suffice for our purposes. We also find that working with differential modules instead of differential polynomials has a tremendously clarifying effect, partly because it improves the parallelism with difference algebra, where there is no good analogue of the cyclic vector theorem even over a field. (See Part IV.)

We find it unfortunate that much literature on complex ordinary differential equations, and nearly all the literature on p-adic ordinary differential equations, is mired in the language of differential polynomials. By switching instead between differential modules and differential polynomials, as appropriate, we will be able to demonstrate strategies that lead to a more systematic development of p-adic theory.

As promised, we offer some results concerning cyclic vectors over rings, due to Katz [**122**].

Theorem 5.7.2. *Let R be a differential local $\mathbb{Q}$-algebra. Suppose that the maximal ideal of R contains an element t such that $d(t) = 1$. Let M be a finite free differential module over R, and let $e_1, \dots, e_n$ be a basis of M. Then*

$$v = \sum_{j=0}^{n-1} \frac{t^j}{j!} \sum_{k=0}^{j} (-1)^k \binom{j}{k} D^k(e_{j+1-k})$$

is a cyclic vector of M.

Proof. For $i, j \geq 0$, put

$$c(i, j) = \sum_{k=0}^{j} (-1)^k \binom{j}{k} D^k(e_{i+j+1-k})$$

with the convention that $e_h = 0$ for $h > n$. It is easily checked that

$$c(i+1, j) = D(c(i, j)) + c(i, j+1).$$

By induction on i, we find that

$$D^i(v) = \sum_{j=0}^{n-1} \frac{t^j}{j!} c(i, j) \equiv e_{i+1} \pmod{t}.$$

Hence $v, D(v), \dots, D^{n-1}(v)$ freely generate M modulo the maximal ideal of R. They thus freely generate M itself by Nakayama's lemma [**84**, Corollary 4.8]. □

Theorem 5.7.3. *Let R be a differential $\mathbb{Q}$-algebra containing an element t for which $d(t) = 1$. Let M be a finite free differential module over R, and let I be a prime ideal of R. Then there exists $f \in R \setminus I$ such that $M \otimes_R R[f^{-1}]$ contains a cyclic vector.*

Proof. Choose a basis $e_1, \dots, e_n$ of M, and define $c(i, j)$ as in the proof of Theorem 5.7.2. For $x \in R$, put

$$c_i(x) = \sum_{j=0}^{n-1} \frac{x^j}{j!} c(i, j).$$

Then there exists a polynomial $P(T) \in R[T]$ such that

$$c_0(x) \wedge \cdots \wedge c_{n-1}(x) = P(x)(e_1 \wedge \cdots \wedge e_n) \qquad (x \in R).$$

Since $c(i, 0) = e_{i+1}$, we have $P(0) = 1$. In particular, the image $Q(T) \in (R/I)[T]$ of $P(T)$ is not identically zero.

Since R contains $\mathbb{Q}$, the images of $t - a$ for $a \in \mathbb{Z}$ are all distinct, so only finitely many of them are roots of $Q(T)$. We can thus choose $a \in \mathbb{Z}$ such that $P(t - a) \notin I$. Since

$$D^i(c_0(t-a)) = c_i(t-a)$$

as in the previous proof, $c_0(t-a)$ is a cyclic vector of $M \otimes_R R[f^{-1}]$ for $f = P(t-a)$, as desired. $\square$

5.8 Taylor series

Definition 5.8.1. Let R be a *topological* differential ring, i.e., a ring equipped with a topology and a derivation such that all operations are continuous. Assume also that R is a $\mathbb{Q}$-algebra. Let M be a topological differential module over R, i.e., a differential module such that all operations are continuous. For $r \in R$ and $m \in M$, we define the *Taylor series* $T(r, m)$ as the infinite sum

$$\sum_{i=0}^{\infty} \frac{r^i}{i!} D^i(m)$$

whenever the sum converges absolutely (i.e., all rearrangements converge to the same value).

Remark 5.8.2. The expression $T(r, m)$ is *de facto* additive in m: if $m_1, m_2 \in M$ then

$$T(r, m_1) + T(r, m_2) = T(r, m_1 + m_2)$$

whenever all three terms are well-defined. For $s \in R$, $T(r, s)$ is also *de facto* multiplicative: if $s_1, s_2 \in R$, then (by the Leibniz rule)

$$T(r, s_1)T(r, s_2) = T(r, s_1 s_2)$$

whenever all three terms are well-defined. More generally, for $m \in M$, $T(r, m)$ is *de facto* semilinear: if $s \in R$, $m \in M$ then

$$T(r, s)T(r, m) = T(r, sm)$$

whenever all three terms are well-defined.

Example 5.8.3. A key instance of the previous remark is the case where R is a completion of a rational function field $F(t)$ and d is the differential operator d/dt. In this case, the ring homomorphism $T(r, \cdot)$ is the substitution $t \mapsto t + r$; note that this makes sense only if $|r| \le 1$.

Remark 5.8.4. Another use for Taylor series is to construct horizontal sections. Note that

$$\begin{aligned} D(T(r, m)) &= \sum_{i=1}^{\infty} d(r) \frac{r^{i-1}}{(i-1)!} D^i(m) + \sum_{i=0}^{\infty} \frac{r^i}{i!} D^{i+1}(m) \\ &= (1 + d(r)) T(r, m) \end{aligned}$$

if everything converges absolutely. In particular, if $d(r) = -1$ then $T(r, m)$ is horizontal.

Notes

The subject of differential algebra is rather well developed; a classic treatment, though possibly too dry to be useful to the casual reader, is the book of Ritt [**188**]. As in abstract algebra in general, the development of differential algebra was partly driven by *differential Galois theory*, i.e., the study of the expression of solutions of differential equations in terms of solutions to ostensibly simpler differential equations. A relatively lively introduction to the latter is [**202**].

Calling an element of a differential module *horizontal* when it is killed by the derivation makes sense if you consider *connections* in differential geometry. In that setting, the differential operator is measuring the extent to which a section of a vector bundle deviates from some prescribed "horizontal" direction identifying points on one fibre with points on nearby fibres.

The history of the cyclic vector theorem is rather complicated. It appears to have been first proved by Loewy [**162**] in the case of meromorphic functions, and (independently) by Cope [**60**] in the case of rational functions. For a detailed historical discussion, see [**53**].

Twisted polynomials were introduced by Ore [**177**]. They are actually somewhat more general than we have discussed; for instance, one can also twist by an endomorphism $\tau : R \to R$ by imposing the relation $Ta = \tau(a)T$. (This enters the realm of the analogue of differential algebra called *difference algebra*, which we will treat in Part IV.) Moreover, one can twist by both an endomorphism and a derivation if they are compatible in an appropriate way, and one can even study differential or difference Galois theory in this setting. A unifying framework for doing so, which is also suitable for considering multiple derivations and automorphisms, was given by André [**4**].

Differential algebra in positive characteristic has a rather different flavor than in characteristic 0; for instance, the pth power of the derivation d/dt on $\mathbb{F}_p(t)$ is the zero map. A brief discussion of the characteristic-p situation is given in [**80**, §III.1].

Exercises

(1) Prove that if M is a locally free differential module over R of rank 1 then $M^\vee \otimes M$ is trivial (as a differential module).

(2) Check that the bijection $\mathrm{Ext}(R, N) \to H^1(N)$ constructed in the proof of Lemma 5.3.3 is indeed a homomorphism.

(3) Check that, in characteristic $p > 0$, the cyclic vector theorem holds for modules of rank at most p but may fail for modules of rank $p + 1$.
(4) Give a counterexample to the cyclic vector theorem for a differential field of characteristic 0 with trivial derivation.
(5) Verify that $R\{T\}$ is indeed a not necessarily commutative ring; the content in this exercise is to check the associativity of multiplication.
(6) Prove the division algorithm (Proposition 5.5.2).

6

Metric properties of differential modules

In this chapter, we study the metric properties of differential modules over nonarchimedean differential rings. The principal invariant that we will identify is a familiar quantity from functional analysis known as the *spectral radius* of a bounded endomorphism. When applied to the derivation acting on a differential module, we obtain a quantity which can be related to the least slope of the Newton polygon of the corresponding twisted polynomial.

We can give meaning to the other slopes as well, by proving that over a complete nonarchimedean differential field any differential module decomposes into components whose spectral radii are computed by the various slopes of the Newton polygon. However, this theorem will provide somewhat incomplete results when we apply it to p-adic differential modules in Part III; we will have to remedy the situation using Frobenius descendants and antecedents.

This chapter provides important foundational material for much of what follows, but on its own it may prove indigestably abstract at first. The reader arriving at this opinion is advised to read Chapter 7 in conjunction with this one, to see how the constructions of this chapter become explicit in a simple but important class of examples.

6.1 Spectral radii of bounded endomorphisms

Before considering differential operators, let us recall the difference between the operator norm and the spectral radius of a bounded endomorphism of an abelian group.

Hypothesis 6.1.1. Throughout this section, let G be a nonzero abelian group equipped with a norm $|\cdot|$, and let $T : G \to G$ be a bounded endomorphism of G. Recall that this means that there exists $c \geq 0$ such that $|T(g)| \leq c|g|$ for all $g \in G$.

Definition 6.1.2. The *operator norm* $|T|_G$ of T is defined to be the least $c \geq 0$ for which $|T(g)| \leq c|g|$ for all $g \in G$, i.e.,

$$|T|_G = \sup_{g \in G, g \neq 0} \{|T(g)|/|g|\}.$$

Recall that, if M is a finite free module over a nonarchimedean ring R, $|\cdot|_M$ is the supremum norm for some basis, and T is an R-linear transformation, then $|T| = |A|$, where A is the matrix of action of T on the chosen basis of M and $|A| = \sup_{ij}\{|A_{ij}|\}$ as in Chapter 4.

One can similarly define the operator norm for a map between two different abelian groups, each equipped with a norm. We may even allow seminorms as long as we take the supremum over elements of the source group which are not in the kernel of the seminorm.

Although the condition that T is bounded is preserved on replacing the norm by a metrically equivalent norm, the operator norm is not preserved. To obtain a less fragile numerical invariant, we introduce the spectral radius.

Definition 6.1.3. The *spectral radius* of T is defined as

$$|T|_{\mathrm{sp},G} = \lim_{s \to \infty} |T^s|_G^{1/s};$$

the existence of the limit follows from the fact $|T^{m+n}|_G \leq |T^m|_G |T^n|_G$ and from the following lemma.

Lemma 6.1.4 (Fekete). *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $a_{m+n} \geq a_m + a_n$ for all m, n. Then the sequence $\{a_n/n\}_{n=1}^{\infty}$ either converges to its supremum or diverges to $+\infty$.*

Proof. Exercise. □

Proposition 6.1.5. *The spectral radius of T depends on the norm $|\cdot|$ only up to metric equivalence.*

Proof. Suppose that $|\cdot|'$ is a norm metrically equivalent to $|\cdot|$. We can then choose $c > 0$ such that $c^{-1}|g| \leq |g|' \leq c|g|$ for all $g \in G$. We then have $|T(g)|/|g| \leq c^2|T(g)|'/|g|'$ for all $g \in G \setminus \{e\}$. Applying this with T replaced by T^s gives $|T^s|_G \leq c^2|T^s|'_G$, so

$$|T^s|_{\mathrm{sp},G} \leq \lim_{s \to \infty} c^{2/s} (|T^s|'_{\mathrm{sp},G})^{1/s}.$$

Since $c^{2/s} \to 1$ as $s \to \infty$, this gives $|T^s|_{\mathrm{sp},G} \leq |T^s|'_{\mathrm{sp},G}$. The reverse inequality is found to hold on reversing the roles of the norms. □

We have the following relationship to the concepts studied in Chapter 4.

Remark 6.1.6. Let F be a field equipped with a norm, and let V be a finite-dimensional vector space over F. Pick a basis for V, and equip V with either the L^2 norm or the supremum norm defined by this basis, according to whether F is archimedean or nonarchimedean. Let A be the matrix of action of T on this basis. Then $|T|_V$ equals the largest singular value of A, whereas $|T|_{\mathrm{sp},V}$ equals the largest norm of an eigenvalue of A (by Proposition 4.2.4 in the archimedean case, and Proposition 4.4.10 in the nonarchimedean case).

Remark 6.1.7. The spectral radius derives its name from the fact that, by a celebrated theorem of Gelfand, for a bounded endomorphism of a commutative Banach algebra over $\mathbb{R}$ or $\mathbb{C}$ the spectral radius computes the radius of the smallest disc containing the entire spectrum of the operator. (This includes the archimedean case of the previous remark.)

When dealing with the spectral radius we will frequently use the following observation.

Lemma 6.1.8. *For any $\epsilon > 0$ there exists $c = c(\epsilon)$ such that for all $s \geq 0$,*

$$|T^s|_G \leq c(|T|_{\mathrm{sp},G} + \epsilon)^s.$$

Proof. For $c = 1$, the claim already holds for all but finitely many s. It thus suffices to increase c in order to cover the finite set of exceptions. □

Remark 6.1.9. Some authors writing about p-adic differential equations (including the present author) tend to refer to the spectral radius as the *spectral norm*. On the one hand, this is somewhat dangerous, because the spectral radius is not in general a norm or even a seminorm, even for matrices over a complete field (exercise). On the other hand, we will be using the spectral radius as a measure of size and so the normlike notation $|\cdot|_{\mathrm{sp},V}$ is useful. We will compromise by keeping this notation but referring to spectral radii rather than spectral norms.

6.2 Spectral radii of differential operators

We now specialize the previous discussion to the case of differential modules.

Definition 6.2.1. By a *nonarchimedean differential ring or field*, we mean a nonarchimedean ring equipped with a *bounded* derivation. For F a nonarchimedean differential field, we can define the operator norm $|d|_F$ and the spectral radius $|d|_{\mathrm{sp},F}$; by hypothesis the former is finite, so the latter is too.

Definition 6.2.2. Let F be a nonarchimedean differential field. By a *normed differential module* over F we mean a vector space V over F equipped with a compatible norm $|\cdot|_V$ and a derivation D with respect to d which is bounded as a endomorphism of the additive group of V. For V nonzero, we may then consider the operator norm $|D|_V$ and the spectral radius $|D|_{\mathrm{sp},V}$.

Remark 6.2.3. If V is finite-dimensional over F and F is complete then the spectral radius does not depend on the norm on V, since by Theorem 1.3.6 any two norms on V compatible with the norm on F are metrically equivalent.

In general, one cannot have differential modules with arbitrarily small spectral radius.

Lemma 6.2.4. *Let F be a nonarchimedean differential field, and let V be a nonzero normed differential module over F. Then*

$$|D|_{\mathrm{sp},V} \geq |d|_{\mathrm{sp},F}.$$

Proof. (This proof was suggested by Liang Xiao.) For $a \in F$ and $v \in V$ nonzero, the Leibniz rule gives

$$D^{s-i}(aD^i(v)) = d^{s-i}(a)D^i(v) + \sum_{j=1}^{s-i} \binom{s-i}{j} d^{s-i-j}(a)D^{i+j}(v) \quad (0 \leq i \leq s).$$

Inverting this system of equations gives an identity of the form

$$d^s(a)v = \sum_{i=0}^{s} c_{s,i} D^{s-i}(aD^i(v))$$

for certain universal constants $c_{s,i} \in \mathbb{Z}$. Consequently,

$$|d^s(a)v|_V \leq \max_{0 \leq i \leq s} \{|D^{s-i}(aD^i(v))|_V\}. \qquad (6.2.4.1)$$

By Lemma 6.1.8, given $\epsilon > 0$ we can choose $c = c(\epsilon)$ such that, for all $s \geq 0$,

$$|D^s|_V \leq c(|D|_{\mathrm{sp},V} + \epsilon)^s.$$

Using (6.2.4.1), we deduce that

$$|d^s(a)v|_V \leq c^2(|D|_{\mathrm{sp},V} + \epsilon)^s |a||v|_V.$$

Dividing by $|a||v|_V$ and taking the supremum over $a \in F$, we obtain

$$|d^s|_F \leq c^2(|D|_{\mathrm{sp},V} + \epsilon)^s.$$

Taking sth roots and then letting $s \to \infty$, we get

$$|d|_{\mathrm{sp},F} \leq |D|_{\mathrm{sp},V} + \epsilon.$$

Since $\epsilon > 0$ was arbitrary, this yields the claim. $\square$

It is sometimes useful to compute in terms of a basis of V over F.

Lemma 6.2.5. *Let F be a complete nonarchimedean differential field, and let V be a nonzero finite differential module over F. Fix a basis $e_1, \dots, e_n$ of V, and let D_s be the matrix of action of D^s on this basis (i.e., $D^s(e_j) = \sum_i (D_s)_{ij} e_i$). Then*

$$|D|_{\mathrm{sp},V} = \max\{|d|_{\mathrm{sp},F}, \limsup_{s\to\infty} |D_s|^{1/s}\}. \tag{6.2.5.1}$$

Proof. (Compare [**49**, Proposition 1.3].) Equip V with the supremum norm defined by $e_1, \dots, e_n$; then $|D^s|_V \geq \max_{i,j} |(D_s)_{i,j}|$. This plus Lemma 6.2.4 imply that the left-hand side of (6.2.5.1) is greater than or equal to the the right-hand side.

Conversely, for any $x \in V$, if we write $x = x_1 e_1 + \cdots + x_n e_n$ with $x_1, \dots, x_n \in F$ then

$$D^s(x) = \sum_{i=1}^{n} \sum_{j=0}^{s} \binom{s}{j} d^j(x_i) D^{s-j}(e_i),$$

so

$$|D^s|_V^{1/s} \leq \max_{0\leq j\leq s} \left\{ |d^j|_F^{1/s} |D_{s-j}|^{1/s} \right\}. \tag{6.2.5.2}$$

Given $\epsilon > 0$, apply Lemma 6.1.8 to choose $c = c(\epsilon)$ such that, for all $s \geq 0$,

$$\begin{aligned} |d^s|_F &\leq c \left(|d|_{\mathrm{sp},F} + \epsilon\right)^s, \\ |D_s|_V &\leq c \left(\limsup_{s\to\infty} |D_s|^{1/s} + \epsilon\right)^s. \end{aligned}$$

Then (6.2.5.2) implies that

$$|D^s|_V^{1/s} \leq c^{2/s} \max \left\{ |d|_{\mathrm{sp},F} + \epsilon,\ \limsup_{s\to\infty} |D_s|^{1/s} + \epsilon \right\}.$$

As in the previous proof, the factor $c^{2/s}$ tends to 1 as $s \to \infty$. From this it follows that the right-hand side of (6.2.5.1) is greater than or equal to the left-hand side minus ϵ; since $\epsilon > 0$ was arbitrary, we get the same inequality with $\epsilon = 0$. $\square$

Remark 6.2.6. In Lemma 6.2.5, if $|D|_{\mathrm{sp},V} > |d|_{\mathrm{sp},F}$ then the limit superior can be replaced by a limit; see the exercises.

Using Lemma 6.2.5, we may infer the following base-change property for the spectral radius. We will need to refine this result later; see Corollary 6.5.5 and Proposition 10.6.6 below.

Corollary 6.2.7. *Let $F \to F'$ be an isometric embedding of complete nonarchimedean differential fields. (In particular, the differential on F' must restrict to the differential on F.) Then, for any nonzero finite differential module V over F,*

$$|D|_{\mathrm{sp},V'} = \max\{|d|_{\mathrm{sp},F'}, |D|_{\mathrm{sp},V}\}.$$

Consequently, if $|d|_{\mathrm{sp},F'} = |d|_{\mathrm{sp},F}$ then $|D|_{\mathrm{sp},V'} = |D|_{\mathrm{sp},V}$ by Lemma 6.2.4.

Proof. Choose a basis of V, and use Lemma 6.2.5 to compute both $|D|_{\mathrm{sp},V}$ and $|D|_{\mathrm{sp},V'}$ in terms of this basis. This yields the desired formula. □

Here is how the spectral radius behaves with respect to basic operations on the category of differential modules.

Lemma 6.2.8. *Let F be a complete nonarchimedean differential field.*

(a) *For a short exact sequence $0 \to V_1 \to V \to V_2 \to 0$ of nonzero finite differential modules over F,*

$$|D|_{\mathrm{sp},V} = \max\{|D|_{\mathrm{sp},V_1}, |D|_{\mathrm{sp},V_2}\}.$$

(b) *For V a nonzero finite differential module over F,*

$$|D|_{\mathrm{sp},V^\vee} = |D|_{\mathrm{sp},V}.$$

(c) *For V_1, V_2 nonzero finite differential modules over F,*

$$|D|_{\mathrm{sp},V_1\otimes V_2} \leq \max\{|D|_{\mathrm{sp},V_1}, |D|_{\mathrm{sp},V_2}\},$$

with equality when $|D|_{\mathrm{sp},V_1} \neq |D|_{\mathrm{sp},V_2}$.

Proof. To prove (a), choose a splitting $V = V_1 \oplus V_2$ of the short exact sequence in the category of vector spaces over F. The action of D on V is then given by $D(v_1, v_2) = (D(v_1) + f(v_2), D(v_2))$ for some F-linear map $f : V_2 \to V_1$ (as in the proof of Lemma 5.3.3). From this, the rest of the proof of (a) is an exercise in the spirit of the proofs of Lemmas 6.2.4 and 6.2.5.

We next recall from Definition 5.3.2 that for any finite differential modules V_1, V_2 over F, the action of D on $V = \mathrm{Hom}_F(V_1, V_2)$ is given by

$$D(f)(v_1) = D(f(v_1)) - f(D(v_1)).$$

The Leibniz rule in this setting is that, for any nonnegative integer s,

$$D^s(f)(v_1) = \sum_{i=0}^{s} (-1)^i \binom{i}{s} D^i \left(f(D^{s-i}(v_1)) \right).$$

We may again (as an exercise) deduce that $|D|_{\mathrm{sp},V} \leq \max\{|D|_{\mathrm{sp},V_1}, |D|_{\mathrm{sp},V_2}\}$. This implies first (b) (since we get $|D|_{\mathrm{sp},V^\vee} \leq |D|_{\mathrm{sp},V}$ by Lemma 6.2.4 and similarly for the opposite inequality) and then the first assertion of (c) (since $V_1 \otimes_F V_2 = \mathrm{Hom}_F(V_1^\vee, V_2)$).

It remains to prove the second assertion of (c). Suppose that $|D|_{\mathrm{sp},V_1} > |D|_{\mathrm{sp},V_2}$. Then by (b) and the first assertion of (c),

$$\begin{aligned} |D|_{\mathrm{sp},V_1} &= \max\{|D|_{\mathrm{sp},V_1}, |D|_{\mathrm{sp},V_2}\} \\ &\geq \max\{|D|_{\mathrm{sp},V_1 \otimes V_2}, |D|_{\mathrm{sp},V_2^\vee}\} \\ &\geq |D|_{\mathrm{sp},V_1 \otimes V_2 \otimes V_2^\vee}. \end{aligned}$$

Moreover, $V_2 \otimes V_2^\vee$ contains a trivial submodule (the trace), so $V_1 \otimes V_2 \otimes V_2^\vee$ contains a copy of V_1. Hence, by (a), $|D|_{\mathrm{sp},V_1 \otimes V_2 \otimes V_2^\vee} \geq |D|_{\mathrm{sp},V_1}$. We thus obtain a chain of inequalities leading to $|D|_{\mathrm{sp},V_1} \geq |D|_{\mathrm{sp},V_1}$; this forces the intermediate equality $|D|_{\mathrm{sp},V_1} = \max\{|D|_{\mathrm{sp},V_1 \otimes V_2}, |D|_{\mathrm{sp},V_2^\vee}\}$. Since $|D|_{\mathrm{sp},V_1} \neq |D|_{\mathrm{sp},V_2} = |D|_{\mathrm{sp},V_2^\vee}$ we have $|D|_{\mathrm{sp},V_1} = |D|_{\mathrm{sp},V_1 \otimes V_2}$, as desired. □

Corollary 6.2.9. *If V_1, V_2 are irreducible finite differential modules over a complete nonarchimedean differential field and if $|D|_{\mathrm{sp},V_1} \neq |D|_{\mathrm{sp},V_2}$ then every irreducible submodule W of $V_1 \otimes V_2$ satisfies $|D|_{\mathrm{sp},W} = \max\{|D|_{\mathrm{sp},V_1}, |D|_{\mathrm{sp},V_2}\}$.*

Proof. Suppose the contrary; we may assume that $|D|_{\mathrm{sp},V_1} > |D|_{\mathrm{sp},V_2}$. The inclusion $W \hookrightarrow V_1 \otimes V_2$ corresponds to a nonzero horizontal section of $W^\vee \otimes V_1 \otimes V_2 \cong (W \otimes V_2^\vee)^\vee \otimes V_1$, which in turn corresponds to a nonzero map $W \otimes V_2^\vee \to V_1$. Since V_1 is irreducible, the map has image V_1; that is, $W \otimes V_2^\vee$ has a quotient isomorphic to V_1.

However, we can contradict this using Lemma 6.2.8. Namely,

$$|D|_{\mathrm{sp},W \otimes V_2^\vee} \leq \max\{|D|_{\mathrm{sp},W}, |D|_{\mathrm{sp},V_2}\} < |D|_{\mathrm{sp},V_1},$$

so each nonzero subquotient of $W \otimes V_2^\vee$ has spectral radius strictly less than $|D|_{\mathrm{sp},V_1}$. □

Remark 6.2.10. By contrast, when $|D|_{\mathrm{sp},V_1} = |D|_{\mathrm{sp},V_2}$ it is entirely possible for an irreducible submodule W of $V_1 \otimes V_2$ to satisfy $|D|_{\mathrm{sp},W} \neq \max\{|D|_{\mathrm{sp},V_1}, |D|_{\mathrm{sp},V_2}\}$. For instance, take V_1 with $|D|_{\mathrm{sp},V_1} > |d|_{\mathrm{sp},F}$, put $V_2 = V_1^\vee$, and let W be the trace component of $V_1 \otimes V_2$.

Remark 6.2.11. It would be convenient to have the analogue of Corollary 6.2.9 also for irreducible subquotients of $V_1 \otimes V_2$. We will prove something slightly weaker later (Corollary 6.6.3).

We now refine the notion of the spectral radius to give a working notion of the spectrum of a differential operator.

Definition 6.2.12. For V a finite differential module over a nonarchimedean differential field F, let $V_1, \dots, V_l$ be the Jordan–Hölder constituents of V (listed with multiplicity). Define the *full spectrum* of V to be the multiset consisting of $|D|_{\mathrm{sp}, V_i}$ with multiplicity $\dim_F V_i$, for $i = 1, \dots, l$. We say V is *pure* if its full spectrum consists of a single element (with multiplicity). We say V is *refined* if V is pure and $|D|_{\mathrm{sp}, V^\vee \otimes V} < |D|_{\mathrm{sp}, V}$. If V and W are two refined finite differential modules over F, we write $V \sim W$ if

$$|D|_{\mathrm{sp}, V} = |D|_{\mathrm{sp}, W} > |D|_{\mathrm{sp}, V^\vee \otimes W}.$$

We will see that this is an equivalence relation (Lemma 6.2.14).

Remark 6.2.13. It may be helpful to keep in mind how the above notions behave when the derivation on F is zero. In this case, a finite differential module over F is simply a finite-dimensional vector space V equipped with a linear transformation. The full spectrum consists of the norms of the eigenvalues in F^{alg} of this linear transformation. The module V is pure if these eigenvalues all have the same norm. The module V is refined if it is pure and, moreover, the ratio of any two eigenvalues is congruent to 1 modulo $\mathfrak{m}_F$. In particular, we can always decompose a finite differential module into pure summands, but not necessarily into refined summands; for instance, one cannot separate a linear transformation over $\mathbb{Q}_p$ with characteristic polynomial $T^2 - p$.

However, we can achieve a refined decomposition after a finite *tamely ramified* extension. We may see this from Definition 3.3.3 as follows. Let E be a finite Galois extension of F containing the eigenvalues of the linear transformation. Then, for any g in the wild inertia subgroup $W_{E/F}$ of $G_{E/F}$ and any eigenvalue λ, we have $|g(\lambda)/\lambda - 1| < 1$. We thus obtain a refined decomposition over the fixed field of $W_{E/F}$.

We can also describe a suitable tamely refined extension F' of F explicitly, by ensuring that it satisfies the following conditions in terms of the characteristic polynomial Q of the linear transformation.

(a) For each root λ of Q we have

$$|\lambda| \in \begin{cases} |(F')^\times| & p = 0, \\ \bigcup_{h \geq 0} |(F')^\times|^{1/p^h} & p > 0, \end{cases}$$

where p is the characteristic of κ_F.

(b) For any two roots λ, μ of Q, we have

$$\overline{\lambda/\mu} \in \kappa_{F'}.$$

Lemma 6.2.14. *Let F be a nonarchimedean differential field. Then the relation $\sim$ on refined finite differential modules over F is an equivalence relation.*

Proof. The reflexivity of $\sim$ holds because we are considering only refined modules. The symmetry of $\sim$ holds because $|D|_{\mathrm{sp}, V^\vee \otimes W} = |D|_{\mathrm{sp}, W^\vee \otimes V}$ by Lemma 6.2.8(b). To check the transitivity, suppose that $V \sim W$ and $W \sim X$. Since $V^\vee \otimes X$ occurs as a direct summand of $(V^\vee \otimes W) \otimes (W^\vee \otimes X)$, by Lemma 6.2.8(a) and (c) we have

$$\begin{aligned} |D|_{\mathrm{sp}, V^\vee \otimes X} &\leq |D|_{\mathrm{sp}, V^\vee \otimes W \otimes W^\vee \otimes X} \\ &\leq \max\{|D|_{\mathrm{sp}, V^\vee \otimes W}, |D|_{\mathrm{sp}, W^\vee \otimes X}\} \\ &< |D|_{\mathrm{sp}, V} = |D|_{\mathrm{sp}, W} = |D|_{\mathrm{sp}, X}. \end{aligned}$$

Hence $V \sim X$. □

Lemma 6.2.15. *Let F be a complete nonarchimedean differential field, and let F' be a finite tamely ramified extension of F. Then d extends uniquely to F', and $|d|_{F'} = |d|_F$.*

Proof. Suppose first that F' is unramified over F. For $\alpha \in F'$, let $P(T) = \sum_i P_i T^i \in F[T]$ be the minimal polynomial of α. Then the unique extension of d to F' is characterized by

$$0 = d(\alpha) P'(\alpha) + \sum_i d(P_i) \alpha^i. \tag{6.2.15.1}$$

If $\alpha \in \mathfrak{o}_{F'}^\times$ then the unramified condition implies that $|P'(\alpha)| = 1$, so that

$$|d(\alpha)| \leq \max_i \{|d(P_i)||\alpha|^i\} \leq |d|_F.$$

In general, we can write any $\alpha \in F'$ as $\beta\gamma$ with $\beta \in \mathfrak{o}_{F'}^\times$ and $\gamma \in F$. We then have $d(\alpha) = \gamma d(\beta) + \beta d(\gamma)$, so that

$$|d(\alpha)| \leq \max\{|\gamma||d(\beta)|, |\beta||d(\gamma)|\} \leq |d|_F |\beta||\gamma| = |d|_F |\alpha|.$$

This proves the claim.

We next verify the claim in the case $F' = F(t^{1/m})$ for some $t \in F$ and some integer m, not divisible by the characteristic p of κ_F, such that $|t|^{1/e} \notin |F^\times|$ for any divisor $e > 1$ of m. Again, the unique extension of d to F' is given by (6.2.15.1). Also, each element of F' has a unique expression of the form $\sum_{i=0}^{m-1} c_i t^{i/m}$ with $c_i \in F$, and the norm on F' is given by

$$\left|\sum_{i=0}^{m-1} c_i t^{i/m}\right| = \max_i\{|c_i||t|^{i/m}\}.$$

It thus suffices to check that $|d(c_i t^{i/m})| \leq |d|_F |c_i t^{i/m}|$ for $i = 0, \dots, m-1$. By the Leibniz rule, this reduces to checking $|d(t^{i/m})| \leq |d|_F |t^{i/m}|$. But

$$|d(t^{i/m})| = |t^{i/m-1} d(t)| \leq |t^{i/m}||t|^{-1}|d|_F|t| = |d|_F|t^{i/m}|,$$

so the claim follows.

To treat the general case, we may choose an integer m not divisible by p and annihilating $I_{E/F}$ and some $t_1, \dots, t_h \in F^\times$ such that $|t_1|, \dots, |t_h|$ generate $|F^\times|/|F^\times|^m$. Put $F'' = F(t_1^{1/m}, \dots, t_h^{1/m})$. By the two previous paragraphs, we have $|d|_{F''} = |d|_F$. (More precisely, when adjoining $t_i^{1/m}$ we split the reasoning into two cases: first, for e the largest divisor of m such that $|t_i|^{1/e}$ is already present, adjoining $t_i^{1/e}$ gives an unramified extension, as in the proof of Proposition 3.3.6, so the operator norm of d does not change. After that, adjoining $t_i^{1/m}$ proceeds as in the second paragraph.) By Proposition 3.3.6, F' is contained in an unramified extension of F'' so the claim follows by applying the first paragraph again. □

6.3 A coordinate-free approach

We note in passing that our definition of the spectral radius is rather sensitive to the choice of the derivation d. (See the exercises for Chapter 9 for an explicit example.) A coordinate-free approach is suggested by the work of Baldassarri and Di Vizio.

Proposition 6.3.1. *Let F be a nonarchimedean differential field. Let $F\{T\}^{(s)}$ be the set of twisted polynomials, of degree at most s, equipped with the seminorm $|P| = |P(d)|_F$ compatible with F (that is, consider $P(d)$ as an operator on F). Let V be a nonzero finite differential module over F, and fix a norm on V compatible with F. Let $L(V)$ be the space of bounded endomorphisms of the additive group of V, equipped with the operator norm. Let $D_s : F\{T\}^{(s)} \to L(V)$ be the map $P \mapsto P(D)$. Then*

$$|D|_{\mathrm{sp},V} \leq |d|_{\mathrm{sp},F} \liminf_{s\to\infty} |D_s|^{1/s}. \tag{6.3.1.1}$$

Conversely, suppose that, for any nonnegative integer s and any $c_0, \dots, c_s \in F$, we have

$$|c_0 + c_1 d + \cdots + c_s d^s|_F = \max\{|c_0|_F, |c_1 d|_F, \ldots, |c_s d^s|_F\}. \qquad (6.3.1.2)$$

Then equality holds in (6.3.1.1)*, and if* $|d|_{\mathrm{sp},F} > 0$ *then the limit inferior on the right is also a limit.*

The condition (6.3.1.2) is satisfied in the situation of greatest interest to us; see Proposition 9.10.2.

Proof. By taking $T^s \in F\{T\}^{(s)}$ we obtain the inequality $|D^s|_V \leq |d^s|_F |D_s|$. Taking sth roots of both sides and then taking limits as $s \to \infty$ yields (6.3.1.1).

Conversely, suppose that (6.3.1.2) holds. Given $\epsilon > 0$, apply Lemma 6.1.8 to choose $c > 0$ such that, for all $s \geq 0$,

$$|D^s|_V \leq c(|D|_{\mathrm{sp},V} + \epsilon)^s.$$

Given a nonnegative integer s such that $|D_s| > \epsilon$, choose $P = \sum_{i=0}^s P_i T^i \in F\{T\}^{(s)}$ nonzero and such that $|P(D)|_V \geq |P(d)|_F(|D_s| - \epsilon)$. (This could only fail to hold if we had $|P(d)|_F = 0$ for all $P \in F\{T\}^{(s)}$, but the presence of constant polynomials eliminates this possibility.) Then

$$\begin{aligned}
\max_{i \leq s}\{|P_i||d|^i_{\mathrm{sp},F}(|D_s| - \epsilon)\} &\leq \max_{i \leq s}\{|P_i d^i|_F(|D_s| - \epsilon)\} \\
&= |P(d)|_F(|D_s| - \epsilon) \\
&\leq |P(D)|_V \\
&\leq \max_{i \leq s}\{|P_i D^i|_V\} \\
&\leq \max_{i \leq s}\{|P_i| c(|D|_{\mathrm{sp},V} + \epsilon)^i\}.
\end{aligned}$$

For the index i which maximizes the right-hand side, we have

$$|d|^i_{\mathrm{sp},F}(|D_s| - \epsilon) \leq c(|D|_{\mathrm{sp},V} + \epsilon)^i.$$

Since $|d|_{\mathrm{sp},F} < |D|_{\mathrm{sp},V} + \epsilon$ by Lemma 6.2.4, we can increase the exponent from i to s on both sides while maintaining the inequality; we may then also include values of s for which $|D_s| < \epsilon$. Taking sth roots and then taking the limit as $s \to \infty$ yields

$$|d|_{\mathrm{sp},F} \limsup_{s \to \infty}(|D_s| - \epsilon)^{1/s} \leq |D|_{\mathrm{sp},V} + \epsilon.$$

Since this holds for any $\epsilon > 0$, we obtain the inequality

$$|D|_{\mathrm{sp},V} \geq |d|_{\mathrm{sp},F} \limsup_{s \to \infty} |D_s|^{1/s}.$$

This forces equality in (6.3.1.1) and forces the limit inferior therein to be a limit provided that $|d|_{\mathrm{sp},F} > 0$. $\square$

6.4 Newton polygons for twisted polynomials

Twisted polynomials admit a partial analogue of the theory of Newton polygons; we will use these polygons in the next section to compute spectral radii of differential operators.

Definition 6.4.1. Let R be a nonarchimedean differential field domain. For $\rho \geq |d|_R$, define the *ρ-Gauss norm* on the twisted polynomial ring $R\{T\}$ by

$$\left| \sum_i P_i T^i \right| = \max_i \{|P_i|\rho^i\};$$

thus it is the same as the ρ-Gauss norm of the untwisted polynomial $\sum_i P_i T^i \in R[T]$. For $r \leq -\log|d|_R$ we obtain a corresponding *r-Gauss valuation* $v_r(P) = -\log|P|_{e^{-r}}$.

Lemma 6.4.2. *For $\rho \geq |d|_R$ the ρ-Gauss norm is multiplicative. Moreover, any polynomial and its formal adjoint have the same ρ-Gauss norm.*

Proof. It suffices to check the statement for $\rho > |d|_R$, as the boundary case may be inferred from the continuity of the map $\rho \mapsto |P|_\rho$ for fixed P. The key observation (and the source of the restriction on ρ) is that, for $P, Q \in R\{T\}$ and $\rho > |d|_R$,

$$|PQ - QP|_\rho \leq \rho^{-1}|d|_R|P|_\rho|Q|_\rho < |P|_\rho|Q|_\rho.$$

(The first inequality is evident in the case $P = T$ and $Q \in R$; the reduction of the general case to this one is an exercise.) This allows us to deduce multiplicativity on $R\{T\}$ from multiplicativity on $R[T]$ (Proposition 2.1.2). The claim about the adjoint follows by a similar argument. □

Definition 6.4.3. We define the *Newton polygon* of $P = \sum_i P_i T^i \in R\{T\}$ by taking the Newton polygon of the corresponding untwisted polynomial $\sum P_i T^i \in R[T]$ and then omitting all slopes greater than or equal to $-\log|d|_R$. We define similarly the multiplicity for slopes less than $-\log|d|_R$. By Lemma 6.4.2 and an argument as in Remark 2.1.7, the multiplicity of a slope $r < -\log|d|_R$ in a product PQ is equal to the sum of the multiplicities of r as a slope of P and as a slope of Q. However, this fails for $r = -\log|d|_R$ because we are only controlling the *left* endpoint of the segment of the Newton polygon with this slope (see the exercises); hence we will omit $-\log|d|_R$ and all larger slopes from the Newton polygon.

By another application of the master factorization theorem (Theorem 2.2.2), we obtain the following.

Theorem 6.4.4. *Let R be a complete nonarchimedean differential domain. Suppose that $S \in R\{T\}$, $r < -\log |d|_R$, and $m \in \mathbb{Z}_{\geq 0}$ satisfy*

$$v_r(S - T^m) > v_r(T^m).$$

Then there exists a unique factorization $S = PQ$ satisfying the following conditions.

(a) *The polynomial $P \in R\{T\}$ has degree $\deg(S) - m$, and its slopes are all less than r.*

(b) *The polynomial $Q \in R\{T\}$ is monic of degree m, and its slopes are all greater than r.*

(c) *We have $v_r(P - 1) > 0$ and $v_r(Q - T^m) > v_r(T^m)$.*

Moreover, for this factorization,

$$\min\{v_r(P - 1), v_r(Q - T^m) - v_r(T^m)\} \geq v_r(S - T^m) - v_r(T^m).$$

Similarly, we can make a factorization $S = QP$ having the same properties (but the factors themselves may differ).

Proof. Use the same procedure as in Theorem 2.2.1. $\square$

Corollary 6.4.5. *Let F be a complete nonarchimedean differential field. If $P \in F\{T\}$ is irreducible, then either it has no slopes or all its slopes are equal to some value less than $-\log |d|_F$.*

Remark 6.4.6. Instead of defining the Newton polygon as above and then truncating, one may prefer to declare all the missing slopes to be equal to $-\log |d|_R$. One can achieve this by modifying the set whose lower convex hull is used to define the Newton polygon, in order to build in the right truncation behavior. The correct modified set is

$$\bigcup_{i=0}^{n} \left\{(x, y) \in \mathbb{R}^2 : x \geq -i,\, y \geq v(P_i) - (x + i) \log |d|_F\right\},$$

provided that d is nonzero. (If $d = 0$ then $-(x+i) \log |d|_F$ must be interpreted to mean 0 if $x + i = 0$ and $+\infty$ if $x + i > 0$.)

6.5 Twisted polynomials and spectral radii

One can use twisted polynomials over nonarchimedean differential fields to detect part of the full spectrum of a normed differential module.

Definition 6.5.1. For V a finite differential module over a nonarchimedean differential field F, define the *visible spectrum* of V to be the submultiset of the full spectrum of V consisting of those values greater than $|d|_F$.

Remark 6.5.2. In the application to regular singularities (Chapter 7), we will consider the case where $|d|_F = |d|_{\mathrm{sp},F}$. In such a case, there is no real loss in restricting to the visible spectrum: the only missing norm is $|d|_F$ itself, and one can infer its multiplicity from the dimension of the module. However, in the applications to p-adic differential equations in Part III we will have $|d|_F > |d|_{\mathrm{sp},F}$, so restriction to the visible spectrum will cause real problems; these will have to be remedied using pullback and pushforward along a Frobenius map.

The key theorem relating spectral radii to Newton polygons is the following.

Theorem 6.5.3 (Christol–Dwork). *Let F be a complete nonarchimedean differential field. For nonconstant $P \in F\{T\}$, put $V = F\{T\}/F\{T\}P$. Let r be the least slope of the Newton polygon of P or $-\log |d|_F$ if no such slope exists. Then*

$$\max\{|d|_F, |D|_{\mathrm{sp},V}\} = e^{-r}.$$

Proof. Let $r_1 \leq \cdots \leq r_k$ be the slopes of P counted with multiplicity, and define $r_{k+1} = \cdots = r_n = -\log |d|_F$. Equip V with the norm

$$\left|\sum_{i=0}^{n-1} a_i T^i\right|_V = \max_i\{|a_i| e^{-r_{n-1}-\cdots-r_{n-i}}\}.$$

As in the proof of Proposition 4.3.10, we then have $|D|_V = e^{-r_1}$ and so $|D|_{\mathrm{sp},V} \leq e^{-r_1}$.

To finish, we must check that if $r_1 < -\log |d|_F$ then $|D|_{\mathrm{sp},V} = e^{-r_1}$. Let δ be the operation defined by

$$\delta\left(\sum_{i=0}^{n-1} a_i T^i\right) = \sum_{i=0}^{n-1} d(a_i) T^i;$$

then $|\delta|_V = |d|_F$, $D-\delta$ is F-linear, and $|D-\delta|_V = e^{-r_1}$. Then, for all positive integers s,

$$|(D-\delta)^s|_V = e^{-r_1 s}, \qquad |D^s - (D-\delta)^s|_V \leq e^{-r_1(s-1)}|d|_F < e^{-r_1 s},$$

so $|D^s|_V = e^{-r_1 s}$ and $|D|_{\mathrm{sp},V} = e^{-r_1}$ as desired. $\square$

Corollary 6.5.4. *Let F be a complete nonarchimedean differential field. For any $P \in F\{T\}$, the visible spectrum of the differential module $F\{T\}/F\{T\}P$ consists of e^{-r}, where r runs over the slope multiset of the Newton polygon of P.*

Proof. Write down a maximal factorization of P; it corresponds to a maximal filtration of $F\{T\}/F\{T\}P$. By Corollary 6.4.5, each factor in the factorization has only a single slope, so Theorem 6.5.3 gives us what we want. $\square$

We now obtain a partial refinement of Corollary 6.5.4. For a further refinement, see Proposition 10.6.6.

Corollary 6.5.5. *Let $F \to F'$ be an isometric embedding of complete nonarchimedean differential fields. For any finite differential module V over F, the visible spectrum of $V \otimes_F F'$ is the submultiset of the visible spectrum of V consisting of those values greater than $|d|_{F'}$.*

Proof. We may reduce to the case where V is irreducible (but $V \otimes_F F'$ need not be). In this case any nonzero element of V is a cyclic vector, so we may apply Corollary 6.5.4 to deduce the claim. $\square$

Using Theorem 6.5.3, we can give a differential version of Proposition 4.4.6.

Proposition 6.5.6. *Let F be a complete nonarchimedean differential field with $|d|_F \leq 1$. Let V be a finite differential module of rank $n > 0$ over F with $|D|_{\mathrm{sp},V} \leq 1$. Let $|\cdot|_V$ be the supremum norm on V defined by a basis $e_1, \dots, e_n$, and suppose that $|D|_V = c \geq 1$. Then there exists a basis of V defining a second supremum norm $|\cdot|'_V$, for which $|D|'_V \leq 1$ and $|x|'_V \leq |x|_V \leq c^{n-1}|x|'_V$ for all $x \in V$.*

Proof. We proceed as in Proposition 4.4.6. Let M be the smallest $\mathfrak{o}_F$-submodule of V containing $e_1, \dots, e_n$ and stable under D. For each i, if $j = j(i)$ is the least integer such that $e_i, D(e_i), \dots, D^j(e_i)$ are linearly dependent then we have $D^j(e_i) = \sum_{h=0}^{j-1} c_h D^h(e_i)$ for some $c_h \in F$. Since $|D|_{\mathrm{sp},V} \leq 1$ and $|d|_F \leq 1$, Theorem 6.5.3 implies that $|c_h| \leq 1$ for each h. Hence M is finitely generated, and thus free, over $\mathfrak{o}_F$.

Let $|\cdot|'_V$ be the supremum norm on V defined by a basis of M. Then $|x|'_V \leq |x|_V$ because $e_1, \dots, e_n \in M$. Conversely, for $i = 1, \dots, n$ and $h = 0, \dots, j(i) - 1$, we have $|D^h(e_i)|_V \leq |D|_V^h \leq c^h \leq c^{n-1}$. Since the $D^h(e_i)$ generate M, this implies that $|x|_V \leq c^{n-1}|x|'_V$ for all $x \in V$. $\square$

6.6 The visible decomposition theorem

Using twisted polynomials, we can split V into components corresponding to the elements of the visible spectrum (see Definition 6.5.1).

Theorem 6.6.1 (Visible decomposition theorem). *Let F be a complete nonarchimedean differential field of characteristic 0, and let V be a finite differential module over F. Then there exists a unique decomposition*

$$V = V_0 \oplus \bigoplus_{s > |d|_F} V_s$$

of differential modules, such that every subquotient of V_s has spectral radius s and every subquotient of V_0 has spectral radius at most $|d|_F$.

Proof. If the derivation on F is zero then we are just considering a vector space equipped with an endomorphism, and the claim becomes an elementary exercise in linear algebra (as noted in Remark 6.2.13). Thus we may assume hereafter that the derivation is nonzero.

We will induct on $\dim(V)$. By Theorem 5.4.2 and our hypothesis that $d \neq 0$, there exists a cyclic vector for V. We thus obtain an isomorphism $V \cong F\{T\}/F\{T\}P$ for some $P \in F\{T\}$. If the Newton polygon of P is empty, we may put $V = V_0$ and be done. Otherwise, let $r < -\log |d|_F$ be the least slope. By applying Theorem 6.4.4 once to P, we obtain a short exact sequence $0 \to V_1 \to V \to V_2 \to 0$ in which (by Theorem 6.5.3) every subquotient of V_1 has spectral radius e^{-r}, and every subquotient of V_2 has spectral radius less than e^{-r}. Applying Theorem 6.4.4 again to P but with the factors in the opposite order, we obtain a short exact sequence $0 \to V_2' \to V \to V_1' \to 0$ in which every subquotient of V_1' has spectral radius e^{-r} and every subquotient of V_2' has spectral radius less than e^{-r}. This yields $V_1 \cap V_2' = 0$, so $V_1 \oplus V_2'$ injects into V. Moreover, $\dim V_1 = \dim V_1'$ and $\dim V_2 = \dim V_2'$ because P and its formal adjoint have the same Newton polygon (Lemma 6.4.2). By a dimension count $V_1 \oplus V_2'$ must equal V. Applying the induction hypothesis to V_2' gives the claim. □

Corollary 6.6.2. *Let F be a complete nonarchimedean differential field of characteristic 0. Let V be a finite differential module over F such that every subquotient of V has spectral radius greater than $|d|_F$. Then $H^0(V) = H^1(V) = 0$.*

Proof. The claim about H^0 is clear: a nonzero element of $H^0(V)$ would generate a differential submodule of V which would be trivial and thus would have spectral radius $|d|_{\mathrm{sp},F} \leq |d|_F$. In the case of H^1, let $0 \to V \to W \to F \to 0$ be a short exact sequence of differential modules. Decompose $W = W_0 \oplus W_1$ according to Theorem 6.6.1, so that every subquotient of W_0 has spectral radius at most $|d|_F$ and every subquotient of W_1 has spectral radius greater than $|d|_F$. The map $V \to W_0$ must vanish (its image is a subquotient of both V and W_0), so $V \subseteq W_1$. But $W_1 \neq W$, as otherwise W could not surject onto a trivial module, so for dimensional reasons we must have $V = W_1$. Hence the sequence splits, proving that $H^1(V) = 0$ by Lemma 5.3.3. □

In this setting we can obtain a refinement of Corollary 6.2.9 in which we allow subquotients rather than just submodules.

Corollary 6.6.3. *Let F be a complete nonarchimedean differential field of characteristic 0. If V_1, V_2 are finite irreducible differential modules over F, $|D|_{\mathrm{sp},V_1} > |d|_F$, and $|D|_{\mathrm{sp},V_1} > |D|_{\mathrm{sp},V_2}$, then every irreducible subquotient W of $V_1 \otimes V_2$ satisfies $|D|_{\mathrm{sp},W} = |D|_{\mathrm{sp},V_1}$.*

Proof. Decompose $V_1 \otimes V_2$ as $V_0 \oplus \bigoplus_{s>|d|_F} V_s$ according to Theorem 6.6.1; we have $V_s = 0$ whenever $s > |D|_{\mathrm{sp},V_1}$. If either V_0 were nonzero or some V_s with $s < |D|_{\mathrm{sp},V_1}$ were nonzero, then $V_1 \otimes V_2$ would have an irreducible submodule of spectral radius less than $|D|_{\mathrm{sp},V_1}$, in violation of Corollary 6.2.9. □

These results are quite sufficient for applications to the study of the singularities of complex meromorphic differential equations, at which we hint in Chapter 7. However, in the p-adic situation we have to decompose V_0 further; we will do this using Frobenius antecedents in Chapter 10.

6.7 Matrices and the visible spectrum

The proof of Theorem 6.5.3 relied on the fact that one can detect the spectral radius of a differential module admitting a cyclic vector by using the characteristic polynomial of the matrix of action of D (see Definition 5.2.1) on the cyclic basis. For some applications we need to extend this to certain bases not necessarily generated by cyclic vectors; for this, the relationship between singular values and eigenvalues considered in Chapter 4 will be crucial.

Lemma 6.7.1. *Let R be a complete nonarchimedean differential domain. Let N be a 2×2 block matrix over R with the following properties.*

(a) *The matrix N_{11} has an inverse A over R.*

(b) *We have $|A| \max\{|d|_F, |N_{12}|, |N_{21}|, |N_{22}|\} < 1$.*

Then there exists a block upper triangular unipotent matrix U over R, such that $|U_{12}| \leq |A| \max\{|N_{12}|, |N_{21}|, |N_{22}|\}$ and $U^{-1}NU + U^{-1}d(U)$ is block lower triangular.

Proof. Put

$$\delta = |A| \max\{|N_{12}|, |N_{21}|, |N_{22}|\} < 1, \qquad \epsilon = |A||N_{12}| \leq \delta.$$

Let X be the block upper triangular nilpotent matrix with $X_{12} = AN_{12}$, put $U = I - X$, and put

$$N' = U^{-1}NU + U^{-1}d(U).$$

Since $U^{-1} = I + X$, we have $N' = N + XN - NX - XNX - d(X)$. In block form,

$$N' = \begin{pmatrix} N_{11} + X_{12}N_{21} & N_{12} - N_{11}X_{12} + X_{12}N_{22} - X_{12}N_{21}X_{12} - d(X_{12}) \\ N_{21} & N_{22} - N_{21}X_{12} \end{pmatrix}.$$

We now claim that

$$\begin{aligned} |N'_{12}| &\leq \epsilon \max\{\delta, |d|_F|A|\}|A|^{-1}, \\ |N'_{21}| &\leq \delta|A|^{-1}, \\ |N'_{22}| &\leq \delta|A|^{-1}. \end{aligned}$$

The second and third lines hold because

$$|(U^{-1}NU - N)_{2j}| = |(-NX)_{2j}| \leq \epsilon\delta|A|^{-1} \qquad (j = 1, 2).$$

The first line holds because $N_{12} - N_{11}X_{12} = 0$, so we can write

$$N'_{12} = X_{12}N_{22} - X_{12}N_{21}X_{12} - d(X_{12}),$$

in which the first two terms have norm at most $\epsilon\delta|A|^{-1}$ and the third has norm at most $|d|_F\epsilon$.

To analyze N'_{11}, we write it as $(I + X_{12}N_{21}A)N_{11}$. Because $|X_{12}N_{21}A| \leq |X_{12}|(|N_{21}||A|) < \epsilon < 1$ the first factor is invertible, and it and its inverse both have norm 1. Hence N'_{11} is invertible, $|N'_{11}| = |N_{11}|$, and $|(N'_{11})^{-1}| = |A|$.

Since $\max\{\delta, |d|_F|A|\} \leq \mu$ for some fixed $\mu < 1$, iterating the construction of N' from N yields a convergent sequence of conjugations whose limit has the desired property. □

We need a version of the argument used in the proof of Theorem 6.5.3 that is no longer restricted to cyclic vectors.

Lemma 6.7.2. *Let F be a complete nonarchimedean differential field. Let V be a nonzero finite differential module over F. Let $e_1, \ldots, e_n$ be a basis of V, and let N be the matrix of the action of D on $e_1, \ldots, e_n$. Suppose that $|N| = \sigma > |d|_F$ and $|N^{-1}| = \sigma^{-1}$. Then the full spectrum of V consists entirely of σ.*

Proof. As in the proof of Theorem 6.5.3, we find that for the supremum norm for $e_1, \ldots, e_n$ we have $|D^s(v)|_V = \sigma^s|v|_V$ for all nonnegative integers s and all $v \in V$. Consequently, for any nonzero differential submodule W of V, we have $|D|_{\mathrm{sp},W} = \sigma$. By Theorem 6.6.1, it follows that every irreducible subquotient of V also has spectral radius σ, as desired. □

By combining what we have so far, we can obtain a further refinement of Lemma 6.7.2 in which we can detect elements of the visible spectrum, which need not all be equal.

Lemma 6.7.3. *Let F be a complete nonarchimedean differential field. Let V be a finite differential module over F. Let $e_1, \dots, e_n$ be a basis of V, and let N be the matrix of action of D on $e_1, \dots, e_n$. Let $\sigma_1 \geq \cdots \geq \sigma_n$ be the singular values of N, and let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of N, arranged so that $|\lambda_1| \geq \cdots \geq |\lambda_n|$. Suppose that the following conditions hold for some $i = 1, \dots, n$ and some $\delta \geq |d|_F$.*

(a) *We have $\sigma_i > \delta$.*

(b) *Either $i = n$ or $\sigma_{i+1} \leq \delta$.*

(c) *We have $\sigma_j = |\lambda_j|$ for $j = 1, \dots, i$.*

Then the elements of the full spectrum that are greater than δ are precisely $\sigma_1, \dots, \sigma_i$.

Proof. We first check that conditions (a)–(c) are invariant under the change of basis

$$N \mapsto U^{-1}NU + U^{-1}d(U) = U^{-1}(N + d(U)U^{-1})U$$

for $U \in \mathrm{GL}_n(\mathfrak{o}_F)$. Note that, by Proposition 4.4.1, N and $N + d(U)U^{-1}$ have the same singular values greater than δ. Thanks to conditions (a)–(c) we may apply Theorem 4.4.2 to deduce that N and $N + d(U)U^{-1}$ also have the same eigenvalue norms for eigenvalues greater than δ. Since neither the singular values nor the eigenvalues are altered upon conjugating by U, we may draw the same conclusion about N and $U^{-1}NU + U^{-1}d(U)$. In particular, the truth of (a)–(c) is not affected by this change of basis.

If $\sigma_1 \leq |d|_F$ then we have nothing to check. If $\sigma_1 = \cdots = \sigma_n > |d|_F$ then Lemma 6.7.2 implies the claim. If neither of these cases applies, we may induct on n: choose i with $\sigma_1 = \cdots = \sigma_i > \sigma_{i+1}$, so that necessarily $\sigma_1 > |d|_F$. View N as a 2×2 block matrix with block sizes $i, n - i$. Apply Lemma 6.7.1 to obtain an upper triangular unipotent block matrix U over $\mathfrak{o}_F$ such that $N' = U^{-1}NU + U^{-1}d(U)$ is lower triangular. We may then reduce to checking the claim with N replaced by the two diagonal blocks of N'. □

To put everything together, we relax the condition on the singular values.

Theorem 6.7.4. *Let F be a complete nonarchimedean differential field. Let V be a finite differential module over F. Let $e_1, \dots, e_n$ be a basis of V, and let N be the matrix of action of D on $e_1, \dots, e_n$. Let $\sigma_1 \geq \cdots \geq \sigma_n$ be the singular values of N, and let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of*

N arranged so that $|\lambda_1| \geq \cdots \geq |\lambda_n|$. Define f_n as in Corollary 4.4.8 and put $\theta = f_n(\sigma_1, \dots, \sigma_n, |\lambda_1|, \dots, |\lambda_n|, |d|_F)$. Suppose that the following conditions hold for some $i = 1, \dots, n$ and some $\delta \geq |d|_F \theta$.

(a) *We have $|\lambda_i| > \delta$.*

(b) *Either $i = n$ or $|\lambda_{i+1}| \leq \delta$.*

Then the elements of the full spectrum greater than δ are precisely $|\lambda_1|, \dots, |\lambda_i|$.

Proof. By Corollary 6.5.5 we can replace F by a larger complete nonarchimedean differential field F', provided that $|d|_{F'} = |d|_F$. In particular we may take F' to be the completion of $F(t)$ for the ρ-Gauss norm for any $\rho > 0$, extending d so that $d(t) = 0$ (exercise). After enlarging F suitably in this manner, by Corollary 4.4.8 we can choose a matrix $U \in \mathrm{GL}_n(F)$ such that the following conditions hold.

(a) We have $|U^{-1}| \leq 1$ and $|U| \leq \theta$.

(b) The first i singular values of $U^{-1}NU$ are $|\lambda_1|, \dots, |\lambda_i|$.

(c) Either $i = n$ or the $(i+1)$th singular value of $U^{-1}NU$ is at most δ.

Since $|U^{-1}d(U)| \leq \theta |d|_F \leq \delta$, by Proposition 4.4.1 the new conditions (b) and (c) hold when $U^{-1}NU$ is replaced by $U^{-1}NU + U^{-1}d(U)$. We may thus apply Lemma 6.7.3 to obtain the desired result. $\square$

6.8 A refined visible decomposition theorem

We give a refinement of the visible decomposition theorem (Theorem 6.6.1). This can be used to obtain the Turrittin–Hukuhara–Levelt decomposition theorem; see Theorem 7.5.1. We first give an extension of Remark 6.2.13.

Lemma 6.8.1. *Let F be a complete nonarchimedean differential field of characteristic 0. Suppose that $P = \sum_i P_i T^i \in F\{T\}$ and $\tilde{P} = \sum_i \tilde{P}_i T^i \in F\{T\}$ are nonconstant twisted polynomials such that, for some $s < -\log |d|_F$, the slopes of P and $\tilde{P}$ are all equal to s. Put $V = F\{T\}/F\{T\}P$ and $W = F\{T\}/F\{T\}\tilde{P}$. Let $Q = \sum_i Q_i U^i \in F[T]$ and $\tilde{Q} = \sum_i \tilde{Q}_i U^i \in F[T]$ be the untwisted polynomials having the same coefficients as P and $\tilde{P}$, respectively. Then $|D|_{\mathrm{sp}, V^\vee \otimes W} < |D|_{\mathrm{sp}, V}$ if and only if for any root $\lambda \in F^{\mathrm{alg}}$ of Q and any root $\mu \in F^{\mathrm{alg}}$ of $\tilde{Q}$ we have $|\lambda/\mu - 1| < 1$.*

Proof. As in the proof of Theorem 6.7.4, we may enlarge F to reduce to the case where $|D|_{\mathrm{sp}, V} = |\eta|$ for some $\eta \in F^\times$. Equip each of V and W with the basis given by $1, \eta^{-1}T, \eta^{-2}T^2, \dots$, equip $V^\vee$ with the dual basis, and use these bases to define supremum norms. As in the proof of Theorem 6.5.3, we have $|D|_V = |D|_W = |\eta|$.

The matrix of action of D on the resulting basis of $V^\vee \otimes W$ has eigenvalues of the form $\mu - \lambda$, where λ is a root of Q and μ is a root of $\tilde{Q}$. By Theorem 6.7.4 we have $|D|_{\mathrm{sp}, V^\vee \otimes W} < |\eta|$ if and only if each eigenvalue has norm strictly less than $|\eta|$, which occurs if and only if $|\lambda/\mu - 1| < 1$ for all λ, μ. □

Theorem 6.8.2 (Refined visible decomposition theorem). *Let F be a complete nonarchimedean differential field of characteristic 0, and let V be a finite differential module over F such that no subquotient of V_s has norm less than or equal to $|d|_F$. Then, for some finite tamely ramified extension F' of F, there exists a (unique) decomposition*

$$V \otimes_F F' = \bigoplus_i V_i$$

of $V \otimes_F F'$ into refined differential modules V_i, such that for $i \neq j$ we have $V_i \not\sim V_j$ in the sense of Definition 6.2.12.

Proof. Uniqueness follows directly since $\sim$ is an equivalence relation (by Lemma 6.2.14). To check existence, by Theorem 6.6.1 we may reduce to the case where V is pure (see Definition 6.2.12). We may also assume that $d \neq 0$, as otherwise the result follows from Remark 6.2.13. Note that for any finite tamely ramified extension F' of F we have $|d|_{F'} = |d|_F$ by Lemma 6.2.15.

Apply Theorem 5.4.2 to choose an isomorphism $V \cong F\{T\}/F\{T\}P$. By Corollary 6.5.4, all the slopes of P are the same. Let $Q \in F[U]$ be the untwisted polynomial with the same coefficients as P. By Remark 6.2.13, we can choose a finite tamely ramified extension F' of F such that Q can be factorized in $F'[U]$ into factors Q_i such that, for any two roots λ, μ of any Q_i, we have $|\lambda/\mu - 1| < 1$. By applying Theorem 2.2.2 (as in Remark 2.2.3) we may correspondingly factorize P in $F'\{T\}$ so that each factor corresponds to a refined differential module. By performing this factorization again with the residual roots in the opposite order and then arguing as in Theorem 6.6.1, we obtain the desired splitting. □

Remark 6.8.3. From the proof of Theorem 6.8.2 one may extract a map from the classes of refined finite differential modules V with $|D|_{\mathrm{sp},V} = s > |d|_F$ to the quotient

$$\frac{\{x \in \mathfrak{o}_{F^{\mathrm{alg}}} : |x| \leq s\}}{\{x \in \mathfrak{o}_{F^{\mathrm{alg}}} : |x| < s\}};$$

namely, given an isomorphism $V \cong F\{T\}/F\{T\}P$, one associates with V the class of the roots of the untwisted polynomial having the same coefficients as P. However, one must check with some care that this map does not depend on the initial choice of cyclic vector. A better approach is to identify certain "test objects" for which one can deduce the map more easily; this has been done by Xiao [**221**].

From the proof of Theorem 6.8.2, we may also deduce the following simple but important observations. They will be used in the proofs of both the Turrittin–Levelt–Hukuhara decomposition (Theorem 7.5.1 below) and the p-adic local monodromy theorem (Theorem 20.1.4).

Proposition 6.8.4. *Let F be a complete nonarchimedean differential field of characteristic* 0*. Let V be a finite differential module over F of rank $n > 0$ which is refined and has spectral radius $s > |d|_F$.*

(a) *For any positive integer m which is nonzero in κ_F, $V^{\otimes m}$ is again refined with spectral radius s.*
(b) *The spectral radius of $(\wedge^n V^\vee) \otimes V^{\otimes n}$ is strictly less than s.*

Proof. The result is elementary (as discussed in Remark 6.2.13) if $d = 0$, so we may assume that $d \neq 0$. By Theorem 5.4.2 we may choose an isomorphism $V \cong F\{T\}/F\{T\}P$. Let Q be the untwisted polynomial with the same coefficients as P, and let μ be a root of Q.

As in the proof of Theorem 6.7.4 we may enlarge F in such a way that we reduce to the case where $|D|_{\mathrm{sp},V} = |\eta|$ for some $\eta \in F^\times$. Equip V with the basis given by $1, \eta^{-1}T, \dots, \eta^{-n+1}T^{n-1}$ and the corresponding supremum norm. Equip $V^{\otimes m}$ and $(\wedge^n V^\vee) \otimes V^{\otimes n}$ with the corresponding induced bases and norms.

On $V^{\otimes m}$ the matrix of action of D has eigenvalues which are m-fold sums of roots of Q. In particular, each eigenvalue λ satisfies $|\lambda/(m\mu) - 1| < 1$. We can then deduce (a) from Theorem 6.7.4.

On $(\wedge^n V^\vee) \otimes V^{\otimes n}$, the matrix of action of D has eigenvalues each of which is an n-fold sum of certain roots of Q minus the sum of all the roots of Q. In particular, each eigenvalue λ satisfies $|\lambda/\mu| < 1$. Finally we deduce (b) from Theorem 6.7.4. □

6.9 Changing the constant field

We will now check that, in many cases, the operation of computing horizontal sections of a differential module commutes with the operation of forming *completed* tensor products.

Proposition 6.9.1. *Let R be a complete nonarchimedean differential domain such that R and* Frac(R) *have the same constant ring R_0 (which is necessarily a complete nonarchimedean field). Let R_0' be a complete field extension of R_0.*

Let R' be the completed tensor product of $R \otimes_{R_0} R'_0$ for the product norm (which is a norm by Lemma 1.3.11). View R' as a differential ring by equipping it with the unique continuous extension of d with R'_0 in its kernel. Then, for any differential module M over R, the natural map

$$H^0(M) \otimes_{R_0} R'_0 \to H^0(M \otimes_R R')$$

is an isomorphism of R'_0-modules.

Proof. We first check injectivity. Since R'_0 is flat over R_0, the map $H^0(M) \otimes_{R_0} R'_0 \to H^0(M \otimes_{R_0} R'_0)$ is bijective by Lemma 5.1.4. Since M is finite over R, we may identify $M \otimes_R R'$ with the completed tensor product of $M \otimes_R (R \otimes_{R_0} R'_0) = M \otimes_{R_0} R'_0$, on which the product seminorm is a norm by Lemma 1.3.11. In particular $M \otimes_{R_0} R'_0$ injects into $M \otimes_R R'$, so the map $H^0(M \otimes_{R_0} R'_0) \to H^0(M \otimes_R R')$ is also injective.

We next check surjectivity in the case where R'_0 is the completion of a finitely generated field extension S_0 of R_0. Then S_0 is of countable dimension over R_0, so the hypothesis of Lemma 1.3.8 is satisfied by $F = R_0$, $V = R'_0$. Let $m_1, m_2, \dots$ be the sequence of elements of R'_0 given by Lemma 1.3.8. As in the proof of Lemma 1.3.11 these define projection maps $\lambda_{j,R} : R' \to R$ and $\lambda_{j,M} : M \otimes_R R' \to M$; these maps are both horizontal.

Given an element $x \in H^0(M \otimes_R R')$, choose a presentation $x = \sum_{k=1}^{s} y_k \otimes z_k$ with $y_k \in M$ and $z_k \in R'$. Then

$$\lambda_{j,M}(x) = \sum_{k=1}^{s} \lambda_{j,R}(z_k) y_k$$

is an element of $H^0(M)$ for each j. Moreover, we can write x as a convergent sum

$$x = \sum_{j=1}^{\infty} \lambda_{j,M}(x) \otimes m_j.$$

Hence x lies in the closure of $H^0(M) \otimes_{R_0} R'_0$ under the product norm. However, since $H^0(M)$ is finite-dimensional over R_0 by Lemma 5.1.5, $H^0(M) \otimes_{R_0} R'_0$ is complete under the product norm, hence closed. Thus $x \in H^0(M) \otimes_{R_0} R'_0$ as desired.

We next check surjectivity in general. Pick any $x \in H^0(M \otimes_R R')$. As noted earlier, we may identify $M \otimes_R R'$ with the completion of $M \otimes_{R_0} R'_0$ under the product norm. We can thus choose a convergent series

$$x = \sum_{k=1}^{\infty} y_k \otimes z_k \qquad (y_k \in M, z_k \in R'_0).$$

Let R_0'' be the completion of the subfield of R_0' generated over R_0 by $z_1, z_2, \dots$ Then x is a horizontal element of the completion of $M \otimes_{R_0} R_0''$, so the previous paragraph shows that $x \in H^0(M) \otimes_{R_0} R_0''$. This proves the claim. □

Notes

Lemma 6.2.5 is tacitly assumed at various places in the literature (including by the present author) but we were unable to locate even an explicit statement, let alone a proof. We thank Liang Xiao for contributing the proof given here.

Proposition 6.3.1 is modeled on some ongoing work of Baldassarri and Di Vizio (a promised sequel to [**11**]), which gives a development of much of the material we are discussing from the point of view of Berkovich analytic spaces. This point of view will probably be vital for the study of differential modules on higher-dimensional spaces. However, there are some key differences from our approach; see the notes for Chapter 9.

Newton polygons for differential operators were considered by Dwork and Robba [**81**, §6.2.3]; the first systematic treatment seems to have been given by Robba [**189**]. Our treatment using Theorem 2.2.2 follows [**42**].

The proof of Theorem 6.5.3 given here is close to the original proof of Christol and Dwork [**49**, Théorème 1.5], save that we avoid a small logical gap in the latter. The gap is in the implication $1 \Longrightarrow 2$; there one made a finite extension of the differential field without accounting for the possibility that this might increase $|d|_F$ to the point where Corollary 6.2.7 fails to show that $|D|_V$ is preserved. (It would be obvious that this does not occur if the finite extension were being made in the constant subfield, but that was not the case.) Compare also [**80**, Lemma VI.2.1].

Proposition 6.5.6 answers a conjecture of Christol and Dwork [**48**, Introduction, Conjecture A]. This conjecture was posed in the context of giving effective convergence bounds, and that is one use to which we will put it here; see Theorem 18.2.1 and its proof.

The use in Lemma 6.7.3 of a well-chosen norm (meaning a norm in which at least some singular values match eigenvalues of the matrix of action of d) is an extension of the notion of the *canonical lattice* (or *Deligne–Malgrange lattice*) introduced by Malgrange in the study of irregular meromorphic connections. See [**165**].

The refined visible decomposition theorem is due to Liang Xiao [**221**]. It was motivated by applications to *refined Swan conductors* of étale sheaves and overconvergent isocrystals, as suggested by the work of Kato [**119**] in the rank 1 case.

We thank Andrea Pulita for the suggestion that Proposition 6.9.1 should be included.

Exercises

(1) Prove Fekete's lemma (Lemma 6.1.4).

(2) (a) Let A, B be *commuting* bounded endomorphisms on an abelian group G equipped with a norm. Prove that

$$|A + B|_{\mathrm{sp},G} \leq |A|_{\mathrm{sp},G} + |B|_{\mathrm{sp},G}.$$

(b) Prove that if the norm on G is nonarchimedean then the inequality in (a) can be improved to

$$|A + B|_{\mathrm{sp},G} \leq \max\{|A|_{\mathrm{sp},G}, |B|_{\mathrm{sp},G}\}$$

and that equality occurs when the maximum is achieved only once.

(c) Prove that both these assertions may fail in the case where A and B do not commute. (Hint: write an identity matrix as a sum of nilpotent matrices.)

(3) Prove that, in Lemma 6.2.5, if $|D|_{\mathrm{sp},V} > |d|_{\mathrm{sp},F}$ then $|D_s|^{1/s}$ converges to a limit as $s \to \infty$. (Hint: again reduce to Lemma 6.1.4.)

(4) Fill in the missing arguments in the proofs of Lemma 6.2.8(a), (b), using Lemma 6.1.8.

(5) Prove the claim from the proof of Lemma 6.4.2 that, for R a nonarchimedean differential domain, $\rho > |d|_R$, and $P, Q \in R\{T\}$, we have $|PQ - QP|_\rho \leq \rho^{-1}|d|_R|P|_\rho|Q|_\rho$. (Hint: reduce to the case where P and Q are monomials and then to the case where $P = T$ and $Q \in R$.)

(6) Exhibit an example to show that, in Definition 6.4.3, if we had not omitted slopes greater than or equal to $-\log|d|_R$ then it would no longer be the case that the multiplicity of a slope r in a product PQ is the sum of the multiplicities of r as a slope of P and as a slope of Q. (Hint: this can already be seen using the t-adic valuation on $\mathbb{Q}(t)$ with $d = d/dt$.)

(7) Let (F, d) be a complete nonarchimedean differential field. Let F_ρ be the completion of $F(t)$ for the ρ-Gauss norm. Prove that there is a unique continuous extension of d to F_ρ with $d(t) = 0$ and that it satisfies

$$|d|_{F_\rho} = |d|_F, \qquad |d|_{\mathrm{sp},F_\rho} = |d|_{\mathrm{sp},F}.$$

(Hint: first check everything on $F[t]$.)

7

Regular singularities

In the next part of the book, which begins with Chapter 8, we will use the results from the previous chapters to make a detailed analysis of ordinary differential equations over nonarchimedean fields of characteristic 0, the motivating case being that of positive residual characteristic. However, before doing so it may be helpful to demonstrate how the results apply in a somewhat simpler setting.

In this chapter, we reconstruct some of the traditional Fuchsian theory of regular singular points of meromorphic differential equations. (The treatment is modeled on [**80**, §3].) We first introduce a quantitative measure of the *irregularity* of a singular point. We then recall how, in the case of a *regular singularity* (i.e., a singularity with irregularity equal to zero), one has an algebraic interpretation, using the notion of *exponents*, of the eigenvalues of the monodromy operator around the singular point. We then describe how to compute formal solutions of meromorphic differential equations and go on to sketch the proof of Fuchs's theorem, that the formal solutions of a regular meromorphic differential equation actually converge in some disc. We finally establish the Turrittin–Levelt–Hukuhara decomposition theorem, which gives a decomposition of an arbitrary formal differential module that is analogous to the eigenspace decomposition of a complex linear transformation. The search for an appropriate p-adic analogue of this result will lead us in Part V to the p-adic local monodromy theorem.

Although this chapter focuses on issues different from much of the rest of the book, it should not be considered entirely optional. It is referenced in the discussions of p-adic exponents in Chapter 13 and of effective convergence bounds in Chapter 18.

Hypothesis 7.0.1. Throughout this chapter, we will view $\mathbb{C}((z))$ as a complete nonarchimedean differential field with valuation given by the z-adic valuation

v_z and derivation given by $d = zd/dz$; note that $|d|_{\mathbb{C}((z))} = 1$. Let K be a field of characteristic 0; although we do not need to think of K as a subfield of $\mathbb{C}$ it is harmless to do so. (The reason is the *Lefschetz principle*: every statement we make about K will refer to at most countably many elements of K, so each individual instance of the statement can be realized over a subfield of K which is countably generated over $\mathbb{Q}$ and hence embeds into $\mathbb{C}$.)

7.1 Irregularity

Definition 7.1.1. Let V be a finite differential module over $\mathbb{C}((z))$, and decompose V according to Theorem 6.6.1. Define the *irregularity* of V as

$$\operatorname{irr}(V) = \sum_{s>1} (-\log s) \dim(V_s).$$

For F a subfield of $\mathbb{C}((z))$ stable under d and V a finite differential module over F, we define the irregularity of V to be the irregularity of $V \otimes_F \mathbb{C}((z))$. We say that V is *regular* if $\operatorname{irr}(V) = 0$.

Theorem 7.1.2. *For any isomorphism $V \cong F\{T\}/F\{T\}P$, the irregularity of V is equal to minus the sum of the slopes of P; consequently, it is always an integer. More explicitly, if $P = T^d + \sum_{i=0}^{d-1} P_i T^i$ then*

$$\operatorname{irr}(V) = \max\left\{0, \max_i\{-v_z(P_i)\}\right\}.$$

Proof. The second assertion follows from Corollary 6.5.4, since in this case $|d|_F = |d|_{\mathrm{sp},F} = 1$. The first assertion follows from the second because, by Theorem 5.4.2, V always admits a cyclic vector. □

Theorem 7.1.2 gives rise to several criteria for regularity.

Corollary 7.1.3. *Let F be any subfield of $\mathbb{C}((z))$ containing z and stable under d, and let V be a finite differential module over F. Then the following conditions are equivalent.*

(a) *The module V is regular, i.e.,* $\operatorname{irr}(V) = 0$.
(b) *For some isomorphism $V \cong F\{T\}/F\{T\}P$ with P monic, P has coefficients in $\mathfrak{o}_F$.*
(c) *For any isomorphism $V \cong F\{T\}/F\{T\}P$ with P monic, P has coefficients in $\mathfrak{o}_F$.*
(d) *There exists a basis of V on which D acts via a matrix over $\mathfrak{o}_F$.*
(e) *For any basis B of V, the $\mathfrak{o}_F$-span of the set $\{D^i(v) : i \in \{0, \dots, \dim(V) - 1\}, v \in B\}$ is stable under D.*

Proof. By Theorem 7.1.2, (a) implies (c). It is obvious that (c) implies (b) and that (b) implies (d). Given (d), let $|\cdot|_V$ be the supremum norm defined by the chosen basis of V; then $|D|_V \leq 1$, which implies (a).

This proves that (a), (b), (c), (d) are all equivalent. To add (e) to the circle of implications, on the one hand note that (e) implies (d). On the other hand, given (a), pick any $v \in B$; then $v, D(v), D^2(v), \dots$ generate a differential submodule W of V for which v is a cyclic vector. Since V is regular, its submodule W is also regular by Lemma 6.2.8; since (a) implies (c), the $\mathfrak{o}_F$-span of $\{D^i(v) : i \in \{0, \dots, \dim(W) - 1\}\}$ is stable under D. This implies (e). □

Remark 7.1.4. One can also view $\mathbb{C}((z))$ as a differential field with the derivation d/dz instead of $z d/dz$. The categories of differential modules for these two choices of derivation are equivalent in an obvious fashion: given the action $z d/dz$, we obtain the action d/dz by dividing by z. If V is a differential module for $z d/dz$ with spectral radius s then the spectral radius of V for d/dz is $s|z|^{-1}$ (exercise). The notion of irregularity translates naturally: for instance, if V is a differential module for d/dz that is isomorphic to $F\{T\}/F\{T\}P$ for some $P = T^n + \sum_{i=0}^{n-1} P_i T^i$ then V is regular if and only if $v_z(P_i) \geq -n + i$ for $i = 1, \dots, n$. For example, for $a, b \in \mathbb{C}$, the differential system corresponding to the hypergeometric differential equation

$$y'' + \frac{c - (a + b + 1)z}{z(1 - z)} y' - \frac{ab}{z(1 - z)} y = 0$$

is regular.

Example 7.1.5. For another example from the classical theory of ordinary differential equations, consider the Bessel equation

$$y'' + \frac{1}{z} y' + \frac{z^2 - n^2}{z^2} y = 0$$

for some parameter n. This gives rise to a regular differential system if we expand around $z = 0$. However, if we expand around ∞, i.e., substitute $1/z$ for z and then expand around $z = 0$, we get an irregular differential system (exercise).

7.2 Exponents in the complex analytic setting

To see why regular singularities are so important in the complex analytic setting (by way of motivation for our p-adic studies), let us consider the monodromy transformation. First we recall a familiar fact.

Theorem 7.2.1 (Cauchy). *Fix $\rho > 0$, and let $R \subset \mathbb{C}[\![z]\!]$ be the ring of power series convergent for $|z| < \rho$. Let N be an $n \times n$ matrix over R. Then the differential system $D(v) = Nv + (d/dz)(v)$ has a basis of horizontal sections.*

Proof. This can be deduced from the fundamental theorem of ordinary differential equations; however, for future reference it will be useful to give a slightly more detailed explanation. (Specifically, we are anticipating Definition 7.3.1.)

Note that there exists a unique $n \times n$ matrix U over $\mathbb{C}[\![z]\!]$ such that $U \equiv I_n \pmod{z}$ and $NU + (d/dz)(U) = 0$; this follows by writing $N = \sum_{i=0}^{\infty} N_i z^i$ and $U = \sum_{i=0}^{\infty} U_i z^i$ and then rewriting the equation $NU + (d/dz)(U) = 0$ as the recurrence

$$(i+1)U_{i+1} = \sum_{j=0}^{i} N_j U_{i-j} \qquad (i = 0, 1, \dots). \tag{7.2.1.1}$$

Following an argument of Cauchy [**80**, Appendix III], we may deduce that the entries of U converge on a disc of positive radius, as follows. (In this argument, we use the L^2 operator norm on matrices over $\mathbb{C}$.) Pick any $\eta \in (0, \rho)$. Since N converges in the open disc of radius ρ, we have $|N_i|\eta^i \to 0$ as $i \to \infty$; in particular, we may choose $c > 1$ with $|N_i|\eta^{i+1} \leq c$ for all i. We then have by induction on i that

$$|U_i|\eta^i \leq c^i \qquad (i \geq 0):$$

namely, this holds for $j = 0$, and (7.2.1.1) implies that

$$|U_{i+1}|\eta^{i+1} \leq \frac{1}{i+1} \sum_{j=0}^{i} |N_j|\eta^{j+1}|U_{i-j}|\eta^{i-j} \leq c^{i+1}.$$

Consequently, the entries of U converge on the open disc of radius η/c.

The previous argument applied to the system $(-N)U^{-T} + (d/dz)(U^{-T}) = 0$ shows that the entries of U^{-1} also converge on an open disc. Consequently, we can find an open neighborhood of $z = 0$ on which the differential system admits a basis of horizontal sections. By translating we may derive the same conclusion around any point of the original disc; since that disc is simply connected, we obtain a basis of horizontal sections over the entire disc. $\square$

Remark 7.2.2. In the p-adic setting we will see that the first step of the proof of Theorem 7.2.1 remains valid, but there is no analogue of the second step (analytic continuation), and indeed the whole conclusion becomes false; see Example 0.4.1.

Let us now consider a punctured disc and look at monodromy.

Notation 7.2.3. In this chapter only, for K a subfield of $\mathbb{C}$, let $K\{z\}$ be the subfield of $K((z))$ consisting of formal Laurent series which represent meromorphic functions on some neighborhood of $z = 0$. The exact choice of that neighborhood may vary with the series.

Definition 7.2.4. Let V be a finite differential module over $\mathbb{C}\{z\}$; choose a basis of V and let N be the matrix of action of D on this basis. On some disc centered at $z = 0$, the entries of N are meromorphic with no poles away from $z = 0$. On any subdisc not containing 0, by Theorem 7.2.1 we obtain a basis of horizontal sections. If we start with a basis of horizontal sections in a neighborhood of some point away from 0 and then analytically continue around a circle, proceeding once counterclockwise around the origin, we end up with a new basis of local horizontal sections. The linear transformation taking the old basis to the new one is called the *monodromy transformation* of V (or of its associated differential system). The *(topological) exponents* of V are defined (modulo translation by $\mathbb{Z}$) to be the multiset of numbers $\alpha_1, \dots, \alpha_n$ for which $e^{-\alpha_1}, \dots, e^{-\alpha_n}$ are the eigenvalues of the monodromy transformation.

The monodromy transformation controls our ability to construct global horizontal sections, by the following statement whose proof is evident.

Proposition 7.2.5. *In Definition 7.2.4, any fixed vector under the monodromy transformation corresponds to a horizontal section defined on some punctured disc rather than on the universal covering space of a punctured disc. As a result, the monodromy transformation is unipotent (i.e., the exponents are all zero) if and only if there exists a basis on which D acts via a nilpotent matrix.*

Definition 7.2.6. In Definition 7.2.4, we say that V is *quasiunipotent* if its exponents are rational; equivalently, the monodromy transformation of V becomes unipotent after V is pulled back along $z \mapsto z^m$ for some positive integer m. This situation arises for Picard–Fuchs modules; see Chapter 22.

Remark 7.2.7. The relationship between the properties of the monodromy transformation and the existence of horizontal sections of the differential module begs the following question: is it possible to extract the monodromy transformation for a differential module, whose definition is purely analytic, from the algebraic data that defines the differential system? The only case in which this is straightforward is that of a regular module, which we consider next.

7.3 Formal solutions of regular differential equations

One can formally imitate the proof of Cauchy's theorem (Theorem 7.2.1) in the regular case, as follows.

Definition 7.3.1. Let $N = \sum_{i=0}^{\infty} N_i t^i$ be an $n \times n$ matrix with entries in $K[\![z]\!]$. A *fundamental solution matrix* for N is an $n \times n$ matrix U with $U \equiv I_n \pmod{z}$ such that $U^{-1}NU + U^{-1}z(d/dz)(U) = N_0$.

Remark 7.3.2. Note that if U is a fundamental solution matrix for N then

$$\begin{aligned} -U^T N^T U^{-T} + U^T z \frac{d}{dz}(U^{-T}) &= -U^T N^T U^{-T} - U^T U^{-T}\left(z\frac{d}{dz}(U^T)\right)U^{-T} \\ &= -U^T N^T U^{-T} - \left(z\frac{d}{dz}(U^T)\right)U^{-T} \\ &= -N_0^T. \end{aligned}$$

That is, U^{-T} is a fundamental solution matrix for $-N^T$. Consequently, by proving a general result about U we also obtain a corresponding result for U^{-T}, and hence for U^{-1}.

To specify when a fundamental solution matrix exists, we need the following definition.

Definition 7.3.3. We say that a square matrix N with entries in a field of characteristic 0 has *prepared eigenvalues* if the eigenvalues $\lambda_1, \dots, \lambda_n$ of N satisfy the following conditions:

$$\begin{gathered} \lambda_i \in \mathbb{Z} \Leftrightarrow \lambda_i = 0, \\ \lambda_i - \lambda_j \in \mathbb{Z} \Leftrightarrow \lambda_i = \lambda_j. \end{gathered}$$

If only the second condition holds, we say that N has *weakly prepared eigenvalues*.

We will also need the following lemma, which will come up again several times later.

Definition 7.3.4. Let N be a nilpotent $n \times n$ matrix over K. The *nilpotency index* of N is the smallest positive integer e such that $N^e = 0$.

Lemma 7.3.5. *Let N_1 and N_2 be matrices of respective sizes $m \times m$ and $n \times n$ over K. Let $\lambda_{1,1}, \dots, \lambda_{1,m}$ and $\lambda_{2,1}, \dots, \lambda_{2,n}$ be the eigenvalues of N_1 and N_2, respectively. Then the eigenvalues of the K-linear endomorphism $X \mapsto N_1X + XN_2$ on the space of $m \times n$ matrices over K are equal to $\lambda_{1,i} + \lambda_{2,j}$ for $i = 1, \dots, m$, $j = 1, \dots, n$. Moreover, if N_1 and N_2 are themselves nilpotent,* *with respective nilpotency indices e_1, e_2, then the nilpotency index of $X \mapsto N_1X + XN_2$ equals $e_1 + e_2 - 1$.*

Proof. There is no harm in enlarging K, so we may assume that it is algebraically closed. Moreover, if $U_1 \in \mathrm{GL}_m(K)$ and $U_2 \in \mathrm{GL}_n(K)$ then it is equivalent to calculate the eigenvalues of the conjugated endomorphism

$$X \mapsto U_1^{-1}(N_1(U_1XU_2^{-1}) + (U_1XU_2^{-1})N_2)U_2 = (U_1^{-1}N_1U_1)X + X(U_2^{-1}N_2U_2).$$

Consequently, we may conjugate N_1 and N_2 into Jordan normal form. By separating X into blocks, we may reduce to the case where N_1 and N_2 consist of single Jordan normal blocks with eigenvalues λ_1 and λ_2. By subtracting λ_1 and λ_2 from the overall endomorphisms, we may reduce to the case $\lambda_1 = \lambda_2 = 0$, for which we want to show that the map $X \mapsto N_1X + XN_2$ is nilpotent with nilpotency index at most $e_1 + e_2 - 1$. This can be done easily by hand but can also be seen as follows: put $f(X) = N_1X + XN_2$, and write the ith composition of f as

$$f^i(X) = \sum_{j=0}^{i} \binom{i}{j} N_1^j X N_2^{i-j}.$$

If $i \geq e_1 + e_2 - 1$ then in each term we have either $j \geq e_1$ or $i - j \geq e_2$, so the whole sum vanishes. If $i = e_1 + e_2 - 2$ then similarly every term vanishes except possibly $\binom{e_1+e_2-2}{e_1-1} N_1^{e_1-1} X N_2^{e_2-1}$. Since we are assuming that K has characteristic 0, the binomial coefficient $\binom{e_1+e_2-2}{e_1-1}$ is nonzero and the matrices $N_1^{e_1-1}, N_2^{e_2-1}$ must be nonzero, so we can make this product nonzero by choosing a suitable matrix X. $\square$

Proposition 7.3.6. *Let $N = \sum_{i=0}^{\infty} N_i t^i$ be an $n \times n$ matrix with entries in $K[\![z]\!]$ such that N_0 has weakly prepared eigenvalues. Then N admits a unique fundamental solution matrix.*

Proof. Let $\lambda_1, \dots, \lambda_n \in K^{\mathrm{alg}}$ be the eigenvalues of N_0. Rewrite the defining equation as $NU + z(d/dz)(U) = UN_0$, then expand U as $\sum_{i=0}^{\infty} U_i t^i$ and write the new defining equation as a recurrence:

$$iU_i = U_iN_0 - N_0U_i - \sum_{j=1}^{i} N_jU_{i-j} \qquad (i > 0). \tag{7.3.6.1}$$

Viewing the map $X \mapsto XN_0 - N_0X$ as a linear transformation on the space of $n \times n$ matrices over $K((t))$, we see by Lemma 7.3.5 that its eigenvalues are the differences $\lambda_j - \lambda_k$ for $j, k = 1, \dots, n$. Likewise, the eigenvalues of $X \mapsto iX - XN_0 + N_0X$ are $i - \lambda_j + \lambda_k$; for i a positive integer, the condition that the λ's are weakly prepared ensures that $i - \lambda_j + \lambda_k$ cannot vanish (indeed, it cannot be an integer unless it equals i). Consequently, given N and $U_0, \dots, U_{i-1}$ there is a unique choice of U_i satisfying (7.3.6.1); this proves the desired result. □

Remark 7.3.7. Suppose that N is an $n \times n$ matrix with entries in $\mathbb{C}[\![z]\!]$ whose constant term has prepared eigenvalues. If we convert the differential system defined by N into a differential module M, the $\mathbb{C}$-span of the columns of the fundamental solution matrix forms a $\mathbb{C}$-submodule of M that is stable under D. In fact, the action of D on M is the linear transformation defined by N_0. Since this transformation may be nonzero, the elements of this span are not all horizontal; however, one can show (exercise) that every horizontal element of M does appear in the span. This justifies the name "fundamental solution matrix".

This formal argument becomes relevant in the complex analytic setting by virtue of the following informal fact. For a proof see [**80**, §III.8, Appendix II] or the exercises.

Theorem 7.3.8 (Fuchs). *Let $N = \sum_{i=0}^{\infty} N_i t^i$ be an $n \times n$ matrix with entries in $\mathbb{C}\{z\}$ such that N_0 has weakly prepared eigenvalues. Then the fundamental solution matrix for N over $\mathbb{C}[\![z]\!]$ also has entries in $\mathbb{C}\{z\}$ (as does its inverse).*

Corollary 7.3.9. *With notation as in Theorem 7.3.8, let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of N_0. Then the eigenvalues of the monodromy transformation of the system $D(v) = Nv + d(v)$ are $e^{-2\pi i\lambda_1}, \dots, e^{-2\pi i\lambda_n}$.*

Proof. In terms of a basis on which the matrix of action of D is N_0, the matrix $\exp(-N_0 \log(z))$ provides a basis of horizontal elements. (The case $N_0 = 0$ is given by Theorem 7.2.1.) □

In order to enforce the condition on prepared eigenvalues, we use what are classically known as *shearing transformations*.

Proposition 7.3.10 (Shearing transformations). *Let N be an $n \times n$ matrix over $K[\![z]\!]$ whose entries have constant term N_0. Let $\alpha_1, \dots, \alpha_m \in K^{\mathrm{alg}}$ be eigenvalues of N forming a single Galois orbit over K. Then there exists $U \in \mathrm{GL}_n(K[z, z^{-1}])$ such that $U^{-1}NU + U^{-1}d(U)$ also has entries in $K[\![z]\!]$, and its matrix of constant terms has the same eigenvalues as N_0 except that every instance of each α_i has been replaced by $\alpha_i + 1$. The same conclusion holds if $\alpha_i - 1$ replaces $\alpha_i + 1$.*

Proof. Exercise. □

Corollary 7.3.11 (Fuchs). *Let V be a regular finite differential module over $\mathbb{C}\{z\}$. Then any horizontal element of $V \otimes_{\mathbb{C}\{z\}} \mathbb{C}((z))$ belongs to V itself; that is, any formal horizontal section is convergent. (This is false in the irregular case; see the notes.)*

Proof. Using Proposition 7.3.10 we may construct a basis of V on which D acts via a matrix in $\mathbb{C}[\![z]\!] \cap \mathbb{C}\{z\}$ whose constant term has weakly prepared eigenvalues. By Remark 7.3.7, any horizontal element of $V \otimes_{\mathbb{C}\{z\}} \mathbb{C}((z))$ is a $\mathbb{C}$-linear combination of column vectors of the fundamental solution matrix; by Theorem 7.3.8, any such linear combination converges in a disc. $\square$

To put everything together, we need the following result.

Proposition 7.3.12. *Let V be a regular finite differential module over $K((z))$. Then there exists a basis of V on which the matrix of action of d has entries in K and prepared eigenvalues.*

Proof. By Proposition 6.5.6 we can find a $K[\![z]\!]$-lattice L in V (i.e., a finitely generated $K[\![z]\!]$-submodule whose $K((z))$-span is the whole of V) which is stable under d. By Proposition 7.3.10 we can modify L in such a way that the constant term of the matrix of action of d on some basis of L has prepared eigenvalues. (Because we cannot separate Galois conjugates, the previous statement relies on the fact that no two distinct Galois conjugates α, β over K may differ by a nonzero integer. This fact holds because the trace from the Galois closure of $K(\alpha, \beta)$ to K vanishes on $\alpha - \beta$ but is injective on K because K is of characteristic 0.) We may then apply Proposition 7.3.6 to deduce the claim. $\square$

7.4 Index and irregularity

We mention very briefly an alternate interpretation of irregularity considered by Malgrange.

Definition 7.4.1. Let F be any subfield of $K((z))$ stable under d, and let V be a finite differential module over F. We say V *has an index* if $\dim_K H^0(V)$ and $\dim_K H^1(V)$ are both finite; in this case we define the *index* of V as $\chi(V) = \dim_K H^0(V) - \dim_K H^1(V)$.

Proposition 7.4.2. *For any finite differential module V over $K((z))$, $\dim_K H^0(V) = \dim_K H^1(V) < \infty$ and so $\chi(V) = 0$.*

Proof. Exercise. $\square$

In the convergent case, the index carries more information.

Theorem 7.4.3. *Let V be a finite differential module over $\mathbb{C}\{z\}$. Then V has an index, and $\chi(V) = -\operatorname{irr}(V)$.*

Proof. See [**164**, Théorème 2.1]. □

7.5 The Turrittin–Levelt–Hukuhara decomposition theorem

One can classify differential modules over $K((z))$ rather simply, provided that one is willing to admit finite field extensions. Note that the results do not descend from $\mathbb{C}((z))$ to $\mathbb{C}\{z\}$, owing to the failure of Corollary 7.3.11 in the irregular case. Nonetheless, even a formal classification is extremely useful; see the notes.

Note the use of the refined visible decomposition theorem (Theorem 6.8.2) in the following proof.

Theorem 7.5.1. *Let V be a finite differential module over $K((z))$. Then there exist a positive integer h and a finite extension K' of K such that $V \otimes_{K((z))} K'((z^{1/h}))$ admits a direct sum decomposition $\oplus_i V_i$ in which each $V_i^\vee \otimes V_i$ is regular.*

Proof. We induct on the dimension of V. Let v be a generator of $\wedge^n V$, write $D(v) = sv$, and let W be the differential module over $K((z))$ having one generator w satisfying $D(v) = n^{-1}sw$. Note that $W^\vee \otimes V$ cannot be refined, as otherwise Proposition 6.8.4(a) would imply that $W^\vee \otimes V$ has the same spectral radius as $\wedge^n(W^\vee \otimes V) \cong (W^\vee)^{\otimes n} \otimes \wedge^n V$, which is trivial.

By Theorem 6.8.2, either $W^\vee \otimes V$ is regular or refined or, for suitable h, K', $(W^\vee \otimes V) \otimes_{K((z))} K'((z^{1/h}))$ splits as a nontrivial direct sum. Since we have ruled out the refined case, either $W^\vee \otimes V$ is regular, and so $V^\vee \otimes V$ is regular, or we may invoke the induction hypothesis. □

Remark 7.5.2. If we insist that the decomposition in Theorem 7.5.1 be minimal (i.e., that it should have as few summands as possible), then it is in fact unique. Consequently, such a minimal decomposition must be respected by any extra structures on V which respect the action of d. For instance, if K carries one or more derivations, and we equip V with actions of these derivations, then the decomposition is preserved by these actions.

From the proof of Theorem 7.5.1 we can immediately read off the following classification of the components appearing in the direct sum decomposition therein.

Theorem 7.5.3. *Let V be a finite differential module over $K((z))$ such that $V^\vee \otimes V$ is regular. Then there exists a differential module W of rank 1 such that $W^\vee \otimes V$ is regular.*

Corollary 7.5.4. *Suppose that K is algebraically closed. Let V be a finite differential module over $K((z))$ such that $V^\vee \otimes V$ is regular. Then there is a unique decomposition $V = \oplus_i V_i$ such that $V_i^\vee \otimes V_j$ has all exponents zero if $i = j$ and all exponents nonzero if $i \neq j$.*

Proof. If V itself is regular, the corollary follows from Proposition 7.3.12 and the fact that a linear transformation over an algebraically closed field decomposes as a direct sum of generalized eigenspaces. In general we may replace V with $W^\vee \otimes V$ for some V as specified in Theorem 7.5.3 and reduce to the regular case. □

The strongest form of the Turrittin–Levelt–Hukuhara decomposition is the following statement, which both eliminates the base extension in Theorem 7.5.1 and incorporates the statement of Theorem 7.5.3.

Definition 7.5.5. Let h be a positive integer, and suppose that $P = \{P_1, \dots, P_h\}$ is a Galois orbit over $K((z))$. Let F be a finite Galois extension of $K((z))$ containing $P_1, \dots, P_h$. Then the differential module of rank h over F with generators $e_1, \dots, e_h$ satisfying

$$d(e_i) = P_i e_i \qquad (i = 1, \dots, h)$$

descends uniquely to a differential module over $K((z))$, which we denote $E(P)$. When the orbit is a singleton, $\{P_1\}$, we also write $E(P_1)$ as shorthand for $E(\{P_1\})$. (See the proof of Theorem 7.5.6 for more on the Galois descent construction.)

Theorem 7.5.6. *Let V be a finite differential module over $K((z))$. Then V admits a direct sum decomposition*

$$V = \bigoplus_i E(P_i) \otimes X_i$$

for some Galois orbits P_i and some regular differential modules X_i.

Proof. Suppose first that the conclusion of Theorem 7.5.1 applies with $K'((z^{1/h})) = K((z))$. Then Theorem 7.5.3 implies that $V = E(P) \otimes X$ for some $P \in K((z))$ and some regular differential module X. In fact, there is a unique choice of $P \in z^{-1}K[z^{-1}]$ for which such an X exists, and the decomposition is also unique.

In general, we get a decomposition of the desired form over $K'((z^{1/h}))$ for some finite extension K' of K and some positive integer h. We can deduce the desired assertion from this using a Galois descent argument. Put $G = \mathrm{Gal}(K'((z^{1/h}))/K((z)))$. For $\tau \in G$, note that we can base change along τ (as in Definition 5.3.2) to define $\tau^* V$.

As above, we choose each P_i to consist entirely of elements of $z^{-1/h}K'[z^{-1/h}]$. For each i and each $\tau \in G$, we must then have $\tau(P_i) = P_j$ for some j. From the uniqueness of the decomposition, we get an isomorphism

$$\psi_\tau : \tau^*(E(P_i) \otimes X_i) \to E(P_j) \otimes X_j.$$

These isomorphisms satisfy a cocycle condition: for $\sigma, \tau \in G$, we have

$$\psi_{\sigma\tau} = \sigma^*(\psi_\tau) \circ \psi_\sigma.$$

We may canonically identify $\tau^* E(P_i)$ with $E(P_j)$ by matching up the generator $w \otimes 1$ with the generator w. We may then canonically identify ψ_τ with a horizontal element of $E(P_j)^\vee \otimes E(P_j) \otimes \tau^* X_i^\vee \otimes X_j$. By projecting onto the trace component of $E(P_j)^\vee \otimes E(P_j)$ we get a horizontal element of $\tau^* X_i^\vee \otimes X_j$, which in turn we identify with a morphism $\psi'_\tau : \tau^* X_i \to X_j$. These maps again satisfy the cocycle condition, so Galois descent allows us to identify each X_i with the base extension of a differential module X'_i over $K((z))$. Under this identification, if P'_i denotes the Galois orbit of P_i, we get a canonical identification

$$(E(P'_i) \otimes X'_i) \otimes_{K((z))} K'((z^{1/h})) \cong \bigoplus_{j : P_j \in P'_i} E(P_j) \otimes X_j.$$

This proves the claim. □

Remark 7.5.7. The proof of Theorem 7.5.6 shows that both the direct sum decomposition and the identification of the tensor factors in each summand can be made canonical, by insisting that each P_i consist of elements of $z^{-1/h}K'[z^{-1/h}]$ for some finite extension K' of K and some positive integer h. However, everything still depends implicitly on the choice of the series parameter z in the field $K((z))$; in many applications, such a choice is not at all obvious.

Notes

The notion of a regular singularity was introduced by Fuchs in the nineteenth century as part of a classification of those differential equations with everywhere-meromorphic singularities on the Riemann sphere which have algebraic solutions. Regular singularities are sometimes referred to as

Fuchsian singularities. Much of our modern understanding of the regularity condition, especially in higher dimensions, comes from the book of Deligne [**68**].

As noted in the text, Corollary 7.3.11 is false for irregular modules. This was originally noticed in numerous examples of particular differential equations (e.g., the Bessel equation at infinity; see Example 20.2.1) and motivated the definition of irregularity in the first place.

The relationship between the interpretations of irregularity using the Newton polygon and using indices is due to Malgrange [**164**]. Our treatment, in which the spectral radius plays a pivotal role, is based on [**80**, §3]; this point of view is ultimately due to Robba.

A complex analytic interpretation of the Newton polygon, in the manner of the relation between irregularity and index, was given by Ramis [**186**]. It involves considering subrings of $\mathbb{C}\{z\}$ composed of functions having certain extra convergence restrictions (*Gevrey functions*) and looking at the index of $z\,d/dz$ after tensoring the given differential module with one of these subrings.

Theorem 7.5.6 is a slight reformulation of classification results due to Turrittin [**212**] (building on earlier work of Hukuhara in the rank 1 case) for the existence aspect and to Levelt [**158**] for the uniqueness aspect. See [**7**, II.3.1] for further discussion; we further recommend [**7**] for higher-dimensional analogues of the results discussed in this chapter. See [**144**] for some additional higher-dimensional analogues, presented in much the same style as in this chapter.

Exercises

(1) Let V be a differential module over $K((z))$ with spectral radius s for zd/dz. Prove that the spectral radius of V for d/dz is $s|z|^{-1}$.

(2) Verify the claims of Example 7.1.5.

(3) Let N be an $n \times n$ matrix over $K[\![z]\!]$ whose constant term has prepared eigenvalues. Prove that any vector $v \in K((z))^n$ with $Nv+z(d/dz)(v) = 0$ is a K-linear combination of columns of the fundamental solution matrix of N. (Hint: change basis first.)

(4) Prove Fuchs's theorem (Theorem 7.3.8). (Hint: let $U = \sum_{i=0}^{\infty} U_i z^i$ be the fundamental solution matrix of N. Relate the norms of U_i and $iU_i - U_i N_0 + N_0 U_i$ by considering the singular values of the map $X \mapsto X - XN_0 + N_0 X$. Then use (7.3.6.1) to obtain

$$\left| U_i - \frac{U_i N_0 - N_0 U_i}{i} \right| \leq \frac{1}{i} \sum_{j=1}^{i} |N_j||U_{i-j}| \leq \max_{1 \leq j \leq i} \{|N_j||U_{i-j}|\}$$

and proceed as in the proof of Theorem 7.2.1.)

(5) Prove Proposition 7.3.10. (Hint: show that, for any space V of constant vectors stable under multiplication by N_0, the set of vectors with entries in $K[\![z]\!]$ whose reductions modulo z belong to V are closed under the operation $v \mapsto Nv + d(v)$. Then take V to be a union of generalized eigenspaces.)

(6) Prove Proposition 7.4.2. (Hint: change basis by preparing eigenvalues and then applying Theorem 7.3.8.)

(7) Let $Q \in K((z))[U]$ be a polynomial whose roots $\lambda_1, \dots, \lambda_n \in K((z))^{\mathrm{alg}}$ satisfy $|\lambda_i/\lambda_j - 1| < 1$ for all i, j. Prove that there exists $\lambda \in K((z))$ such that $|\lambda_i/\lambda - 1|$ for all i. Then show that this fails without the hypothesis that K has characteristic 0.

Part III

p-adic Differential Equations on Discs and Annuli

8

Rings of functions on discs and annuli

In Part III we focus our attention specifically on p-adic ordinary differential equations (although most of our results apply also to complete nonarchimedean fields of residual characteristic 0). To do this with maximal generality, one would need first to introduce a category of geometric spaces over which to work. This would require a fair bit of discussion of either rigid analytic geometry, in the manner of Tate, or nonarchimedean analytic geometry in the manner of Berkovich, neither of which we want either to assume or introduce. Fortunately, since we only need to consider one-dimensional spaces, we can manage by working completely algebraically and considering differential modules over appropriate rings.

In this chapter, we introduce those rings and collect their basic algebraic properties. This includes the fact that they carry Newton polygons analogous to those for polynomials. Another key fact is that there is a form of the approximation lemma (Lemma 1.3.7) valid over some of these rings.

Notation 8.0.1. Throughout this part, let K be a field of characteristic 0 that is complete for a nontrivial nonarchimedean norm $|\cdot|$. (The assumption of characteristic 0 is not used in this chapter; it will become crucial when we start discussing differential modules again.) Let p denote the characteristic of the residue field κ_K. We do not assume $p > 0$ (as the case $p = 0$ may be useful for some applications), but when $p > 0$ we do require the norm to be normalized in such a way that $|p| = p^{-1}$.

Definition 8.0.2. By a *piecewise affine* function on an interval I we will mean a continuous function $f : I \to \mathbb{R}$ such that I can be covered with intervals of positive length, on each of which f restricts to an affine function (i.e., a function of the form $ax + b$ for some $a, b \in \mathbb{R}$). By the compactness of a closed interval (see Lemma 8.0.4), it is equivalent to require that each point of I

admit a one-sided neighborhood, on each side within I, on which f restricts to an affine function. (If I is infinite, we allow the possibility of having infinitely many different slopes unless we say otherwise explicitly.)

Example 8.0.3. An example specifically excluded by Definition 8.0.2 is the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1/2 & x = 0, \\ 1/2 - 1/(N+1) + N/(N+2)\,(1/N - x) & x \in \left[1/(N+1), 1/N\right], \\ & N = 1, 2, \dots; \end{cases}$$

this function is piecewise affine on $(0, 1]$ and continuous on $[0, 1]$, but there is no one-sided neighborhood of 0 on which f is affine.

Lemma 8.0.4. *Let $I = [\alpha, \beta]$ be a (bounded) closed interval. Let S be a set of closed subintervals of I which cover every one-sided neighborhood of every point in I. (That is, for any $\gamma \in [\alpha, \beta)$, there exist $\delta \in (\gamma, \beta]$ and $J \in S$ such that $[\gamma, \delta] \subseteq J$ and, similarly, for any $\gamma \in (\alpha, \beta]$, there exist $\delta \in [\alpha, \gamma]$ and $J \in S$ such that $[\delta, \gamma] \subseteq J$.) Then there exists a finite subset of S with union I.*

Proof. Exercise. □

8.1 Power series on closed discs and annuli

We start by introducing some rings that should be thought of as the analytic functions on a closed disc $|t| \leq \beta$ or a closed annulus $\alpha \leq |t| \leq \beta$. As noted above, this is more properly done in a framework of p-adic analytic geometry, but nevertheless we choose to avoid this framework.

Definition 8.1.1. For $\alpha, \beta > 0$, put

$$K\langle \alpha/t, t/\beta \rangle = \left\{ \sum_{i \in \mathbb{Z}} c_i t^i \in K[\![t, t^{-1}]\!] : \lim_{i \to \pm\infty} |c_i| \rho^i = 0 \quad (\rho \in [\alpha, \beta]) \right\}.$$

That is, consider the set of formal bidirectional power series which converge whenever one inserts a value for t with $|t| \in [\alpha, \beta]$ or, in other words, when $\alpha/|t|$ and $|t|/\beta$ are both at most 1; it suffices to check for $\rho = \alpha$ and $\rho = \beta$. Although formal bidirectional power series do not form a ring, the subset $K\langle \alpha/t, t/\beta \rangle$ does form a ring under the expected operations. (In this and all similar notation, we will omit β when it is equal to 1.)

Definition 8.1.2. If $\alpha = 0$, the only reasonable interpretation of the previous definition is to require $c_i = 0$ for $i < 0$. When there are no negative powers of t, it is redundant to require convergence for $\rho < \beta$. In other words, we shall define

$$K\langle 0/t, t/\beta\rangle = K\langle t/\beta\rangle = \left\{ \sum_{i=0}^{\infty} c_i t^i \in K[\![t]\!] : \lim_{i\to\infty} |c_i|\beta^i = 0 \right\}.$$

One can also write $K\langle \alpha/t\rangle$, for which the implied value of β is ∞. More succinctly put, we identify $K\langle \alpha/t, t/\beta\rangle$ with $K\langle \beta^{-1}/t^{-1}, t^{-1}/\alpha^{-1}\rangle$. We do not allow both α and β to equal 0, as this case is exceptional: no condition is then imposed on the power series except that negative indices cannot occur. This results in the ring $K[\![t]\!]$.

Remark 8.1.3. Note that if $\alpha \le \gamma \le \beta \le \delta$ then, within $K\langle \gamma/t, t/\beta\rangle$,

$$K\langle \alpha/t, t/\beta\rangle \cap K\langle \gamma/t, t/\delta\rangle = K\langle \alpha/t, t/\delta\rangle.$$

This will serve as the basis for some gluing arguments later.

Definition 8.1.4. We will also occasionally use the intermediate ring

$$K[\![t/\beta]\!]_0 = \left\{ \sum_{i=0}^{\infty} c_i t^i \in K[\![t]\!] : \sup_i\{|c_i|\beta^i\} < \infty \right\};$$

these are the power series which converge and take bounded values on the open disc $|t| < \beta$. (The notation will make more sense once we have also defined $K[\![t/\beta]\!]_\delta$ for $\delta > 0$; see Definition 18.4.1.) Note that, for any $\delta \in (0, \beta)$,

$$K\langle t/\beta\rangle \subset K[\![t/\beta]\!]_0 \subset K\langle t/\delta\rangle.$$

We will most often use this construction with $\beta = 1$, in which case we can also write

$$K[\![t]\!]_0 = \mathfrak{o}_K[\![t]\!] \otimes_{\mathfrak{o}_K} K.$$

Definition 8.1.5. An analogue of the previous construction for an annulus is

$$K\langle \alpha/t, t/\beta]\!]_0 = \left\{ \sum_{i\in\mathbb{Z}} c_i t^i : c_i \in K,\ \lim_{i\to-\infty} |c_i|\alpha^i = 0,\ \sup_i\{|c_i|\beta^i\} < \infty \right\};$$

these are the Laurent series which converge and take bounded values on the half-open annulus $\alpha \le |t| < \beta$. For any $\delta \in [\alpha, \beta)$, this ring satisfies

$$K\langle \alpha/t, t/\beta\rangle \subset K\langle \alpha/t, t/\beta]\!]_0 \subset K\langle \alpha/t, t/\delta\rangle.$$

One can also use the boundedness condition on both sides, defining

$$K[\![\alpha/t, t/\beta]\!]_0 = \left\{ \sum_{i\in\mathbb{Z}} c_i t^i : c_i \in K,\ \sup_i\{|c_i|\alpha^i\} < \infty,\ \sup_i\{|c_i|\beta^i\} < \infty \right\}.$$

Remark 8.1.6. The rings of bounded series behave well only in the case where K is discretely valued; otherwise, they are not even noetherian (exercise). For general K it is better to work with rings of analytic elements; see Section 8.5.

8.2 Gauss norms and Newton polygons

The rings $K\langle\alpha/t, t/\beta\rangle$ behave quite like polynomial rings (or Laurent polynomial rings, in the case $\alpha \neq 0$) in one variable. The next few statements are all instances of this analogy.

Definition 8.2.1. From the definition of $K\langle\alpha/t, t/\beta\rangle$, we see that it carries a well-defined ρ-Gauss norm

$$\left|\sum_i c_i t^i\right|_\rho = \max_i\{|c_i|\rho^i\}$$

for any $\rho \in [\alpha, \beta]$. For $\rho = \alpha = 0$, this reduces to simply $|c_0|$. The fact that this is a multiplicative norm follows as in Proposition 2.1.2. In the additive version (Gauss valuation) one takes $r \in [-\log\beta, -\log\alpha]$ and puts

$$v_r\left(\sum_i c_i t^i\right) = \min_i\{v(c_i) + ri\},$$

where $v(c) = -\log|c|$. There is also a β-Gauss norm on $K\langle\alpha/t, t/\beta]\!]_0$, although it must be defined as a supremum that may fail to be achieved if K is not discretely valued.

Definition 8.2.2. One may define the *Newton polygon* for an element $x = \sum_i x_i t^i \in K\langle\alpha/t, t/\beta\rangle$ as the boundary of the lower convex hull of the set

$$\{(-i, v(x_i)) : i \in \mathbb{Z}, x_i \neq 0\},$$

retaining only those slopes within $[-\log\beta, -\log\alpha]$.

Proposition 8.2.3. *Let $x = \sum_i x_i t^i \in K\langle\alpha/t, t/\beta\rangle$ be nonzero.*

(a) *The Newton polygon of x has finite width.*

(b) *The function $r \mapsto v_r(x)$ on $[-\log\beta, -\log\alpha]$ is continuous, piecewise affine, and concave. Moreover, even if $\alpha = 0$ there are only finitely many different slopes.*

(c) *The function $\rho \mapsto |x|_\rho$ on $[\alpha, \beta]$ is continuous and log-convex. The log-convexity means that if $\rho, \sigma \in [\alpha, \beta]$, $c \in [0, 1]$, and $\tau = \rho^c\sigma^{1-c}$ then*

$$|x|_\tau \leq |x|_\rho^c |x|_\sigma^{1-c}.$$

(d) *If* $\alpha = 0$ *then* $v_r(x)$ *is increasing on* $[-\log\beta, +\infty)$*; in other words, for all* $\rho \in [0, \beta]$, $|x|_\rho \leq |x|_\beta$.

Part (c) should be thought of as a nonarchimedean analogue of the Hadamard three-circle theorem.

Proof. We have (a) because there is a least i for which $|c_i|\alpha^i$ is maximal, and there is a greatest j for which $|c_j|\beta^j$ is maximal. This implies (b) because, as in the polynomial case, we may interpret $v_r(x)$ as the y-intercept of the supporting line of the Newton polygon of slope r. This in turn implies (c), and (d) is a remark made earlier. □

Remark 8.2.4. The analogue of Proposition 8.2.3 for $x \in K\langle\alpha/t, t/\beta\rrbracket_0$ also holds *except* that if K is not discretely valued then $v_r(x)$ need not be piecewise affine in a one-sided neighborhood of $r = -\log\beta$ (exercise).

When dealing with the ring $K\langle\alpha/t, t/\beta\rangle$, the following completeness property will be extremely useful.

Proposition 8.2.5. *The rings* $K\langle\alpha/t, t/\beta\rangle$ *and* $K\langle\alpha/t, t/\beta\rrbracket_0$ *are Fréchet complete for norms* $|\cdot|_\rho$ *for all* $\rho \in [\alpha, \beta]$. *That is, if* $\{x_n\}_{n=0}^\infty$ *is a sequence which is simultaneously Cauchy under* $|\cdot|_\rho$ *for all* $\rho \in [\alpha, \beta]$ *then it is convergent. (By Proposition 8.2.3 it suffices to check the Cauchy property at* $\rho = \alpha, \beta$*, or just at* $\rho = \beta$ *in the case* $\alpha = 0$.*)*

Proof. Exercise. □

The completeness property is used in the construction of multiplicative inverses, for instance.

Lemma 8.2.6

(a) *A nonzero element* $f \in K\langle t/\beta\rangle$ *is a unit if and only if there exists* $c \in K^\times$ *such that* $|f - c|_\beta < |f|_\beta$.

(b) *A nonzero element* $f \in K\langle\alpha/t, t/\beta\rangle$ *is a unit if and only if there exist* $c \in K^\times$ *and* $i \in \mathbb{Z}$ *such that* $|f - ct^i|_\rho < |f|_\rho$ *for* $\rho = \alpha, \beta$.

(c) *If* $\alpha < \beta$ *then a nonzero element* $f \in K\langle\alpha/t, t/\beta\rrbracket_0$ *is a unit if and only if there exist* $c \in K^\times$ *and* $i \in \mathbb{Z}$ *such that* $|f - ct^i|_\rho < |f|_\rho$ *for all* $\rho \in [\alpha, \beta)$.

Proof. If f is a unit then the Newton polygon of f has no slope in $[\alpha, \beta]$; this implies the forward implication in all cases. For the reverse implication in cases (a) and (b), it suffices to check that if $|f - 1|_\rho < |f|_\rho$, for $\rho = \alpha$ in case (a) or for $\rho = \alpha, \beta$ in case (b), then f is a unit. This holds because the series

$$\sum_{j=0}^{\infty}(1-f)^j \tag{8.2.6.1}$$

converges by Proposition 8.2.5, and its limit is the inverse of f.

For the reverse implication in case (c), we note that the series (8.2.6.1) converges in $K\langle\alpha/t, t/\delta\rangle$ for each $\delta \in [\alpha, \beta)$. Moreover, the terms of the sum have bounded β-norm, so the limit does also. $\square$

8.3 Factorization results

We need a number of results to the effect that elements of one ring that we are considering can be factored into "positive" and "negative" parts. The basic result of this form may be viewed as a form of the Weierstrass preparation theorem.

Proposition 8.3.1 (Weierstrass preparation). *Assume one of the following two sets of conditions.*

(a) *Put $R = K\langle\alpha/t, t/\beta\rangle$ or $R = K\langle\alpha/t, t/\beta]\!]_0$. Given $f = \sum_{i\in\mathbb{Z}} f_i t^i \in R$ and $\rho \in [\alpha, \beta]$, suppose that there is a unique $m \in \mathbb{Z}$ maximizing $|f_m|\rho^m$.*

(b) *Put $R = K\langle\alpha/t, t/\alpha]\!]_0$ and put $\rho = \alpha$. Given $f = \sum_{i\in\mathbb{Z}} f_i t^i \in R$, suppose that the supremum of $|f_i|\rho^i$ is achieved by at least one i (this is only guaranteed to hold if K is discretely valued), and let m be the least such i.*

Then there is a unique factorization $f = f_m t^m gh$, with

$$g \in R \cap K[\![t]\!], \qquad h \in R \cap K[\![t^{-1}]\!],$$

such that $|g|_\rho = |g_0| = 1$ and $|h-1|_\rho < 1$.

Proof. As in Theorem 2.2.1, we invoke the master factorization theorem (Theorem 2.2.2). This gives a factorization of the desired form in the completion of R with respect to $|\cdot|_\rho$.

However, within this completion, define the subring $R' = K\langle\rho/t, t/\beta\rangle$ if $R = K\langle\alpha/t, t/\beta\rangle$ or $R' = K\langle\rho/t, t/\beta]\!]_0$ if $R = K\langle\alpha/t, t/\beta]\!]_0$. Then h is a unit in $R' \cap K[\![t^{-1}]\!] = K\langle\rho/t\rangle$ and hence in R'. We thus have $g = fh^{-1} \in R' \cap K[\![t]\!] = R \cap K[\![t]\!]$. We may similarly deduce that $h \in R \cap K[\![t^{-1}]\!]$. $\square$

In light of the finite-width property of the Newton polygon, the following should not be a surprise. (One can only replace $K\langle\alpha/t, t/\beta\rangle$ with $K\langle\alpha/t, t/\beta]\!]_0$ if K is discretely valued; see the exercises.)

Proposition 8.3.2 (More Weierstrass preparation). *For $f \in K\langle \alpha/t, t/\beta \rangle$ with $\beta < +\infty$, there exists a polynomial $P \in K[t]$ and a unit $g \in K\langle \alpha/t, t/\beta \rangle^{\times}$ such that $f = Pg$. In particular, $K\langle \alpha/t, t/\beta \rangle$ is a principal ideal domain (even if $\beta = +\infty$).*

Proof. If $\alpha > 0$ then we may apply Proposition 8.3.1(b) to f since the Newton polygon has only finitely many slopes. We may thus factor f in $R = K\langle \alpha/t, t/\alpha \rrbracket_0$ as $f_m t^m gh$ with $f_m \in K$, $m \in \mathbb{Z}$, $g \in R \cap K\llbracket t \rrbracket = K\llbracket t/\alpha \rrbracket_0$, $h \in R \cap K\llbracket t^{-1} \rrbracket = K\langle \alpha/t \rangle$, $|g|_\alpha = |g_0| = 1$, and $|h - 1|_\alpha < 1$. In particular h is a unit in $K\langle \alpha/t \rangle$ by Lemma 8.2.6, so

$$h^{-1} f \in K\langle \alpha/t, t/\beta \rangle \cap t^m K\llbracket t/\alpha \rrbracket_0 \subseteq t^m K\langle t/\beta \rangle.$$

If $\alpha = 0$, the same conclusion holds with $h = 1, m = 0$.

Next, factor $h^{-1} f$ in $K\langle \beta^{-1}/t^{-1}, t^{-1}/\beta^{-1} \rrbracket_0$ and argue as above, to get an "associate element" of f in $K\langle \alpha/t, t/\beta \rangle$ which belongs to $t^m K\langle t/\beta \rangle \cap t^n K\langle \alpha/t \rangle$ for some integer n. This element must now belong to $K[t]$ if $\alpha = 0$ or to $K[t, t^{-1}]$ if $\alpha > 0$; in either case we may deduce the claim. □

We next wish to generalize the previous considerations to matrices, but this will require a bit more care.

Lemma 8.3.3. *Let A be an invertible $n \times n$ matrix over $K\langle \alpha/t, t/\beta \rangle$. Then A can be factored as a product of elementary matrices.*

Proof. We first note that we can perform a sequence of elementary row operations after which A_{11} becomes a unit and A_{i1} becomes zero for $i = 2, \dots, n$. Namely, using Proposition 8.3.2, multiply each row by a suitable unit to replace each A_{i1} by an element of $K[t]$. Then perform the Euclidean algorithm in $K[t]$ on the A_{i1} to achieve the desired result.

To finish, it suffices to note that we can now perform the usual Gauss–Jordan elimination over A: repeatedly apply the previous paragraph to construct a sequence of row operations putting A into upper triangular form and then perform column operations to eliminate entries above the diagonal. □

Lemma 8.3.4. *Let A be an invertible $n \times n$ matrix over $K\langle t/\rho \rangle$ (resp. over $K\langle \rho/t, t/\rho \rangle$.) Then there exists $U \in \mathrm{GL}_n(K[t])$ (resp. $U \in \mathrm{GL}_n(K[t, t^{-1}])$) such that $|UA - I_n|_\rho < 1$.*

Proof. Note that in the notation of Definition 4.0.3, conjugating an elementary matrix of type (c) (scaling a row) by one of type (a) (swapping rows) produces another of type (c), whereas conjugating an elementary matrix of type (b) (adding a multiple of one row to another) by one of type (c) produces another of type (b). Consequently, any sequence of elementary row operations has the same effect as another sequence in which all steps of type (a) and (b) happen before all steps of type (c).

By Lemma 8.3.3 we can perform a sequence of elementary row operations on A that produces the identity matrix. By this, plus the previous paragraph, we can perform another sequence of elementary row operations, of types (a) and (b), on A that produces a diagonal matrix D. By appending additional row operations of type (c) with $c \in K^\times$ (resp. $c \in K^\times t^{\mathbb{Z}}$), we can force $|D - I_n|_\rho < 1$.

Let $R_1, \dots, R_m$ be elementary matrices as above with $R_1 \cdots R_m A = D$; note that those R_j of type (a) and (c) are already elementary matrices over $K[t]$ (resp. over $K[t, t^{-1}]$), the case of type (c) being handled by the previous paragraph. Put

$$\delta = \max\{1, |A|^{-1}\} \prod_{j=1}^{m} \max\{1, |R_j|^{-1}\}.$$

For R_j of type (b), let R'_j be an elementary matrix of type (b) over $K[t]$ (resp. over $K[t, t^{-1}]$) with $|R_j - R'_j|_\rho < \delta^{-1}$; for R_j not of type (b), put $R'_j = R_j$. Then $|R'_1 \cdots R'_m A - I_n|_\rho < 1$, proving the claim. □

We can now deduce a Weierstrass preparation theorem for matrices.

Proposition 8.3.5. *Let A be an invertible $n \times n$ matrix over $K\langle \alpha/t, t/\beta \rangle$. Then there exist $U \in \mathrm{GL}_n(K\langle t/\beta \rangle)$ and $V \in \mathrm{GL}_n(K\langle \alpha/t \rangle)$ such that $A = UV$.*

Proof. Pick any $\rho \in [\alpha, \beta]$. By Lemma 8.3.4, we can find $U_1 \in \mathrm{GL}_n(K[t, t^{-1}])$ such that $|U_1 A - I_n|_\rho < 1$. By applying the master factorization theorem (Theorem 2.2.2) in the (noncommutative) ring of $n \times n$ matrices over $K\langle \rho/t, t/\rho \rangle$ and then arguing as in Proposition 8.3.1, we can factor $U_1 A$ as $U_2 V_2$ with $U_2 \in \mathrm{GL}_n(K\langle t/\beta \rangle)$ and $V_2 \in \mathrm{GL}_n(K\langle \alpha/t \rangle)$. We may then set $U = U_1^{-1} U_2$ and $V = V_2$. □

Our main application of Proposition 8.3.5 is the following gluing lemma, which we will invoke frequently and often implicitly.

Lemma 8.3.6 (Gluing lemma). *Suppose that $\alpha \le \gamma \le \beta \le \delta$. Let M_1 be a finite free module over $K\langle \alpha/t, t/\beta \rangle$, let M_2 be a finite free module over $K\langle \gamma/t, t/\delta \rangle$, and suppose that we are given an isomorphism*

$$\psi : M_1 \otimes K\langle \gamma/t, t/\beta \rangle \cong M_2 \otimes K\langle \gamma/t, t/\beta \rangle.$$

Then we can find a finite free module M over $K\langle \alpha/t, t/\delta \rangle$ and isomorphisms $M_1 \cong M \otimes K\langle \alpha/t, t/\beta \rangle$, $M_2 \cong M \otimes K\langle \gamma/t, t/\delta \rangle$ inducing ψ. Moreover, M is determined by this requirement up to unique isomorphism.

Proof. We will explain only the case $\alpha > 0$; the case $\alpha = 0$ is similar. Choose bases $v_{1,1}, \dots, v_{1,n}$ and $v_{2,1}, \dots, v_{2,n}$ of M_1 and M_2, respectively. Let A be the change-of-basis matrix from the $v_{1,i}$ to the $v_{2,i}$, viewing both as bases of $M_1 \otimes K\langle\gamma/t, t/\beta\rangle \cong M_2 \otimes K\langle\gamma/t, t/\beta\rangle$ via ψ. By Proposition 8.3.5 we can factor A as UV with $U \in \mathrm{GL}_n(K\langle t/\beta\rangle)$ and $V \in \mathrm{GL}_n(K\langle\gamma/t\rangle)$.

We can then construct a finite free module M over $K\langle\alpha/t, t/\delta\rangle$ equipped with a basis $v_1, \dots, v_n$ such that the change-of-basis matrices from this basis to the $v_{1,i}$ and to the $v_{2,i}$ are U^{-1} and V, respectively. This is the desired module. □

Remark 8.3.7. If $\alpha \le \gamma \le \beta < \delta$, one can similarly glue together a finite free module over $K\langle\alpha/t, t/\beta\rangle$ and a finite free module over $K\langle\gamma/t, t/\delta]\!]_0$, whose base extensions to $K\langle\gamma/t, t/\beta\rangle$ are isomorphic; the result is a finite free module over $K\langle\alpha/t, t/\delta]\!]_0$. As we will not use this fact, we omit further details.

8.4 Open discs and annuli

Although we have been discussing closed discs so far, it is quite natural also to consider open discs. One important reason is that the antiderivative of an analytic function on the closed disc of radius β is only defined on the *open* disc of radius β (see exercises for Chapter 9).

Definition 8.4.1. Define the ring

$$K\{t/\beta\} = \left\{\sum_{i=0}^{\infty} c_i t^i : c_i \in K, \lim_{i\to\infty} |c_i|\rho^i = 0 \quad (\rho \in (0,\beta))\right\};$$

these are the power series convergent on the open disc $|t| < \beta$, with no boundedness restriction. Note that we can write

$$K\{t/\beta\} = \bigcap_{\delta\in(0,\beta)} K\langle t/\delta\rangle;$$

in particular, for any $\delta \in (0,\beta)$,

$$K[\![t/\beta]\!]_0 \subset K\{t/\beta\} \subset K\langle t/\delta\rangle.$$

Definition 8.4.2. An analogue of the previous construction for an annulus is

$$K\langle\alpha/t, t/\beta\} = \left\{\sum_{i\in\mathbb{Z}} c_i t^i : c_i \in K, \lim_{i\to-\infty} |c_i|\alpha^i = 0, \lim_{i\to+\infty} |c_i|\rho^i = 0 \quad (\rho \in (0,\beta))\right\};$$

these are the Laurent series convergent on the half-open annulus $\alpha \le |t| < \beta$.

These rings behave worse than their closed counterparts (see the exercises), so we will only make occasional use of them. More often, we will work with the following definition, which is motivated by considerations from rigid analytic geometry.

Definition 8.4.3. Consider the region $|t| \in I$, for any interval $I \subseteq [0, +\infty)$; this could be a closed or open disc or a closed, open, or half-open annulus. By a *coherent locally free module M* on this region, we will mean a sequence of finite free modules M_i over $K\langle \alpha_i/t, t/\beta_i\rangle$, with $[\alpha_1, \beta_1] \subseteq [\alpha_2, \beta_2] \subseteq \cdots$ an increasing sequence of closed intervals with union I, together with isomorphisms $M_{i+1} \otimes K\langle \alpha_i/t, t/\beta_i\rangle \cong M_i$. Using Lemma 8.3.6 we check that the construction is canonically independent of the choice of sequence. However, the resulting object is not *finite* in the sense of admitting a finite generating set over the entire annulus, except when K is spherically complete. For a special case, see Proposition 16.1.4; the notes give further discussion.

8.5 Analytic elements

An intermediate construction between open and closed discs is the ring $K\llbracket t/\beta \rrbracket_0$ of bounded power series but, as noted previously, it behaves badly if K is not discretely valued. Another intermediate construction that behaves somewhat better is the following.

Definition 8.5.1. Define the ring $K\llbracket t/\beta \rrbracket_{\mathrm{an}}$ by starting with the subring of $K(t)$ consisting of rational functions with no poles in the disc $|t| < \beta$ and then completing for the β-Gauss norm. This is the ring of *analytic elements* on the open disc $|t| < \beta$; it satisfies

$$K\langle t/\beta\rangle \subset K\llbracket t/\beta \rrbracket_{\mathrm{an}} \subset K\llbracket t/\beta \rrbracket_0.$$

Analogously, we define the ring $K\langle \alpha/t, t/\beta\rrbracket_{\mathrm{an}}$ of analytic elements on the half-open annulus $\alpha \leq |t| < \beta$ as follows. Start with the subring of $K(t)$ consisting of rational functions with no poles in the annulus $\alpha \leq |t| < \beta$. Then take the Fréchet completion for the ρ-Gauss norms with $\alpha \leq \rho \leq \beta$.

One may also define the ring $K\llbracket \alpha/t, t/\beta \rrbracket_{\mathrm{an}}$ of analytic elements on the open annulus $\alpha < |t| < \beta$. For this, we start with the subring of $K(t)$ consisting of rational functions with no poles in the annulus $\alpha < |t| < \beta$ and then again take the Fréchet completion for the ρ-Gauss norms with $\alpha \leq \rho \leq \beta$. In the case $\alpha = \beta$ this construction gives a field; we will see this field again in Definition 9.4.1.

For analytic elements, we have analogues of many properties asserted for analytic functions.

Proposition 8.5.2. *Let R be $K⟦t/\beta⟧_{\mathrm{an}}$, $K\langle\alpha/t, t/\beta⟧_{\mathrm{an}}$, or $K⟦\alpha/t, t/\beta⟧_{\mathrm{an}}$.*

(a) *For any $x \in R$, the function $r \mapsto -\log|x|_{e^{-r}}$ is continuous and concave in r.*
(b) *For any $x \in R$, the Newton polygon of x has finite width.*
(c) *Any $x \in R$ can be written as a polynomial $P \in K[t]$ times a unit in R.*
(d) *The ring R is a principal ideal domain.*

Proof. Since we are taking the Fréchet completion for norms over a closed interval, the convergence of any Cauchy sequence must be uniform in the different norms. We may thus deduce (a) from the corresponding assertions when $x \in K(t)$ with no poles in the appropriate disc or annulus.

We will check (b) for $K⟦t/\beta⟧_{\mathrm{an}}$; the other cases are analogous. Let $x = \sum_{i=0}^{\infty} x_i t^i \in K⟦t/\beta⟧_{\mathrm{an}}$ be any nonzero ring element. We can then choose a rational function $f \in K(t)$, with no poles in the disc $|t| < \beta$, such that $|x-f|_\beta < |x|_\beta$. By (a), $|x|_\rho$ is continuous in ρ, so we also have $|x-f|_\rho < |x|_\rho$ for ρ in a neighborhood of β. Consequently, those slopes of x sufficiently close to β occur with the same multiplicities as the corresponding slopes of f. But f is a rational function, so it has no slopes in some punctured neighborhood of β. This proves that x has no slopes in some punctured neighborhood of β either, so its Newton polygon has finite width.

To check (c) and (d), we may use the same proof as in Proposition 8.3.2. □

Corollary 8.5.3. *The ring $\cup_{\alpha<\beta} K\langle\alpha/t, t/\beta⟧_{\mathrm{an}}$ is a field.*

Proof. By Proposition 8.5.2 every element of $K\langle\alpha/t, t/\beta⟧_{\mathrm{an}}$ can be written as a unit in $K\langle\alpha/t, t/\beta⟧_{\mathrm{an}}$ times a polynomial. It thus suffices to observe that, for any $P \in K[t]$, we can choose α so that none of the roots of P lie in the annulus $\alpha \leq |t| < \beta$; for such an α, P is a unit in $K\langle\alpha/t, t/\beta⟧_{\mathrm{an}}$. □

We will use the following optimal approximation property.

Lemma 8.5.4. *Let R be $K\langle t/\beta\rangle$, $K\langle\beta/t, t/\beta\rangle$, or $K⟦t/\beta⟧_{\mathrm{an}}$. Let F be the completion of* Frac(R) *under $|\cdot|_\beta$. Then, for any $f \in F$, there exists $g \in R$ minimizing $|f-g|_\beta$.*

Proof. We may assume $f \notin R$, as otherwise $g = f$ fulfills the lemma. Put $c = \inf\{|f-g|_\beta : g \in R\}$. Since R is complete under $|\cdot|_\beta$ we have $c > 0$. Since $K(t)$ is dense in F we can choose $h \in K(t)$ such that $|f-h|_\beta < c$. Let $P \in K[t]$ be the (monic) denominator of h. By Theorem 2.2.1 we may factor P as P_1P_2 in such a way that no irreducible factor of P_1 in $K[t]$ is a unit in R whereas every irreducible factor of P_2 in $K[t]$ is a unit in R. (The point is that each irreducible polynomial has only a single slope in its Newton polygon, and whether the polynomial is a unit is determined entirely by this slope.) Since the claims for f and fP_2 are equivalent, we may assume that $P_2 = 1$. Using the division algorithm, write $hP = gP + S$ with $g, S \in K[t]$ and $\deg(S) < \deg(P)$. We claim that this choice of g works; that is, for any $g' \in R$ we have $|f - g'|_\beta \geq |f - g|_\beta$.

Suppose first that $R = K\langle t/\beta\rangle$; in this case $R \cap K[t]$ is dense in R. Given $g' \in R$, we may choose $g'' \in K[t]$ such that $|g' - g''|_\beta < c$. Applying Lemma 2.3.1 to the instance $(h - g'')P = (g - g'')P + S$ of the division algorithm yields

$$|(h - g'')P|_\beta = \max\{|(g - g'')P|_\beta, |S|_\beta\} \geq |S|_\beta.$$

Therefore $|h - g''|_\beta \geq |S/P|_\beta = |h - g|_\beta$; since $|f - h|_\beta, |g' - g''|_\beta < c \leq |h - g|_\beta$, we also have $|f - g'|_\beta \geq |f - g|_\beta$ as claimed.

Suppose next that $R = K\langle \beta/t, t/\beta\rangle$; in this case $R \cap K[t, t^{-1}]$ is dense in R. Given $g' \in R$, we may choose $g'' \in K[t]$ and $m \geq 0$ such that $|g' - g''t^{-m}|_\beta < c$. Using the division algorithm to divide $(ht^m - g'')P$ by P now returns the same remainder as does dividing $t^m S$ by P. We may argue as in the previous case once we have checked that, for $S \in K[t]$ with $\deg(S) < \deg(P)$, the remainder S' upon dividing tS by P satisfies $|S'|_\beta = \beta|S|_\beta$. We proceed as in the proof of Lemma 2.3.1, as follows. Put $d = \deg(P)$ and let S_{d-1} be the coefficient of t^{d-1} in S. If $|S|_\beta = |S_{d-1}t^{d-1}|_\beta$ then the constant term of S' is equal to the remainder upon dividing $S_{d-1}t^d$ by P, which has norm $\beta|S|_\beta$. Otherwise $|S' - t(S - S_{d-1}t^{d-1})|_\beta < \beta|S|_\beta$, so the claim follows again.

Finally, suppose that $R = K[\![t/\beta]\!]_{\mathrm{an}}$. Given $g' \in R$, we may choose $g'', Q \in K[t]$ such that Q is monic with all roots of norm β and $|g' - g''/Q|_\beta < c$. Using the division algorithm to divide $(hQ - g'')P$ by P now returns the same remainder as does dividing QS by P. If we denote the latter by S', we may now argue as in the previous cases once we have shown that $|S'|_\beta = |S|_\beta$. Put $d = \deg(P)$; in this case, all the roots of P have norm less than β, so $|P - t^d|_\beta < 1$. Hence $|S'|_\beta = |S''|_\beta$, where S'' is the remainder upon dividing QS by t^d. But it is clear that $|S''|_\beta = |S|_\beta$: we may check this after dividing S and t^d by any common factors of t, at which point $|S|_\beta$ and $|S''|_\beta$ are achieved as the respective constant terms of S and S''. □

Remark 8.5.5. Note that the proof of Lemma 8.5.4 actually shows something a bit stronger: the constructed element $g \in R$ continues to minimize $|f - g|_\beta$ even if we replace K with a complete extension L.

Remark 8.5.6. If K is discretely valued then the conclusion of Lemma 8.5.4 also holds for $R = K\llbracket t \rrbracket_0$ and $\beta = 1$, since we can then write F as the completion of $K\llbracket t \rrbracket_0[t^{-1}]$.

8.6 More approximation arguments

We now give some variants of the approximation lemmas (Lemmas 1.3.7 and 1.5.5) that involve rings of power series.

Lemma 8.6.1. *Let R be $K\langle t \rangle$, $K\langle 1/t, t \rangle$, $K\llbracket t \rrbracket_{\mathrm{an}}$, or (if K is discretely valued) $K\llbracket t \rrbracket_0$ equipped with the 1-Gauss norm. Let F be the completion of* $\mathrm{Frac}(R)$ *under $|\cdot|_1$. Let M be a finite free R-module. Let $|\cdot|_M$ be a norm on M, compatible with R, obtained by restriction from the supremum norm defined by some basis of $M \otimes_R F$. Then $|\cdot|_M$ is the supremum norm defined by some basis of M.*

Proof. (Thanks to Liang Xiao for his help with this proof.) Let N be the $\mathfrak{o}_F$-span of a basis of $M \otimes_R F$ whose supremum norm restricts to $|\cdot|_M$. Put $R_1 = \{r \in R : |r|_1 \leq 1\}$ and $M_1 = \{x \in M : |x|_M \leq 1\}$. Note that $\mathfrak{m}_K R_1$ coincides with the subring of R consisting of series whose coefficients all belong to $\mathfrak{m}_K$. (If $R = K\llbracket t \rrbracket_0$ this holds only if K is discretely valued.) Hence the ring $R_1/\mathfrak{m}_K R_1$ is either $\kappa_K\llbracket t \rrbracket$ or a localization of $\kappa_K[t]$, so in any case it is a principal ideal domain.

Note that $M_1/\mathfrak{m}_K M_1$ embeds into $N/\mathfrak{m}_K N$, so in particular it is torsion-free as a module over $R_1/\mathfrak{m}_K R_1$. Since $\mathrm{Frac}(R)$ is dense in F, we can choose a basis $y_1, \dots, y_n$ of N which is also a basis of $M \otimes_R \mathrm{Frac}(R)$. We can then find an $f \in R$ which is nonzero and such that M contains $fy_1, \dots, fy_n$; by multiplying by a unit in K, we may normalize f so that $|f|_1 = 1$. Then $fy_1, \dots, fy_n$ project to elements of $M_1/\mathfrak{m}_K M_1$ which form a basis of $N/\mathfrak{m}_K N$ over κ_F.

We can also find a $g \in R$ which is nonzero and such that M is contained in the R-span of $g^{-1}y_1, \dots, g^{-1}y_n$, which we again normalize so that $|g|_1 = 1$. This means that $M_1/\mathfrak{m}_K M_1$ is contained in the $(R_1/\mathfrak{m}_K R_1)$-submodule of $N/\mathfrak{m}_K N$ generated by $g^{-1}y_1, \dots, g^{-1}y_n$. Since $R_1/\mathfrak{m}_K R_1$ is a principal ideal domain, it follows that $M_1/\mathfrak{m}_K M_1$ is finitely generated, torsion-free, and hence finite free as a module over $R_1/\mathfrak{m}_K R_1$. By the previous paragraph, any basis of $M_1/\mathfrak{m}_K M_1$ over $R_1/\mathfrak{m}_K R_1$ freely generates $N/\mathfrak{m}_K N$ over κ_F and hence freely generates N over $\mathfrak{o}_F$.

Let $x_1, \dots, x_n \in M_1$ lift a basis of $M_1/\mathfrak{m}_K M_1$ over $R_1/\mathfrak{m}_K R_1$. For any $x \in M_1$ we have a unique representation $x = r_1x_1 + \cdots + r_nx_n$ with

$r_1, \dots, r_n \in \mathfrak{o}_F$. By Lemma 8.5.4 and Remark 8.5.6 we may choose $r_i' \in R$ to maximize $|r_i - r_i'|_1$. Choose $\lambda \in K$ so that $|\lambda| = \max_i\{|r_i - r_i'|_1\}$; if $\lambda \neq 0$ then the image of $\lambda^{-1}(x - r_1'x_1 - \cdots - r_n'x_n)$ in $M_1/\mathfrak{m}_K M_1$ fails to be in the $(R_1/\mathfrak{m}_K R_1)$-span of the images of $x_1, \dots, x_n$. This yields a contradiction, so we must have $\lambda = 0$ and $r_i' = r_i \in R \cap \mathfrak{o}_F = R_1$ for $i = 1, \dots, n$. Consequently, $x_1, \dots, x_n$ form a basis of M with supremum norm $|\cdot|_M$, as desired. □

It is more difficult to deal with the case of $K[\![t]\!]_0$ when K is not discretely valued. Although we will not use that case in what follows, for completeness we mention one result that applies to it.

Lemma 8.6.2. *Suppose that K is spherically complete with value group $\mathbb{R}$. Let M be a finite free module over $K[\![t/\beta]\!]_0$ for some $\beta > 0$. Let $|\cdot|_M$ be a supremum-equivalent norm on M that is compatible with $K[\![t/\beta]\!]_0$. For $j = 1, 2, \dots,$ let $|\cdot|_j$ be the quotient seminorm on M/t^jM induced by $|\cdot|_M$. Suppose that, for any $m \in M$,*

$$|m|_M = \lim_{j\to\infty} |m|_j. \tag{8.6.2.1}$$

Then $|\cdot|_M$ is the supremum norm for some basis of M.

Proof. By rescaling we reduce immediately to the case $\beta = 1$. Put $R = K[\![t]\!]_0$ and $R_1 = \mathfrak{o}_K[\![t]\!]$, so that R_1 is the set of $r \in R$ having norm at most 1. Let M_1 be the set of $x \in M$ having $|x|_M \leq 1$; then M_1 is a R_1-submodule of M. Since $|\cdot|_M$ is compatible with R, for any positive integer j we have $M_1 \cap t^jM = t^jM_1$.

By assumption we can find a basis $m_1', \dots, m_n'$ of M which determines a supremum norm equivalent to $|\cdot|_M$. That is, there exist $c_1, c_2 > 0$ such that, for any $a_1, \dots, a_n \in R$,

$$c_1 \max_i\{|a_i|_R\} \leq |a_1m_1' + \cdots + a_nm_n'|_M \leq c_2 \max_i\{|a_i|_R\}.$$

As in the proof of Lemma 1.3.5 it follows that, for any positive integer j and any $a_1', \dots, a_n' \in R/t^jR$,

$$c_1 \max_i\{|a_i'|_{R/t^jR}\} \leq |a_1'm_1' + \cdots + a_n'm_n'|_j \leq c_2 \max_i\{|a_i'|_{R/t^jR}\}. \tag{8.6.2.2}$$

In particular, each $|\cdot|_j$ is a norm.

By Lemma 1.5.5, $|\cdot|_1$ is the supremum norm defined by some basis $m_{1,1}, \dots, m_{n,1}$ of M/tM. By Lemma 1.5.4 (applied by viewing each M/t^jM as a K-vector space), for $i = 1, \dots, n$ and $j = 1, 2, \dots$ we can construct $m_{i,j+1} \in M/t^{j+1}M$ lifting $m_{i,j} \in M/t^jM$ and satisfying $|m_{i,j+1}|_{j+1} = |m_{i,j}|_j$.

For each i, the inverse limit of the $m_{i,j}$ determines an element m_i of $M \otimes_R K[\![t]\!]$. However, by (8.6.2.2), if we write m_i as a $K[\![t]\!]$-linear combination of $m'_1, \dots, m'_n$ then each coefficient is a power series bounded in norm by c_1^{-1} so $m_i \in M$. By (8.6.2.1), $|m_i|_M = 1$.

For any $x \in M$ with $|x|_M \leq 1$, in each $M/t^j M$ we can write $x = a'_{1,j} m_1 + \cdots + a'_{n,j} m_n$ with $a'_{1,j}, \dots, a'_{n,j} \in R/t^j R$. We will prove by induction on j that $|a'_{1,j}|, \dots, |a'_{n,j}| \leq 1$. This is true for $j = 1$ because the $m_1, \dots, m_n$ were chosen so that the supremum norm they define matches $|\cdot|_1$. Given the claim for $j-1$, lift each $a'_{i,j-1} \in R_1/t^{j-1}R_1$ to $b_{i,j} \in R_1/t^j R_1$ by making the last coefficient 0, so that the norm is preserved. Then $t^{1-j}(x - \sum_i b_{i,j} m_i)$ is an element of M/tM of norm at most 1, so it is an $\mathfrak{o}_K$-linear combination of $m_1, \dots, m_n$. This completes the induction.

We thus obtain a representation $x = a_1 m_1 + \cdots + a_n m_n$ with $a_1, \dots, a_n \in R_1$. Since any element of M can be written as an element of K times an element of M of norm at most 1, we deduce that $m_1, \dots, m_n$ is a basis of M and that the supremum norm defined by this basis coincides with $|\cdot|_M$. $\square$

Question 8.6.3. *Is condition* (8.6.2.1) *necessary?*

Notes

The Hadamard three-circles theorem (Proposition 8.2.3(c)) is a special case of the fact that the *Shilov boundary* of the annulus $\alpha \leq |t| \leq \beta$ consists of the two circles $|t| = \alpha$ and $|t| = \beta$. For a considerable amplification of this remark, including a full-blown theory of harmonic functions on Berkovich analytic curves, see [**205**]. For an alternative presentation, restricted to the Berkovich projective line but otherwise more detailed, see [**12**].

The gluing lemma (Lemma 8.3.6) is a special case of the gluing property of coherent sheaves on affinoid rigid analytic spaces, as specified in the theorems of Kiehl and Tate [**31**, Theorems 8.2.1/1 and 9.4.2/3]. The factorization argument in the proof, however, is older still; it is the nonarchimedean version of what is called a *Birkhoff factorization* over an archimedean field. Similarly, Definition 8.4.3 corresponds to the definition of a locally free coherent sheaf on the corresponding rigid or Berkovich analytic space. Such a sheaf is only guaranteed to be freely generated by global sections in the case where K is spherically complete [**128**, Theorem 3.14]; in fact, a previous result of Lazard [**156**] implies that this property, even when restricted to modules of rank 1, is in fact equivalent to the spherical completeness of K.

We again thank Liang Xiao for his help with the proof of Lemma 8.6.1.

Exercises

(1) Prove Lemma 8.0.4. (Hint: for each point of I find an open neighborhood covered by one or two elements of S. Then reduce to the usual compactness property of a bounded closed interval.)

(2) Verify the assertions of Remark 8.2.4. (Hint: a typical example where piecewise affinity fails is $\sum_{n=1}^{\infty} p^{1/n} t^n$.)

(3) Prove Proposition 8.2.5. (Hint: it may be easiest to first construct the limit using a single $\rho \in [\alpha, \beta]$ and then show that this can also be done for the other ρ.)

(4) Prove that if K is discretely valued then, for any $f \in K\langle \alpha/t, t/\beta \rrbracket_0$, there exists a polynomial $P \in K[t]$ and a unit $g \in K\langle \alpha/t, t/\beta \rrbracket_0^\times$ such that $f = Pg$. (Hint: the Newton polygon has finite width in this case, so one may argue as in Proposition 8.3.2.)

(5) Prove that the ring $K\{t\}$ is not noetherian. (Hint: pick a sequence of points in the open unit disc converging to the boundary, and consider the ideal of functions vanishing on all but finitely many of these points.)

(6) Prove that if K is not trivially or discretely valued then $K\llbracket t \rrbracket_0$ is not noetherian. (Hint: proceed as in the previous exercise but choose the points so that the Newton polygon of a function vanishing on all the points has finite height.)

(7) Prove that if K is discretely valued then $\mathfrak{o}_K\langle t \rangle = \mathfrak{o}_K\llbracket t \rrbracket \cap K\langle t \rangle$ is noetherian. Otherwise it is not, because then $\mathfrak{o}_K$ itself is not noetherian.

(8) Prove that each maximal ideal of $\mathfrak{o}_K\langle t \rangle$ is generated by $\mathfrak{m}_K$ together with some $P \in \mathfrak{o}_K[t]$ whose reduction modulo $\mathfrak{m}_K$ is irreducible in $\kappa_K[t]$.

(9) State and prove an analogue of the gluing lemma (Lemma 8.3.6) for gluing together finite free modules over $K\llbracket 1/t, t \rrbracket_{\mathrm{an}}$ and $K\llbracket t \rrbracket_0$, using an isomorphism over the completion of $K\llbracket t \rrbracket_0 \otimes_{K[t]} K(t)$ under $|\cdot|_1$, to obtain a module over $K\llbracket t \rrbracket_{\mathrm{an}}$.

9

Radius and generic radius of convergence

In this chapter, we begin to approach a fundamental question peculiar to the study of nonarchimedean differential modules. It was pointed out in Chapter 0 that a differential module over a nonarchimedean disc can fail to have horizontal sections even in the absence of singularities. The radius of convergence of local horizontal sections is thus an important numerical invariant, whose control is a key factor in being able to produce solutions of p-adic differential equations.

Unfortunately, the radius of convergence is often difficult to compute directly. One of Dwork's fundamental insights is that one can get much better control over the radius of convergence around a so-called *generic point*. The properties of the *generic radius of convergence* can then be used to infer information about the actual convergence of horizontal sections. For instance, Dwork's *transfer theorem* asserts that the radius of convergence of a differential module over a nonarchimedean disc is no less than the generic radius of convergence at the boundary of the disc.

However, both the radius of convergence and the generic radius of convergence are rather coarse invariants. Just as the notion of the spectral radius is refined by the notion of the full spectrum, we can introduce *subsidiary radii of convergence* and *subsidiary generic radii of convergence*, which detect whether some local horizontal sections at a point converge further than others. We will devote much effort in the remainder of this part of the book to analyzing the behavior of these refined invariants.

Hypothesis 9.0.1. Throughout this chapter we will view $K\langle\alpha/t, t/\beta\rangle$, $K[\![\alpha/t, t/\beta]\!]_{\mathrm{an}}$, and so forth as differential rings with derivation $d = d/dt$, formal differentiation in the variable t, unless otherwise specified (as in Section 9.6).

9.1 Differential modules have no torsion

We start with some simple but critical observations about the categories of differential modules over the rings of power series introduced in Chapter 8.

Lemma 9.1.1. *Let R be one of the following rings: $K\langle t/\beta\rangle$, $K[\![t/\beta]\!]_{\mathrm{an}}$, $K\langle\alpha/t, t/\beta\rangle$, $K\langle\alpha/t, t/\beta]\!]_{\mathrm{an}}$, $K[\![\alpha/t, t/\beta]\!]_{\mathrm{an}}$, or (if K is discretely valued) $K[\![t/\beta]\!]_0$ or $K\langle\alpha/t, t/\beta]\!]_0$. Then R has no nonzero proper differential ideal.*

Proof. If I is a nonzero differential ideal then, by Proposition 8.3.2 (or Proposition 8.5.2), I contains a nonzero element $P \in K[t]$. But then I also contains $d^{\deg(P)}(P)$, which is a nonzero element of K (because K is of characteristic 0), so I must be the unit ideal. □

Proposition 9.1.2. *For R as in Lemma 9.1.1, any finite differential module over R is free. Consequently, the finite free differential modules over R form an abelian category.*

Proof. Let M be a finite differential module over R. If $m \in M$ is annihilated by the nonzero element $r \in R$ then $0 = D(rm) = rD(m) + d(r)m$, and so $D(m)$ is annihilated by r^2. Consequently the torsion submodule T of M is also a differential module. Also, T is finite over R because R is a principal ideal domain (by either Proposition 8.3.2 or Proposition 8.5.2) and hence noetherian. Hence the annihilator I of T is a nonzero ideal of R. It is also a differential ideal: if $r \in I$ then, for any $m \in T$, $0 = D(rm) = rD(m) + d(r)m = d(r)m$, so $d(r) \in I$. By Lemma 9.1.1, I must be the trivial ideal. Hence $T = 0$, so M is torsion-free; since R is a principal ideal domain, M must also be free. □

Corollary 9.1.3. *For R as in Lemma 9.1.1 and for M a finite differential module over R, any finite set of horizontal sections which is linearly independent over K forms part of a basis of M.*

Proof. Let S be a finite set of horizontal sections which is linearly independent over K. By Lemma 5.1.5, S is also linearly independent over R. In this case, S determines an injective morphism from a trivial differential module to M. By Proposition 9.1.2 the image of this map must be a direct summand of M as an R-module; this yields the desired result. □

Corollary 9.1.4. *For R as in Lemma 9.1.1, let M be a finite differential module over R, of rank n and admitting a set S of n horizontal sections linearly independent over K. Then M is trivial and $H^0(M)$ is the K-span of S.*

Since we also wish to deal with open discs and annuli, we must formally define differential modules on them.

Definition 9.1.5. Consider the region $|t| \in I$, for any interval $I \subseteq [0, +\infty)$; this region could be a closed or open disc or a closed, open, or half-open annulus. By a *finite differential module M* on this region we will mean a coherent locally free module in the sense of Definition 8.4.3, in which each module M_i carries the structure of a differential module over $K\langle \alpha_i/t, t/\beta_i\rangle$ and each isomorphism $M_{i+1} \otimes K\langle \alpha_i/t, t/\beta_i\rangle \cong M_i$ is horizontal. By Proposition 9.1.2 we see that these again form an abelian category. (It would probably be better to call these *coherent differential modules* since they need not be generated by a finite set of sections.)

9.2 Antidifferentiation

One initially surprising fact about p-adic analysis is that the relationship between differentiation and boundary convergence is reversed from the archimedean case: whereas d carries $K\langle \alpha/t, t/\beta\rangle$ into itself, antidifferentiation does not. Instead, one has merely the following.

Lemma 9.2.1. *For any $x = \sum_i x_i t^i \in K\{\alpha/t, t/\beta\}$ with $x_{-1} = 0$, there exists $y \in K\{\alpha/t, t/\beta\}$ for which $d(y) = x$.*

Proof. Exercise. □

Corollary 9.2.2. *Let M be the trivial differential module over $K\langle t/\beta\rangle$ (resp. over $K\langle \alpha/t, t/\beta\rangle$). Then, for any δ with $0 \leq \delta < \beta$ (resp. any γ, δ with $\alpha < \gamma \leq \delta < \beta$), the map $H^1(M) \to H^1(M \otimes K\langle t/\delta\rangle)$ (resp. $H^1(M) \to H^1(M \otimes K\langle \gamma/t, t/\delta\rangle)$) is the zero map.*

This gives us an explicit description of unipotent differential modules. (Recall that these are the successive extensions of trivial modules.)

Lemma 9.2.3. *Let M be a finite unipotent differential module for the derivation $t(d/dt)$ over $K\langle \alpha/t, t/\beta\rangle$ with $0 < \alpha < \beta$. Then, for any γ, δ with $\alpha < \gamma \leq \delta < \beta$, $M \otimes K\langle \gamma/t, t/\delta\rangle$ admits a basis on which the matrix of action of D is nilpotent and has entries in K.*

Proof. Let $v_1, \dots, v_n$ be a basis of M such that, letting M_i denote the span of $v_1, \dots, v_i$, we have that M_i is stable under D and M_i/M_{i-1} is trivial for $i = 1, \dots, n$. We proceed by induction on n; that being said, we may assume that the matrix of action of D on $v_1, \dots, v_{n-1}$ is upper triangular nilpotent with entries in K.

We now write $D(v_n) = c_1 v_1 + \cdots + c_{n-1} v_{n-1}$. If $c_{n-1} \notin K$, we can choose α', β' with $\alpha < \alpha' < \gamma \leq \delta < \beta' < \beta$ and then (by Lemma 9.2.1) choose

$x \in K\langle \alpha'/t, t/\beta' \rangle$ such that $c_{n-1} - t(dx/dt) \in K$. If we replace v_n by $v'_n = v_n - xv_{n-1}$ then $D(v'_n) = c'_1 v_1 + \cdots + c'_{n-1} v_{n-1}$, with $c'_{n-1} \in K$. Similarly, if $c_{n-1}, \ldots, c_{n-i+1} \in K$ but $c_{n-i} \notin K$, we can change basis to bring c_{n-i} into K while possibly changing $c_1, \ldots, c_{n-i-1}$ but not $c_{n-i+1}, \ldots, c_{n-1}$. Repeating this process, we ultimately obtain a basis of the desired form. □

9.3 Radius of convergence on a disc

Definition 9.3.1. Let M be a finite differential module over $K\langle t/\beta \rangle$, $K[\![t/\beta]\!]_{\mathrm{an}}$, or $K[\![t/\beta]\!]_0$. Define the *radius of convergence* of M around 0, denoted $R(M)$, to be the supremum of the set of $\rho \in (0, \beta)$ such that $M \otimes K\langle t/\rho \rangle$ has a basis of horizontal elements; we refer to those elements as *local horizontal sections* of M. For M a finite differential module on the open disc of radius β around $t = 0$, define $R(M)$ as the supremum of $R(M \otimes K\langle t/\gamma \rangle)$ over all $\gamma < \beta$. For $\gamma \leq \beta$, note that

$$R(M \otimes K\langle t/\gamma \rangle) = \begin{cases} \gamma & \gamma \leq R(M), \\ R(M) & \gamma > R(M). \end{cases}$$

Example 9.3.2. In general, if $p > 0$ then it is possible to have $R(M) < \beta$; that is, there is no p-adic analogue of the fundamental theorem of ordinary differential equations (as was noted in Example 0.4.1). For instance, consider the module $M = K\langle t/\beta \rangle v$ with $D(v) = v$; for $\beta > p^{-1/(p-1)}$ we have $R(M) = p^{-1/(p-1)}$ because that is the radius of convergence of the exponential series. (This is essentially Example 0.4.1 again.)

However, the local form of the fundamental theorem of ordinary differential equations has the following analogue.

Proposition 9.3.3 (p-adic Cauchy theorem). *Let M be a finite differential module over $K\langle t/\beta \rangle$, $K[\![t/\beta]\!]_{\mathrm{an}}$, or $K[\![t/\beta]\!]_0$. Then $R(M) > 0$.*

Proof. By shrinking β we reduce to the case over $K\langle t/\beta \rangle$. One can give a direct proof of this, but instead we will deduce it from Dwork's transfer theorem (Theorem 9.6.1 below). We will give a direct proof of a slightly stronger result later (Proposition 18.1.1); see also the notes. □

Here are some easy consequences of the definition of the radius of convergence; note the parallels with properties of the spectral radius (Lemma 6.2.8).

Lemma 9.3.4. *Let M, M_1, M_2 be finite differential modules over $K\langle t/\beta \rangle$, $K[\![t/\beta]\!]_{\mathrm{an}}$, or $K[\![t/\beta]\!]_0$.*

(a) *If* $0 \to M_1 \to M \to M_2 \to 0$ *is exact then*

$$R(M) = \min\{R(M_1), R(M_2)\}.$$

(b) *We have*

$$R(M^\vee) = R(M).$$

(c) *We have*

$$R(M_1 \otimes M_2) \geq \min\{R(M_1), R(M_2)\},$$

with equality when $R(M_1) \neq R(M_2)$.

Proof. For (a), it is clear that $R(M) \leq \min\{R(M_1), R(M_2)\}$; we must check that equality holds. Choose $\lambda < \min\{R(M_1), R(M_2)\}$, so that $M_1 \otimes K\langle t/\lambda\rangle$ and $M_2 \otimes K\langle t/\lambda\rangle$ are both trivial. For any $\lambda' < \lambda$, the map $H^1(M_2^\vee \otimes M_1) \to H^1((M_2^\vee \otimes M_1) \otimes K\langle t/\lambda'\rangle)$ is zero by Corollary 9.2.2, so $M \otimes K\langle t/\lambda'\rangle$ is trivial by Lemma 5.3.3. Since we can make λ and λ' as close to $\min\{R(M_1), R(M_2)\}$ as we like, we find that $R(M) \geq \min\{R(M_1), R(M_2)\}$.

For (b), we obtain $R(M^\vee) \geq R(M)$ from the fact that if $M \otimes K\langle t/\lambda\rangle$ is trivial, then so is its dual $M^\vee \otimes K\langle t/\lambda\rangle$. Since M and $M^\vee$ enter symmetrically, we get $R(M^\vee) = R(M)$.

For (c), the inequality is clear from the fact that the tensor product of two trivial modules over $K\langle t/\lambda\rangle$ is also trivial. The last assertion follows as in the proof of Lemma 6.2.8(c). □

Example 9.3.5. Assume that $p > 0$, and let M be the differential module of rank 1 over $K\langle t/\beta\rangle$ defined by $D(v) = \lambda v$ with $\lambda \in K$. Then it is an exercise to show that

$$R(M) = \min\{\beta, p^{-1/(p-1)}|\lambda|^{-1}\}.$$

A special case is provided by an important example of Dwork; see Example 17.1.4.

9.4 Generic radius of convergence

In general, the radius of convergence of a differential module is difficult to compute. To make its computation easier, we introduce a related but simpler invariant.

Definition 9.4.1. For $\rho > 0$, let F_ρ be the completion of $K(t)$ under the ρ-Gauss norm $|\cdot|_\rho$. Put $d = d/dt$ on $K(t)$; then d extends by continuity to F_ρ, and we have

$$|d|_{F_\rho} = \rho^{-1}, \qquad |d|_{\mathrm{sp},F_\rho} = \lim_{n\to\infty} |n!|^{1/n}\rho^{-1} = \begin{cases} \rho^{-1} & p = 0, \\ p^{-1/(p-1)}\rho^{-1} & p > 0. \end{cases}$$

(See Proposition 9.10.2 below for a refinement of this assertion.) It is a common convention to define

$$\omega = \begin{cases} 1 & p = 0, \\ p^{-1/(p-1)} & p > 0, \end{cases}$$

so that we may write $|d|_{\mathrm{sp},F_\rho} = \omega\rho^{-1}$.

Remark 9.4.2. Note that F_ρ coincides with the ring $K\llbracket\rho/t, t/\rho\rrbracket_{\mathrm{an}}$ of analytic elements on the circle $|t| = \rho$ (see Definition 8.5.1). As a result, F_ρ is commonly known as the *field of analytic elements* of norm ρ.

We will also make a related construction in the case $\rho = 1$. For a way to unify the two constructions, see Definition 9.10.1.

Definition 9.4.3. Put $\mathcal{E} = K\langle 1/t, t\rrbracket_0$ or, in other words, let $\mathcal{E}$ be the completion of $\mathfrak{o}_K((t)) \otimes_{\mathfrak{o}_K} K$ for the 1-Gauss norm $|\cdot|_1$. If K is discretely valued, the supremum in the Gauss norm is achieved; consequently $\mathcal{E}$ is a field and its residue field is equal to $\kappa_K((t))$. However, none of this applies unless K is discretely valued. (This is the same issue that arises in Remark 8.2.4.) In any case $\mathcal{E}$ is complete under $|\cdot|_1$, there is an isometric map $F_1 \to \mathcal{E}$ carrying t to t, and the supremum is achieved for elements of $\mathcal{E}$ in the image of that map; this at least gives an embedding of $\kappa_K((t))$ into the quotient $\{x \in \mathcal{E} : |x|_1 \leq 1\}/\{x \in \mathcal{E} : |x|_1 < 1\}$.

Definition 9.4.4. Let (V, D) be a nonzero finite differential module over F_ρ or $\mathcal{E}$. We define the *generic radius of convergence* (or for short, the *generic radius*) of V to be

$$R(V) = \omega |D|^{-1}_{\mathrm{sp},V};$$

note that $R(V) > 0$. We will see later (in Proposition 9.7.5) that this does indeed compute the radius of convergence of horizontal sections of V on a generic disc. (If V is the zero module, set $R(V) = \rho$.)

Remark 9.4.5. Note that the map $F_1 \to \mathcal{E}$ is isometric, and $|d|_{\mathrm{sp},\mathcal{E}} = \omega = |d|_{\mathrm{sp},F_1}$. Consequently, for any finite differential module V over F_1, Corollary 6.2.7 implies that $|D|_{\mathrm{sp},V\otimes\mathcal{E}} = |D|_{\mathrm{sp},V}$.

We can translate some basic properties of the spectral radius (Lemma 6.2.8) into properties of generic radii of convergence, leading to the following analogue of Lemma 9.3.4. Alternatively, one can first check Proposition 9.7.5 and then simply invoke Lemma 9.3.4 itself around a generic point.

Lemma 9.4.6. *Let V, V_1, V_2 be nonzero finite differential modules over F_ρ.*

(a) *For an exact sequence $0 \to V_1 \to V \to V_2 \to 0$,*

$$R(V) = \min\{R(V_1), R(V_2)\}.$$

(b) *We have*

$$R(V^\vee) = R(V).$$

(c) *We also have*

$$R(V_1 \otimes V_2) \geq \min\{R(V_1), R(V_2)\},$$

with equality when $R(V_1) \neq R(V_2)$.

Definition 9.4.7. In some situations it is more natural to consider the *intrinsic generic radius of convergence*, or for short the *intrinsic radius*, defined as

$$IR(V) = \rho^{-1}R(V) = \frac{|d|_{\mathrm{sp},F_\rho}}{|D|_{\mathrm{sp},V}} \in (0, 1].$$

(The upper bound of 1 comes from Lemma 6.2.4.) To emphasize the difference, we may refer to the unadorned generic radius of convergence defined earlier as the *extrinsic* generic radius of convergence. (See Proposition 9.7.6 and the notes for some reasons why the intrinsic radius deserves its name.)

Remark 9.4.8. For I an interval, M a differential module on the annulus $|t| \in I$, and $\rho \in I$, it is unambiguous to refer to the generic radius of convergence $R(M \otimes F_\rho)$ of M at radius ρ. This is defined by first making a base change to $K\langle \alpha/t, t/\beta \rangle$ for some closed subinterval $[\alpha, \beta]$ of I containing ρ. Since the resulting module $M \otimes F_\rho$ does not depend on the choice of $[\alpha, \beta]$, neither does its generic radius.

9.5 Some examples in rank 1

Assume $p > 0$ for these examples. See also Example 9.9.3.

Example 9.5.1. In Example 9.3.5 we have $D^s(v) = \lambda^s v$ for all nonnegative integers s. By Lemma 6.2.5 we have

$$R(M \otimes F_\beta) = \min\{\beta, p^{-1/(p-1)}|\lambda|^{-1}\} = R(M).$$

Another important class of examples is given as follows.

Example 9.5.2. For $\lambda \in K$, let V_λ be the differential module of rank 1 over F_ρ defined by $D(v) = \lambda t^{-1} v$. It is an exercise to show that $IR(V_\lambda) = 1$ if and only if $\lambda \in \mathbb{Z}_p$.

We can classify Example 9.5.2 further as follows.

Proposition 9.5.3. *We have $V_\lambda \cong V_{\lambda'}$ if and only if $\lambda - \lambda' \in \mathbb{Z}$.*

Proof. Note that $V_\lambda \cong V_{\lambda'}$ if and only if $V_{\lambda-\lambda'}$ is trivial, so we may reduce to the case $\lambda' = 0$. By Example 9.5.2, V_λ is nontrivial whenever $\lambda \notin \mathbb{Z}_p$; by direct inspection, V_λ is trivial whenever $\lambda \in \mathbb{Z}$.

It remains to deduce a contradiction, assuming that V_λ is trivial, $\lambda \in \mathbb{Z}_p$, and $\lambda \notin \mathbb{Z}$. There is no harm in enlarging K now, so we may assume that K contains a scalar of norm ρ; by rescaling, we may reduce to the case $\rho = 1$. We now have $f \in F_1^\times$ such that $t(df/dt) = \lambda f$; by multiplying by an element of $K^\times$ we can force $|f|_1 = 1$.

Let λ_1 be an integer such that $\lambda \equiv \lambda_1 \pmod p$. Then

$$\left| \frac{d(ft^{-\lambda_1})}{dt} \right|_1 = |(\lambda - \lambda_1) f t^{-\lambda_1 - 1}|_1 \leq p^{-1}.$$

Using the embedding $F_1 \hookrightarrow \mathcal{E}$, we may expand $f = \sum_{i \in \mathbb{Z}} f_i t^i$ with $\max_i\{|f_i|\} = 1$. The previous calculation then forces $|f_i| \leq p^{-1}$ unless $i \equiv \lambda_1 \equiv \lambda \pmod p$.

By considering the reduction of f modulo p^n and arguing similarly, we find that $|f_i| \leq p^{-1}$ unless $i \equiv \lambda \pmod{p^n}$ for all n. But, since $\lambda \notin \mathbb{Z}$, this means that the image of f in $\kappa_K((t))$ cannot have any terms at all, a contradiction. $\square$

9.6 Transfer theorems

A fundamental relationship between the radius of convergence and the generic radius of convergence is given by the following theorem. In the language of Dwork, it is a *transfer theorem*, because it transfers convergence information from one disc to another. (Note that Proposition 9.3.3, which asserts that $R(M) > 0$, is an immediate corollary.)

Theorem 9.6.1 (Dwork). *For any nonzero finite differential module M over $K\langle t/\beta \rangle$ or $K[[t/\beta]]_{\mathrm{an}}$, $R(M) \geq R(M \otimes F_\beta)$. That is, the radius of convergence is equal to at least the generic radius.*

Proof. Suppose that $\lambda < \beta$ and $\lambda < \omega|D|^{-1}_{\mathrm{sp},M}$. Fix a supremum norm $|\cdot|_M$ on M that is compatible with $|\cdot|_\lambda$. We claim that, for any $x \in M$, the Taylor series

$$y = \sum_{i=0}^{\infty} \frac{(-t)^i}{i!} D^i(x) \tag{9.6.1.1}$$

converges under $|\cdot|_M$. To see this, pick $\epsilon > 0$ such that $\lambda\omega^{-1}(|D|_{\mathrm{sp},M} + \epsilon) < 1$. By Lemma 6.1.8 there exists $c > 0$ such that $|D^i(x)|_M \leq c(|D|_{\mathrm{sp},M} + \epsilon)^i |x|_M$ for all i. The ith term of the sum defining y thus has norm at most $\lambda^i \omega^{-i} c(|D|_{\mathrm{sp},M} + \epsilon)^i |x|_M$, which tends to 0 as $i \to \infty$.

By differentiating the series expression as in Remark 5.8.4 we find that $D(y) = 0$, so y is a horizontal section of $M \otimes K\langle t/\lambda\rangle$. If we run this construction over a basis of M, we obtain horizontal sections of $M \otimes K\langle t/\lambda\rangle$ whose reductions modulo t also form a basis; these sections are thus K-linearly independent and so form a basis of $M \otimes K\langle t/\lambda\rangle$ by Proposition 9.1.2. This proves the claim. □

Here is a simple example of how one may apply the transfer theorem; it is conditional on one result, Theorem 11.3.2.

Example 9.6.2. Recall the hypergeometric differential equation

$$y'' + \frac{c - (a+b+1)z)}{z(1-z} y' - \frac{ab}{z(1-z)} y = 0,$$

considered in Chapter 0. In general, one solution in the ring $K((z))$ is given by the hypergeometric series

$$F(a, b; c; z) = \sum_{i=0}^{\infty} \frac{a(a+1)\cdots(a+i)b(b+1)\cdots(b+i)}{c(c+1)\cdots(c+i)i!} z^i.$$

Let us now restrict to the case $a, b, c \in \mathbb{Z}_p \cap \mathbb{Q}$. Let m be the denominator of c. In the ring $K((z^{1/m}))$, the general solution is

$$AF(a, b; c; z) + Bz^{1-c} F(a+1-c, b+1-c; 2-c; z) \qquad (A, B \in K), \tag{9.6.2.1}$$

and it converges for $|z| < 1$ (see the exercises for Chapter 0).

We now pass to the associated differential module M of rank 2 which is defined over the ring $K\langle \alpha/z, z]\!]_{\mathrm{an}}$ for any $\alpha > 0$. From (9.6.2.1) we see that, for any $\beta \in (0, 1)$, $M \otimes K\langle \alpha/z, z/\beta\rangle$ has a filtration in which one quotient is a trivial module and the other has the form $D(w) = \lambda z^{-1} w$ for some $\lambda \in \mathbb{Z}_p \cap \mathbb{Q}$. From Example 9.5.2 and Lemma 9.4.6 we deduce that $R(M \otimes F_\rho) = 1$ for

$\rho \in (0, 1)$. By Theorem 11.3.2(a), $R(M \otimes F_\rho)$ is continuous at $\rho = 1$ so that $R(M \otimes F_1) = 1$ also.

We now expand in power series around another value, $z_0 \in \mathfrak{o}_K$, that is not congruent to 0 or 1 modulo $\mathfrak{m}_K$. We get another differential module N over $K[\![z - z_0]\!]_{\mathrm{an}}$ such that $M \otimes F_1 \cong N \otimes F_1$ and so, in particular, $R(N \otimes F_1) = 1$. By Theorem 9.6.1, $R(N) = 1$; that is, the general solution of the hypergeometric differential equation at $z = z_0$ converges in the disc $|z - z_0| < 1$.

Remark 9.6.3. In Example 9.6.2 one cannot directly prove that $R(N \otimes F_1) = 1$ using Theorem 6.5.3, because of the limitation on slopes therein. An alternate approach that works is to construct a Frobenius structure on N; see Part V.

9.7 Geometric interpretation

As promised, here is a construction that explains the terminology "generic radius of convergence".

Definition 9.7.1. Let L be a complete extension of K. A *generic point* of L of norm ρ relative to K is an element $t_\rho \in L$, with $|t_\rho| = \rho$, such that there is no $z \in K^{\mathrm{alg}}$ with $|z - t_\rho| < \rho$. For instance, t itself is a generic point of F_ρ of norm ρ.

Remark 9.7.2. If $t_\rho \in L$ is a generic point then evaluation at t_ρ gives an isometry $K[t] \to L$ for the ρ-Gauss norm on $K[t]$. To see this, it suffices to check after replacing K by a completed algebraic closure and enlarging L to contain this enlarged K. Then any $P \in K(t)$ can be factored as $c \prod_i (t - z_i)$ for some $z_i \in K$, and for each i we have $|z_i - t_\rho| \geq \rho$ because t_ρ is a generic point. Consequently,

$$\begin{aligned} |P(t_\rho)| &= |c| \prod_i |t_\rho - z_i| \\ &= |c| \prod_i \max\{\rho, |z_i|\} \\ &= |c| \prod_i |t - z_i|_\rho \\ &= |P|_\rho. \end{aligned}$$

Definition 9.7.3. Let L be a complete extension of K. For any $t_\rho \in L$ with $|t_\rho| = \rho$, the substitution $t \mapsto t_\rho + (t - t_\rho)$ induces an isometric map $K[t] \to L\langle (t - t_\rho)/\rho \rangle$. However, if (and only if) t_ρ is a generic point then the composition of this map with the reduction modulo $t - t_\rho$ is again an isometry, by Remark 9.7.2. Hence in this case the map $K[t] \to L\langle (t - t_\rho)/\rho \rangle$ extends to an isometry $F_\rho \to L[\![(t - t_\rho)/\rho]\!]_{\mathrm{an}}$.

Remark 9.7.4. In Berkovich's theory of nonarchimedean analytic geometry, the geometric interpretation of the above construction is that the analytic space corresponding to F_ρ is obtained from the closed disc of radius ρ by removing the open disc of radius ρ centered around each point of K^{alg}. As a result, it still contains any open disc of radius ρ that does not meet K^{alg}.

Proposition 9.7.5. *Let V be a finite differential module over F_ρ, and put $V' = V \otimes_{F_\rho} L[\![(t - t_\rho)/\rho]\!]_{\mathrm{an}}$. Then the generic radius of convergence of V is equal to the radius of convergence of V'.*

Proof. Let G_λ be the completion of $L(t - t_\rho)$ for the λ-Gauss norm; then the map $F_\rho \to G_\lambda$ is an isometry for any $\lambda < \rho$. By Corollary 6.2.7,

$$|D|_{\mathrm{sp}, V \otimes G_\lambda} = \max\{|d|_{\mathrm{sp}, G_\lambda}, |D|_{\mathrm{sp}, V}\} = \max\{\omega\lambda^{-1}, |D|_{\mathrm{sp}, V}\}.$$

On one hand, this implies $R(V) \leq R(V')$ by the application of Theorem 9.6.1 to $V \otimes L\langle (t - t_\rho)/\lambda \rangle$ for a sequence of values of λ converging to ρ.

On the other hand, pick any $\lambda < R(V')$; then $V \otimes G_\lambda$ is a trivial differential module, so the spectral radius of D on it is $\omega\lambda^{-1}$. We thus have

$$|D|_{\mathrm{sp}, V} \leq \omega\lambda^{-1},$$

so $R(V) \geq \lambda$. This yields $R(V) \geq R(V')$. □

Here is an example illustrating both the use of the geometric interpretation and a good transformation property of the intrinsic normalization.

Proposition 9.7.6. *Let m be a nonzero integer not divisible by p, and let $f_m : F_\rho \to F_{\rho^{1/m}}$ be the substitution $t \mapsto t^m$. Then, for any finite differential module V over F_ρ, $IR(V) = IR(f_m^*(V))$.*

Proof. Let ζ_m be a primitive mth root of unity in K^{alg}. Then the claim follows from the geometric interpretation plus the fact that

$$|t - t_\rho \zeta_m^i| < c\rho \text{ for some } i \in \{0, \dots, m-1\} \Leftrightarrow |t^m - t_\rho^m| < c\rho^m \quad (c \in (0, 1)), \tag{9.7.6.1}$$

whose proof is left as an exercise. □

Remark 9.7.7. Be aware that in Proposition 9.7.6 we are not quite performing the standard base change. In explicit terms, if d_m denotes the derivation with respect to t on $F_{\rho^{1/m}}$ and d_1 denotes the derivation with respect to t on F_ρ extended via f_m then

$$d_m = mt^{m-1} d_1.$$

We will encounter this issue again when we perform Frobenius pullback and pushforward in Chapter 10. One may view it as an argument in favor of a coordinate-free perspective (Proposition 6.3.1).

Remark 9.7.8. A similar construction can be made for $\mathcal{E}$. Let L be the completion of $\mathfrak{o}_K((t_1)) \otimes_{\mathfrak{o}_K} K$ for the 1-Gauss norm. Then the substitution $t \mapsto t_1 + (t - t_1)$ induces an isometry $\mathfrak{o}_K((t)) \to \mathfrak{o}_L[\![t - t_1]\!]$ for the 1-Gauss norm that extends to an isometric embedding of $\mathcal{E}$ into the completion of $\mathfrak{o}_L[\![t - t_1]\!] \otimes_{\mathfrak{o}_L} L$ for the 1-Gauss norm.

9.8 Subsidiary radii

It is sometimes important to consider not only the generic radius of convergence but also some secondary invariants.

Definition 9.8.1. Let V be a finite differential module over F_ρ. Let $V_1, \dots, V_m$ be the Jordan–Hölder constituents of V. We define the multiset of *subsidiary generic radii of convergence*, for short the *subsidiary radii*, to consist of $R(V_i)$ with multiplicity $\dim V_i$ for $i = 1, \dots, m$. We will always list these in increasing order, as $s_1 \leq \cdots \leq s_n$, so that $s_1 = R(V)$. We also have *intrinsic subsidiary generic radii of convergence*, obtained by multiplying the subsidiary radii by ρ^{-1}.

Remark 9.8.2. If we replace F_ρ by $\mathbb{C}((z))$ in the definition of intrinsic subsidiary generic radii of convergence, then the negative logarithm of the product of the radii equals the irregularity of V. Thus our analysis of the variation of subsidiary radii, in the remainder of this part, will also imply results about the variation of irregularity. See [**144**] for an application of this.

Remark 9.8.3. It is not immediate from the definition how to interpret the subsidiary radii. We will give an interpretation in a later chapter (Theorem 11.9.2).

9.9 Another example in rank 1

We introduce one more important example in rank 1, in the case $p > 0$.

Definition 9.9.1. For $p > 0$, the *Artin–Hasse exponential series* is the formal power series

$$E_p(t) = \exp\left(\sum_{i=0}^{\infty} \frac{t^{p^i}}{p^i}\right).$$

Proposition 9.9.2. *We have $E_p(t) \in (\mathbb{Z}_p \cap \mathbb{Q})[\![t]\!]$.*

Proof. This follows from the formal identity

$$E_p(x) = \prod_{n \geq 1,\, p \nmid n} (1 - x^n)^{-\mu(n)/n}, \tag{9.9.2.1}$$

in which $\mu(n)$ is the Möbius function. That is, $\mu(n)$ equals $(-1)^e$ if n is the product of $e \geq 0$ distinct primes, and it equals 0 otherwise. □

Example 9.9.3 (Matsuda). Let h be a nonnegative integer, and suppose that K contains a primitive p^{h+1}th root of unity ζ. Let M_h be the differential module of rank 1 on the whole t-line, the action of D on a generator v being given by

$$D(v) = \sum_{i=0}^{h} \left(\zeta^{p^i} - 1 \right) t^{p^i - 1} v;$$

note that M_1 is isomorphic to the module of Example 9.3.5 for $\lambda = \zeta - 1$. On the open disc $|t| < 1$, M_h admits the horizontal section

$$\frac{E_p(t)}{E_p(\zeta t)} v = \exp\left(\sum_{i=0}^{h} \frac{1 - \zeta^{p^i}}{p^i} t^{p^i} \right) v \tag{9.9.3.1}$$

by Proposition 9.9.2, so $R(M_h) \geq 1$. Since this horizontal section is bounded it also gives a horizontal section in the open unit disc around a generic point of norm 1, so $R(M_h \otimes F_1) = 1$. (This can also be seen by arguing that M_h is trivial on any disc of the form $|t| < \rho$ for $\rho < 1$, so that $R(M_h \otimes F_\rho) = 1$, and then using the continuity of the generic radius of convergence as in Example 9.6.2.) By Proposition 9.7.6 the same remains true if we pull back along the map $t \mapsto ct^m$, for any positive integer m not divisible by p and any $c \in \mathfrak{o}_K^\times$. (This can be generalized further; see the notes.)

Remark 9.9.4. Note that

$$\left(\frac{E_p(t)}{E_p(\zeta t)} \right)^p = \exp((1 - \zeta) pt) \frac{E_p(t^p)}{E_p(\zeta^p t^p)}.$$

Consequently, on some disc of radius greater than 1, $M_h^{\otimes p}$ is isomorphic to the pullback of M_{h-1} along $t \mapsto t^p$.

Remark 9.9.5. It is possible to prove directly that, for $\rho \in [1, +\infty)$ sufficiently close to 1,

$$IR(M_h \otimes F_\rho) = \rho^{-p^h} \tag{9.9.5.1}$$

(exercise). We can also do this using variational properties of the generic radius; see the exercises for Chapter 11.

9.10 Comparison with the coordinate-free definition

It is worth reconciling our definition of the generic radius of convergence with the coordinate-free formula in Proposition 6.3.1. In the process, we can give a more unified treatment of F_ρ and $\mathcal{E}$.

Definition 9.10.1. Let F be a complete nonarchimedean field of characteristic 0 equipped with a bounded derivation d. For $u \in F$, we say that d is of *rational type* for the parameter u if the following conditions hold.

(a) We have $d(u) = 1$ and $|ud|_F = 1$.
(b) For each positive integer n, $|d^n/n!|_F \leq |d|_F^n$.

For instance, this holds for $F = F_\rho$ with $d = d/dt$ and $u = t$, and similarly for $F = \mathcal{E}$ if K is discretely valued. Note also that condition (b) is vacuously true if κ_F is of characteristic 0.

Proposition 9.10.2. *Set notation as in Definition 9.10.1. Then, for any nonnegative integer n and any $c_0, \dots, c_n \in F$, we have*

$$\left| \sum_{i=0}^{n} c_i \frac{u^i}{i!} d^i \right|_F = \max_i \{|c_i|\}. \tag{9.10.2.1}$$

Proof. The left-hand side of (9.10.2.1) is less than or equal to the right-hand side because, by hypothesis, $|u^i d^i/i!|_F \leq |u|^i |d|_F^i \leq 1$ for all i. Conversely, let $j \in \{0, \dots, n\}$ be the minimal index for which $|c_j| = \max_i\{|c_i|\}$; since

$$c_i \frac{u^i}{i!} d^i(u^j) = \binom{j}{i} c_i u^j$$

and F is of characteristic 0, we have

$$\left| c_i \frac{u^i}{i!} d^i(u^j) \right| \begin{cases} = |c_j u^j| & (i = j), \\ < |c_j u^j| & (i < j), \\ = 0 & (i > j). \end{cases}$$

Hence $|\sum_{i=0}^n c_i u^i d^i(u^j)/i!| = |c_j u^j|$, so the left-hand side of (9.10.2.1) is greater than or equal to the right-hand side. $\square$

Corollary 9.10.3. *Set notation as in Definition 9.10.1. Then, for any nonzero finite differential module V over F, we have*

$$|D|_{\mathrm{sp},V} = |d|_{\mathrm{sp},F} \lim_{s\to\infty} |D_s|^{1/s},$$

where D_s is defined as in Proposition 6.3.1.

Proof. This follows from Proposition 6.3.1, since (6.3.1.2) holds by virtue of Proposition 9.10.2. □

Notes

As noted in [**80**, Appendix III], which can be consulted for more information, the p-adic Cauchy theorem (Proposition 9.3.3) was originally proved by Lutz [**163**]. (Note that this publication long predates Dwork's work.) See Proposition 18.1.1 for a related result.

The idea of restricting a p-adic differential module to a generic disc originated in the work of Dwork [**75**], although in retrospect the base change involved is quite natural in Berkovich's framework of nonarchimedean analytic geometry. For instance, the Berkovich space associated with a closed unit disc contains generic points corresponding to each disc of each possible radius less than or equal to 1; the "residue field" of the generic point of the disc $|t| \leq \rho$ is precisely F_ρ. (This point of view was adopted by Baldassarri and Di Vizio in [**11**] and by Baldassarri in [**10**].) Our definition of the generic radius of convergence is taken from Christol and Dwork [**49**].

The intrinsic generic radius of convergence (the original terminology) was introduced in [**140**], where it is called the "toric normalization" in light of Proposition 9.7.6.

Previously the subsidiary radii (the original terminology) have not been studied much; the one reference we found is the work of Young [**222**]. We will give Young's interpretation of the subsidiary radii as the radii of convergence of certain horizontal sections, in a refined form, as Theorem 11.9.2.

Our description of the Artin–Hasse exponential follows Robert [**191**, §7.2]. Matsuda's example, from [**168**], is an explicit instance of a general construction introduced by Robba [**190**]. In turn, Matsuda's example can be greatly generalized as shown by Pulita [**185**], building on work of Chinellato. One obtains an analogous construction from any Lubin–Tate group over $\mathbb{Q}_p$; Matsuda's example arises from the multiplicative group. (The introduction to [**185**] provides a detailed historical discussion.)

A slightly different notion of the generic radius of convergence was introduced by Baldassarri and Di Vizio [**11**] and studied further by Baldassarri [**10**]. The difference between that definition and ours appears only in the case where our definition gets truncated. For instance, for a differential module on a closed disc of radius β, our generic radius of convergence at radius $\rho \in (0, \beta)$ is only allowed to take values up to ρ. However, Baldassarri and Di Vizio allow values up to β since the domain of definition of the original module includes a generic disc of radius β.

The approach of Baldassarri and Di Vizio gives a more refined invariant than ours, since ours can be recovered by truncation. Thus, their approach is likely to have certain advantages in applications. One serious disadvantage is that, at the moment, there is no good theory of subsidiary generic radii of convergence in the Baldassarri–Di Vizio framework. The correct definition is presumably that suggested by Theorem 11.9.2, but it is not clear how to prove any useful properties for it.

The notion of a derivation of rational type was introduced in [**145**, §1.4] as a way to isolate those features of the field F_ρ which are needed in the spectral theory of differential modules. For instance, it is shown in [**145**] that the rational type is preserved under unramified or tamely ramified field extensions. The coordinate-free interpretation of the generic radius of convergence (Corollary 9.10.3) is useful in the study of the irregularity of higher-dimensional flat meromorphic connections [**144**].

Exercises

(1) Prove Lemma 9.2.1. Then exhibit an example showing that the cokernel of d/dt on $K\langle\alpha/t, t/\beta\rangle$ is not spanned over K by t^{-1}. That is, antidifferentiation with respect to t is not well defined on $K\langle\alpha/t, t/\beta\rangle$.

(2) Prove Example 9.3.5. (Hint: use the exponential series to construct a horizontal section.)

(3) Prove Example 9.5.2. (Hint: consider the cases $\lambda \in \mathbb{Z}_p$, $\lambda \in \mathfrak{o}_K - \mathbb{Z}_p$, and $\lambda \notin \mathfrak{o}_K$ separately.)

(4) In Example 9.5.2 give an explicit formula for $IR(V_\lambda)$ in terms of ρ and the minimum distance from λ to an integer.

(5) Prove (9.7.6.1). (Hint: you may find it easier to start with the cases where $m > 0$ and where $m = -1$ and then deduce the general result from them.)

(6) The following considerations illustrate the pitfalls of using td/dt instead of d/dt in the p-adic setting.
   (a) Verify that $|td/dt|_{\mathrm{sp},F_\rho} \neq |t|_\rho |d/dt|_{\mathrm{sp},F_\rho}$.
   (b) Show that the inequality (6.3.1.1) of Proposition 6.3.1 can be strict for $F = F_\rho$ and $d = td/dt$. (Hint: use Example 9.5.2.)

(7) With notation as in Proposition 9.7.6, show that all the intrinsic subsidiary radii of V match those of $f_m^*(V)$, not just the generic radii.

(8) Here is an "off-centered" analogue of Proposition 9.7.6 suggested by Liang Xiao (compare with Theorem 10.8.2 below). Let m be a nonzero integer not divisible by p. Given $\rho \in (0, 1]$, let $f_m : F_\rho \to F_\rho$ be the

map $t \mapsto (t+1)^m - 1$. Then, for any finite differential module V over F_ρ, $R(V) = R(f_m^*(V))$. (As in the previous exercise, one also obtains equality for the other subsidiary radii.)

(9) Prove (9.9.2.1). (Hint: take logarithms.)

(10) Prove (9.9.5.1) by analyzing an explicit horizontal section around a generic point. A similar argument is given in the proof of [**39**, Proposition 1.5.1]. (Hint: use the equality $|1-\zeta| = p^{-p^{-h+1}/(p-1)}$ from Example 2.1.6.)

10

Frobenius pullback and pushforward

In this chapter, we introduce Dwork's technique of Frobenius descent to analyze the generic radius of convergence and subsidiary radii of a differential module, in the range where Newton polygons do not apply. In one direction we introduce a somewhat refined form of the *Frobenius antecedents* introduced by Christol and Dwork. These fail to apply in an important boundary case; we remedy this by introducing the new notion of *Frobenius descendants*, which covers the boundary case.

Using these results, we are able to improve a number of results from Chapter 6 in the special case of differential modules over F_ρ. For instance we get a full decomposition by spectral radius, extending the visible decomposition theorem (Theorem 6.6.1) and the refined visible decomposition theorem (Theorem 6.8.2). We will use these results again to study the variation of subsidiary radii, and decomposition by subsidiary radii, in the remainder of this part.

Notation 10.0.1. Throughout this chapter we retain Hypothesis 9.0.1. We also continue to use F_ρ to denote the completion of $K(t)$ for the ρ-Gauss norm viewed as a differential field with respect to $d = d/dt$, unless otherwise specified.

Notation 10.0.2. Throughout this chapter we also assume $p > 0$ unless otherwise specified.

10.1 Why Frobenius descent?

Remark 10.1.1. It may be helpful to review the current state of affairs, in order to clarify why we need to descend along a Frobenius morphism.

Let V be a finite differential module over F_ρ. Then the possible values of the spectral radius $|D|_{\mathrm{sp},V}$ are the real numbers greater than or equal to $|d|_{\mathrm{sp},F_\rho} = p^{-1/(p-1)}\rho^{-1}$, corresponding to generic radii of convergence less than or equal to ρ. However, in calculating the spectral radius using the Newton polygon of a twisted polynomial, we cannot distinguish values less than or equal to the operator norm $|d|_{F_\rho} = \rho^{-1}$. In particular, we cannot use this technique to prove a decomposition theorem for differential modules that separates components of spectral radius between $p^{-1/(p-1)}\rho^{-1}$ and ρ^{-1}.

One way in which one might want to get around this is to consider not d but a high power of d, in particularly the p^nth power. The trouble with this is that iterating a derivation does not give another derivation but something much more complicated. Thus, instead we will try to differentiate with respect to t^{p^n}, rather than t, in order to increase the spectral radius into the range where Newton polygons become useful.

10.2 pth powers and roots

We first make some calculations in answer to the following question: if two p-adic numbers are close together, how close are their pth powers or their pth roots? (See also [**80**, §V.6] and [**42**, Proposition 4.6.4].)

Remark 10.2.1. We observed previously, in (9.7.6.1), using slightly different notation, that when m is a positive integer coprime to p, for ζ_m a primitive mth root of unity we have

$$|t - \eta\zeta_m^i| < \lambda|\eta| \text{ for some } i \in \{0, \dots, m-1\} \Leftrightarrow |t^m - \eta^m| < \lambda|\eta|^m \qquad (\lambda \in (0,1)).$$

This breaks down for $m = p$, because $|1 - \zeta_p| = p^{-1/(p-1)} < 1$ by Example 2.1.6. Instead, we have the following bounds.

Lemma 10.2.2. *Pick $t, \eta \in K$.*

(a) *For $\lambda \in (0,1)$, if $|t - \eta| \leq \lambda|\eta|$ then*

$$|t^p - \eta^p| \leq \max\{\lambda^p, p^{-1}\lambda\}|\eta^p| = \begin{cases} \lambda^p|\eta^p| & \lambda \geq p^{-1/(p-1)}, \\ p^{-1}\lambda|\eta^p| & \lambda \leq p^{-1/(p-1)}. \end{cases}$$

(b) *Suppose that $\zeta_p \in K$. If $|t^p - \eta^p| \leq \lambda|\eta^p|$ then there exists $m \in \{0, \dots, p-1\}$ such that*

$$|t - \zeta_p^m\eta| \leq \min\{\lambda^{1/p}, p\lambda\}|\eta| = \begin{cases} \lambda^{1/p}|\eta| & \lambda \geq p^{-p/(p-1)}, \\ p\lambda|\eta| & \lambda \leq p^{-p/(p-1)}. \end{cases}$$

Moreover, if $\lambda \geq p^{-p/(p-1)}$, we may always take $m = 0$.

We will use repeatedly, and without comment, the fact that

$$\lambda \mapsto \max\{\lambda^p, p^{-1}\lambda\}, \qquad \lambda \mapsto \min\{\lambda^{1/p}, p\lambda\}$$

are strictly increasing functions from $[0, 1]$ to $[0, 1]$ that are inverse to each other.

Proof. There is no harm in assuming that $\zeta_p \in K$ for both parts. For (a), factor $t^p - \eta^p$ as $(t-\eta)\prod_{m=1}^{p-1}(t - \eta\zeta_p^m)$, and write

$$t - \eta\zeta_p^m = (t - \eta) + \eta(1 - \zeta_p^m).$$

If $|t - \eta| \geq p^{-1/(p-1)}|\eta|$ then $t - \eta$ is the dominant term; otherwise $\eta(1 - \zeta_p^m)$ dominates. This gives the claimed bounds.

For (b), consider the Newton polygon of

$$t^p - \eta^p - c = \sum_{i=0}^{p-1} \binom{p}{i} \eta^i (t - \eta)^{p-i} - c,$$

viewed as a polynomial in $t - \eta$. Suppose that $|c| = \lambda|\eta^p|$. If $\lambda \geq p^{-p/(p-1)}$ then the terms $(t - \eta)^p$ and c dominate, and all roots have norm $\lambda^{1/p}|\eta|$. Otherwise the terms $(t - \eta)^p$, $p(t - \eta)\eta^{p-1}$, and c dominate, so one root has norm $p\lambda|\eta|$ and the others are larger. Repeating with η replaced by $\zeta_p^m\eta$ for $m = 0, \ldots, p-1$ gives p distinct roots. This accounts for all the roots. □

Corollary 10.2.3. *Let $T : K[\![t^p - \eta^p]\!] \to K[\![t - \eta]\!]$ be the substitution*

$$t^p - \eta^p \mapsto \sum_{i=0}^{p-1} \binom{p}{i} \eta^i (t - \eta)^{p-i}.$$

(a) *If $f \in K\langle (t^p - \eta^p)/(\lambda|\eta^p|)\rangle$ for some $\lambda \in (0, 1)$ then $T(f) \in K\langle (t - \eta)/(\lambda'|\eta|)\rangle$ for $\lambda' = \min\{\lambda^{1/p}, p\lambda\}$.*

(b) *If $T(f) \in K\langle (t - \eta)/(\lambda|\eta|)\rangle$ for some $\lambda \in (p^{-1/(p-1)}, 1)$ then $f \in K\langle (t^p - \eta^p)/(\lambda'|\eta^p|)\rangle$ for $\lambda' = \lambda^p$.*

(c) *Suppose that K contains a primitive pth root of unity ζ_p. For $m = 0, \ldots, p-1$, let $T_m : K[\![t^p - \eta^p]\!] \to K[\![t - \zeta_p^m\eta]\!]$ be the substitution $t^p - \eta^p \mapsto \sum_{i=0}^{p-1} \binom{p}{i} \zeta_p^{im}\eta^i (t - \zeta_p^m\eta)^{p-i}$. If for some $\lambda \in (0, p^{-1/(p-1)}]$ one has $T_m(f) \in K\langle (t - \zeta_p^m\eta)/(\lambda|\eta|)\rangle$, for $m = 0, \ldots, p-1$, then $f \in K\langle (t^p - \eta^p)/(\lambda'|\eta^p|)\rangle$ for $\lambda' = p^{-1}\lambda$.*

10.3 Frobenius pullback and pushforward operations

We now define Frobenius pullback and pushforward operations, and show how they affect the generic radius of convergence.

Definition 10.3.1. Let F'_ρ be the completion of $K(t^p)$ for the ρ^p-Gauss norm, viewed as a subfield of F_ρ and equipped with the derivation $d' = d/d(t^p)$. We then have

$$d = \frac{d(t^p)}{dt} d' = pt^{p-1} d'.$$

Given a finite differential module (V', D') over F'_ρ, we may view $\varphi^* V' = V' \otimes_{F'_\rho} F_\rho$ as a differential module over F_ρ with

$$D(v \otimes f) = pt^{p-1} D'(v) \otimes f + v \otimes d(f).$$

Note that this is *not* the usual base-change operation, because the restriction of d to F'_ρ is not d'; this is the same situation as that encountered in Remark 9.7.7.

Lemma 10.3.2. *Let (V', D') be a finite differential module over F'_ρ. Then*

$$IR(\varphi^* V') \geq \min\{IR(V')^{1/p}, pIR(V')\}.$$

Proof. For any $\lambda < IR(V')$, any complete extension L of K, and any generic point $t_\rho \in L$ relative to K of norm ρ, $V' \otimes L\langle (t^p - t_\rho^p)/(\lambda \rho^p)\rangle$ admits a basis of horizontal sections. By Corollary 10.2.3(a), $(\varphi^* V') \otimes L\langle (t - t_\rho)/(\min\{\lambda^{1/p}, p\lambda\}\rho)\rangle$ does likewise. $\square$

Here is a important example, for which Lemma 10.3.2 gives a strict inequality.

Definition 10.3.3. For $m = 0, \dots, p-1$, let W_m be the differential module over F'_ρ with one generator v such that

$$D(v) = \frac{m}{p} t^{-p} v.$$

(The generator v is taken to behave as $t^m = (t^p)^{m/p}$.) From the Newton polygon associated to v, we may read off $IR(W_m) = p^{-p/(p-1)}$ for $m \neq 0$ since this is within the range of applicability of Theorem 6.5.3. Note that the inequality of Lemma 10.3.2 is strict in this case, since $\varphi^* W_m$ is trivial and so has intrinsic radius 1.

Definition 10.3.4. For V a differential module over F_ρ, define the *Frobenius descendant* of V as the module $\varphi_* V$ obtained from V by restriction along $F'_\rho \to F_\rho$. The module $\varphi_* V$ is viewed as a differential module over F'_ρ with differential $D' = p^{-1} t^{1-p} D$. Note that this operation commutes with duals, but not with tensor products because of a rank mismatch: the rank of $\varphi_* V$ is p times that of V. See, however, Lemma 10.3.6(f) below.

Remark 10.3.5. The definition of the Frobenius descendant extends to differential modules over $K\langle \alpha/t, t/\beta \rangle$ for $\alpha > 0$ but not for $\alpha = 0$, since we must divide by a power of t to express D' in terms of D. The underlying problem is that (in geometric terms) the map φ ramifies at the point $t = 0$. We will see one way to deal with this problem in the discussion of off-centered Frobenius descendants (Section 10.8).

The operation φ_* enjoys a number of properties which are useful and reasonably easy to verify.

Lemma 10.3.6

(a) *For V a differential module over F_ρ, there are canonical isomorphisms*

$$\psi_m : (\varphi_* V) \otimes W_m \cong \varphi_* V \qquad (m = 0, \dots, p-1).$$

(b) *For V a differential module over F_ρ, a submodule U of $\varphi_* V$ is itself the Frobenius descendant of a submodule of V if and only if $\psi_m(U \otimes W_m) = U$ for $m = 0, \dots, p-1$.*

(c) *For V' a differential module over F'_ρ, there is a canonical isomorphism*

$$\varphi_* \varphi^* V' \cong \bigoplus_{m=0}^{p-1} (V' \otimes W_m).$$

(d) *For V a differential module over F_ρ, there is a canonical isomorphism*

$$\varphi^* \varphi_* V \cong V^{\oplus p}.$$

(e) *For V a differential module over F_ρ, there are canonical bijections*

$$H^i(V) \cong H^i(\varphi_* V) \qquad (i = 0, 1).$$

(f) *For differential modules V_1, V_2 over F_ρ, there is a canonical isomorphism*

$$\varphi_* V_1 \otimes \varphi_* V_2 \cong (\varphi_*(V_1 \otimes V_2))^{\oplus p}.$$

Proof. Exercise. □

10.4 Frobenius antecedents

An important counterpart to the construction of Frobenius descendants is the construction of Frobenius antecedents; this inverts the pullback operation φ^* when the intrinsic radius is sufficiently large.

Definition 10.4.1. Let (V, D) be a finite differential module over F_ρ. A *Frobenius antecedent* of V is a differential module (V', D') over F'_ρ such that $IR(V') > p^{-p/(p-1)}$, together with an isomorphism $V \cong \varphi^* V'$. By Lemma 10.3.2 a necessary condition for the existence of a Frobenius antecedent is that $IR(V) > p^{-1/(p-1)}$; Theorem 10.4.2 below implies that this condition is also sufficient.

Theorem 10.4.2 (after Christol and Dwork). *Let (V, D) be a finite differential module over F_ρ such that $IR(V) > p^{-1/(p-1)}$. Then there exists a unique Frobenius antecedent V' of V. Moreover, $IR(V') = IR(V)^p$.*

Proof. We will assume that $\zeta_p \in K$, as otherwise we could check everything by adjoining ζ_p and then performing a Galois descent at the end.

We first check existence. Since $|D|_{\mathrm{sp},V} < \rho^{-1}$, we may define an action of $\mathbb{Z}/p\mathbb{Z}$ on V using a Taylor series:

$$\zeta_p^m(x) = \sum_{i=0}^{\infty} \frac{(\zeta_p^m t - t)^i}{i!} D^i(x) \qquad (x \in V, m \in \mathbb{Z}/p\mathbb{Z}).$$

Note that the maps $P_j : V \to V$ defined by

$$P_j(v) = \frac{1}{p} \sum_{i=0}^{p-1} \zeta_p^{-ij} \zeta_p^i(v) \qquad (j = 0, \dots, p-1)$$

are F'_ρ-linear projectors onto the generalized eigenspaces for the characters of $\mathbb{Z}/p\mathbb{Z}$. Note also that these eigenspaces are permuted by multiplication by t, so they must all have the same dimension. We may conclude that the fixed space V' is an F'_ρ-subspace of V and the natural map $\varphi^* V' \to V$ is an isomorphism. (This calculation amounts to a simple instance of the Hilbert–Noether theorem, i.e., of Galois descent.)

By applying the $\mathbb{Z}/p\mathbb{Z}$-action to a basis of horizontal sections of V in the generic disc $|t - t_\rho| \leq \lambda\rho$ for some $\lambda \in (p^{-1/(p-1)}, IR(V))$ and then invoking Corollary 10.2.3(b), we may construct horizontal sections of V' in the generic disc $|t^p - t_\rho^p| \leq \lambda^p \rho^p$. Hence $IR(V') \geq IR(V)^p > p^{-p/(p-1)}$.

To check uniqueness, suppose that $V \cong \varphi^* V' \cong \varphi^* V''$ with $IR(V')$, $IR(V'') > p^{-p/(p-1)}$. By Lemma 10.3.6(c) we have

$$\varphi_* V \cong \oplus_{m=0}^{p-1} (V' \otimes W_m) \cong \oplus_{m=0}^{p-1} (V'' \otimes W_m).$$

For $m = 1, \dots, p-1$, we have $IR(W_m) = p^{-p/(p-1)}$; since $IR(V') > IR(W_m)$, we have $IR(V' \otimes W_m) = p^{-p/(p-1)}$ by Lemma 9.4.6(c). Since $IR(V'') > p^{-p/(p-1)}$, the factor $V'' \otimes W_0$ must be contained in $V' \otimes W_0$ and vice versa. Hence $V' \cong V''$.

For the last assertion, note that the proof of existence gives $IR(V') \geq IR(V)^p$, whereas Lemma 10.3.2 gives the reverse inequality. $\square$

Corollary 10.4.3. *Let V' be a differential module over F'_ρ such that $IR(V') > p^{-p/(p-1)}$. Then V' is the Frobenius antecedent of $\varphi^* V'$, so $IR(V') = IR(\varphi^* V')^p$.*

Proof. By Lemma 10.3.2, $IR(\varphi^* V') \geq IR(V')^{1/p} > p^{-1/(p-1)}$, so $\varphi^* V'$ has a unique Frobenius antecedent by Theorem 10.4.2. Since $IR(V') > p^{-p/(p-1)}$, V' is that antecedent. $\square$

The construction of Frobenius antecedents carries over to discs and annuli as follows.

Theorem 10.4.4. *Let M be a finite differential module over $K\langle \alpha/t, t/\beta \rangle$ (we may allow $\alpha = 0$) such that $IR(M \otimes F_\rho) > p^{-1/(p-1)}$ for $\rho \in [\alpha, \beta]$ (or, equivalently, for $\rho = \alpha$ and $\rho = \beta$). Then there exists a unique differential module M' over $K\langle \alpha^p/t^p, t^p/\beta^p \rangle$ such that $M = M' \otimes K\langle \alpha/t, t/\beta \rangle$ and $IR(M' \otimes F'_\rho) > p^{-p/(p-1)}$ for $\rho \in [\alpha, \beta]$; this M' also satisfies $IR(M' \otimes F'_\rho) = IR(M \otimes F_\rho)^p$ for $\rho \in [\alpha, \beta]$. (This theorem also holds with $K\langle \alpha/t, t/\beta \rangle$, $K\langle \alpha^p/t^p, t^p/\beta^p \rangle$ replaced with $K[\![\alpha/t, t/\beta]\!]_{\mathrm{an}}$, $K[\![\alpha^p/t^p, t^p/\beta^p]\!]_{\mathrm{an}}$, respectively, by a similar proof.)*

Proof. Define the projectors P_j as in the proof of Theorem 10.4.2, and let M' be the image of P_0. By arguing as in Theorem 10.4.2 we may show that $M' \otimes K\langle \alpha/t, t/\beta \rangle[t^{-1}] \cong M \otimes K\langle \alpha/t, t/\beta \rangle[t^{-1}]$. However, the quotient of M by the image of $M' \otimes K\langle \alpha/t, t/\beta \rangle$ is a differential module and so cannot have any t-torsion, by Proposition 9.1.2. Hence $M' \otimes K\langle \alpha/t, t/\beta \rangle \cong M$, and we may continue as in Theorem 10.4.2. $\square$

10.5 Frobenius descendants and subsidiary radii

We saw in Lemma 10.3.2 that we can only weakly control the behavior of the generic radius of convergence under Frobenius pullback. Under Frobenius pushforward, we can do much better; we can control not only the generic radius of convergence but also the subsidiary radii. This will lead to a refinement of Lemma 10.3.2; see Corollary 10.5.4. (Example 11.7.2 gives an explicit case.)

Theorem 10.5.1. *Let V be a finite differential module over F_ρ with intrinsic subsidiary radii $s_1 \leq \cdots \leq s_n$. Then the intrinsic subsidiary radii of $\varphi_* V$ comprise the multiset*

$$\bigcup_{i=1}^n \begin{cases} \{s_i^p,\ p^{-p/(p-1)}\ (p-1 \text{ times})\} & s_i > p^{-1/(p-1)}, \\ \{p^{-1} s_i\ (p \text{ times})\} & s_i \leq p^{-1/(p-1)} \end{cases}$$

(note that in the upper expression the second element occurs $p-1$ times altogether and in the lower expression the first element occurs p times altogether). In particular,

$$IR(\varphi_* V) = \min\{p^{-1} IR(V), p^{-p/(p-1)}\}.$$

Proof. It suffices to consider V as irreducible. First suppose that $IR(V) > p^{-1/(p-1)}$. Let V' be the Frobenius antecedent of V (as per Theorem 10.4.2); note that V' is also irreducible. By Lemma 10.3.6(c), $\varphi_* V \cong \oplus_{m=0}^{p-1}(V' \otimes W_m)$. Since each W_m has rank 1, $V' \otimes W_m$ is also irreducible. Since $IR(V') = IR(V)^p$ by Theorem 10.4.2 and $IR(V' \otimes W_m) = p^{-p/(p-1)}$ for $m \neq 0$ by Lemma 9.4.6(c), we have the claim.

Next suppose that $IR(V) \leq p^{-1/(p-1)}$. We show first that

$$IR(\varphi_* V) \geq p^{-1} IR(V) = \max\{IR(V)^p, p^{-1} IR(V)\}.$$

For t_ρ a generic point of radius ρ and $\lambda \in (0, p^{-1/(p-1)})$, the module $\varphi_* V \otimes L\langle (t^p - t_\rho^p)/(p^{-1}\lambda\rho^p)\rangle$ splits as the direct sum of $V \otimes L\langle (t - \zeta_p^m t_\rho)/(\lambda\rho)\rangle$ over $m = 0, \dots, p-1$. If $\lambda < IR(V)$, by applying Corollary 10.2.3(c) we obtain $IR(\varphi_* V) \geq p^{-1}\lambda$.

Next, let W' be any irreducible subquotient of $\varphi_* V$; then $IR(W') \geq IR(\varphi_* V)$, so Lemma 10.3.2 gives on the one hand

$$IR(\varphi^* W') \geq \min\{IR(W')^{1/p}, pIR(W')\} \geq \min\{IR(\varphi_* V)^{1/p}, pIR(\varphi_* V)\} \geq IR(V). \tag{10.5.1.1}$$

On the other hand $\varphi^* W'$ is a subquotient of $\varphi^* \varphi_* V$, which by Lemma 10.3.6(d) is isomorphic to $V^{\oplus p}$. Since V is irreducible, each Jordan–Hölder constituent of $\varphi^* W'$ must be isomorphic to V, yielding $IR(\varphi^* W') = IR(V)$. That forces each inequality in (10.5.1.1) to be an equality; thus, in particular, $IR(W')$ and $IR(\varphi_* V)$ have the same image under the injective map $s \mapsto \min\{s^{1/p}, ps\}$. We conclude that $IR(W') = IR(\varphi_* V) = p^{-1} IR(V)$, proving the claim. □

Remark 10.5.2. One might be tempted to think that the verification that $IR(\varphi_* V) \geq p^{-1} IR(V)$ within the proof of Theorem 10.5.1 should carry over to the case $IR(V) > p^{-1/(p-1)}$, in which case it would lead to the false conclusion $IR(\varphi_* V) \geq IR(V)^p$. What breaks down in this case is that pushing forward a basis of local horizontal sections of V gives only $\dim V$ local horizontal sections of $\varphi_* V$; what these span is precisely the Frobenius antecedent of V.

Corollary 10.5.3. *Let $s_1 \leq \cdots \leq s_n$ be the intrinsic subsidiary radii of V.*

(a) *For i such that $s_i \leq p^{-1/(p-1)}$, the product of the ip smallest intrinsic subsidiary radii of $\varphi_* V$ is equal to $p^{-ip} s_1^p \cdots s_i^p$.*

(b) *For i such that either $i = n$ or $s_{i+1} \geq p^{-1/(p-1)}$, the product of the $ip + (p-1)(n-i)$ smallest intrinsic subsidiary radii of $\varphi_* V$ is equal to $p^{-in} s_1^p \cdots s_i^p$.*

In particular, the product of the intrinsic subsidiary radii of $\varphi_* V$ is $p^{-np} s_1^p \cdots s_n^p$.

Note that both conditions apply when $s_i = p^{-1/(p-1)}$; this will be important later (see Remark 11.6.2).

As promised, we will now obtain a refined version of Lemma 10.3.2.

Corollary 10.5.4. *Let V' be a differential module over F'_ρ such that $IR(V') \neq p^{-p/(p-1)}$. Then $IR(\varphi^* V') = \min\{IR(V')^{1/p}, pIR(V')\}$. (This fails when $IR(V') = p^{-p/(p-1)}$, e.g., for $V' = W_m$.)*

Proof. In the case $IR(V') > p^{-p/(p-1)}$, this holds by Corollary 10.4.3. Otherwise, by Lemma 10.3.6(c), $\varphi_* \varphi^* V' \cong \oplus_{m=0}^{p-1} (V' \otimes W_m)$ and $IR(V' \otimes W_m) = IR(V')$ since $IR(V') < IR(W_m)$. Hence, by Theorem 10.5.1,

$$IR(V') = IR(\varphi_* \varphi^* V') = \min\{p^{-1} IR(\varphi^* V'), p^{-p/(p-1)}\}.$$

We get a contradiction if the right-hand side equals $p^{-p/(p-1)}$, so it must be the case that $IR(V') = p^{-1} IR(\varphi^* V') \leq p^{-p/(p-1)}$, proving the claim. □

10.6 Decomposition by spectral radius

As our first application of Frobenius descent, we will extend the visible decomposition theorem, and its refined form, in the special case of a differential module over F_ρ. We then deduce some consequences analogous to the consequences of the visible decomposition theorem. To do this, we must use Frobenius descendants to cross the bound on the spectral radius. This cannot be done using Frobenius antecedents alone, as they give no information in the boundary case $IR(V) = p^{-1/(p-1)} \rho$.

In this section, we may suppress the hypothesis $p > 0$, since the case $p = 0$ is already covered by the original decomposition theorems.

Proposition 10.6.1. *Let V_1, V_2 be irreducible finite differential modules over F_ρ, with $IR(V_1) \neq IR(V_2)$. Then $H^1(V_1 \otimes V_2) = 0$.*

Proof. We may assume that $IR(V_2) > IR(V_1)$; note that $IR(V_1^\vee) = IR(V_1)$ by Lemma 9.4.6(b). If $p = 0$, or if $p > 0$ and $IR(V_1) < p^{-1/(p-1)}$, then any short exact sequence $0 \to V_2 \to V \to V_1^\vee \to 0$ splits by the visible decomposition theorem (Theorem 6.6.1), yielding the desired vanishing by Lemma 5.3.3. Thus we may assume hereafter that $p > 0$.

Suppose that $p > 0$ and $IR(V_1) = p^{-1/(p-1)}$. Let V_2' be the Frobenius antecedent of V_2; it also is irreducible, and $IR(V_2') = IR(V_2)^p > p^{-p/(p-1)}$ by Theorem 10.4.2. By Theorem 10.5.1, each irreducible subquotient W of $\varphi_* V_1$ satisfies $IR(W) = p^{-p/(p-1)}$; hence $H^1(W \otimes V_2') = 0$ by the previous case, so $H^1(\varphi_* V_1 \otimes V_2') = 0$ by the snake lemma.

By Lemma 10.3.6(a), (c),

$$\begin{aligned}\varphi_* V_1 \otimes \varphi_* V_2 &\cong \oplus_{m=0}^{p-1} (\varphi_* V_1 \otimes W_m \otimes V_2') \\ &\cong (\varphi_* V_1 \otimes V_2')^{\oplus p}.\end{aligned}$$

This yields $H^1(\varphi_* V_1 \otimes \varphi_* V_2) = 0$. By Lemma 10.3.6(f), $\varphi_*(V_1 \otimes V_2)$ is a direct summand of $\varphi_* V_1 \otimes \varphi_* V_2$, so $H^1(\varphi_*(V_1 \otimes V_2)) = 0$. By Lemma 10.3.6(e), $H^1(V_1 \otimes V_2) = H^1(\varphi_*(V_1 \otimes V_2)) = 0$.

In the general case, $1 \geq IR(V_2) > IR(V_1)$. If $IR(V_1) > p^{-1/(p-1)}$ then Theorem 10.4.2 implies that V_1, V_2 have Frobenius antecedents V_1', V_2'. In addition, for any extension $0 \to V_1 \to V \to V_2^\vee \to 0$, the module V satisfies $IR(V) > p^{-1/(p-1)}$ by Lemma 9.4.6, so Theorem 10.4.2 implies that the whole sequence is itself the pullback of an extension $0 \to V_1' \to V' \to (V_2')^\vee \to 0$. To show that V always splits, it suffices to do so for V'; that is, we may reduce from V_1, V_2 to V_1', V_2'. By repeating this enough times, we reach the situation where $IR(V_1) \leq p^{-1/(p-1)}$. We may then apply the previous cases. □

From here, the proof of the following theorem is purely formal.

Theorem 10.6.2 (Strong decomposition theorem). *Let V be a finite differential module over F_ρ. Then there exists a decomposition*

$$V = \bigoplus_{s \in (0,1]} V_s,$$

where every subquotient W_s of V_s satisfies $IR(W_s) = s$.

Proof. We induct on $\dim V$; we need consider only reducible V. Choose a short exact sequence $0 \to U_1 \to V \to U_2 \to 0$ with U_2 irreducible. Decompose U_1 as $\oplus_{s \in (0,1]} U_{1,s}$, where every subquotient W_s of $U_{1,s}$ satisfies $IR(W_s) = s$. For each $s \neq IR(U_2)$, we have $H^1(U_2^\vee \otimes U_{1,s}) = 0$ by repeated application of Proposition 10.6.1 plus the snake lemma. By Lemma 5.3.3 we have

$$V = V' \oplus \bigoplus_{s \neq IR(U_2)} U_{1,s},$$

where $0 \to U_{1,IR(U_2)} \to V' \to U_2 \to 0$ is exact. □

As with the visible decomposition theorem, we obtain the following corollaries.

Corollary 10.6.3. *Let V be a finite differential module over F_ρ whose intrinsic subsidiary radii are all less than* 1. *Then $H^0(V) = H^1(V) = 0$.*

Proof. It is clear that $H^0(V)$ vanishes, as otherwise V would have a submodule with intrinsic generic radius of convergence equal to 1. To see that $H^1(V) = 0$, it suffices by Lemma 5.3.3 to show that any short exact sequence $0 \to V \to W \to X \to 0$ with X trivial is split. This follows by applying Theorem 10.6.2 to W: in the resulting decomposition, V and X must project into distinct summands, so $W \cong V \oplus X$. □

Corollary 10.6.4. *With $V = \oplus_{s \in (0,1]} V_s$ as in Theorem 10.6.2, we have $H^i(V) = H^i(V_1)$ for $i = 0, 1$.*

This suggests that difficulties in computing H^0 and H^1 arise in the case of intrinsic generic radius 1. We will pursue a closer study of this case in Chapter 13.

Using the strong decomposition theorem, we obtain a refined version of Corollary 6.2.9 that extends Corollary 6.6.3.

Corollary 10.6.5. *If V_1 and V_2 are irreducible finite differential modules over F_ρ and $IR(V_1) < IR(V_2)$ then every irreducible subquotient W of $V_1 \otimes V_2$ satisfies $IR(W) = IR(V_1)$.*

Proof. Decompose $V_1 \otimes V_2 = \oplus_{s \in (0,1]} V_s$ according to Theorem 10.6.2; by Lemma 9.4.6(c), $V_s = 0$ whenever $s < IR(V_1)$. If some V_s with $s > IR(V_1)$ were nonzero then $V_1 \otimes V_2$ would have an irreducible submodule of intrinsic radius greater than $IR(V_1)$, in violation of Corollary 6.2.9. □

We should also mention the following related result extending Corollary 6.2.7 and Corollary 6.5.5.

Proposition 10.6.6. *Let $F_\rho \to E$ be an isometric embedding of complete nonarchimedean differential fields such that $|d|_{F_\rho} = |d|_E$ and $|d|_{\mathrm{sp},F_\rho} = |d|_{\mathrm{sp},E}$. Then, for any finite differential module V over F_ρ, the full spectra of V and $V \otimes_{F_\rho} E$ are equal.*

Proof. We may assume that V is irreducible and has spectral radius s. We first check the case where E is a finite extension of F_ρ; we may assume that E is Galois over F_ρ. Let W be any submodule of $V \otimes_{F_\rho} E$; then $V \otimes_{F_\rho} E$ is a quotient of the direct sum of the Galois conjugates of W, all of which have the same spectral radius. Hence W and $V \otimes_{F_\rho} E$ have the same spectral radius, which by Corollary 6.2.7 is s. Moreover, every constituent of $V \otimes_{F_\rho} E$ is a conjugate of W and so has spectral radius s. This proves the claim.

In the general case, we may use the previous paragraph to add a primitive pth root of unity ζ_p to F_ρ in the case $p > 0$. If $p = 0$ or $s > \rho$ then Theorem 6.5.3 implies the desired result. Otherwise we may apply Theorem 10.5.1, replacing F_ρ by F'_ρ and E by the fixed field under the action of $\mathbb{Z}/p\mathbb{Z}$ (given by a Taylor series). Repeating this finitely many times, we arrive back at the case where Theorem 6.5.3 becomes applicable. □

We end this discussion by extending the refined visible decomposition theorem (Theorem 6.8.2) to the full spectrum.

Theorem 10.6.7 (Refined strong decomposition theorem). *Let V be a finite differential module over F_ρ such that no subquotient of V has intrinsic radius equal to* 1. *Then, for some finite tamely ramified extension E of F_ρ, $V \otimes_{F_\rho} E$ admits a (unique) direct sum decomposition*

$$V \otimes_{F_\rho} E = \bigoplus_i V_i,$$

with each V_i refined, such that for $i \neq j$ we have $V_i \not\sim V_j$ in the sense of Definition 6.2.12.

Proof. By Theorem 10.6.2 we may reduce immediately to the case where V is pure and $IR(V) < 1$. If $p = 0$, or if $p > 0$ and $IR(V) < p^{-1/(p-1)}$, then Theorem 6.8.2 gives the claim.

Next we consider the case $IR(V) = p^{-1/(p-1)}$; by Theorem 10.5.1, $\varphi_* V$ is again pure with $IR(\varphi_* V) = p^{-p/(p-1)}$. By the previous paragraph, for some finite tamely extension E' of F'_ρ we obtain a decomposition of $(\varphi_* V) \otimes_{F'_\rho} E'$ into nonequivalent refined submodules. For V_1, V_2 appearing in this decomposition, we declare V_1 and V_2 to be *weakly equivalent* if V_1 is equivalent to $V_2 \otimes_{E'} (W_m \otimes_{F'_\rho} E')$ for some m. This is again an equivalence relation; by Lemma 10.3.6(b), if we group the summands of $(\varphi_* V) \otimes_{F'_\rho} E'$ into weak equivalence classes then the resulting decomposition descends to $V \otimes_{F_\rho} E$ for $E = F_\rho \otimes_{F'_\rho} E'$.

Let X be a summand of $V \otimes_{F_\rho} E$ in this decomposition; it remains to check that X is refined. By Lemma 10.3.6(f) we have

$$(\varphi_* X^\vee) \otimes (\varphi_* X) \cong (\varphi_*(X^\vee \otimes X))^{\oplus p}.$$

Hence, by the previous construction, $\varphi_*(X^\vee \otimes X)$ decomposes as a direct sum in which each summand can be twisted by a suitable $W_m \otimes_{F'_\rho} E'$ to raise its intrinsic radius above $p^{-p/(p-1)}$. However, by Lemma 10.3.6(a), $\varphi_*(X^\vee \otimes X)$

is isomorphic to its own twist by any $W_m \otimes_{F'_\rho} E'$. Hence on the one hand its intrinsic subsidiary radii must comprise a multiset of $p\,\mathrm{rank}(X)^2$ elements in which exactly $\mathrm{rank}(X)^2$ of the elements are greater than $p^{-p/(p-1)}$. On the other hand, since $IR(X^\vee \otimes X) \geq IR(X) = p^{-1/(p-1)}$ by Lemma 9.4.6(c), Theorem 10.5.1 implies that the intrinsic subsidiary radii of $\varphi_*(X^\vee \otimes X)$ include at most $\mathrm{rank}(X)^2$ elements greater than $p^{-p/(p-1)}$. In fact, equality occurs if and only if *all* the intrinsic subsidiary radii of $X^\vee \otimes X$ are greater than $p^{-1/(p-1)}$. Hence X is refined.

Finally, we handle the case $p^{-1/(p-1)} < IR(V) < 1$ by induction on the smallest integer h such that $IR(V) \leq p^{-p^{-h}/(p-1)}$. The case $h = 0$ is handled by the arguments above. If $h > 0$ then, by Theorem 10.4.2, V has a Frobenius antecedent W for which $IR(W) = IR(V)^{1/p} \leq p^{-p^{-h+1}/(p-1)}$. By the induction hypothesis, W admits a decomposition into nonequivalent refined submodules; pulling back by a Frobenius morphism then gives the desired decomposition of V, by Corollary 10.5.4. □

10.7 Integrality of the generic radius

The relationship between the generic radius of convergence and Newton polygons in the visible range (given in Theorem 6.5.3) suggests that the generic radius of convergence should satisfy some sort of integrality property. On the one hand we can infer such a property using Frobenius antecedents; on the other hand, a certain price must be paid.

Theorem 10.7.1. *Let V be a finite differential module over F_ρ with intrinsic subsidiary radii $s_1 \leq \cdots \leq s_n$. Let m be the largest integer such that $s_m = IR(V)$. Then, for any nonnegative integer h,*

$$s_1 < p^{-p^{-h}/(p-1)} \quad \Longrightarrow \quad s_1^m \in |K^\times|^{p^{-h}} \rho^{\mathbb{Z}}.$$

Proof. For $h = 0$ we can read this off from the case of a Newton polygon (i.e., we can invoke Theorem 6.5.3). To reduce from h to $h-1$, if $IR(V) > p^{-1/(p-1)}$ then replace V by its Frobenius antecedent (Theorem 10.4.2); if $IR(V) = p^{-1/(p-1)}$, apply φ_* and invoke Corollary 10.5.3. □

Here is an example to show that the exponent p^{-h} in the conclusion $s_1^m \in |K^\times|^{p^{-h}} \rho^{\mathbb{Z}}$ of Theorem 10.7.1 is not spurious.

Example 10.7.2. Suppose that $\pi \in K$ satisfies $|\pi| = p^{-1/(p-1)}$. Pick $\lambda \in K^\times$ and $0 < \alpha \leq \beta$ such that, for $\rho \in [\alpha, \beta]$,

$$p^{1/(p-1)} < |\lambda| \rho^{-p} < p^{p/(p-1)}.$$

Let M be the differential module over $K\langle \alpha/t, t/\beta\rangle$ that is generated by v satisfying $D(v) = -p\pi\lambda t^{-p-1}v$. Then $M \cong \varphi^* M'$, where M' is the differential module over $K\langle \alpha^p/t^p, t^p/\beta^p\rangle$ with generator w and $D'(w) = -\pi\lambda(t^p)^{-2}w$. We deduce that

$$|D'|_{M'\otimes F'_\rho} = p^{-1/(p-1)}|\lambda|\rho^{-2p} > \rho^{-p}.$$

Hence we have

$$IR(M' \otimes F'_\rho) = |\lambda|^{-1}\rho^p,$$
$$IR(M \otimes F_\rho) = |\lambda|^{-1/p}\rho,$$

where the first equality follows by Theorem 6.5.3 and the second follows from the first by Corollary 10.4.3.

Remark 10.7.3. Another way to understand Example 10.7.2 is by means of the Dwork exponential series; see Definition 17.1.3.

Question 10.7.4. *What is the correct extension of Theorem 10.7.1 for the remaining subsidiary radii? (This should not be difficult.)*

10.8 Off-center Frobenius antecedents and descendants

Since pushing forward along a Frobenius morphism does not work well on a disc (Remark 10.3.5), we must also consider "off-center" Frobenius antecedents and descendants. Although this can be done rather more generally, we will stick to one case that is sufficient for our purposes.

Definition 10.8.1. For $\rho \in (p^{-1/(p-1)}, 1]$, let F''_ρ be the completion of $K((t-1)^p - 1)$ under the ρ^p-Gauss norm. Note that this coincides with the restriction of the ρ-Gauss norm on $K(t)$, because $|((t-1)^p - 1) - t^p|_\rho < |t^p|_\rho$. (One could allow $K((t-\mu)^p - \mu^p)$ for any $\mu \in K$ of norm 1, but there is no significant loss of generality in rescaling t to reduce to the case $\mu = 1$.) For brevity, write $u = (t-1)^p - 1$. Equip F''_ρ with the derivation

$$d'' = \frac{d}{du} = \frac{1}{du/dt}d.$$

Given a differential module V'' over F''_ρ, we may view $\psi^* V'' = V'' \otimes F_\rho$ as a differential module over F_ρ. Given a differential module V over F_ρ, we may view the restriction $\psi_* V$ of V along $F''_\rho \to F_\rho$ as a differential module over F''_ρ.

The main point here is that $du/dt \in K\langle t\rangle^\times$. Consequently, we can extend both ψ^* and ψ_* not just to annuli but also discs. This is needed to establish the monotonicity property for subsidiary radii (Theorem 11.3.2(d)).

We may apply Lemma 10.2.2 with η replaced by $\eta + 1$, keeping in mind that $|\eta + 1| = 1$ for $|\eta| < 1$. This has the net effect that everything that holds for φ also holds for ψ, except that the intrinsic generic radius of convergence must be replaced by the extrinsic one. Rather than rederive everything, we simply state the analogues of Theorems 10.4.2 and 10.5.1 and leave the proofs as exercises.

Theorem 10.8.2. *Let (V, D) be a finite differential module over F_ρ such that $R(V) > p^{-1/(p-1)}$. Then there exists a unique differential module (V'', D'') over F''_ρ equipped with an isomorphism $V \cong \psi^* V''$ such that $R(V'') > p^{-p/(p-1)}$. For this V'', one has in fact $R(V'') = R(V)^p$.*

Theorem 10.8.3. *Let V be a finite differential module over F_ρ with extrinsic subsidiary radii $s_1, \dots, s_n$. Then the extrinsic subsidiary radii of $\psi_* V$ comprise the multiset*

$$\bigcup_{i=1}^{n} \begin{cases} \{s_i^p,\ p^{-p/(p-1)}\ (p-1 \text{ times})\} & s_i > p^{-1/(p-1)}, \\ \{p^{-1} s_i\ (p \text{ times})\} & s_i \leq p^{-1/(p-1)}. \end{cases}$$

Remark 10.8.4. Note that one cannot expect Theorem 10.8.3 to hold for $\rho < p^{-1/(p-1)}$, as in that case $p^{-p/(p-1)}$ is too large to appear as a subsidiary radius of $\psi_* V$.

Notes

Lemma 10.2.2 is taken from [**128**, §5.3] with some typos corrected.

The Frobenius antecedent theorem of Christol and Dwork [**49**, Théorème 5.4] is slightly weaker than the one given here: it only applies for $IR(V) > p^{-1/p}$. The discrepancy is due to the introduction of cyclic vectors, which create some regular singularities which can only eliminated under the stronger hypothesis. Much closer to the statement of Theorem 10.4.2 is [**128**, Theorem 6.13]; however, uniqueness is only asserted there when $IR(V') \geq IR(V)^p$.

The concept of the Frobenius descendant, and the results deduced using it, are original. This includes Theorem 10.5.1 and its off-center analogue (Theorem 10.8.3), the strong decomposition theorem (Theorem 10.6.2), and the refined strong decomposition theorem (Theorem 10.6.7). The latter was suggested by Liang Xiao; see the notes for Chapter 6 for the motivation for this.

The notions of Frobenius antecedents and descendants extend to derivations of rational type, with one caveat: the assertion of Theorem 10.7.1 does not carry over. See [**145**, Theorem 1.4.21] for the correct statement.

Exercises

(1) Prove Lemma 10.3.6.

(2) Prove that, for any finite differential module V' over F'_ρ with $IR(V') > p^{-p/(p-1)}$, $H^0(V') = H^0(\varphi^* V')$. (The example $V' = W_m$ shows that the bound on $IR(V')$ cannot be relaxed.)

(3) Derive Theorem 10.8.2 by imitating the proof of Theorem 10.4.2.

(4) Derive Theorem 10.8.3 by imitating the proof of Theorem 10.5.1.

11
Variation of generic and subsidiary radii

In this chapter, we apply the tools developed in the preceding chapters to study the variation of the generic radius of convergence, and of the subsidiary radii, associated with a differential module on a disc or annulus. We have already seen some instances where this study is needed to deduce consequences about the convergence of solutions of p-adic differential equations (Examples 9.6.2 and 9.9.3).

The statements we will formulate are modeled on statements governing the variation of the Newton polygon of a polynomial over a ring of power series as we vary the choice of Gauss norm on the power series ring. The guiding principle is that, in the visible spectrum, one should be able to relate the variation of subsidiary radii to the variation of Newton polygons via matrices of action of the derivation on suitable bases. This includes the relationship between subsidiary radii and Newton polygons for cyclic vectors (Theorem 6.5.3), but trying to use that approach directly creates no end of difficulties because cyclic vectors only exist in general for differential modules over fields. We will implement the guiding principle in a somewhat more robust manner than before, using the discussion of matrix inequalities in Chapter 6.

To this principle we must add the techniques of descent along a Frobenius morphism introduced in Chapter 10, including the off-centered variant. This allows us to overcome the limitation to the visible spectrum.

As corollaries of this analysis, we deduce some facts about the true radius of convergence of a differential module on a disc. We also establish a geometric interpretation of subsidiary radii in terms of the convergence of local horizontal sections around a generic point, extending a result of Young. We will build further on this work when we discuss decomposition theorems in Chapter 12.

Throughout this chapter we retain Notation 10.0.1 but we do not assume that $p > 0$. We will continue to use the convention

$$\omega = \begin{cases} 1 & p = 0, \\ p^{-1/(p-1)} & p > 0. \end{cases}$$

11.1 Harmonicity of the valuation function

For $f \in K\langle\alpha/t, t/\beta\rangle$ and $r \in [-\log\beta, -\log\alpha]$, the function $r \mapsto v_r(f)$ is continuous, piecewise affine, and (by Proposition 8.2.3(b)) concave in r. However, one can make an even more precise statement; for simplicity, we will write this out explicitly only for $r = 0$.

Definition 11.1.1. For $\overline{\mu}$ in some extension of k, let μ be a lift of $\overline{\mu}$ in some complete extension L of K. For $\alpha \leq 1 \leq \beta$, define the substitution

$$T_\mu : K\langle\alpha/t, t/\beta\rangle \to L[\![t]\!]_{\mathrm{an}}, \qquad t \mapsto t + \mu.$$

(This map extends to $K[\![\alpha/t, t/\beta]\!]_0$ if $\alpha < 1 < \beta$.) The function $r \mapsto v_r(T_\mu(f))$ on $[0, +\infty)$ is continuous and piecewise affine; moreover, its right-hand slope at $r = 0$ does not depend on the choice of the field L or the lift μ of $\overline{\mu}$. We call this slope $s_{\overline{\mu}}(f)$. For $1 < \beta$ (resp. $\alpha < 1$), define $s_\infty(f)$ (resp. $s_0(f)$) to be the left-hand (resp. right-hand) slope of the function $r \mapsto v_r(f)$ at $r = 0$.

Then we have the following harmonicity property.

Proposition 11.1.2. *For $0 \leq \alpha < 1 < \beta$ and $f \in K\langle\alpha/t, t/\beta\rangle$ nonzero, we have*

$$s_\infty(f) = \sum_{\overline{\mu} \in \kappa_K^{\mathrm{alg}}} s_{\overline{\mu}}(f).$$

Proof. Without loss of generality we may assume that $|f|_1 = 1$. The quotient of $\mathfrak{o}_{F_1} \cap K\langle\alpha/t, t/\beta\rangle$ by the ideal generated by $\mathfrak{m}_K$ is isomorphic to $\kappa_K[t, t^{-1}]$; let $\overline{f}$ be the image of f in this quotient. Then $s_{\overline{\mu}}$ is the order of vanishing of $\overline{f}$ at $\overline{\mu}$, whereas s_∞ is the pole order of $\overline{f}$ at ∞. The desired equality then follows from the fact that a rational function has as many zeroes as poles (counted including the multiplicity of each pole). $\square$

Remark 11.1.3. Note that $s_{\overline{\mu}}(f) \geq 0$ for $\overline{\mu} \neq 0$; thus Proposition 11.1.2 does indeed recover the concavity inequality $s_\infty \geq s_0$. Also, $s_{\overline{\mu}}(f) = 0$ if $\overline{\mu} \notin \kappa_K^{\mathrm{alg}}$ because the zeroes and poles of a rational function with coefficients in κ_K must be algebraic over κ_K.

11.2 Variation of Newton polygons

Before proceeding to differential modules, we study the variation of the Newton polygon of a polynomial over $K\langle\alpha/t, t/\beta\rangle$ or $K[\![\alpha/t, t/\beta]\!]_{\mathrm{an}}$ when measured with respect to different Gauss valuations. We begin with this both because it motivates the statements of the results for differential modules and also because it will be used heavily in the proofs of those statements.

Theorem 11.2.1. *Let $P \in K[\![\alpha/t, t/\beta]\!]_{\mathrm{an}}[T]$ be a polynomial of degree n. For $r \in [-\log\beta, -\log\alpha]$, put $v_r(\cdot) = -\log|\cdot|_{e^{-r}}$. Let $\mathrm{NP}_r(P)$ be the Newton polygon of P under v_r. Let $f_1(P,r), \dots, f_n(P,r)$ be the slopes of $\mathrm{NP}_r(P)$ (listed with multiplicity) in increasing order. For $i = 1, \dots, n$, put $F_i(P,r) = f_1(P,r) + \cdots + f_i(P,r)$.*

(a) *(Linearity) For $i = 1, \dots, n$, the functions $f_i(P,r)$ and $F_i(P,r)$ are continuous and piecewise affine in r. Moreover, even if $\alpha = 0$ there are only finitely many different slopes.*

(b) *(Integrality) If $i = n$ or $f_i(r_0) < f_{i+1}(r_0)$ then the slopes of $F_i(P,r)$ in some neighborhood of $r = r_0$ belong to $\mathbb{Z}$. Consequently, the slopes of each $f_i(P,r)$ and $F_i(P,r)$ belong to*

$$\frac{1}{1}\mathbb{Z} \cup \cdots \cup \frac{1}{n}\mathbb{Z}.$$

(c) *(Superharmonicity) Suppose that $\alpha < 1 < \beta$. For $i = 1, \dots, n$, let $s_{\infty,i}(P)$ and $s_{0,i}(P)$ be the left-hand and right-hand slopes of $F_i(P,r)$ at $r = 0$. For $\overline{\mu} \in (\kappa_K^{\mathrm{alg}})^\times$, let $s_{\overline{\mu},i}(P)$ be the right-hand slope of $F_i(T_\mu(P), r)$ at $r = 0$. Then*

$$s_{\infty,i}(P) \geq \sum_{\overline{\mu} \in \kappa_K^{\mathrm{alg}}} s_{\overline{\mu},i}(P),$$

with equality if $i = n$ or $f_i(P,0) < f_{i+1}(P,0)$.

(d) *(Monotonicity) Suppose that P is monic and $\alpha = 0$. For $i = 1, \dots, n$, the slopes of $F_i(P,r)$ are nonnegative.*

(e) *(Concavity) Suppose that P is monic. For $i = 1, \dots, n$, the function $F_i(P,r)$ is concave.*

Proof. Write $P = \sum_{i=0}^n P_i T^i$ with $P_i \in K[\![\alpha/t, t/\beta]\!]_{\mathrm{an}}$. By Proposition 8.5.2 the function $v_r(P_i)$ is continuous and concave in r and piecewise affine with slopes in $\mathbb{Z}$. Moreover, even if $\alpha = 0$ there are only finitely many different slopes.

For $s \in \mathbb{R}$ and $r \in [-\log\beta, -\log\alpha]$, put

$$v_{s,r}(P) = \min_i\{v_r(P_i) + is\};$$

that is, $v_{s,r}(P)$ is the y-intercept of the supporting line of $\mathrm{NP}_r(P)$ of slope s. Since $v_{s,r}(P)$ is the minimum of finitely many functions of the pair (r,s), each of which is continuous, piecewise affine with only finitely many different slopes, and concave, it also enjoys these properties. (Compare Remark 2.1.7.)

Note that $F_i(P,r)$ is the difference between the y-coordinates of the points of $\mathrm{NP}_r(P)$ having x-coordinates $i-n$ and $-n$. That is,

$$F_i(P,r) = \sup_s \{v_{s,r}(P) - (n-i)s\} - v_r(P_n). \tag{11.2.1.1}$$

The supremum in (11.2.1.1) is achieved by some s whose denominator is bounded by n. For any given r_0, for r in some neighborhood of r_0 there can be only finitely many values of s with denominator bounded by n achieving the supremum in (11.2.1.1) for at least one value of r in the neighborhood. Consequently, $F_i(P,r)$ is continuous and piecewise affine with only finitely many different slopes, proving (a).

If $i = n$ or $f_i(P,r_0) < f_{i+1}(P,r_0)$ then the point of $\mathrm{NP}_{r_0}(P)$ having x-coordinate $i-n$ is a vertex, and likewise for r in some neighborhood of r_0. In that case, for r near r_0,

$$F_i(P,r) = v_r(P_{n-i}) - v_r(P_n), \tag{11.2.1.2}$$

proving (b).

Assume that $\alpha < 1 < \beta$. Then Proposition 11.1.2 implies that

$$s_\infty(P_i) = \sum_{\overline{\mu}\in\kappa_K^{\mathrm{alg}}} s_{\overline{\mu}}(P_i) \qquad (i = 0,\dots,n).$$

If $i = n$ or $f_i(P,0) < f_{i+1}(P,0)$ then the above equation imply plus (11.2.1.2) that the desired inequality is in fact an equality. Otherwise, let j,k be the least and greatest indices for which $f_j(P,0) = f_i(P,0) = f_k(P,0)$; then $j \leq i < k$, and the convexity of the Newton polygon implies that

$$F_i(P,r) \leq \frac{k-i}{k-j+1} F_{j-1}(P,r) + \frac{i-j+1}{k-j+1} F_k(P,r), \tag{11.2.1.3}$$

with equality for $r=0$. From this plus piecewise affinity we deduce that

$$\begin{aligned} s_{\infty,i}(P) &\geq \frac{k-i}{k-j+1} s_\infty(P_{n-j+1}) + \frac{i-j+1}{k-j+1} s_\infty(P_{n-k}),\\ s_{\overline{\mu},i}(P) &\leq \frac{k-i}{k-j+1} s_{\overline{\mu}}(P_{n-j+1}) + \frac{i-j+1}{k-j+1} s_{\overline{\mu}}(P_{n-k}) \qquad (\overline{\mu} \in \kappa_K^{\mathrm{alg}}), \end{aligned}$$

yielding (c).

Assume that $\alpha = 0$ and that P is monic. Then each $v_r(P_i)$ is a nondecreasing function of r, as is each $v_{s,r}(P)$. Since $v_r(P_n) = 0$, $F_i(P,r)$ is nondecreasing by (11.2.1.1), proving (d).

To prove (e), one can reduce to working locally around $r = 0$ and then deduce the claim from (c) and (d) (because the latter implies that $s_{\overline{\mu},i}(P) \geq 0$ for $\overline{\mu} \neq 0$). However, one can also prove (e) directly as follows. Assume that P is monic, so that $P_n = 1$ and (11.2.1.1) reduces to

$$F_i(P, r) = \sup_s \{v_{s,r}(P) - (n-i)s\}.$$

It is not immediately clear from this that $F_i(P, r)$ is concave, since we are taking the supremum rather than the infimum of a collection of concave functions. To get around this, pick $r_1, r_2 \in [-\log \beta, -\log \alpha]$ and put $r_3 = ur_1 + (1-u)r_2$ for some $u \in [0, 1]$. For $j \in \{1, 2\}$, choose an s_j achieving the supremum in (11.2.1.1) for $r = r_j$. Put $s_3 = us_1 + (1-u)s_2$; then, using the concavity of $v_{s,r}(P)$ in both s and r, we have

$$\begin{aligned} F_i(P, r_3) &\geq v_{s_3,r_3}(P) - (n-i)s_3 \\ &\geq u(v_{s_1,r_1}(P) - (n-i)s_1) + (1-u)(v_{s_2,r_2}(P) - (n-i)s_2) \\ &= uF_i(P, r_1) + (1-u)F_i(P, r_2). \end{aligned}$$

This yields concavity for $F_i(P, r)$, proving (e). □

Remark 11.2.2. A more geometric interpretation of the previous proof can be given by writing each P_i as $\sum_j P_{i,j}t^j$ and considering the lower convex hull of the set of points $\{(-i, -j, v(P_{i,j}))\}$ in $\mathbb{R}^3$. We leave elaboration of this point to the reader.

Remark 11.2.3. If $i = n$ or $f_i(P, r_0) < f_{i+1}(P, r_0)$ then (11.2.1.2) implies that

$$f_1(P, r_0) + \cdots + f_i(P, r_0) \in v(K^\times) + \mathbb{Z}r_0.$$

This fact does not analogize to subsidiary radii, because one has to replace $v(K^\times)$ by its p-divisible closure. See Theorem 10.7.1 and Example 10.7.2.

Remark 11.2.4. The conclusions of Theorem 11.2.1 carry over if we replace $f_i(P, r)$ by $\min\{f_i(P, r), ar + b\}$ for any fixed $a, b \in \mathbb{R}$ (except for (b), for which we need $a, b \in \mathbb{Z}$). This is so because

$$\sum_{j=1}^{i} \min\{f_i(P, r), ar + b\} = \sup_{s \leq ar+b} \{v_{s,r}(P) - (n-i)s\} - v_r(P_n);$$

in other words, the height of the relevant point is determined by the supporting lines of slope less than or equal to $ar+b$, rather than by all slopes. Note that, in the notation of the proof of Theorem 11.2.1(e), the inequality $s_j \leq ar_j + b$ for $j = 1, 2$ implies the same for $j = 3$, so the proof of concavity goes through.

For bounded elements, we obtain a similar but slightly weaker conclusion.

Theorem 11.2.5. *Let $P \in K[\![\alpha/t, t/\beta]\!]_0[T]$ be a polynomial of degree n. Then the conclusions of Theorem 11.2.1 continue to hold except that, if K is not discrete, in (a) the functions $f_i(P, r)$ and $F_i(P, r)$ are only piecewise affine on the interior of $[-\log\beta, -\log\alpha]$ (with possibly infinitely many slopes).*

Proof. The revised statement of (a) holds by Remark 8.2.4. For the other parts, one may simply apply Theorem 11.2.1 over the ring $K\langle\gamma/t, t/\delta\rangle[T]$ for all γ, δ with $\alpha < \gamma \le \delta < \beta$ (in the case $\alpha \ne 0$) or $0 = \alpha = \gamma \le \delta < \beta$ (in the case $\alpha = 0$). □

11.3 Variation of subsidiary radii: statements

In order to state the analogue of Theorem 11.2.1 for the subsidiary radii of a differential module on a disc or annulus, we must set some corresponding notation.

Notation 11.3.1. Let M be a finite free differential module of rank n over $K\langle\alpha/t, t/\beta\rangle$, $K\langle\alpha/t, t/\beta]\!]_{\mathrm{an}}$, or $K[\![\alpha/t, t/\beta]\!]_{\mathrm{an}}$. For $\rho \in [\alpha, \beta]$, let $R_1(M, \rho), \dots, R_n(M, \rho)$ be the extrinsic subsidiary radii of $M \otimes F_\rho$ in increasing order, so that $R_1(M, \rho) = R(M \otimes F_\rho)$ is the generic radius of convergence of $M \otimes F_\rho$. For $r \in [-\log\beta, -\log\alpha]$, define

$$f_i(M, r) = -\log R_i(M, e^{-r}),$$

so that $f_i(M, r) \ge r$ for all r. Put $F_i(M, r) = f_1(M, r) + \cdots + f_i(M, r)$.

We now have the following results, whose proofs are distributed across the remainder of this chapter (Lemmas 11.5.1, 11.6.1, 11.6.3, and 11.7.1). Note that there is an overall sign discrepancy with Theorem 11.2.1, so that concavity becomes convexity and so forth. There are also some exceptions made in cases where $f_i(M, r) = r$.

Theorem 11.3.2. *Let M be a finite free differential module of rank n over $K\langle\alpha/t, t/\beta\rangle$, $K\langle\alpha/t, t/\beta]\!]_{\mathrm{an}}$, or $K[\![\alpha/t, t/\beta]\!]_{\mathrm{an}}$.*

(a) *(Linearity) For $i = 1, \dots, n$, the functions $f_i(M, r)$ and $F_i(M, r)$ are continuous and piecewise affine. Moreover, even if $\alpha = 0$ there are only finitely many different slopes.*

(b) *(Integrality) If $i = n$ or $f_i(M, r_0) > f_{i+1}(M, r_0)$ then the slopes of $F_i(M, r)$ in some neighborhood of r_0 belong to $\mathbb{Z}$. Consequently, the slopes of each $f_i(M, r)$ and $F_i(M, r)$ belong to*

$$\frac{1}{1}\mathbb{Z} \cup \cdots \cup \frac{1}{n}\mathbb{Z}.$$

(c) *(Subharmonicity) Suppose that $\alpha < 1 < \beta$ and that $f_i(M, 0) > 0$. For $i = 1, \dots, n$, let $s_{\infty,i}(M)$ and $s_{0,i}(M)$ be the left-hand and right-hand slopes of $F_i(M, r)$ at $r = 0$. For $\overline{\mu} \in (\kappa_K^{\mathrm{alg}})^\times$, let $s_{\overline{\mu},i}(M)$ be the right-hand slope of $F_i(T_\mu^*(M), r)$ at $r = 0$. Then*

$$s_{\infty,i}(M) \leq \sum_{\overline{\mu} \in \kappa_K^{\mathrm{alg}}} s_{\overline{\mu},i}(M),$$

with equality if either $i = n$ and $f_n(M, 0) > 0$ or $i < n$ and $f_i(M, 0) > f_{i+1}(M, 0)$.

(d) *(Monotonicity) Suppose that $\alpha = 0$. For $i = 1, \dots, n$, for any point r_0 where $f_i(M, r_0) > r_0$ the slopes of $F_i(M, r)$ are nonpositive in some neighborhood of r_0. (Remember that if $\alpha = 0$ then $f_i(M, r) = r$ for r sufficiently large, by Proposition 9.3.3; see also Proposition 11.8.1.)*

(e) *(Convexity) For $i = 1, \dots, n$, the function $F_i(M, r)$ is convex.*

Remark 11.3.3. Note that $f_i(M, r)$ and $F_i(M, r)$ are defined using the extrinsic normalization. However, if we switch to the intrinsic normalization then everything in Theorem 11.3.2 stays the same except for (d), in which the upper bound on the slopes in a neighborhood of r_0 changes from 0 to -1.

Remark 11.3.4. Suppose instead that M is a finite free differential module of rank n over $K\langle \alpha/t, t/\beta]\!]_0$ (resp. over $K[\![\alpha/t, t/\beta]\!]_0$). We may then apply Theorem 11.3.2 to $M \otimes K\langle \gamma/t, t/\delta\rangle$ for all γ, δ with $\alpha < \gamma \leq \delta < \beta$ (in the case $\alpha \neq 0$ if we are working over $K[\![\alpha/t, t/\beta]\!]_0$) or $\alpha = \gamma \leq \delta < \beta$ (otherwise). This implies that all the conclusions of Theorem 11.3.2 continue to hold for M itself, except that in (a) the functions $f_i(M, r)$ and $F_i(M, r)$ are defined only on $(-\log \beta, -\log \alpha]$ (resp. on $(-\log \beta, -\log \alpha)$). One can also show that they extend continuously, and continue to be piecewise affine (with finitely many slopes) if K is discrete, on the whole of $[-\log \beta, -\log \alpha]$; see Remark 11.6.5.

11.4 Convexity for the generic radius

As a prelude to tackling Theorem 11.3.2, we give a quick proof of subharmonicity, monotonicity, and convexity (parts (c)–(e) of Theorem 11.3.2) for the function f_1 corresponding to the generic radius of convergence. This argument applies to both discs and annuli and can be used in place of the full strength of Theorem 11.3.2 for many purposes; indeed, this is true for numerous results which predate Theorem 11.3.2. See the notes for further details.

Proof of Theorem 11.3.2(c), (d), (e) for $i = 1$. Choose a basis of M, and let D_s be the matrix of action of D^s on this basis. Then recall from Lemma 6.2.5 that

$$R_1(M, \rho) = \min\{\rho,\ \omega \liminf_{s\to\infty} |D_s|_\rho^{-1/s}\}.$$

For each s, the function $r \mapsto -\log|D_s|_{e^{-r}}^{-1/s}$ is convex in r by Proposition 8.2.3(b). This implies the convexity of

$$f_1(M, r) = \max\left\{r,\ -\log\omega + \limsup_{s\to\infty}(-\log|D_s|_{e^{-r}}^{-1/s})\right\}.$$

Similarly, we may deduce (c) by applying Proposition 11.1.2 to each D_s. If $\alpha = 0$ then the function $r \mapsto -\log|D_s|_{e^{-r}}^{-1/s}$ is nonincreasing, yielding (d). □

Remark 11.4.1. To improve upon this result, we will try to read off the generic radius of convergence, and maybe even the other subsidiary radii, from the Newton polygon of a cyclic vector. In order to do this, we have to circumvent two obstacles.

(a) In general one can only construct cyclic vectors for differential modules over differential fields, not over differential rings. (While Theorem 5.7.3 produces cyclic vectors over certain rings, it only does so locally for the Zariski topology, which appears to be insufficient for this purpose.)

(b) If $p > 0$ some subsidiary radii may be greater than $p^{-1/(p-1)}\rho$, in which case Newton polygons will not detect them.

The first problem will be addressed by using a cyclic vector over a fraction field to establish linearity, integrality, and subharmonicity and then using a carefully chosen lattice to deduce monotonicity and convexity. The second problem will be addressed using Frobenius descendants.

11.5 Measuring small radii

In this section, we address concern (a) from Remark 11.4.1, using both cyclic vectors and matrix inequalities.

Lemma 11.5.1. *For any $i \in \{1, \dots, n\}$ and any r_0 such that $f_i(M, r_0) > r_0 - \log\omega$, Theorem 11.3.2 holds in a neighborhood of r_0.*

Proof. If M is defined over R, put $F = \operatorname{Frac} R$. Choose a cyclic vector for $M \otimes_R F$ to obtain an isomorphism $M \otimes_R F \cong F\{T\}/F\{T\}P$ for some monic twisted polynomial P over F. We may then apply Corollary 6.5.4 and Theorem 11.2.1 to deduce (a), (b), (c) of Theorem 11.3.2.

To deduce (d), we may work in a right-hand neighborhood of a single value r_0 of r. There is no harm in enlarging K (by Proposition 10.6.6), so we may assume that $v(K^\times) = \mathbb{R}$. Then we may reduce to the case $r_0 = 0$ by replacing t by λt for some $\lambda \in K^\times$.

Since $v(K^\times) = \mathbb{R}$, we may pick $c_1, \dots, c_n \in K$ such that

$$-\log |c_j| = \min\{-\log \omega - f_j(M, 0),\ 0\} \qquad (j = 1, \dots, n).$$

Let $S \in F[U]$ be the untwisted polynomial with the same coefficients as P, and let $\mu_1, \dots, \mu_n$ be the roots of S. By Corollary 6.5.4,

$$|c_i| = \max\{|\mu_i|, 1\} = \max\{\omega e^{f_j(M,0)}, 1\} \qquad (i = 1, \dots, n).$$

We now construct a basis of $M \otimes_R F$ as in Theorem 6.5.3. Let B_0 be the basis of $M \otimes_R F$ given by

$$c_{n-1}^{-1} \cdots c_{n-j}^{-1} T^j \qquad (j = 0, \dots, n-1).$$

Let N_0 be the matrix of action of D on B_0; it is a conjugated companion matrix, of the form appearing in Proposition 4.3.10, corresponding to S. In particular, the singular values of N_0 are $|c_1|, \dots, |c_{n-1}|, |\mu_1 \cdots \mu_n/(c_1 \cdots c_{n-1})|$. The latter equals $|c_n|$ if $|c_n| > 1$ and otherwise is less than or equal to 1.

By Lemma 8.6.1 the supremum norm defined by B_0 is also defined by some basis B_1 of $M \otimes_R K[\![t]\!]_{\mathrm{an}}$. Let N_1 be the matrix of action of D on B_1. Theorem 6.7.4 implies that, for r close to 0, the visible spectrum of $M \otimes_R F_{e^{-r}}$ is the multiset of those norms of eigenvalues of the characteristic polynomial of N_1 which exceed e^{-r}. We may then deduce (d) from Theorem 11.2.1(d). (Alternatively, one may replace K by a spherical completion, then tensor M with $K[\![t]\!]_0$, and use Lemma 8.6.2.)

We can deduce (e) from (c) and (d), as noted in the proof of Theorem 11.2.1(e). It can also be proved directly as follows. We may again assume that $r_0 = 0$. We may also assume that $\alpha < 1 < \beta$, as otherwise there is nothing to check at r_0. Define B_0 as above. This time, apply Lemma 8.6.1 to construct a basis B_1' of $M \otimes_R K\langle 1/t, t\rangle$ defining the same supremum norm as B_0. We may approximate B_1' with a basis B_1 of $M \otimes_R K\langle \gamma/t, t/\delta\rangle$ for some $\alpha \le \gamma < 1 < \delta \le \beta$ defining the same supremum norm with respect to $|\cdot|_1$. (Here we use the facts that $K\langle \alpha/t, t/\beta\rangle$ is dense in $K\langle 1/t, t\rangle$ and that any element of $K\langle \alpha/t, t/\beta\rangle$ which becomes a unit in $K\langle 1/t, t\rangle$ is already a unit in $K\langle \gamma/t, t/\delta\rangle$ for some $\alpha \le \gamma < 1 < \delta \le \beta$. The latter holds because if the Newton polygon of an element of $K\langle \gamma/t, t/\delta\rangle$ has no slope equal to 0, then it also has no slopes in some neighborhood of 0.) Let N_1 be the matrix of action of D on B_1. Applying Theorem 6.7.4 to N_1, we may deduce (e) from Theorem 11.2.1(e). □

11.6 Larger radii

Next we address concern (b) from Remark 11.4.1, considering the cases $f_i(M, r_0) > r_0$ and $f_i(M, r_0) = r_0$ separately. We temporarily omit monotonicity, as it requires a slightly different argument. (We must also say more about Theorem 11.3.2(a); see Remark 11.6.4 below.)

Lemma 11.6.1. *For any $i \in \{1, \dots, n\}$ and any r_0 such that $f_i(M, r_0) > r_0$, statements (a)–(c), (e) of Theorem 11.3.2 hold in a neighborhood of r_0.*

Proof. This holds by Lemma 11.5.1 in the case $p = 0$, so we may assume $p > 0$ throughout the proof. For each nonnegative integer j, we will prove the claim for r_0 such that $f_i(M, r_0) > r_0 + (1/p^j(p-1)) \log p$, by induction on j; the base case $j = 0$ is precisely Lemma 11.5.1, so we may assume $j > 0$ hereafter. As in the proof of Lemma 11.5.1, we may reduce to the case $r_0 = 0$.

Let $R'_1(\rho^p), \dots, R'_{pn}(\rho^p)$ be the subsidiary radii of $\varphi_* M \otimes F'_\rho$ in increasing order. (The normalization is chosen in this way because the series variable in F'_ρ is t^p, which has norm ρ^p.) Put $g_i(pr) = -\log R'_i(e^{-pr})$. By Theorem 10.5.1, the list $g_1(pr), \dots, g_{pn}(pr)$ consists of

$$\bigcup_{i=1}^{n} \begin{cases} \{pf_i(M,r),\ \ pr + \frac{p}{p-1}\log p \ \ (p-1 \text{ times})\} & f_i(M,r) \le r + \frac{1}{p-1}\log p, \\ \{\log p + (p-1)r + f_i(M,r) \ \ (p \text{ times})\} & f_i(M,r) \ge r + \frac{1}{p-1}\log p. \end{cases}$$

Thus we may deduce (a) from the induction hypothesis.

To check (b), (c), (e), it suffices to handle the cases where $i = n$ or $f_i(M, 0) > f_{i+1}(M, 0)$. (As in the proof of Theorem 11.2.1(c), we may linearly interpolate to establish convexity and subharmonicity in the other cases.) In these cases, as in Corollary 10.5.3, we have at least one of $f_i(M, 0) > (1/(p-1)) \log p$, in which case in some neighborhood of $r = 0$ we have

$$g_1(pr) + \cdots + g_{pi}(pr) = pF_i(M, r) + pi \log p + (p-1)ipr, \quad (11.6.1.1)$$

or $f_{i+1}(M, 0) < (1/(p-1)) \log p$, or $i = n$, in which last case we have in some neighborhood of $r = 0$

$$g_1(pr) + \cdots + g_{pi+(p-1)(n-i)}(pr) = pF_i(M, r) + pn \log p + (p-1)npr. \quad (11.6.1.2)$$

Moreover, $f_i(M, 0) > (1/p^j(p-1))) \log p$ if and only if $g_{pi}(0) > (1/p^{j-1}(p-1)) \log p$.

If $f_i(M,0) > (1/(p-1))\log p$, apply (11.6.1.1) and the induction hypothesis to write piecewise

$$\begin{aligned} F_i(M,r) &= (1/p)(g_1(pr)+\cdots+g_{pi}(pr)-pi\log p-(p-1)ipr) \\ &= (1/p)(m(pr)+*) \\ &= mr+(1/p)* \end{aligned}$$

for some $m \in \mathbb{Z}$. (Note that $*$ is not guaranteed to be in $pv(K^\times)$; this explains Example 10.7.2.) Otherwise, we may apply (11.6.1.2) to write piecewise

$$\begin{aligned} F_i(M,r) &= (1/p)(g_1(pr)+\cdots+g_{pi+(p-1)(n-i)}(pr)-pn\log p-(p-1)npr) \\ &= (1/p)(m(pr)+*) \\ &= mr+(1/p)* \end{aligned}$$

for some $m \in \mathbb{Z}$. □

Remark 11.6.2. In the proof of Lemma 11.6.1, note the importance of the fact that the domains of applicability of (11.6.1.1) and (11.6.1.2) overlap: if $f_i(M,0) = (1/(p-1))\log p$ then (11.6.1.1) is valid for $r=0$ but possibly not for nearby r values.

The case $f_i(M,r_0) = r_0$ remains inaccessible even using descent along a Frobenius morphism, so we make an *ad hoc* argument.

Lemma 11.6.3. *For any $i \in \{1,\dots,n\}$ and any r_0 such that $f_i(M,r_0)=r_0$, Theorem 11.3.2 holds in a neighborhood of r_0.*

Proof. As in the proof of Lemma 11.5.1, it suffices to consider the case $r_0 = 0$. We first check continuity. For this, note that the proofs of Lemma 11.5.1 and 11.6.1 show that, for any $c>0$, the function $\max\{f_i(M,r), r+c\}$ is continuous at $r=0$. Consequently, for any $\epsilon>0$, we can find $0<\delta<\epsilon/2$ such that

$$|\max\{f_i(M,r), r+\epsilon/4\}| < \epsilon/2 \qquad (|r|<\delta).$$

For such r, $-\epsilon < -\delta < r \leq f_i(M,r) < \epsilon$; this yields continuity.

Next we check convexity. Using Remark 11.2.4, the proofs of Lemmas 11.5.1 and 11.6.1 show that, for any $c > 0$, the function $\sum_{j=1}^i \max\{f_j(M,r), r+c\}$ is convex. (The key point is that the domain over which this holds does not depend on c.) Since this function tends to $F_i(M,r)$ as c tends to 0, we may deduce (e).

We now check piecewise affinity by induction on i. Given that $f_1(M,r),\dots,f_{i-1}(M,r)$ are affine in a one-sided neighborhood of $r=0$, say $[-\delta,0]$, and given that $f_i(M,0)=0$, it suffices to check the linearity of $f_i(M,r) - r$ in some $[-\delta', 0]$. By (e), the set of $r \in [-\delta, 0]$ for which $f_i(M,r) - r \leq 0$ is connected. Since $f_i(M,r) - r \geq 0$ always, it follows that if $f_i(M, r_0) = 0$ for a single $r_0 \in [-\delta, 0]$ then $f_i(M,r) = 0$ for $r \in [-r_0, 0]$ so, in particular, $f_i(M,r) - r$ is linear in a one-sided neighborhood of 0. Otherwise the slopes of $f_i(M,r) - r$ in $[-\delta, 0)$ form a sequence of discrete values which are negative and nondecreasing (by (e)). This sequence must then stabilize, so $f_i(M,r) - r$ is again linear in a one-sided neighborhood of 0. This proves (a).

To prove (b), note that when $f_i(M, 0) = 0$ the input hypothesis can only hold if $i = n$. Suppose that we wish to check the integrality of the right slope of $F_n(M,r)$ (the argument for the left-hand slope is analogous). If $f_1(M,r) - r, \dots, f_n(M,r) - r$ are identically zero in a right-hand neighborhood of 0 then we have nothing to check. Otherwise, let j be the greatest integer such that $f_j(M,r) - r$ is not identically zero in a right-hand neighborhood of 0; we then deduce (b) by applying Lemma 11.6.1 with i replaced by j.

Since (c) and (d) make no assertion at $r = 0$ in the case $f_i(0) = 0$, we are done. □

Remark 11.6.4. Lemmas 11.6.1 and 11.6.3 fail to establish the last assertion of Theorem 11.3.2(a), i.e., if $\alpha = 0$ then each $f_i(M,r)$ has only finitely many differential slopes. However, this is easy to see from parts (b) and (e) of the theorem. Namely, each $F_i(M,r)$ has slopes which are discrete and nondecreasing but also bounded above by i because $f_i(M,r) = r$ for r large (by Proposition 9.3.3). Hence each $F_i(M,r)$ has only finitely many different slopes, as does each $f_i(M,r)$.

Remark 11.6.5. Suppose that M is a finite free differential module of rank n over $K\langle\alpha/t, t/\beta]\!]_0$ (resp. over $K[\![\alpha/t, t/\beta]\!]_0$). As noted in Remark 11.3.4, Theorem 11.3.2 implies that the functions $f_i(M,r)$ and $F_i(M,r)$ are continuous and piecewise affine on $(-\log\beta, -\log\alpha]$ (resp. on $(-\log\beta, -\log\alpha)$). However, by imitating the proofs of Lemmas 11.5.1, 11.6.1, and 11.6.3, using Theorem 11.2.5 in place of Theorem 11.2.1, we see that $f_i(M,r)$ and $F_i(M,r)$ extend continuously to $[-\log\beta, -\log\alpha]$. Moreover, if K is discretely valued and $\beta = 1$, the limits of the $e^{-f_i(M,r)}$ as $r \to 0^+$ are the subsidiary radii of $M \otimes \mathcal{E}$.

11.7 Monotonicity

To complete the proof of Theorem 11.3.2 we must prove (d) for $p > 0$ without the restriction $f_i(M, r_0) > r_0 - \log\omega$. The reason why we do not have (d) as part of Lemma 11.6.1 is that passing from M to $\varphi_* M$ introduces a singularity at $t = 0$ (Remark 10.3.5), so we cannot hope to infer monotonicity on $\varphi_* M$. To fix this, we must use off-center Frobenius descendants.

Lemma 11.7.1. *If $\alpha = 0$ and $f_i(M, r_0) > r_0$ then the slope of $f_i(M, r)$ in a right-hand neighborhood of r_0 is nonpositive.*

Proof. We may assume $p > 0$, as otherwise Lemma 11.5.1 implies the result. We proceed as in the proof of Lemma 11.6.1 but using the off-center Frobenius ψ instead of φ. Again, we may assume that $r_0 = 0$ and that $i = n$ or $f_i(M, 0) > f_{i+1}(M, 0)$ (reducing to the latter case by linear interpolation).

Let $R''_1(\rho^p), \dots, R''_n(\rho^p)$ be the subsidiary radii of $\psi_* M \otimes F''_\rho$ in increasing order. Put $g_i(pr) = -\log R''_i(e^{-pr})$. By Theorem 10.8.3, if $f_i(M, 0) > (1/(p-1)) \log p$ then

$$g_1(pr) + \cdots + g_{ip}(pr) = pF_i(M, r) + ip \log p,$$

whereas if $f_{i+1}(M, 0) < (1/(p-1)) \log p$ or $i = n$ then

$$g_1(pr) + \cdots + g_{ip+(p-1)(n-i)}(pr) = pF_i(M, r) + np \log p.$$

Moreover, $f_i(M, 0) > (1/p^j(p-1)) \log p$ if and only if $g_{ip}(0) > (1/p^{j-1}(p-1)) \log p$. To conclude, we may proceed as in Lemma 11.6.1. $\square$

Example 11.7.2. To see in action the discrepancy between the behavior of the centered and off-center Frobenius descendants, we consider an example suggested by Liang Xiao. (All verifications are left as an exercise.) Take $\beta > 1$, and let M be the differential module over $K\langle t/\beta \rangle$ with a single generator v satisfying $D(v) = t^{p-1} v$. Pick any $\alpha \in (0, 1)$, so that we may form $\varphi_* M$ on $K\langle \alpha/t^p, t^p/\beta \rangle$. Then $\varphi_* M$ splits as $\oplus_{m=0}^{p-1} (M' \otimes W_m)$, where M' has a single generator v' satisfying $D'(v') = p^{-1} v$, and W_m is defined as in Definition 10.3.3. One then computes, for $m \neq 0$ and $\overline{\mu} \in \kappa_K^{\mathrm{alg}}$,

$$\begin{aligned}
s_{\infty,1}(M') &= 0, \\
s_{\overline{\mu},1}(M') &= 0, \\
s_{\infty,1}(M' \otimes W_m) &= 0, \\
s_{0,1}(M' \otimes W_m) &= 1, \\
s_{-m,1}(M' \otimes W_m) &= -1, \\
s_{\overline{\mu},1}(M' \otimes W_m) &= 0 \qquad (\overline{\mu} \neq 0, -m).
\end{aligned}$$

This yields

$$\begin{aligned}
s_{\infty,p}(\varphi_* M) &= 0, \\
s_{0,p}(\varphi_* M) &= p - 1, \\
s_{\overline{\mu},p}(\varphi_* M) &= -1 \qquad (\overline{\mu} \in \mathbb{F}_p^\times), \\
s_{\overline{\mu},p}(\varphi_* M) &= 0 \qquad (\overline{\mu} \notin \mathbb{F}_p),
\end{aligned}$$

and in turn

$$\begin{aligned} s_{\infty,1}(M) &= -p+1, \\ s_{0,1}(M) &= 0, \\ s_{\overline{\mu},1}(M) &= -1 \qquad (\overline{\mu} \in \mathbb{F}_p^\times), \\ s_{\overline{\mu},1}(M) &= 0 \qquad (\overline{\mu} \notin \mathbb{F}_p). \end{aligned}$$

11.8 Radius versus generic radius

As promised, we can recover some information about the radius of convergence from the properties of the generic radius of convergence.

Proposition 11.8.1. *Let M be a finite differential module over $K\langle t/\beta\rangle$ or $K[\![t/\beta]\!]_{\mathrm{an}}$ for some $\beta > 0$. Then the radius of convergence of M equals e^{-r}, for r the smallest value in $[-\log\beta, +\infty)$ such that $f_1(r) = r$. Consequently $f_1(r') = r'$ for all $r' \geq r$.*

Proof. By Theorem 9.6.1 the radius of convergence of M is on the one hand at least the generic radius of convergence of $M \otimes F_{e^{-r}}$, which by hypothesis equals e^{-r}. On the other hand, if $\lambda > e^{-r}$ then by hypothesis $f_1(-\log\lambda) > -\log\lambda$, or in other words $R(M \otimes F_\lambda) < \lambda$. This means that $M \otimes K\langle t/\lambda\rangle$ cannot be trivial, so the radius of convergence cannot exceed λ. This proves the desired result. □

Corollary 11.8.2. *Let M be a finite differential module over $K\langle t/\beta\rangle$ or $K[\![t/\beta]\!]_{\mathrm{an}}$ for some $\beta > 0$. Then the radius of convergence of M belongs to the divisible closure of the multiplicative value group of K.*

Proof. By Theorem 11.3.2(a), (b) and Theorem 10.7.1, the function $f_1(r)$ is piecewise of the form $ar + b$ with $a \in \mathbb{Q}$ and $b \in \mathbb{Q}v(K^\times)$. By Proposition 11.8.1 the radius of convergence of M equals e^{-r}, for r the smallest value such that $f_1(r) = r$. To the left of this r, f_1 must be piecewise affine with slope $\neq 1$; by comparing the left and right limits at r we deduce that $r = ar + b$ for some rational $a \neq 1$ and some $b \in \mathbb{Q}v(K^\times)$. Since this gives $r = b/(a-1)$, we deduce the claim. □

One should be able to control the denominators better, as implied in the following question. (The $p = 0$ analogue is easy; see the exercises.)

Question 11.8.3. *Assume that $p > 0$. Let M be a finite differential module over $K\langle t/\beta\rangle$ for some $\beta > 0$. Does there necessarily exist $j \in \{1, \dots, \mathrm{rank}(M)\}$ such that the jth power of the radius of convergence of M belongs to the p-divisible closure of the multiplicative value group of K?*

We also have a criterion to establish when the radius of convergence equals the generic radius.

Corollary 11.8.4. *Let M be a finite differential module over $K\langle t/\beta\rangle$ or $K[\![t/\beta]\!]_{\mathrm{an}}$ for some $\beta > 0$ such that, for some $\alpha \in (0, \beta)$, $R(M \otimes F_\rho)$ is constant for $\rho \in [\alpha, \beta]$. Then $R(M) = R(M \otimes F_\rho)$.*

Proof. The hypothesis implies that $f_1(M, r)$ is constant in a right neighborhood of $r = -\log\beta$. By Theorem 11.3.2(d), $f_1(M, r)$, which is piecewise affine, remains constant until it becomes equal to r. By Proposition 11.8.1, we deduce the claim. □

11.9 Subsidiary radii as radii of optimal convergence

The subsidiary generic radii of convergence can be interpreted as the radii of convergence of a well-chosen basis of local horizontal sections at a generic point. The argument is a variation on Corollary 11.8.4.

Definition 11.9.1. Let M be a differential module of rank n over $K\langle t/\beta\rangle$ or $K[\![t/\beta]\!]_{\mathrm{an}}$ or on the open disc of radius β. For $i = 1, \dots, n$, the ith *radius of optimal convergence* of M at 0 is the supremum of those $\lambda \in [0, \beta)$ for which there exist $n - i$ linearly independent horizontal sections of $M \otimes K\langle t/\lambda\rangle$. (Remember that by Corollary 9.1.3 it is equivalent to require linear independence of the horizontal sections over K or over $K\langle t/\lambda\rangle$.)

Note that there exists a basis of local horizontal sections $s_1, \dots, s_n$ of M such that s_i has radius of convergence equal to the ith radius of optimal convergence of M at 0: once $s_{i+1}, \dots, s_n$ have been chosen, there must be at least a one-dimensional space of choices left for s_i. Such a basis is sometimes called an *optimal basis* of local horizontal sections. (It might be more consistent with our earlier terminology to refer to the radii of optimal convergence as the *subsidiary radii of convergence*, but we have refrained from doing so to avoid confusion with the subsidiary generic radii of convergence, which we commonly abbreviate to *subsidiary radii*.)

The following generalizes Proposition 9.7.5.

Theorem 11.9.2 (after Young). *Let (V, D) be a differential module over F_ρ of dimension n with subsidiary radii $r_1 \leq \cdots \leq r_n$. Let L be a complete extension of K, let t_ρ be a generic point of L relative to K of norm ρ, and put $V' = V \otimes_{F_\rho} L[\![(t - t_\rho)/\rho]\!]_{\mathrm{an}}$. Then the radii of optimal convergence of V' are also $r_1 \leq \cdots \leq r_n$.*

Proof. We first produce a basis $s_1, \ldots, s_n$ for which s_i converges in the open disc of radius r_i around t_ρ for $i = 1, \ldots, n$. For this, we may apply Theorem 10.6.2 to decompose V into components each with a single subsidiary radius and thus reduce to the case $r_1 = \cdots = r_n = r$. By the geometric interpretation of the generic radius (Proposition 9.7.5), each Jordan–Hölder constituent of V admits a basis of local horizontal sections on a generic disc of radius r. By Lemma 6.2.8(a) the same is true for V itself.

It remains to check that there cannot exist $n - i$ linearly independent local horizontal sections converging on a disc of radius strictly greater than r_i. We will prove this by induction on n. Let m be the largest integer such that $r_1 = r_m$. Let V_1 be the component of V of subsidiary radius r_1, so that $\dim V_1 = m$. We will check that no local horizontal section of V_1 at t_ρ can have radius of convergence strictly greater than r_1.

Put

$$f_i(r) = f_i(V_1 \otimes_{F_\rho} L[\![(t-t_\rho)/\rho]\!]_{\mathrm{an}}, r) \qquad (i = 1, \ldots, m; \quad r \in [-\log \rho, +\infty)).$$

By Theorem 11.3.2(c) the $f_i(r)$ are constant in a neighborhood of $r = -\log \rho$. By Theorem 11.3.2(c), (e) and induction on i,

$$f_i(r) = \begin{cases} -\log r_i & 0 < r \leq -\log r_i, \\ r & r \geq -\log r_i. \end{cases}$$

However, if there were a local horizontal section of V_1 at t_ρ which converged on a closed disc of radius λ for some $\lambda \in (r_1, \rho)$ then $V_1 \otimes_{F_\rho} L\langle (t - t_\rho)/\lambda \rangle$ would have a trivial submodule, and so it would have λ as one of its subsidiary radii. This would force $f_n(r) = r$ for $r = -\log \lambda < -\log r_i$, a contradiction.

We conclude that any local horizontal section of V that projects nontrivially onto V_1 has radius of convergence at most r_1. If $i \leq m$, we are done; otherwise, any linearly independent set of $n - i$ local horizontal sections converging in an open disc of radius greater than r_i must project to zero in V_1. We may thus reduce to applying the induction hypothesis to the complementary component. □

Notes

The harmonicity property of functions on annuli (Proposition 11.1.2) may be best viewed as a theory of subharmonic functions on one-dimensional Berkovich analytic spaces. Such a theory has been developed by Thuillier [**205**] with a view towards applications in Arakelov theory. A related development is the work of Favre and Jonsson concerning potential theory on the valuative tree, with applications to the theory of plurisubharmonic singularities on complex surfaces. See [**87**] and [**88**] and also a more recent paper by Boucksom, Favre, and Jonsson [**33**] giving some higher-dimensional generalizations.

For the function $f_1(M, r) = F_1(M, r)$ representing the generic radius of convergence, Christol and Dwork established convexity [**49**, Proposition 2.4] (using essentially the same short proof as that given here) and continuity at endpoints [**49**, Théorème 2.5] (see also [**80**, Appendix I]) in the case of a module over a full annulus. In the case of analytic elements, continuity and piecewise affinity were conjectured by Dwork and proved by Pons [**181**, Théorème 2.2]. The analogous results for the higher $F_i(M, r)$ are original.

When restricted to intrinsic subsidiary radii less than ω, Theorem 11.9.2 is a result of Young [**222**, Theorem 3.1]. Young's proof is an explicit calculation using twisted polynomials and cyclic vectors.

As suggested earlier (see the notes for Chapter 9), Young's definition of the radii of optimal convergence suggests an analogue in the framework of Baldassarri and Di Vizio. For instance, given a differential module on a closed disc of radius β, the radii of optimal convergence at a generic point of radius $\rho \in (0, \beta)$ should be defined in terms of an optimal basis of local horizontal sections which are allowed to extend all the way across the disc of radius β rather than just across the disc of radius ρ. However, we do not know how to prove the appropriate analogue of Theorem 11.3.2 for these quantities, since working over F_ρ loses all information beyond radius ρ. Accomplishing this is an important open problem.[1]

Exercises

(1) Give an example to show that, in Theorem 11.2.1, f_2 need not be concave (even though f_1 and $f_1 + f_2$ are concave).

(2) Verify Example 11.7.2.

(3) Give another proof of (9.9.5.1) as follows. First find one value ρ_0 for which Theorem 6.5.3 implies that $IR(M \otimes F_{\rho_0}) = \rho_0^{-b}$. Then use Theorem 11.3.2 to show that (9.9.5.1) holds for $\rho \in [1, \rho_0]$.

(4) State, then answer affirmatively (using Theorem 6.5.3), the analogue of Question 11.8.3 for $p = 0$.

[1] Added in proof: A solution to this problem has been announced by Baldassarri.

12

Decomposition by subsidiary radii

In the previous chapter we established a number of important variational properties of the subsidiary radii of a differential module over a disc or annulus. In this chapter, we continue the analysis by showing that, under suitable conditions, one can separate a differential module into components of different subsidiary radii. That is, we can globalize the decompositions by spectral radius provided by the strong decomposition theorem, if a certain numerical criterion is satisfied.

As in the previous chapter, our discussion begins with some observations about power series, in this case identifying criteria for invertibility. We use these in order to set up a Hensel lifting argument to give the desired decompositions; again we must start with the visible (see Definition 6.5.1) case and then extend using Frobenius descendants. We end up with a number of distinct statements, covering open and closed discs and annuli as well as analytic elements.

As a corollary of these results we recover an important theorem of Christol and Mebkhout. That result gives a decomposition by subsidiary radii on an annulus in a neighborhood of a boundary radius at which the module is *solvable*, that is, all the intrinsic subsidiary radii tend to 1. (It is not necessary to assume that the annulus is closed at this boundary.) One may view our results as a collection of quantitative refinements of the Christol–Mebkhout theorem.

Note that nothing is this chapter is useful if the intrinsic subsidiary radii are *everywhere* equal to 1. We will tackle this case in Chapter 13.

Throughout this chapter, besides Notation 10.0.1 we also retain Notation 11.3.1.

12.1 Metrical detection of units

One can identify the units in rings such as $K\langle\alpha/t, t/\beta\rangle$ rather easily in terms of power series coefficients (Lemma 8.2.6). However, for the present application we need an alternate characterization based on metric data, i.e., Gauss norms.

Definition 12.1.1. For $f \in K\langle\alpha/t, t/\beta\rangle$ with $\alpha \leq 1 \leq \beta$, define the *discrepancy* of f at $r = 0$ as the sum

$$\mathrm{disc}(f, 0) = \sum_{\overline{\mu}\in(\kappa_K^{\mathrm{alg}})^\times} s_{\overline{\mu}}(f);$$

note that $\mathrm{disc}(f, 0) \geq 0$ because it is a sum of nonnegative terms. We define $\mathrm{disc}(f, r)$ for general $r \in [-\log\beta, -\log\alpha]$ by rescaling: assume without loss of generality that K contains a scalar c of norm e^{-r}, let $T_c : K\langle\alpha/t, t/\beta\rangle \to K\langle(\alpha e^r)/t, t/(\beta e^r)\rangle$ be the substitution $t \mapsto ct$, and then put

$$\mathrm{disc}(f, r) = \mathrm{disc}(T_c(f), 0).$$

Lemma 12.1.2. *For $x \in K\langle t/\beta\rangle$ nonzero and $c \in K$ of norm β, x is a unit if and only if* $s_0(T_c(x)) = \mathrm{disc}(x, -\log\beta) = 0$.

Proof. We may reduce to the case $\beta = 1$ and $|x|_1 = 1$. In this case, by Lemma 8.2.6, x is a unit if and only if its image modulo $\mathfrak{m}_K$ in $\kappa_K[t]$ is a unit. As noted in Proposition 11.1.2, the order of vanishing of this image at $\overline{\mu} \in \kappa_K^{\mathrm{alg}}$ is precisely $s_{\overline{\mu}}(x)$; this proves the claim. □

There is also a variant for bounded series and analytic elements which is slightly simpler.

Lemma 12.1.3. *For $x \in K[\![t/\beta]\!]_0$ or $x \in K[\![t/\beta]\!]_{\mathrm{an}}$ nonzero, x is a unit if and only if* $s_0(T_c(x)) = 0$.

Proof. Again, we reduce to the case $\beta = 1$ and $|x|_1 = 1$. Since $|x|_\rho$ is nonincreasing, $s_0(T_c(x)) = 0$ if and only if $|x|_\rho = 1$ for all $\rho \in (0, 1)$. This is true if and only if the constant term of x has norm 1, which happens if and only if x is a unit in $\mathfrak{o}_K[\![t]\!]$ or $\mathfrak{o}_K[\![t]\!] \cap F_1$ (since these are both local rings). □

For annuli, it is more convenient to prove a weak criterion first.

Lemma 12.1.4. *For $x \in \cup_{\alpha\in(0,\beta)} K\langle\alpha/t, t/\beta\rangle$ nonzero, x is a unit if and only if we have* $\mathrm{disc}(x, -\log\beta) = 0$.

Proof. We may again reduce to the case $\beta = 1$ and $|x|_1 = 1$. In this case, by Lemma 8.2.6, x is a unit if and only if its image modulo $\mathfrak{m}_K$ in $\kappa_K[t, t^{-1}]$ is a unit. We then argue as in Lemma 12.1.2. □

One may then deduce the following.

Lemma 12.1.5. *For $\alpha > 0$ and for $x \in K\langle\alpha/t, t/\beta\rangle$ nonzero, x is a unit if and only if the function $r \mapsto v_r(x)$ is affine on $[-\log\beta, -\log\alpha]$ and* $\operatorname{disc}(x, -\log\alpha) = \operatorname{disc}(x, -\log\beta) = 0$.

Proof. Note that, by Proposition 11.1.2, $r \mapsto v_r(x)$ is affine on $[-\log\beta, -\log\alpha]$ if and only if $\operatorname{disc}(x, r) = 0$ for $r \in (-\log\beta, -\log\alpha)$. We may thus reformulate the desired result as follows: x is a unit if and only if $\operatorname{disc}(x, r) = 0$ for all $r \in [-\log\beta, -\log\alpha]$.

If x is a unit then $\operatorname{disc}(x, r) = 0$ for all $r \in [-\log\beta, -\log\alpha]$ by Lemma 12.1.4. Conversely, given the latter condition, to check that x is a unit, it suffices by Remark 8.1.3 to check that x is a unit in $K\langle\alpha_i/t, t/\beta_i\rangle$ for a finite collection of closed intervals $[\alpha_i, \beta_i]$ with union $[\alpha, \beta]$. However, Lemma 12.1.4 implies that one can cover a one-sided neighborhood of any element of $[\alpha, \beta]$ with such an interval; the compactness of $[\alpha, \beta]$ (Lemma 8.0.4) then yields the claim. □

Remark 12.1.6. Another statement in this vein is the fact that $\cup_{\alpha\in(0,\beta)} K\langle\alpha/t, t/\beta]\!]_{\mathrm{an}}$ is a field (Corollary 8.5.3).

12.2 Decomposition over a closed disc

We consider decomposition by subsidiary radii first in the case of a closed disc. The numerical criterion in this case involves an analogue of the discrepancy function from the previous section.

Definition 12.2.1. Let M be a finite differential module over $K\langle\alpha/t, t/\beta\rangle$ with $\alpha \leq 1 \leq \beta$. Define the *ith discrepancy of M at $r = 0$* as

$$\operatorname{disc}_i(M, 0) = -\sum_{\overline{\mu}\in(\kappa_K^{\mathrm{alg}})^\times} s_{\overline{\mu},i}(M);$$

it is always nonnegative by Theorem 11.3.2(d). Extend the definition to define $\operatorname{disc}_i(M, r)$ for general $r \in [-\log\beta, -\log\alpha]$ as in Definition 12.1.1.

Theorem 12.2.2. *Let M be a finite differential module over $K\langle t/\beta\rangle$ of rank n. Suppose that the following conditions hold for some $i \in \{1, \dots, n-1\}$.*

(a) *We have $f_i(M, -\log\beta) > f_{i+1}(M, -\log\beta)$.*

(b) *The function $F_i(M, r)$ is constant for r in a neighborhood of $-\log\beta$.*

(c) *We have* $\operatorname{disc}_i(M, -\log\beta) = 0$.

Then there is a direct sum decomposition of M inducing, for each $\rho \in (0, \beta]$, the decomposition of $M \otimes F_\rho$ separating the first i subsidiary radii from the others.

Before proving Theorem 12.2.2 we record some observations which will simplify the proof.

Remark 12.2.3. To prove Theorem 12.2.2 it suffices to lift to M the decomposition of $M \otimes F_\beta$ separating the first i subsidiary radii. Namely, suppose that $M_1 \oplus M_2$ is this decomposition and that $M_1 \otimes F_\beta$ accounts for the first i subsidiary radii of $M \otimes F_\beta$. By Theorem 11.3.2(a), for r in a neighborhood of $-\log \beta$, $M_1 \otimes F_{e^{-r}}$ accounts for the first i subsidiary radii of $M \otimes F_{e^{-r}}$. Consequently, for r in a neighborhood of $-\log \beta$ we have $f_j(M, r) = f_j(M_1, r)$ for $j = 1, \dots, i$ and $f_j(M, r) = f_{j-i}(M_2, r)$ for $j = i+1, \dots, n$.

By Theorem 11.3.2(d), (e), the function $F_i(M_1, r)$ is convex and nonincreasing as long as $f_i(M_1, r) > r$. For r in a neighborhood of $-\log \beta$, $F_i(M_1, r) = F_i(M, r)$ by the previous paragraph, and $F_i(M, r)$ is constant by hypothesis. Hence $F_i(M_1, r)$ must remain constant until the first value of r_1 for which $f_i(M_1, r_1) = r_1$. Then, for $-\log \beta \le r \le r_1$, $f_i(M_1, r) = F_i(M_1, r) - F_{i-1}(M_1, r)$ is nondecreasing by Theorem 11.3.2(d), whereas for $-\log \beta \le r$ $f_1(M_2, r)$ is nonincreasing until it becomes equal to r and then stays equal to r thereafter. Hence $f_i(M_1, r) > f_1(M_2, r)$ for $-\log \beta \le r < r_1$ and $r_1 = f_i(M_1, r_1) \ge f_1(M_2, r_1) \ge r_1$; thus also $f_i(M_1, r) \ge r = f_1(M_2, r)$ for $r \ge r_1$. Consequently, $M_1 \otimes F_{e^{-r}}$ accounts for the first i subsidiary radii of $M \otimes F_{e^{-r}}$ for all r.

We next make a trivial but quite useful observation.

Lemma 12.2.4. *Let R, S, T be subrings of a common ring U with $S \cap T = R$. Let M be a finite free R-module. Then the intersection $(M \otimes_R S) \cap (M \otimes_R T)$ inside $M \otimes_R U$ is equal to M itself.*

This also holds when M is only locally free; see the exercises.

Remark 12.2.5. Lemma 12.2.4 allows us to replace K by a complete extension L in the course of proving Theorem 12.2.2; thus, inside the completion of $L(t)$ for the β-Gauss norm we have

$$F_\beta \cap L\langle t/\beta \rangle = K\langle t/\beta \rangle$$

(exercise). Thus, obtaining matching decompositions of $M \otimes_{K\langle t/\beta \rangle} F_\beta$ and $M \otimes_{K\langle t/\beta \rangle} L\langle t/\beta \rangle$ gives a corresponding decomposition of M itself.

For annuli, we will use the related fact that, for any $\rho \in [\alpha, \beta]$, inside the completion of $L(t)$ for the ρ-Gauss norm we have

$$F_\rho \cap L\langle \alpha/t, t/\beta \rangle = K\langle \alpha/t, t/\beta \rangle$$

(exercise).

We also need a lemma about polynomials over $K\langle t \rangle$.

Lemma 12.2.6. *Let $P = \sum_i P_i T^i$ and $Q = \sum_i Q_i T^i$ be polynomials over $K\langle t \rangle$ satisfying the following conditions.*

(a) *We have $|P - 1|_1 < 1$.*

(b) *For $m = \deg(Q)$, Q_m is a unit and $|Q|_1 = |Q_m|_1$.*

Then P and Q generate the unit ideal in $K\langle t \rangle[T]$.

Proof. We may assume without loss of generality that $Q_m = 1$. The hypothesis that $|Q|_1 = |Q_m|_1 = 1$ implies that all the slopes of the Newton polygon of Q under $|\cdot|_1$ are nonnegative. Hence, by Lemma 2.3.1, if $R \in K\langle t \rangle[T]$ and S is the remainder upon dividing R by Q then $|S|_1 \leq |R|_1$.

Let S_i denote the remainder upon dividing $(1 - P)^i$ by Q. By the previous paragraph, the series $\sum_{i=0}^\infty S_i$ converges and its limit S satisfies $PS \equiv 1 \pmod{Q}$. This proves the claim. □

We are now ready to make the key step of establishing Theorem 12.2.2 in the visible range (see Definition 6.5.1).

Lemma 12.2.7. *Theorem 12.2.2 holds if $f_i(-\log \beta) > -\log \omega - \log \beta$.*

Proof. By invoking Remark 12.2.5 to justify enlarging K and then rescaling, we may reduce to the case $\beta = 1$. We may also assume that K has value group $\mathbb{R}$.

Set the notation as in that part of the proof of Lemma 11.5.1 dealing with Theorem 11.3.2(d). Let $Q(T)$ be the characteristic polynomial of N_1, so that the Newton polygon of Q computes $f_1(M, r), \dots, f_i(M, r)$ in a neighborhood of $r = 0$ (from the same part of the proof of Lemma 11.5.1). Condition (a) of Theorem 12.2.2 implies that these Newton polygons all have a vertex, with x-coordinate $n - i$, whose position is determined by the coefficient of T^{n-i} in Q. Conditions (b) and (c) then imply that this coefficient satisfies the hypothesis of Lemma 12.1.2 and so is a unit in $K\langle t \rangle$. We can thus apply Theorem 2.2.1 to factor Q as $Q_2 Q_1$, so that the roots of Q_1 are the i largest roots of Q under $|\cdot|_1$.

Use the basis B_1 to identify M with $K\langle t \rangle^n$, so that we may view N_1 as a $K\langle t \rangle$-linear endomorphism of M. By Lemma 12.2.6 applied after rescaling, Q_1 and Q_2 generate the unit ideal in $K\langle t \rangle[T]$. Hence M splits as a direct sum

$M_1' \oplus M_2'$ of modules (but not differential modules) with $M_i' = \ker(Q_i(N_1))$. We now invoke some of the approximation lemmas, as follows. Equip M with the supremum norm compatible with $|\cdot|_1$ defined by B_1, and then equip M_1', M_2' with the induced quotient norms; these are both supremum-equivalent by Lemma 1.3.5. By Lemma 1.3.7, for any $c > 1$ we can approximate these norms to within a factor of at most c by supremum norms defined by bases of $M_1' \otimes_{K\langle t\rangle} F_1$ and $M_2' \otimes_{K\langle t\rangle} F_1$. By Lemma 8.6.1 the latter norms are also defined by bases $B_{1,1}, B_{1,2}$ of M_1', M_2'.

Let U be the change-of-basis matrix from B_1 to $B_{1,1} \cup B_{1,2}$, so that $U^{-1}N_1U$ is a block diagonal matrix and $|U|_1, |U^{-1}|_1 \leq c$. Then the matrix of action of D on $B_{1,1} \cup B_{1,2}$ is $U^{-1}N_1U + U^{-1}d(U)$. If we write this in block form as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

then, by taking c sufficiently close to 1, we may force the following conditions to hold.

(a) The matrix A is invertible and $|A^{-1}|_1 \max\{|d|_1, |B|_1, |C|_1, |D|_1\} < 1$.
(b) The Newton slopes of A under $|\cdot|_1$ account for the first i subsidiary radii of $M \otimes_{K\langle t\rangle} F_1$.

As in the proof of Lemma 6.7.3 we may now use Lemma 6.7.1 to produce a submodule of M accounting for the last $n - i$ subsidiary radii of $M \otimes_{K\langle t\rangle} F_1$. By repeating this argument for $M^\vee$, we obtain a submodule of M accounting for the first i subsidiary radii of $M \otimes_{K\langle t\rangle} F_1$. By Remark 12.2.3 this suffices to prove the desired result. $\square$

To prove Theorem 12.2.2 in general, we must use Frobenius descendants again.

Proof of Theorem 12.2.2. The claim holds by Lemma 12.2.7 if $p = 0$, so we may assume $p > 0$ throughout the proof. It suffices to prove that, for $\beta = 1$, Theorem 12.2.2 holds if $f_i(M, 0) > 1/(p^j(p-1)) \log p$ for each nonnegative integer j; we again proceed by induction on j, the base case $j = 0$ being provided by Lemma 12.2.7.

Suppose that $f_i(M, 0) > 1/(p^j(p-1)) \log p$ for some $j > 0$. Let $M_1' \oplus M_2'$ be the decomposition of $\varphi_* M$ separating the subsidiary radii less than or equal to $e^{-pf_i(M,0)}$ into M_1' (which exists by the induction hypothesis). This might not be induced by a decomposition of M_1, because some factors of subsidiary radius $p^{-p/(p-1)}$ that are needed in M_2' are instead grouped into M_1'. To correct this, consider instead the decomposition

$$(\psi_0(M_1') \cap \cdots \cap \psi_{p-1}(M_1')) \oplus (\psi_0(M_2') + \cdots + \psi_{p-1}(M_2')),$$

where ψ_m is defined as in Lemma 10.3.6(a). By Lemma 10.3.6(b) this decomposition is induced by a decomposition of M. By Theorem 10.5.1, it has the desired effect. □

Remark 12.2.8. As in Lemma 11.5.1 one can prove Lemma 12.2.7 using Lemma 8.6.2 in place of Lemma 8.6.1, at the expense of some extra complications. First, one must replace K by a spherical completion with value group $\mathbb{R}$ and algebraically closed residue field (by invoking Theorem 1.5.3). One then obtains the desired decomposition only over $K[\![t]\!]_0$. By gluing with the original decomposition (defined over F_1), we recover a decomposition over $K[\![t]\!]_{\mathrm{an}}$. To remove poles in the disc $|t-\mu| < 1$ we apply the same argument with M replaced by $T_\mu^*(M)$ (in the sense of Definition 11.1.1).

12.3 Decomposition over a closed annulus

Over a closed annulus, one has a decomposition theorem of a shape somewhat different from that over a closed disc. Fortunately, the proof is essentially the same as for Theorem 12.2.2.

Theorem 12.3.1. *Let M be a finite differential module over $K\langle\alpha/t, t/\beta\rangle$ of rank n, for some $0 < \alpha \leq \beta$. Suppose that the following conditions hold for some $i \in \{1, \dots, n-1\}$.*

(a) *We have $f_i(M, r) > f_{i+1}(M, r)$ for $-\log\beta \leq r \leq -\log\alpha$.*
(b) *The function $F_i(M, r)$ is affine for $-\log\beta \leq r \leq -\log\alpha$.*
(c) *We have $\mathrm{disc}_i(M, -\log\beta) = \mathrm{disc}_i(M, -\log\alpha) = 0$.*

Then there is a direct sum decomposition of M inducing, for each $\rho \in [\alpha, \beta]$, the decomposition of $M \otimes F_\rho$ separating the first i subsidiary radii from the others.

First we prove a lemma which looks somewhat more like Theorem 12.2.2.

Lemma 12.3.2. *Let M be a finite differential module over $K\langle\alpha/t, t/\beta\rangle$ of rank n. Suppose that the following conditions hold for some $i \in \{1, \dots, n-1\}$.*

(a) *We have $f_i(M, -\log\beta) > f_{i+1}(M, -\log\beta)$.*
(b) *We have $\mathrm{disc}_i(M, -\log\beta) = 0$.*

Then, for some $\gamma \in [\alpha, \beta)$, there is a direct sum decomposition of $M \otimes K\langle\gamma/t, t/\beta\rangle$ inducing, for each $\rho \in [\gamma, \beta]$, the decomposition of $M \otimes F_\rho$ separating the first i subsidiary radii from the others.

Proof. Using Remark 12.2.5 again, we may enlarge K and then reduce to the case $\beta = 1$. Moreover, it suffices to consider the case where

$f_i(M, 0) > -\log \omega$, as we may reduce the general case to this one as in the proof of Theorem 12.2.2.

Again set the notation as in that part of the proof of Lemma 11.5.1 dealing with Theorem 11.3.2(d), but take $\beta = 1$; as in the part dealing with Theorem 11.3.2(e), we can find a basis B_1 of $M \otimes_{K\langle \alpha/t,t\rangle} K\langle \gamma/t, t\rangle$ for some $\gamma \in [\alpha, 1)$ defining the same supremum norm as B_0. Let N_1 be the matrix of action of D on B_1. By conditions (a) and (b) of the lemma plus Lemma 12.1.4, the coefficient of T^{n-i} in the characteristic polynomial of N_1 is a unit in $K\langle \gamma/t, t\rangle$ for some $\gamma \in [\alpha, 1)$. We may thus continue as in the proof of Lemma 12.2.7. □

To prove Theorem 12.3.1 from Lemma 12.3.2, we proceed as in the proof of Lemma 12.1.5. However, we must first give an alternate formulation of the hypotheses of the theorem.

Remark 12.3.3. In the statement of Theorem 12.3.1, given condition (a) we may reformulate conditions (b) and (c) together as the following condition.

(a) We have $\mathrm{disc}_i(M, r) = 0$ for $-\log \beta \leq r \leq -\log \alpha$.

To see this, note that if $\alpha < 1 < \beta$ then condition (a) implies that equality holds in Theorem 11.3.2(c), so we have $s_{0,i}(M) - s_{\infty,i}(M) = \mathrm{disc}_i(M, 0)$. Consequently $F_i(M, r)$ is affine in a neighborhood of 0 if and only if $\mathrm{disc}_i(M, 0) = 0$. By rescaling, we obtain the desired equivalence. (Compare the proof of Lemma 12.1.5.)

Proof of Theorem 12.3.1. The case $\alpha = \beta$ proceeds exactly as in Theorem 12.2.2, so we will assume $\alpha < \beta$ hereafter. By Remark 12.3.3, if M satisfies the given hypothesis then so does $M \otimes K\langle \gamma/t, t/\delta\rangle$ for each closed subinterval $[\gamma, \delta] \subseteq [\alpha, \beta]$. For each $\rho \in (\alpha, \beta]$, Lemma 12.3.2 implies that, for some $\gamma \in [\alpha, \rho)$, $M \otimes K\langle \gamma/t, t/\rho\rangle$ admits a decomposition with the desired property. Similarly, for each $\rho \in [\alpha, \beta)$ and for some $\gamma \in (\rho, \beta]$, $M \otimes K\langle \rho/t, t/\gamma\rangle$ admits a decomposition with the desired property.

By the compactness of $[\alpha, \beta]$ (Lemma 8.0.4), we can cover $[\alpha, \beta]$ with finitely many intervals $[\gamma_i, \delta_i]$ for which $M \otimes K\langle \gamma_i/t, t/\delta_i\rangle$ admits a decomposition with the desired property. Since the decomposition of $M \otimes K\langle \gamma_i/t, t/\delta_i\rangle$ is uniquely determined by the induced decomposition over F_ρ for any single $\rho \in [\gamma_i, \delta_i]$, these decompositions agree on overlaps of the covering intervals. By the gluing lemma (Lemma 8.3.6), we obtain a decomposition of M itself. □

Remark 12.3.4. As in Remark 12.2.8, to prove Lemma 12.3.2 one may use Lemma 8.6.2 in place of Lemma 8.6.1. Again, one enlarges K to be spherically complete with value group $\mathbb{R}$ and algebraically closed residue field. One then notes that $F = \cup_{\gamma<1} K\langle\gamma/t, t\rrbracket_{\mathrm{an}}$ is a field (Corollary 8.5.3), so one can approximate the basis B_0 of $M \otimes_{K\langle\alpha/t,t\rangle} F_1$ with a basis B_1 of $M \otimes_{K\langle\alpha/t,t\rangle} F$ defining the same supremum norm. One then obtains a decomposition of $M \otimes_{K\langle\alpha/t,t\rangle} K\langle\gamma/t, t\rrbracket_{\mathrm{an}}$ for some γ, and one can remove the unwanted poles as in Remark 12.2.8.

12.4 Decomposition over an open disc or annulus

Over open discs, we have similar decomposition theorems but without the discrepancy conditions at endpoints.

Theorem 12.4.1. *Let M be a finite differential module of rank n over the open disc of radius β. Suppose that the following conditions hold for some $i \in \{1, \dots, n-1\}$ and some $\gamma \in (0, \beta)$.*

(a) *The function $F_i(M, r)$ is constant for $-\log\beta < r \leq -\log\gamma$.*

(b) *We have $f_i(M, r) > f_{i+1}(M, r)$ for $-\log\beta < r \leq -\log\gamma$.*

Then M admits a unique decomposition separating the first i subsidiary radii of $M \otimes F_\rho$ for $\rho \in [\gamma, \beta)$.

Proof. As in Remark 12.3.3, note that (a) and subharmonicity (Theorem 11.3.2(c)) imply that $\mathrm{disc}_i(M, \delta) = 0$ for $\delta \in (\gamma, \beta)$. Thus, for any such δ, we may apply Theorem 12.2.2 to $M \otimes K\langle t/\delta\rangle$; doing so for all such δ (or a sequence increasing to β) yields the desired result. □

Similarly, for open annuli we obtain a decomposition theorem without a discrepancy condition at endpoints.

Theorem 12.4.2. *Let M be a finite differential module of rank n over the open annulus of inner radius α and outer radius β. Suppose that the following conditions hold for some $i \in \{1, \dots, n-1\}$.*

(a) *The function $F_i(M, r)$ is affine for $-\log\beta < r < -\log\alpha$.*

(b) *We have $f_i(M, r) > f_{i+1}(M, r)$ for $-\log\beta < r < -\log\alpha$.*

Then M admits a unique decomposition separating the first i subsidiary radii of $M \otimes F_\rho$ for any $\rho \in (\alpha, \beta)$.

Proof. By Remark 12.3.3 the conditions of Theorem 12.4.2 are satisfied by $M \otimes K\langle\gamma/t, t/\delta\rangle$ whenever $\alpha < \gamma \leq \delta < \beta$. Gluing together the resulting decompositions yields the desired result. □

Remark 12.4.3. One can also obtain a decomposition theorem for a half-open annulus, by covering the half-open annulus with an open annulus and a closed annulus and then gluing together the decompositions given by Theorems 12.3.1

and 12.4.2. Similarly, one can obtain decomposition theorems on more exotic subspaces of the affine line by gluing; the reader knowledgeable enough to be interested in such statements should at this point have no trouble formulating and deriving them. See also Remark 12.5.3 below.

12.5 Partial decomposition over a closed disc or annulus

We next state decomposition theorems which apply to a closed disc or annulus without requiring a discrepancy condition. The price one must pay is that one must work with analytic elements.

Theorem 12.5.1. *Let M be a finite differential module of rank n over $K\llbracket t/\beta\rrbracket_{\mathrm{an}}$. Suppose that the following conditions hold for some $i \in \{1, \dots, n-1\}$.*

(a) *The function $F_i(M, r)$ is constant in a neighborhood of $r = -\log\beta$.*

(b) *We have $f_i(M, -\log\beta) > f_{i+1}(M, -\log\beta)$.*

Then M admits a direct sum decomposition separating the first i subsidiary radii of $M \otimes F_\rho$ for $\rho \in (0, \beta]$.

Proof. As for Theorem 12.2.2, except that Lemma 12.1.2 is replaced by Lemma 12.1.3. □

Theorem 12.5.2. *Let M be a finite differential module of rank n over $K\llbracket\alpha/t, t/\beta\rrbracket_{\mathrm{an}}$, for some $\alpha < \beta$. Suppose that the following conditions hold for some $i \in \{1, \dots, n-1\}$.*

(a) *The function $F_i(M, r)$ is affine for $-\log\beta \le r \le -\log\alpha$.*

(b) *We have $f_i(M, r) > f_{i+1}(M, r)$ for $-\log\beta \le r \le -\log\alpha$.*

Then M admits a direct sum decomposition separating the first i subsidiary radii of $M \otimes F_\rho$ for $\rho \in [\alpha, \beta]$.

Proof. To obtain a decomposition of $M \otimes K\langle\gamma/t, t/\beta\rrbracket_{\mathrm{an}}$ for some $\gamma \in (\alpha, \beta)$, we may proceed as in Theorem 12.3.1, Lemma 12.1.4 being replaced by Remark 12.4.3. We will omit further details. □

Remark 12.5.3. Readers familiar with affinoid algebras should be able to extend the results of this section to cases where M is defined over the ring of analytic elements for a disc contained in a one-dimensional affinoid space. (We leave even the definition of this ring as an unstated exercise.)

If K is discrete, one can also extend to modules defined over $K\llbracket t/\beta\rrbracket_0$ or $K\langle\alpha/t, t/\beta\rrbracket_0$. However, if K is not discretely valued then one runs into various difficulties associated with the fact that $\mathcal{E}$ is no longer a field; compare Remark 8.2.4.

Remark 12.5.4. If $p = 0$, it should be possible to get a decomposition theorem over $K[\![\alpha/t, t/\beta]\!]_0$ even without assuming that $f_i(M, r) > f_{i+1}(M, r)$ for $r \in \{-\log\beta, -\log\alpha\}$; a statement along these lines (for K discretely valued) appears in [**144**]. However, this fails completely if $p > 0$; one can generate numerous counterexamples using the theory of isocrystals (Chapter 23).

12.6 Modules solvable at a boundary

One of the most important special cases of our decomposition theorems occurs in the following setting, which occurs frequently in applications.

Definition 12.6.1. Let M be a finite differential module on the half-open annulus with closed inner radius α and open outer radius β. We say that M is *solvable at* β if $R(M \otimes F_\rho) \to \beta$ as $\rho \to \beta^-$ or, equivalently, if $IR(M \otimes F_\rho) \to 1$ as $\rho \to \beta^-$. (In a similar definition the roles of the inner and outer radius are reversed; we will not refer to that definition here.)

Lemma 12.6.2. *Let M be a finite differential module on the half-open annulus with closed inner radius α and open outer radius β which is solvable at β. There exist $b_1 \geq \cdots \geq b_n \in [0, +\infty)$ such that, for $\rho \in [\alpha, \beta)$ sufficiently close to β, the intrinsic subsidiary radii of $M \otimes F_\rho$ are $(\rho/\beta)^{b_1}, \ldots, (\rho/\beta)^{b_n}$. Moreover, if $i = n$ or $b_i > b_{i+1}$ then $b_1 + \cdots + b_i \in \mathbb{Z}$.*

Proof. For $r \to (-\log\beta)^+$, $F_i(M, r) - ir$ is a convex function (Theorem 11.3.2(e)) with slopes in a discrete subset of $\mathbb{R}$ (Theorem 11.3.2(a), (b)). Moreover, it is nonnegative and its limit is 0; this implies that the slopes are all nonnegative. Hence these slopes must eventually stabilize; that is, each $f_i(M, r)$ becomes linear in a neighborhood of $-\log\beta$. This provides the existence of $b_1, \ldots, b_n$; by Theorem 11.3.2(b), if $i = n$ or $b_i > b_{i+1}$ then $b_1 + \cdots + b_i \in \mathbb{Z}$. $\square$

Definition 12.6.3. Let M be a finite differential module on the half-open annulus with closed inner radius α and open outer radius β which is solvable at β. The quantities $b_1, \ldots, b_n$ defined by Lemma 12.6.2 are called the *differential slopes* of M at β. (They are also called *ramification numbers*; the reason for this will become clear when we consider quasiconstant differential modules in Chapter 19. See specifically Theorem 19.4.1.)

We now recover a decomposition theorem of Christol and Mebkhout; see the notes for further discussion. We will see several applications of this result later in the book.

Theorem 12.6.4 (Christol–Mebkhout). *Let M be a finite differential module on the half-open annulus with closed inner radius α and open outer radius β which is solvable at β. Then, for any sufficiently large $\gamma \in [\alpha, \beta)$, the restriction of M to the open annulus with inner radius γ and outer radius β splits uniquely as a direct sum $\bigoplus_{b \in [0,+\infty)} M_b$, such that for each $b \in [0, +\infty)$ and for all $\rho \in [\gamma, \beta)$ the intrinsic subsidiary radii of $M_b \otimes F_\rho$ are all equal to $(\rho/\beta)^b$.*

Proof. By Lemma 12.6.2, this is a case where Theorem 12.4.2 may be applied. □

Remark 12.6.5. For some differential modules for which one has fairly explicit series expansions for local horizontal sections, one may be able to establish solvability at a boundary by explicit estimates. (A related strategy appears in Example 9.6.2.) However, it is more common for solvability to be established by proving the existence of a Frobenius structure; this notion will be introduced in Chapter 17.

Remark 12.6.6. Be aware that, in Theorem 12.6.4, if M is the restriction of a differential module over the closed annulus with inner radius α and outer radius β, or over the ring of analytic elements on the open annulus, the decomposition of M given by the theorem need not descend back to this original structure. Compare Remark 12.5.4.

12.7 Solvable modules of rank 1

We now give a partial classification of modules of rank 1 on an open annulus which are solvable at a boundary. Assume that $p > 0$.

Definition 12.7.1. Fix a coherent system of p-power roots of 1 in K^{alg}; that is, for each h, the chosen p^{h+1}th root of unity should be a pth root of the chosen p^hth root of unity. For $c \in \mathfrak{o}_K^\times$ and n a positive integer write $n = mp^h$ with m coprime to p, and let $M_{n,c}$ be the pullback of the module M_h defined in Example 9.9.3 along the map $t \mapsto ct^{-m}$ (using the chosen p^{h+1}th root of unity); this module is solvable at 1.

Theorem 12.7.2. *Let b be a positive integer. Assume that K contains the p^hth roots of unity for all $h \leq \log_p b$. Let M denote a finite differential module of rank 1 on a half-open annulus with open outer radius 1 which is solvable at 1 with differential slope b. Then there exist $c_1, \dots, c_b \in \{0\} \cup \mathfrak{o}_K^\times$ and nonnegative integers $j_1, \dots, j_b$ such that*

$$M \otimes (\varphi^{j_1})^*(M_{1,c_1}) \otimes \cdots \otimes (\varphi^{j_b})^*(M_{b,c_b})$$

has differential slope 0.

We will refine this statement a little later by eliminating the Frobenius pullbacks (Theorem 17.1.6).

Proof. By Theorem 11.3.2(b) we have $b \in \mathbb{Z}$. It thus makes sense to proceed by induction on b; we may assume that $b > 0$. Pick $0 < \alpha < \beta < 1$ such that, for some nonnegative integer j,

$$p^{-p^{-j+1}/(p-1)} < IR(M \otimes F_\alpha) = \alpha^b < IR(M \otimes F_\beta) = \beta^b < p^{-p^j/(p-1)}.$$

By Theorem 10.4.4, M admits a j-fold Frobenius antecedent N on $K\langle \alpha^{p^j}/t^{p^j}, t^{p^j}/\beta^{p^j}\rangle$. Pick a generator v of N, and put $D(v) = nv$. Then $|n|_{\rho^{p^j}} = p^{-1/(p-1)}\rho^{-p^j(b+1)}$ for $\rho \in [\alpha, \beta]$, so in this range n is dominated by a term of the form $n_{-b-1}(t^{p^j})^{-b-1}$ with $|n_{b-1}| = p^{-1/(p-1)}$. Let ζ be the chosen pth root of unity, write $b = p^h m$ with m coprime to p, and take

$$c_b = \frac{n_{-b-1}}{m(\zeta - 1)};$$

then

$$IR((N \otimes M_{b,c_b}) \otimes F_{\rho^{p^j}}) > \rho^{p^j b} \qquad (\rho \in [\alpha, \beta]).$$

We may use the log-concavity of the intrinsic radius (Theorem 11.3.2(e)) to deduce that the differential slope of $N \otimes M_{b,c_b}$ is strictly less than b. Thus the induction hypothesis gives the desired result. □

Corollary 12.7.3. *Let M denote a finite differential module of rank 1 on a half-open annulus with open outer radius 1 which is solvable at 1 with differential slope $b > 0$. If $M^{\otimes p}$ has differential slope 0 then b is not divisible by p.*

Proof. There is no harm in enlarging K in such a way that the hypotheses of Theorem 12.7.2 are satisfied; thus we may assume that $M \cong (\varphi^{j_1})^*(M_{1,c_1}) \otimes \cdots \otimes (\varphi^{j_b})^*(M_{b,c_b})$. By Remark 9.9.4 and Theorem 10.4.2, for $c_b \in \mathfrak{o}_K^\times$:

- if b is coprime to p then $M_{b,c_b}^{\otimes p}$ has differential slope 0;
- if b is divisible by p then $M_{b,c_b}^{\otimes p}$ has differential slope b/p.

Moreover, these differential slopes are preserved under Frobenius pullback, by Theorem 10.4.2. This implies the desired result as follows. If b were divisible by p then $(\varphi^{j_b})^*(M_{b,c_b})^{\otimes p}$ would have differential slope b/p whereas $(\varphi^{j_1})^*(M_{1,c_1})^{\otimes p} \otimes \cdots \otimes (\varphi^{j_{b-1}})^*(M_{b-1,c_{b-1}})^{\otimes p}$ would have differential slope strictly less than b/p. By Lemma 9.4.6(c), $M^{\otimes p}$ would have differential slope $b/p > 0$, a contradiction. □

12.8 Clean modules

It is reasonable to ask whether the refined strong decomposition theorem (Theorem 10.6.7) admits an analogue over a disc or annulus. The following discussion provides a possible answer to this question, though not a definitive one.

Lemma 12.8.1. *Let M be a finite free differential module of rank n over $K[\![\alpha/t, t/\beta]\!]_{\mathrm{an}}$. Suppose that $F_n(M, r)$ is affine on $[-\log\beta, -\log\alpha]$. Let $j \in \{0, \dots, n\}$ be the smallest integer such that $f_i(M, r)$ is affine on $[-\log\beta, -\log\alpha]$ for all $i > j$. If $0 < j < n$ then $f_j(M, r) > f_{j+1}(M, r)$ for all $r \in (-\log\beta, -\log\alpha)$; if $j = n$ then $f_j(M, r) > r$ for all $r \in (-\log\beta, -\log\alpha)$.*

Proof. There is nothing to check if $j = 0$, and $j = 1$ is impossible, so we may assume that $j > 1$. Note that $F_j(M, r) = F_n(M, r) - f_{j+1}(M, r) - \cdots - f_n(M, r)$ is affine by hypothesis, and that $F_{j-1}(M, r)$ is convex by Theorem 11.3.2(e), so $f_j(M, r) = F_j(M, r) - F_{j-1}(M, r)$ is concave. Moreover $f_j(M, r)$ is bounded below by the affine function $f_{j+1}(M, r)$ if $j < n$ or by the affine function r if $j = n$. Hence if this inequality becomes an equality for any interior point of $[-\log\beta, -\log\alpha]$ then it must hold identically, contrary to the choice of j. This proves the claim. □

Definition 12.8.2. Let M be a finite differential module of rank n over $K[\![\alpha/t, t/\beta]\!]_{\mathrm{an}}$. We say that M is *clean* if $F_n(M, r)$ and $F_{n^2}(M^\vee \otimes M, r)$ are both affine on $[-\log\beta, -\log\alpha]$.

Theorem 12.8.3. *Let M be a finite clean differential module of rank n over $K[\![\alpha/t, t/\beta]\!]_{\mathrm{an}}$.*

(a) *For $i \in \{1, \dots, n\}$, $F_i(M, r)$ is affine (and so $f_i(M, r)$ is affine).*
(b) *For $i \in \{1, \dots, n-1\}$, either $f_i(M, r) = f_{i+1}(M, r)$ for all $r \in (-\log\beta, -\log\alpha)$ or $f_i(M, r) > f_{i+1}(M, r)$ for all $r \in (-\log\beta, -\log\alpha)$. In addition, either $f_n(M, r) = r$ for all $r \in (-\log\beta, -\log\alpha)$ or $f_i(M, r) > r$ for all $r \in (-\log\beta, -\log\alpha)$.*
(c) *For $i \in \{1, \dots, n-1\}$, if $f_i(M, r) > f_{i+1}(M, r)$ for $r \in \{-\log\beta, -\log\alpha\}$ then M admits a direct sum decomposition separating the first i subsidiary radii of $M \otimes F_\rho$, for $\rho \in [\alpha, \beta]$.*

Proof. To check (a) we may replace K by a field with algebraically closed residue field and value group $\mathbb{R}$. Take j as in Lemma 12.8.1, and suppose by way of contradiction that $j > 0$ (which in turn forces $j > 1$). Pick a point $r_0 \in (-\log\beta, -\log\alpha)$ at which $f_j(M, r)$ fails to be affine or, equivalently, where $F_{j-1}(M, r)$ fails to be affine; we may rescale to reduce to the case $r_0 = 0$.

Then, by Theorem 11.3.2(c), on the one hand we must have $s_{\overline{\mu},j-1}(M) < 0$ for some $\overline{\mu} \in \kappa_K^\times$.

On the other hand, we have $f_j(M,0) > 0$ if $j > 0$ (by Lemma 12.8.1) and hence $s_{\overline{\mu},j}(M) = 0$ (by Theorem 11.3.2(c) again). Similarly, if $h \in \{0,\dots,n^2\}$ denotes the smallest index for which $f_i(M^\vee \otimes M, r) = r$ identically for $i > h$, we have $f_h(M^\vee \otimes M, 0) > 0$, if $h > 0$, and $s_{\overline{\mu},h}(M^\vee \otimes M) = 0$.

Choose a lift μ of $\overline{\mu}$ in K, and set $N = T_\mu^*(M)$ as a differential module over $K[\![t]\!]_{\mathrm{an}}$. Note that $F_h(N^\vee \otimes N, r)$ is constant for r in a right-hand neighborhood of 0; $f_h(N^\vee \otimes N, 0) > 0$ if $h > 0$; and $f_i(N^\vee \otimes N, r) = r$ for $i > h$. By Theorem 12.5.1 there exists a direct sum decomposition $P_0 \oplus P_1$ of $N^\vee \otimes N$ such that, for each $\rho \in (0,1)$, $P_0 \otimes F_\rho$ accounts for the first h subsidiary radii of $(N^\vee \otimes N) \otimes F_\rho$. By Theorem 9.6.1, P_1 restricts to a trivial differential module over the open unit disc.

For any $\rho \in (0,1)$, any direct sum decomposition of $N \otimes F_\rho$ is defined by projectors which are horizontal elements of $(N^\vee \otimes N) \otimes F_\rho$. For ρ sufficiently close to 1 the subsidiary radii of $P_0 \otimes F_\rho$ are all strictly less than ρ (by Lemma 12.8.1 again), so the projectors must belong to $P_1 \otimes F_\rho$. Since P_1 is trivial on the open unit disc, the projectors must also extend to horizontal elements of $N^\vee \otimes N$ over the open unit disc. That is, they define a direct sum decomposition of N over the open unit disc.

It follows that, over the open unit disc, N admits a direct sum decomposition $\oplus_i N_i$ in which, for each i and each $\rho \in (0,1)$ sufficiently close to 1, $N_i \otimes F_\rho$ has only a single subsidiary radius. Namely, given any decomposition not satisfying this condition, we can apply Theorem 10.6.2 and then the previous paragraph to obtain a finer decomposition. (This can only be repeated as many times as the rank of N.)

Let S be the set of indices i for which $\lim_{r\to 0^+} f_1(N_i, r) \geq f_j(M,0)$. Since $f_j(M,0) > 0$, in a neighborhood of $r = 0$ we have that $f_1(N_i, r)$ is affine and nonincreasing for each $i \in S$, by Theorem 11.3.2(a), (d), and $F_j(N,r)$ is a positive linear combination of these functions. However, the right-hand slope of $F_j(N,r)$ at $r = 0$ is $s_{\overline{\mu},j}(M) = 0$, so $f_1(N_i, r)$ must be constant in a neighborhood of $r = 0$ for each $i \in S$. In a neighborhood of $r = 0$, $F_{j-1}(N,r)$ is a nonnegative linear combination of the $f_1(N_i, r)$ for $i \in S$, so it also has right-hand slope 0 at $r = 0$. But this contradicts the fact that $s_{\overline{\mu},j-1}(M) < 0$.

This contradiction leads to the conclusion that $j = 0$, which implies (a). Given (a), we may deduce (b) from the fact that $f_i(M,r)$ and $f_{i+1}(M,r)$ are affine functions on $[-\log\beta, -\log\alpha]$ satisfying $f_i(M,r) \geq f_{i+1}(M,r)$ (or, in the case $i = n$, the same argument with $f_{i+1}(M,r)$ replaced by r). Given (a) and (b), the hypothesis of (c) implies that $f_i(M,r) > f_{i+1}(M,r)$ for all $r \in [-\log\beta, -\log\alpha]$, so the claim follows from Theorem 12.5.2. $\square$

Notes

Some results described here have also been obtained recently by Christol [**46**]. However, since we did not have access to Christol's definitive manuscript at the time of this writing, we are unable to provide detailed references.

Our results on modules solvable at a boundary are originally due to Christol and Mebkhout [**51, 52**]. In particular, Lemma 12.6.2 for the generic radius is [**51**, Théorème 4.2.1] and the decomposition theorem (which implies Lemma 12.6.2 in general) is [**52**, Corollaire 2.4–1]. However, the proof technique of Christol and Mebkhout is significantly different from ours: they construct the desired decomposition by exhibiting convergent sequences for a certain topology on the ring of differential operators. This does not appear to give quantitative results; that is, the range over which the decomposition occurs is not controlled, although we are not sure whether this is an intrinsic limitation of the method. (Keep in mind that the our approach here crucially uses our method of Frobenius descendants, which was not available when [**51, 52**] were written.)

Note also that Christol and Mebkhout work directly with a differential module on an open annulus as a ring-theoretic object; this requires a freeness result of the following form. If K is spherically complete, any coherent locally free module on the half-open annulus with closed inner radius α and open outer radius β is induced by a finite free module over the ring $\cap_{\rho\in[\alpha,\beta)} K\langle\alpha/t, t/\rho\rangle$. (That is, any coherent locally free sheaf on this annulus is freely generated by global sections.) See the notes for Chapter 8 for further discussion.

A partial extension of the work of Christol and Mebkhout can be found in a paper of Pons [**181**]. However, in the absence of a theory of Frobenius descendants, Pons was forced to impose somewhat awkward hypotheses in order to avoid the exceptional value $p^{-1/(p-1)}\rho$ for the generic radius of convergence.

For the attribution of Theorem 12.7.2, see the notes for Chapter 17.

The notion of a clean differential module is original and was motivated partly by discussions with Liang Xiao. It is a first attempt to model in p-adic differential theory a higher-rank analogue of Kato's notion of cleanness for a rank-1 étale sheaf [**119**]. Similar considerations in residual characteristic 0 appear in [**144**].

Exercises

(1) Prove an analogue of Lemma 12.2.4 in which M is required only to be locally free (in the sense of Remark 5.3.4).

(2) Let L be a complete extension of K. Prove that, for any $\beta > 0$, within the completion of $L(t)$ for the β-Gauss norm we have

$$F_\beta \cap L\langle t/\beta\rangle = K\langle t/\beta\rangle, \qquad F_\beta \cap L\langle \beta/t, t/\beta\rangle = K\langle \beta/t, t/\beta\rangle.$$

(Hint: use Remark 8.5.5.)

(3) Use the previous exercise to deduce that, for any $\rho \in [\alpha, \beta]$, within the completion of $L(t)$ for the ρ-Gauss norm we have

$$F_\rho \cap L\langle \alpha/t, t/\beta\rangle = K\langle \alpha/t, t/\beta\rangle.$$

(4) (a) Prove that, for $x \in K[\![t]\!]_0$ nonzero, the following are equivalent: (i) the element x is a unit; (ii) the function $v_r(x)$ is constant in a neighborhood of $r = 0$; (iii) $v_r(x)$ is constant for all $r \in [0, +\infty)$.

(b) Prove that, for $x \in \cup_{\alpha\in(0,1)} K\langle \alpha/t, t]\!]_0$ nonzero, x is a unit if and only if the function $r \mapsto v_r(x)$ is affine in some neighborhood of $r = 0$. In fact, this always happens if K is discretely valued, since the Newton polygon of any $x \in K\langle \alpha/t, t]\!]_0$ has finite width in this case (see the exercises for Chapter 8). Hence, in this case the ring $\cup_{\alpha\in(0,1)} K\langle \alpha/t, t]\!]_0$ is a field. We will encounter this field again under the name of the *bounded Robba ring*; see Definition 15.1.2.

13

p-adic exponents

In this chapter we study p-adic differential modules in a situation left untreated by our preceding analysis, namely when the intrinsic generic radius of convergence is equal to 1 everywhere. (This condition is commonly called the *Robba condition*.) This setting is loosely analogous to the study of regular singularities of formal meromorphic differential modules considered in Chapter 7; in particular, there is a meaningful theory of *p-adic exponents* in this setting.

However, some basic considerations indicate that p-adic exponents must necessarily be more complicated than the exponents considered in Chapter 7. For instance, the p-adic analogue of the Fuchs theorem (Theorem 7.3.8) can fail unless we impose a further condition: that the differences between exponents must not be p-adic Liouville numbers.

With this in mind we may proceed to construct p-adic exponents for differential modules satisfying the Robba condition. Such modules carry an action of the group of p-power roots of unity via Taylor series; under favorable circumstances the module splits into isotypical components for the characters of this group. We may identify these characters with elements of $\mathbb{Z}_p$, and these give the exponents.

Throughout this chapter, we retain Notation 10.0.1 and assume that $p > 0$.

13.1 p-adic Liouville numbers

Definition 13.1.1. For $\lambda \in K$, the *type* of λ, denoted $\operatorname{type}(\lambda)$, is the radius of convergence of the power series

$$\sum_{m=0, m \neq \lambda}^{\infty} \frac{t^m}{\lambda - m}.$$

This cannot exceed 1, as there are infinitely many m for which $|\lambda - m| = \max\{|\lambda|, 1\}$. (For instance, we may take $m \in p\mathbb{Z}_p$ if $|\lambda| \geq 1$ and $m \in 1 + p\mathbb{Z}_p$ if $|\lambda| < 1$.) Moreover, if $\lambda \notin \mathbb{Z}_p$, then $|\lambda - m|$ is bounded below, so $\operatorname{type}(\lambda) = 1$. Thus we will concentrate mostly on $\lambda \in \mathbb{Z}_p$.

Definition 13.1.2. We say that $\lambda \in K$ is a *p-adic Liouville number* if either λ or $-\lambda$ has type less than 1 and a *p-adic non-Liouville number* otherwise. The explicit reference to λ and $-\lambda$ is not superfluous, as they may have different types (exercise).

The following alternative characterization of type may be helpful.

Definition 13.1.3. For $\lambda \in \mathbb{Z}_p$, let $\lambda^{(m)}$ be the unique integer in $\{0, \dots, p^m - 1\}$ congruent to λ modulo p^m.

Proposition 13.1.4. *For $\lambda \in \mathbb{Z}_p$ not a nonnegative integer,*

$$-\frac{1}{\log_p \operatorname{type}(\lambda)} = \liminf_{m \to +\infty} \frac{\lambda^{(m)}}{m}. \tag{13.1.4.1}$$

In particular, λ has type 1 if and only if $\lambda^{(m)}/m \to +\infty$ as $m \to +\infty$.

Proof. It suffices to check that, for $0 < \eta < 1$, we have

$$\limsup_{m \to \infty} (m + \lambda^{(m)} \log_p \eta) = -\infty \tag{13.1.4.2}$$

when $\eta < \operatorname{type}(\lambda)$ and

$$\limsup_{m \to \infty} (m + \lambda^{(m)} \log_p \eta) = +\infty \tag{13.1.4.3}$$

when $\eta > \operatorname{type}(\lambda)$. Namely (13.1.4.2) implies that $m + \lambda^{(m)} \log_p \eta \leq 0$ for all large m, and so $\liminf_{m \to \infty} \lambda^{(m)}/m \geq -1/(\log_p \eta)$, whereas (13.1.4.3) implies that $m + \lambda^{(m)} \log_p \eta \geq 0$ for infinitely many m, and so $\liminf_{m \to \infty} \lambda^{(m)}/m \leq -1/(\log_p \eta)$.

Suppose first that $\operatorname{type}(\lambda) > \eta > 0$; then, as $s \to \infty$, $\eta^s/|\lambda - s| \to 0$ or equivalently $v_p(\lambda - s) + s \log_p \eta \to -\infty$. (Here v_p denotes the renormalized valuation with $v_p(p) = 1$.) Since λ is not a nonnegative integer we have $\lambda^{(m)} \to +\infty$ as $m \to +\infty$, and so

$$v_p(\lambda - \lambda^{(m)}) + \lambda^{(m)} \log_p \eta \to -\infty.$$

The left-hand side does not increase if we replace $v_p(\lambda - \lambda^{(m)})$ by m, so we may deduce (13.1.4.2).

Suppose next that $\mathrm{type}(\lambda) < \eta < 1$; then we may choose a sequence s_j such that, as $j \to \infty$, $v_p(\lambda - s_j) + s_j \log_p \eta \to +\infty$. Put $m_j = v_p(\lambda - s_j)$, so that $s_j \geq \lambda^{(m_j)}$. Then

$$m_j + \lambda^{(m_j)} \log_p \eta \to +\infty,$$

yielding (13.1.4.3). □

The alternate characterization is convenient for such verifications as the fact that rational numbers are non-Liouville (exercise), or the following stronger result [**80**, Proposition VI.1.1], whose proof we omit.

Proposition 13.1.5. *Any element of $\mathbb{Z}_p$ algebraic over $\mathbb{Q}$ is non-Liouville.*

Later, we will encounter the p-adic Liouville property in yet another, apparently different, form. (See the exercises for an alternate proof of this lemma.)

Lemma 13.1.6. *For λ not a nonnegative integer, we have the following equality of formal power series:*

$$\sum_{m=0}^{\infty} \frac{x^m}{\lambda(1-\lambda)(2-\lambda)\cdots(m-\lambda)} = e^x \sum_{m=0}^{\infty} \frac{(-x)^m}{m!} \frac{1}{\lambda - m}.$$

Proof. The coefficient of x^m on the right-hand side is a sum of the form $\sum_{i=0}^{m} c_i/(i-\lambda)$ for some $c_i \in \mathbb{Q}$. It is thus a rational function of λ of the form $P(\lambda)/(\lambda(1-\lambda)\cdots(m-\lambda))$, where P has coefficients in $\mathbb{Q}$ and degree at most m. To check that in fact $P(\lambda) = 1$ identically, we need only check this for $\lambda = 0, \ldots, m$.

In other words, to check the original identity it suffices to check after multiplying both sides by $\lambda - i$ and evaluating at $\lambda = i$ for each nonnegative integer i. On the left-hand side we obtain

$$\sum_{m=i}^{\infty} \frac{-x^m}{(-1)^{i-1} i!(m-i)!}.$$

On the right-hand side we obtain

$$e^x \frac{(-x)^i}{i!},$$

which is the same thing. □

Corollary 13.1.7. *If* $\lambda \in K$ *is not a nonnegative integer and* $\mathrm{type}(\lambda) = 1$ *then the series*

$$\sum_{m=0}^{\infty} \frac{x^m}{\lambda(1-\lambda)(2-\lambda)\cdots(m-\lambda)}$$

has radius of convergence at least $p^{-1/(p-1)}$.

13.2 p-adic regular singularities

We now consider a p-adic analogue of Theorem 7.3.8. Unlike its archimedean analogue, it requires a hypothesis on exponents beyond simply that they are weakly prepared (which means that no two eigenvalues of the constant matrix differ by a nonzero integer).

Definition 13.2.1. We say that a finite set is *p-adic non-Liouville* if its elements are p-adic non-Liouville numbers. We say the set has *p-adic non-Liouville differences* if the difference between any two elements of the set is a p-adic non-Liouville number.

Theorem 13.2.2 (**p-adic Fuchs theorem for discs**). *Let* $N = \sum_{i=0}^{\infty} N_i t^i$ *be an* $n \times n$ *matrix over* $K\langle t/\beta\rangle$ *for some* $\beta > 0$. *Assume that* N_0 *has weakly prepared eigenvalues with p-adic non-Liouville differences. Then there exists* $\gamma > 0$ *such that the fundamental solution matrix for* N, *which exists and is unique by Proposition 7.3.6, has its entries in* $K\langle t/\gamma\rangle$ *(the same holds for its inverse).*

Proof. Recall that the fundamental solution matrix U is computed by the recursion (7.3.6.1):

$$N_0 U_i - U_i N_0 + i U_i = -\sum_{j=1}^{i} N_j U_{i-j} \qquad (i > 0).$$

There is no harm in enlarging K to include the eigenvalues $\lambda_1, \dots, \lambda_n$ of N_0. By Lemma 7.3.5 the map $X \mapsto i + N_0 X - X N_0$ has eigenvalues $\lambda_g - \lambda_h + i$ for $g, h \in \{1, \dots, n\}$. View the operator $X \mapsto N_0 X - X N_0$ as a linear transformation T on the space V of $n \times n$ matrices over K. The matrix of action of $(i + T)^{-1}$ on some basis of V can be written as a matrix of cofactors of $i + T$ divided by the determinant of $i + T$. If we fix the basis of V then the entries of $i + T$ are bounded independently of i, as are the cofactors; we thus obtain a bound of the form

$$|U_i|\beta^i \leq c|N|_\beta \max_{j<i}\{|U_j|\beta^j\} \prod_{g,h=1}^{n} |\lambda_g - \lambda_h + i|^{-1}$$

for some $c > 0$ not depending on i. (For a somewhat more careful argument in this vein, see Proposition 18.1.1.)

Thus, to conclude the theorem it suffices to verify that, for each $g, h \in \{1, \dots, n\}$, the quantity $\lambda = \lambda_g - \lambda_h$ has the property that

$$\prod_{i=1}^{m} \max\{1, |\lambda - i|^{-1}\}$$

grows at worst exponentially. If $\lambda \notin \mathbb{Z}_p$ then $|\lambda - i|^{-1}$ is bounded above and the claim is verified. Otherwise Corollary 13.1.7 and the hypothesis that λ is a p-adic non-Liouville number give the desired estimate. □

By a slight modification of the argument (which we omit), one may obtain the following result of Clark [**54**, Theorem 3], which may viewed as a p-adic analogue of Corollary 7.3.11.

Theorem 13.2.3 (Clark). *Let M be a finite differential module over $K\langle t/\beta\rangle$ for the derivation td/dt, with a regular singularity at* 0 *whose exponents are p-adic non-Liouville numbers. Then, for any $x \in M$ and $y \in M \otimes_{K\langle t/\beta\rangle} K[\![t]\!]$ such that $D(y) = x$, we have $D(y) \in M \otimes_{K\langle t/\beta\rangle} K\langle t/\rho\rangle$ for some $\rho > 0$.*

Remark 13.2.4. The conclusion of Theorem 13.2.2 remains true, with the same proof, if it is assumed only that the pairwise differences between eigenvalues of N_0 all have type greater than 0. However, it is possible for the conclusion to fail otherwise. To construct a counterexample put $a = b = 1$, choose $c \in \mathbb{Z}_p$ with $\mathrm{type}(-c) = 0$, and consider the differential module associated with the hypergeometric differential equation (0.3.2.2). Then the eigenvalues of N_0 are 0 and c, whose difference has type 0; correspondingly, the hypergeometric series (0.3.2.1), i.e.,

$$F(a, b; c; z) = \sum_{i=0}^{\infty} \frac{a(a+1)\cdots(a+i)b(b+1)\cdots(b+i)}{c(c+1)\cdots(c+i)i!} z^i,$$

gives rise to a formal horizontal section with radius of convergence 0.

13.3 The Robba condition

Given a finite differential module on an annulus for the derivation td/dt, we would like to be able to tell whether it extends over a disc with a regular singularity. It turns out that when the exponents of that singularity are constrained to lie in $\mathbb{Z}_p$ (e.g., if they are rational numbers with denominators prime to p), one gets a strong necessary condition from the generic radius of convergence function.

Definition 13.3.1. Let M be a finite differential module on the disc or annulus $|t| \in I$, for I an interval. We say that M satisfies the *Robba condition* if $IR(M \otimes F_\rho) = 1$ for all nonzero $\rho \in I$.

Proposition 13.3.2. *Let M be a finite differential module on the open disc of radius β for the derivation td/dt satisfying the Robba condition in some annulus. Then the eigenvalues of the action of D on M/tM belong to $\mathbb{Z}_p$.*

Proof. Let $N = \sum_{i=0}^{\infty} N_i t^i$ be the matrix of action of D on some basis of M. Suppose that N_0 has an eigenvalue $\lambda \notin \mathbb{Z}_p$; there is no harm in enlarging K to force $\lambda \in K$. Choose $v \in M$ such that the image of v in M/tM is a nonzero eigenvector of N_0 of eigenvalue λ. Let D' be the derivation corresponding to $d' = d/dt$ instead of td/dt. Then with notation as in Example 9.5.2 we have, for any $\rho < \beta$,

$$\max\left\{\limsup_{s\to\infty} |(D')^s v|_\rho^{1/s}, |d'|_{\mathrm{sp},F_\rho}\right\} \geq |D'|_{\mathrm{sp},V_\lambda,\rho} > p^{-1/(p-1)}\rho,$$

so that $IR(M \otimes F_\rho) < 1$ by Lemma 6.2.5. □

We will establish a partial converse to Proposition 13.3.2 later (Theorem 13.7.1). In the interim, we mention the following easy result.

Proposition 13.3.3. *Let M be a finite differential module, on the open disc of radius β for the derivation td/dt, such that the matrix of action N_0 of D on some basis of M has entries in K. Then M satisfies the Robba condition if and only if N_0 has eigenvalues in $\mathbb{Z}_p$.*

Proof. Exercise, or see [**80**, Corollary IV.7.6]. □

13.4 Abstract p-adic exponents

In the previous section, we considered a finite differential module on an annulus for the derivation td/dt and saw that the Robba condition was necessary for extending the module over a disc with a regular singularity at $t = 0$. Moreover, the exponents of that regular singularity must belong to $\mathbb{Z}_p$. We may then ask whether it is possible to identify these exponents by looking only at the original annulus.

The answer to this question is complicated by the fact that the exponents are only well defined as elements of the quotient $\mathbb{Z}_p/\mathbb{Z}$. This means that we cannot hope to identify them using purely p-adic considerations; in fact, we must use *archimedean* considerations to identify them. Here are those considerations.

Definition 13.4.1. We will say that $A, B \in \mathbb{Z}_p^n$ are *equivalent* (or sometimes *strongly equivalent*) if there exists a permutation σ of $\{1, \dots, n\}$ such that $A_i - B_{\sigma(i)} \in \mathbb{Z}$ for $i = 1, \dots, n$. This is evidently an equivalence relation.

Definition 13.4.2. We say that $A, B \in \mathbb{Z}_p^n$ are *weakly equivalent* if there exists a constant $c > 0$, a sequence $\sigma_1, \sigma_2, \dots$ of permutations of $\{1, \dots, n\}$, and signs $\epsilon_{i,m} \in \{\pm 1\}$ such that

$$(\epsilon_{i,m}(A_i - B_{\sigma_m(i)}))^{(m)} \le cm \qquad (i = 1, \dots, n; m = 1, 2, \dots).$$

In other words, $A_i - B_{\sigma_m(i)}$ has a representative modulo p^m of size at most cm. Again, this is clearly an equivalence relation and equivalence implies weak equivalence.

Lemma 13.4.3. *If $A, B \in \mathbb{Z}_p$ (regarded as 1-tuples) are weakly equivalent then they are equivalent.*

Proof. For some $c > 0$, we have

$$|\epsilon_{1,m+1}(\epsilon_{1,m+1}(A - B))^{(m+1)} - \epsilon_{1,m}(\epsilon_{1,m}(A - B))^{(m)}| \le 2cm + c,$$

and the left-hand side is an integer divisible by p^m. For m large enough, we have $p^m > 2cm + c$ and so

$$\epsilon_{1,m+1}(\epsilon_{1,m+1}(A - B))^{(m+1)} = \epsilon_{1,m}(\epsilon_{1,m}(A - B))^{(m)}.$$

Hence, for m large enough, $\epsilon_{1,m}$ is constant and $\epsilon_{1,m}(A - B)$ is a constant nonnegative integer. □

Corollary 13.4.4. *Suppose that $A \in \mathbb{Z}_p^n$ is weakly equivalent to hA for some positive integer h. Then $A \in (\mathbb{Z}_p \cap \mathbb{Q})^n$.*

Proof. We are given that, for some $c > 0$, some permutations σ_m, and some signs $\epsilon_{i,m}$,

$$(\epsilon_{i,m}(A_i - hA_{\sigma_m(i)}))^{(m)} \le cm.$$

The order of σ_m divides $n!$, so we have

$$(\pm(A_i - h^{n!}A_i))^{(m)} \le n!cm$$

for some choice of sign (depending on i, m). That is, for each i the 1-tuple consisting of $(h^{n!} - 1)A_i$ is weakly equivalent to zero. By Lemma 13.4.3, $(h^{n!} - 1)A_i \in \mathbb{Z}$, so $A_i \in \mathbb{Z}_p \cap \mathbb{Q}$. □

Proposition 13.4.5. *Suppose that $A, B \in \mathbb{Z}_p^n$ are weakly equivalent and that B has p-adic non-Liouville differences. Then A and B are equivalent.*

Proof. There is no harm in replacing B by an equivalent tuple in which $B_i - B_j \in \mathbb{Z}$ if and only if $B_i = B_j$. For some c and σ_m, we have, for all m,

$$(\pm(A_i - B_{\sigma_m(i)}))^{(m)} \leq cm,$$
$$(\pm(A_i - B_{\sigma_{m+1}(i)}))^{(m+1)} \leq c(m+1),$$

and so

$$(\pm(B_{\sigma_m(i)} - B_{\sigma_{m+1}(i)}))^{(m)} \leq 2cm + c.$$

By hypothesis, the difference $B_{\sigma_m(i)} - B_{\sigma_{m+1}(i)}$ is either zero or a p-adic non-Liouville number which is not an integer; for m large, the previous inequality is inconsistent with the second option, by Proposition 13.1.4, so $B_{\sigma_m(i)} = B_{\sigma_{m+1}(i)}$. That is, for m large we may take $\sigma_m = \sigma$ for some fixed σ, so

$$(\pm(A_i - B_{\sigma(i)}))^{(m)} \leq cm \qquad (m = 1, 2, \dots).$$

By Lemma 13.4.3 $A_i - B_{\sigma(i)} \in \mathbb{Z}$, so A and B are equivalent. □

Example 13.4.6. It is easy to give examples of sets which are weakly equivalent but not equivalent; here is an example of Dwork (from [**161**, Example 2.3.2]). Let γ be an increasing function on the nonnegative integers such that, for some $\epsilon > 0$, we have $\gamma(i+1)p^{-\gamma(i)} \geq \epsilon$ for all $i \geq 0$. Put

$$\alpha = \sum_{i=0}^{\infty} p^{\gamma(2i)}, \qquad \beta = \sum_{i=0}^{\infty} p^{\gamma(2i+1)}.$$

One may verify (exercise) that the pairs $(\alpha, -\beta)$ and $(\alpha - \beta, 0)$ are weakly equivalent but not equivalent. In this case $\alpha, \beta, \alpha + \beta, \alpha - \beta$ are all p-adic Liouville numbers.

13.5 Exponents for annuli

We now give an abstract definition of the p-adic exponents of a differential module on an annulus satisfying the Robba condition, after a motivating remark.

Remark 13.5.1. Let N be a differential module over $K\langle t/\beta\rangle$ for the derivation td/dt, such that the matrix of action, N_0, of D on N has entries in K and eigenvalues in $\mathbb{Z}_p$. By Proposition 13.3.3, N satisfies the Robba condition.

Our best hope at this point would be to prove that any differential module M over $K\langle\alpha/t, t/\beta\rangle$ satisfying the Robba condition is isomorphic to such an N. This turns out to be rather too much to ask for, owing to difficulties arising with Liouville numbers (not to mention the closed boundary), but for the moment let us postulate the existence of an isomorphism $N \otimes K\langle\alpha/t, t/\beta\rangle \cong M$ and see what it tell us.

Assume (for clarity) that K contains the group μ_{p^∞} of all p-power roots of unity (i.e., all roots of unity of orders that are powers of p) in K^{alg}. Since M and N satisfy the Robba condition, a Taylor series construction gives an action of μ_{p^∞}; recall that we have used the p-power action already in the construction of Frobenius antecedents (Theorem 10.4.2). The K-valued characters of μ_{p^∞} may be naturally identified with $\mathbb{Z}_p$, by identifying $a \in \mathbb{Z}_p$ with the map carrying $\zeta \in \mu_{p^\infty}$ to ζ^a; we can use the structure of N to decompose M into character spaces for this action. Namely, perform a shearing transformation if needed (Proposition 7.3.10) to ensure that N_0 has prepared eigenvalues. Then choose the basis of N so that the matrix N_0 splits into blocks corresponding to individual eigenspaces V_j for the action of D (so, in particular, the K-span of the chosen basis of N equals $\oplus_j V_j$). Let $\lambda_j \in \mathbb{Z}_p$ be the eigenvalue corresponding to V_j. We may then identify M with $(\oplus_j V_j) \otimes_K K\langle\alpha/t, t/\beta\rangle$; under this identification $t^i V_j$ is an eigenspace with character $\lambda_j + i$.

The strategy for constructing p-adic exponents is then to turn this argument on its head, by first constructing the eigenspaces for the actions of the finite subgroups of μ_{p^∞} and then extracting elements of these which stabilize at a basis of M. One might expect an obstruction to arise because the full action of $1 + \mathfrak{m}_K$ by Taylor series need not be semisimple, but this does not occur; see the exercises for a statement that may help to give the reason.

Following the strategy discussed in the previous remark, we now introduce the definition of exponents.

Definition 13.5.2. Let M be a finite differential module of rank n over $K\langle\alpha/t, t/\beta\rangle$ satisfying the Robba condition. For $m > 0$, let μ_{p^m} denote the group of pth roots of unity in K^{alg}, and put $\mu_{p^\infty} = \cup_{m>0}\mu_{p^m}$. For $\zeta \in \mu_{p^\infty}$, define the action of ζ on $M \otimes_{K\langle\alpha/t,t/\beta\rangle} K(\zeta)\langle\alpha/t, t/\beta\rangle$ via the Taylor series

$$\zeta(x) = \sum_{i=0}^{\infty} \frac{(\zeta t - t)^i}{i!} D^i(x);$$

this series converges because of the Robba condition. Fix a basis $e_1, \dots, e_n$ of M. An *exponent* for M is an element $A \in \mathbb{Z}_p^n$ for which there exists a sequence $\{S_{m,A}\}_{m=1}^{\infty}$ of $n \times n$ matrices over $K\langle\alpha/t, t/\beta\rangle$ satisfying the following conditions.

(a) For $j = 1, \dots, n$ and $m = 1, 2, \dots$, put $v_{m,A,j} = \sum_i (S_{m,A})_{ij} e_i$. Then, for all $\zeta \in \mu_{p^m}$, $\zeta(v_{m,A,j}) = \zeta^{A_j} v_{m,A,j}$.

(b) For some $k > 0$, we have $|S_{m,A}|_\rho \leq p^{km}$ for all m and all $\rho \in [\alpha, \beta]$. (This condition will be used to obtain some sort of convergence for the $S_{m,A}$.)

(c) We have $\left|\det(S_{m,A})\right|_\rho \geq 1$ for all m and all $\rho \in [\alpha, \beta]$. (This condition prevents the $S_{m,A}$ from converging to zero.)

We make the following observations. w

(i) The property of being an exponent does not depend on the choice of basis, although the matrices S_m do depend on the basis.

(ii) If A is an exponent for M then so is any $B \in \mathbb{Z}_p^n$ equivalent to A.

(iii) To obtain (b), it is enough to have $|S_{m,A}|_\rho \leq cp^{km}$ for some $c, k > 0$, as we can enlarge k to absorb c. In fact it is enough to check this just for $\rho = \alpha, \beta$, by the Hadamard three circles theorem (Proposition 8.2.3(c)).

Before constructing exponents in general, we note the following extension of Remark 13.5.1.

Proposition 13.5.3. *Let M be a differential module of rank n over $K\langle t/\beta\rangle$ for the derivation $t\,d/dt$, such that the eigenvalues of D on M/tM are in $\mathbb{Z}_p$. Then, for any $\alpha \in (0, \beta)$, these eigenvalues form an exponent for $M \otimes_{K\langle t/\beta\rangle} K\langle\alpha/t, t/\beta\rangle$ (which satisfies the Robba condition by Proposition 13.3.3).*

Proof. By applying shearing transformations (Proposition 7.3.10) as needed, we may reduce to the case where the eigenvalues of D on M/tM are prepared. Let $e_1, \dots, e_n$ be a basis of M reducing modulo t to a basis of M/tM consisting of bases for the generalized eigenspaces of D. For $j = 1, \dots, n$ and $m = 1, 2, \dots$, let A_j be the eigenvalue corresponding to e_j, and put

$$v_{m,A,j} = p^{-m} \sum_{\zeta \in \mu_{p^m}} \zeta^{-A_j} \zeta(e_j).$$

Then the resulting matrices $S_{m,A}$ have the desired property (exercise). □

For the general construction of exponents, we need the following lemma due to Dwork and Robba. It will be easiest to postpone its proof until Chapter 18, when we will derive it as a corollary of some explicit convergence bounds on solutions of p-adic differential systems (Corollary 18.2.5).

Lemma 13.5.4. *Let V be a finite differential module of rank n over F_ρ for the derivation d/dt, such that $IR(V) = 1$. Choose a basis of V and, for $i = 1, 2, \ldots$, let N_i be the matrix of action of $D^i/i!$ on this basis. Then*

$$|N_i|_\rho \rho^i \leq p^{(n-1)\lfloor \log_p i \rfloor} \max\{1, |N_1|_\rho^{n-1}\} \qquad (i = 1, 2, \ldots).$$

We now give the general construction of exponents, using a discrete Fourier transform for the group $\mathbb{Z}/p^m\mathbb{Z}$. (Compare [**79**, Lemma 3.1 and Corollary 3.3] or [**161**, Proposition 3.1.1].)

Theorem 13.5.5. *Let M be a finite differential module of rank n over $K\langle \alpha/t, t/\beta \rangle$ satisfying the Robba condition. Then there exists an exponent for M.*

Proof. Fix a basis $e_1, \ldots, e_n$ of M. For any $A \in \mathbb{Z}_p^n$ and any positive integer m, we wish to define the matrix $S_{m,A}$ such that $v_{m,A,j}$ corresponds to the projection of e_j into the eigenspace of μ_{p^m} for the character A_j. To achieve this, we let $E(\zeta)$ be the matrix of action of $\zeta \in \mu_{p^\infty}$, and put

$$S_{m,A} = p^{-m} \sum_{\zeta \in \mu_{p^m}} E(\zeta) \operatorname{Diag}(\zeta^{-A_1}, \ldots, \zeta^{-A_n}).$$

This matrix is invariant under $\mathrm{Gal}(K(\mu_{p^\infty})/K)$ and so has entries in $K\langle \alpha/t, t/\beta \rangle$. By the vector interpretation, it satisfies condition (a) of Definition 13.5.2. (Another way to see this is to check the identity

$$E(\zeta)\zeta(S_{m,A}) = S_{m,A} \operatorname{Diag}(\zeta^{A_1}, \ldots, \zeta^{A_n}) \qquad (\zeta \in \mu_{p^m})$$

and use the formula for change of basis in a difference module. See Remark 14.1.3 below.)

We will check that $S_{m,A}$ satisfies (b) of Definition 13.5.2 using Lemma 13.5.4. For each $\zeta \in \mu_{p^m}$, we have $|\zeta - 1| \leq p^{-p^{-m+1}/(p-1)}$ by Example 2.1.6. If we write

$$E(\zeta) = \sum_{i=0}^{\infty} (\zeta - 1)^i t^i N_i$$

then, under $|\cdot|_\rho$, the ith summand is bounded under $|\cdot|_\rho$ by $\max\{1, |N_1|_\rho^{n-1}\} p^{c(m,i)}$ for

$$c(m,i) = (n-1)\log_p i - \frac{ip^{-m+1}}{p-1} = (n-1)m + (n-1)\log_p(ip^{-m}) - \frac{ip^{-m}}{p(p-1)}.$$

Here $c(m,i) - (n-1)m$ is a function of ip^{-m} which is continuous on $(0, +\infty)$ and tends to $-\infty$ as ip^{-m} tends to either 0 or $+\infty$, so it is bounded above independently of m. Hence $|E(\zeta)|_\rho \leq p^{km}$ for some $k > 0$, implying a similar bound for $S_{m,A}$ (but with a larger k).

We next choose A to satisfy (c) of Definition 13.5.2. Note that

$$v_{m,A,1} \wedge \cdots \wedge v_{m,A,n} = \det(S_{m,A}) e_1 \wedge \cdots \wedge e_n$$

and that, for $b \in \{0, \ldots, p-1\}^n$,

$$v_{m,A,j} = \sum_{b=0}^{p-1} v_{m+1,A+p^m b,j} \qquad (j = 1, \ldots, n).$$

(In words this second equation states that each eigenspace for the action of μ_{p^m} is the direct sum of p eigenspaces for the action of $\mu_{p^{m+1}}$.) Hence, for any $A \in \mathbb{Z}_p^n$ and any $m \geq 0$, we have

$$\det(S_{m,A}) = \sum_{b \in \{0,\ldots,p-1\}^n} \det(S_{m+1,A+p^m b}). \tag{13.5.5.1}$$

Write $S_{m,A} = \sum_{i \in \mathbb{Z}} S_{m,A,i} t^i$. By (13.5.5.1) we can choose b such that $\left|\det(S_{m+1,A+p^m b,0})\right| \geq \left|\det(S_{m,A,0})\right|$. Since $S_{0,A}$ is the identity matrix, this allows us to choose A such that the matrices $S_{m,A}$ satisfy $\left|\det(S_{m,A,0})\right| \geq \left|\det(S_{0,A,0})\right| = 1$ for all m. Since $\left|\det(S_{m,A})\right|_\rho \geq \left|\det(S_{m,A,0})\right|$ for all $\rho \in [\alpha, \beta]$, this yields (c) of Definition 13.5.2. $\square$

We also have the following limited uniqueness result for the exponents; here we see the first appearance of a non-Liouville condition. We also must begin to assume that the annulus has positive width. (Compare [**79**, Theorem 4.4].)

Theorem 13.5.6. *Assume that $\alpha < \beta$. Let M be a finite differential module of rank n over $K\langle \alpha/t, t/\beta \rangle$ satisfying the Robba condition. Then any two exponents for M are weakly equivalent. In particular, if M admits an exponent A with non-Liouville differences then (by Lemma 13.4.3) any other exponent for M is equivalent to A.*

Proof. Fix a basis for M, let A, B be two exponents for M, and let $S_{m,A}$, $S_{m,B}$ be the corresponding sequences of matrices (for the same constant $k > 0$). For $j = 1, \ldots, n$, put $v_{m,j,A} = \sum_i (S_{m,A})_i e_i$ and $v_{m,j,B} = \sum_i (S_{m,B})_i e_i$. Let $T_m = S_{m,A} S_{m,B}^{-1}$ be the change-of-basis matrix between the bases $v_{m,j,A}$ and $v_{m,j,B}$ of M. Since $|S_{m,A}^{-1}|_\rho \leq |S_{m,A}|_\rho^{n-1} \left|\det(S_{m,A})\right|_\rho^{-1}$ (from a description of the inverse matrix using cofactors), we have

$$|T_m|_\rho \leq |S_{m,B}|_\rho |S_{m,A}|_\rho^{n-1} \left|\det(S_{m,A})\right|_\rho^{-1} \leq p^{nkm} \qquad (\rho \in [\alpha, \beta]). \tag{13.5.6.1}$$

We will make progress by levering (in a metaphorical sense) the upper bound (13.5.6.1) against various lower bounds involving T_m, using the log-convexity of the Gauss norm as a function of the radius, i.e., the Hadamard three circles theorem (Proposition 8.2.3(c)).

To begin with, we have

$$|\det(T_m)|_\rho \geq \left|\det(S_{m,A})\right|_\rho^{-1} \geq \left|S_{m,A}\right|_\rho^{-n} \geq p^{-nkm} \qquad (\rho \in [\alpha, \beta]).$$

Put $\gamma = \sqrt{\alpha\beta}$. From the additive formula for the determinant of T_m, there must be a permutation σ_m of $\{1, \dots, n\}$ such that

$$\prod_{i=1}^{n} |T_{i,\sigma_m(i)}|_\gamma \geq p^{-nkm}.$$

We now use (13.5.6.1) to isolate a single factor, yielding

$$|T_{i,\sigma_m(i)}|_\gamma \geq p^{-nkm} \prod_{j \neq i} |T_{j,\sigma_m(j)}|_\gamma^{-1} \geq p^{-n^2 km} \qquad (i = 1, \dots, n). \tag{13.5.6.2}$$

Write $T_{i,\sigma_m(i)} = \sum_{j \in \mathbb{Z}} T_{i,j} t^j$ with $T_{i,j} \in K$; we can then choose $j = j(i, m)$ so that $|T_{i,j(i)} t^j|_\gamma = |T_{i,\sigma_m(i)}|_\gamma$. We repeat the leverage process to limit the size of j. Put $\eta = \sqrt{\beta/\alpha} > 1$. In the case $j \geq 0$, combine (13.5.6.2) with the case $\rho = \beta$ of (13.5.6.1) to get

$$\eta^j = \left(\frac{\beta}{\gamma}\right)^j = \frac{|T_{i,j(i)} t^j|_\beta}{|T_{i,j(i)} t^j|_\gamma} \leq p^{(n^2+n)km};$$

in the case $j \leq 0$, combine (13.5.6.2) with the case $\rho = \alpha$ of (13.5.6.1) to get

$$\eta^{-j} = \left(\frac{\gamma}{\alpha}\right)^{-j} = \frac{|T_{i,j(i)} t^j|_\alpha}{|T_{i,j(i)} t^j|_\gamma} \leq p^{(n^2+n)km}.$$

In either case we deduce that

$$|j| \log_p \eta \leq (n^2 + n)km.$$

Finally, note that

$$\mathrm{Diag}(t^{B_1}, \dots, t^{B_n}) T_m \, \mathrm{Diag}(t^{-A_1}, \dots, t^{-A_n})$$

must have entries in $K\langle \alpha^{p^m}/t^{p^m}, t^{p^m}/\beta^{p^m} \rangle$ by condition (a) of Definition 13.5.2. Thus the integer $j = j(i, m)$ is a representative of $B_i - A_{\sigma_m(i)}$ modulo p^m which is bounded in size by a constant times m. This implies that A and B are weakly equivalent, as desired. □

Corollary 13.5.7. *Assume that $\alpha < \beta$. Let M be a finite differential module of rank n over $K\langle \alpha/t, t/\beta\rangle$ satisfying the Robba condition. Suppose that, for some α, β with $\alpha \leq \alpha' < \beta' \leq \beta$, $M \otimes_{K\langle \alpha/t, t/\beta\rangle} K\langle \alpha'/t, t/\beta'\rangle$ admits an exponent A with p-adic non-Liouville differences. Then A is also an exponent for M.*

Proof. By Theorem 13.5.5 there exists an exponent B for M. By Theorem 13.5.6, A and B are weakly equivalent and hence equivalent by Lemma 13.4.3 since A has p-adic non-Liouville differences. Hence A is also an exponent for M. □

In general, it is quite difficult to compute the p-adic exponents of a differential module. However, one can at least check the following compatibility, which will lead to an important instance in which the exponent of a differential module can be controlled. See Corollary 13.6.2.

Lemma 13.5.8. *Let M be a finite differential module of rank n over $K\langle \alpha/t, t/\beta\rangle$ satisfying the Robba condition, and let $\varphi : K\langle \alpha/t, t/\beta\rangle \to K\langle \alpha^{1/q}/t, t/\beta^{1/q}\rangle$ be the substitution $t \mapsto t^q$. If A is an exponent of M then qA is an exponent of $\varphi^* M$.*

13.6 The p-adic Fuchs theorem for annuli

We now come to the question of whether one can really invert the construction of Remark 13.5.1, i.e., whether any differential module satisfying the Robba condition is isomorphic to a differential module over a disc. We can hope to treat this question only in the case where the module has an exponent with p-adic non-Liouville differences: in this case the exponent is unique up to equivalence, by Theorem 13.5.6, so there is no ambiguity about how to fill in the hole in the annulus. The fact that no other conditions are necessary is the content of the following remarkable theorem of Christol and Mebkhout.

Theorem 13.6.1 (Christol–Mebkhout). *Let M be a finite differential module on an open annulus for the derivation td/dt satisfying the Robba condition and admitting an exponent on some closed subannulus of positive width with p-adic non-Liouville differences. Then M admits a basis on which the matrix of action of D has entries in K and eigenvalues representing the exponent of M (and hence belonging to $\mathbb{Z}_p$). Consequently, M admits a canonical decomposition*

$$M = \bigoplus_{\lambda \in \mathbb{Z}_p/\mathbb{Z}} M_\lambda$$

in which each M_λ has exponent identically equal to λ.

The proof is loosely analogous to that of Theorem 13.5.6 but instead of comparing two different sequences of matrices we compare one sequence with itself.

Proof. Let α, β be the inner and outer radii of the original annulus. It suffices to construct a basis of the desired form over the closed annulus $\alpha' \leq |t| \leq \beta'$ for every pair α', β' with $\alpha < \alpha' < \beta' < \beta$. Choose α'', β'' with $\alpha < \alpha'' < \alpha' < \beta' < \beta'' < \beta$ and $\alpha'/\alpha'' = \beta''/\beta' = \eta > 1$, so that M is represented by a finite free module over $K\langle \alpha''/t, t/\beta'' \rangle$.

By hypothesis, M admits an exponent A on some closed subannulus of positive width having p-adic non-Liouville differences; by replacing A with an equivalent exponent we can force it also to have no nonzero integer differences. By Corollary 13.5.7, A is also an exponent for M on any closed subannulus of positive width, including $\alpha'' \leq |t| \leq \beta''$. We may thus fix a basis of $M \otimes K\langle \alpha''/t, t/\beta'' \rangle$ and then define a sequence $S_{m,A}$ as in Definition 13.5.2.

For $m' \geq m$, set $T_{m',m} = S_{m,A}^{-1} S_{m',A}$. As in (13.5.6.1) we have

$$|T_{m',m}|_\rho, |T_{m',m}^{-1}|_\rho \leq p^{nkm} \qquad (\rho \in [\alpha'', \beta'']).$$

Our first goal is to show that $T_{m',m}$ is close to its constant term. Write $T_{m',m} = \sum_{h \in \mathbb{Z}} T_{m',m,h} t^h$; note that the (i,j)th entry of $T_{m',m,h}$ can only be nonzero if $h \equiv A_i - A_j \pmod{p^m}$. Since A has p-adic non-Liouville differences, for any $c > 0$ and for m sufficiently large the congruence $h \equiv A_i - A_j \pmod{p^m}$ forces either $h = A_i - A_j = 0$ or $|h| \geq cm$. If $h > 0$ we then have

$$|T_{m',m,h} t^h|_{\alpha'} \leq |T_{m',m,h} t^h|_{\beta'} \leq |T_{m',m,h} t^h|_{\beta''} \eta^{-cm} \leq \left(p^{nk} \eta^{-c}\right)^m$$

and, if $h < 0$,

$$|T_{m',m,h} t^h|_{\beta'} \leq |T_{m',m,h} t^h|_{\alpha'} \leq |T_{m',m,h} t^h|_{\alpha''} \eta^{-cm} \leq \left(p^{nk} \eta^{-c}\right)^m.$$

Pick any $\lambda \in (0, p^{-nk})$. Since $\eta > 1$, we may choose c in the above argument such that $p^{nk} \eta^{-c} \leq \lambda$; we then deduce that, for some m_0,

$$|T_{m',m} - T_{m',m,0}|_\rho \leq \lambda^m \qquad (\rho \in [\alpha', \beta'], m \geq m_0).$$

This suggests renormalizing the $T_{m',m}$ to make them convergent. Specifically, we define $U_{m',m} = T_{m',m} T_{m',m,0}^{-1}$, so that $U_{m',m}$ has constant term I_n. Then

$$\begin{aligned} \left| I_n - T_{m',m,0} T_{m',m}^{-1} \right|_\rho &\leq \left| T_{m',m} - T_{m',m,0} \right|_\rho \left| T_{m',m}^{-1} \right|_\rho \\ &\leq p^{nkm} \lambda^m < 1 \qquad (\rho \in [\alpha', \beta'], m \geq m_0), \end{aligned}$$

and so $|I_n - U_{m',m}| \leq p^{nkm} \lambda^m < 1$; in particular,

$$|T_{m',m,0}^{-1}|_\rho \leq |U_{m',m}|_\rho |T_{m',m}^{-1}|_\rho \leq p^{nkm} \qquad (\rho \in [\alpha', \beta'], m \geq m_0).$$

Consider now three indices $m'' > m' > m$. We have $T_{m',m} T_{m'',m'} = T_{m'',m}$, of course; moreover, this multiplicativity is approximately preserved upon taking constant terms. Namely, the identity

$$T_{m'',m,0} = T_{m',m,0} T_{m'',m',0} + \sum_{h \neq 0} T_{m',m,-h} T_{m'',m',h}$$

yields the bound

$$\begin{aligned} |T_{m'',m,0} - T_{m',m,0} T_{m'',m',0}|_\rho &\leq |T_{m'',m'} - T_{m'',m',0}|_\rho |T_{m',m}|_\rho \\ &\leq \lambda^{m'} p^{nkm} \qquad (\rho \in [\alpha', \beta'], m \geq m_0). \end{aligned}$$

We now write

$$\begin{aligned} U_{m'',m} - U_{m',m} &= T_{m'',m} T^{-1}_{m'',m,0} - T_{m',m} T^{-1}_{m',m,0} \\ &= T_{m',m} T^{-1}_{m',m,0} (T_{m',m,0} T_{m'',m'} - T_{m'',m,0}) T^{-1}_{m'',m,0} \\ &= T_{m',m} T^{-1}_{m',m,0} (T_{m',m,0} T_{m'',m',0} - T_{m'',m,0}) T^{-1}_{m'',m,0} \\ &\quad + T_{m',m} (T_{m'',m'} - T_{m'',m',0}) T^{-1}_{m'',m,0}. \end{aligned}$$

Using this last expression, we obtain the bound

$$|U_{m'',m} - U_{m',m}|_\rho \leq \lambda^{m'} p^{4nkm} \qquad (\rho \in [\alpha', \beta'], m \geq m_0).$$

We conclude that, for any fixed $m \geq m_0$, the sequence $\{U_{m',m}\}_{m'=m}^{\infty}$ is Cauchy. It thus has limit U with the property that $S_{m,A} U$ is the matrix that effects the change to a basis of $M \otimes K\langle \alpha'/t, t/\beta' \rangle$ of the desired form. This completes the proof. □

The hypothesis of Theorem 13.6.1 is difficult to verify in general, because of the indirect nature of the definition of exponents. However, we do have the following important case where the condition can be verified.

Corollary 13.6.2. *Let M be a finite differential module on the open annulus with inner radius α and outer radius β satisfying the Robba condition. Let q be a power of p, and suppose that the intervals $(\alpha^{1/q}, \beta^{1/q})$ and (α, β) have nonempty intersection. Suppose moreover that on some annulus there exists an isomorphism $\varphi_K^* \varphi^* M \cong M$, where $\varphi_K : K \to K$ is an isometry and φ is the substitution $t \mapsto t^q$. Then any exponent for M consists of rational numbers; consequently, the conclusion of Theorem 13.6.1 applies.*

Proof. Choose α', β' with $\alpha \leq \alpha' < \beta' \leq \beta$ such that the intervals $((\alpha')^{1/q}, (\beta')^{1/q})$ and (α', β') have nonempty intersection. On the intersection, M has exponent A by Theorem 13.5.5; A extends to be an exponent of both M and $\varphi_K^* \varphi^* M$ on their entire domains of definition by Corollary 13.5.7.

The rationality of A holds by Lemma 13.5.8 and Corollary 13.4.4. By Theorem 13.5.6 and Lemma 13.4.3, any exponent for M is equivalent to A. We may thus apply Theorem 13.6.1. □

Remark 13.6.3. Corollary 13.6.2 is critical in what follows; it gives rise to a quasiunipotence result (Lemma 21.2.1) which can be used to establish the p-adic local monodromy theorem (Theorem 20.1.4).

One other consequence of Theorem 13.6.1 that can be stated without reference to exponents is the following. (In fact, it can also be proved without exponents; see Remark 13.7.3.)

Corollary 13.6.4. *Let M be a finite differential module on an open annulus for the derivation td/dt satisfying the Robba condition. Suppose that the restriction M' of M to some smaller open annulus is either trivial or unipotent. Then the same is true for M.*

Proof. If M' is unipotent then it admits the exponent 0 by Remark 13.5.1. However, any exponent A of M restricts to an exponent of M' and so is weakly equivalent to 0 by Proposition 13.5.6. By Lemma 13.4.3, A is equivalent to 0 so Theorem 13.6.1 implies that M is unipotent. If now M' is trivial then M is forced to be trivial also. □

13.7 Transfer to a regular singularity

As an application of Theorem 13.6.1 we obtain a transfer theorem in the presence of a regular singularity, in the spirit of Theorem 9.6.1 but with a somewhat weaker estimate. For the necessity of weakening the estimate, see Example 13.7.2 below.

Theorem 13.7.1. *Let M be a finite differential module of rank n, on the open unit disc for the derivation td/dt, with a regular singularity at $t = 0$ whose exponents are in $\mathbb{Z}_p$ and have p-adic non-Liouville differences. Then on the open disc of radius $\lim_{\rho\to 1^-} R(M \otimes F_\rho)^n$, M admits a basis on which the matrix of action of D has entries in $\mathbb{Z}_p$. In particular, if M is solvable at 1 then this basis is defined over the whole open unit disc.*

Proof. Using shearing transformations (as in Proposition 7.3.10), we may reduce to the case where the exponents of M at the regular singularity $t = 0$ are prepared. Note that, for any $\rho \in (0, 1)$, $M \otimes K\langle t/\rho\rangle$ admits a basis. By Theorem 13.2.2 the fundamental solution matrix in terms of this basis has entries defined over some disc of positive radius. From this and

Proposition 13.3.3, it follows that $R(M \otimes F_\rho) = \rho$ for $\rho \in (0, 1)$ sufficiently small.

Let λ be the supremum of $\rho \in (0, 1)$ for which $R(M \otimes F_\rho) = \rho$. Note that the function $f_1(r) = -\log R(M \otimes F_{e^{-r}})$ is convex by Theorem 11.3.2(e), is equal to r for r sufficiently large by the previous paragraph, and is also equal to r for $r = -\log\lambda$ by continuity (Theorem 11.3.2(a)). Consequently, $f_1(r) = r$ for all $r \geq -\log\lambda$.

Choose $\alpha, \beta \in (0, \lambda)$, with $\alpha < \beta$, such that the fundamental solution matrix of M (with respect to some basis on the closed disc of radius β) converges in the open disc of radius β. Let N_0 be the matrix of action of D on the basis B_0 given by the columns of the fundamental solution matrix; by construction, N_0 has entries in K whose eigenvalues are prepared and form a tuple $A \in \mathbb{Z}_p^n$ with p-adic non-Liouville differences. On the open annulus of inner radius α and outer radius β, M admits the exponent A by Remark 13.5.1. By Corollary 13.5.7, A is also an exponent for M on the open annulus of inner radius α and outer radius λ. By Theorem 13.6.1, on that annulus we obtain another basis B_1 of M on which D acts via a matrix N_1 with entries in K and eigenvalues equal to A.

Let U be the change-of-basis matrix from B_0 to B_1; it is an invertible $n \times n$ matrix over $K\{\alpha/t, t/\beta\}$ and satisfies the equation

$$U^{-1}N_0U + U^{-1}t\frac{d}{dt}(U) = N_1,$$

or $N_0U - UN_1 + t(d/dt)(U) = 0$. Since N_0, N_1 have entries in K, if we write $U = \sum_i U_i t^i$ then $N_0U_i - U_iN_1 + iU_i = 0$ for each $i \in \mathbb{Z}$. By Lemma 7.3.5 the map $U_i \mapsto N_0U_i - U_iN_1 + iU_i$ on $n \times n$ matrices over K has eigenvalues each of which is i plus a difference between two elements of A. In particular, since A is prepared these eigenvalues can never vanish for $i \neq 0$, so $U = U_0$ has entries in K. (Compare the uniqueness argument in Proposition 7.3.6.) It follows that B_0 is in fact a basis of M on the open disc of radius λ, on which D acts via a matrix over K. Since that matrix has eigenvalues in $\mathbb{Z}_p$ we can conjugate over K to put the matrix in Jordan normal form, in which case the entries will belong to $\mathbb{Z}_p$.

It remains to give a lower bound for λ. By Theorem 11.3.2, for $r \in [0, -\log\lambda]$ the function f_1 is continuous, convex, and piecewise affine, with slopes belonging to

$$\frac{1}{1}\mathbb{Z} \cup \cdots \cup \frac{1}{n}\mathbb{Z}.$$

Since the slope for $r > -\log\lambda$ is equal to 1, the slopes for $r \leq -\log\lambda$ cannot exceed 1. Moreover, there cannot be a slope equal to 1 in this range as otherwise it would occur as the left-hand slope at $r = -\log \lambda$, so there would exist $\rho > \lambda$ for which $R(M \otimes F_\rho) = \rho$, contrary to the definition of λ. Consequently f_1 has all slopes less than or equal to $(n-1)/n$ for $r \in [0, -\log \lambda]$, yielding

$$-\log \lambda = f_1(-\log \lambda) \leq f_1(0) + \frac{n-1}{n}(-\log \lambda).$$

From this we deduce that $\lambda \geq \lim_{\rho \to 1^-} R(M \otimes F_\rho)^n$, as desired. □

Example 13.7.2. The following example shows that the exponent n in Theorem 13.7.1 cannot be replaced by 1. Pick $\lambda \in K^\times$ with $|\lambda| > 1$. Let M be the differential module of rank 2 over the open unit disc for the operator $t\,d/dt$, whose action of D on a basis is given by

$$\begin{pmatrix} 0 & \lambda \\ t & 0 \end{pmatrix}.$$

Since this matrix modulo t is nilpotent, the exponents of M are both zero; in particular, they are in $\mathbb{Z}_p$ with p-adic non-Liouville differences.

For $\rho \in (|\lambda|^{-1}, 1)$ we may apply Theorem 6.5.3 to compute $|t^{-1}D|_{\mathrm{sp}, M \otimes F_\rho} = |\lambda|^{1/2}\rho^{-1/2}$ and so $R(M \otimes F_\rho) = p^{-1/(p-1)}|\lambda|^{-1/2}\rho^{1/2}$. To compute $R(M \otimes F_\rho)$ for $\rho \leq |\lambda|^{-1}$, we note that the function $f_1(M, r)$ is affine of slope $1/2$ for $0 \leq r \leq \log|\lambda|$ and of slope 1 for r large. By Theorem 11.3.2(a), (b), (e), f_1 is piecewise affine and its slopes on $[0, +\infty)$ must lie in $[1/2, 1] \cap (\mathbb{Z} \cup (1/2)\mathbb{Z})$. Consequently $f_1(M, r)$ must have slope 1/2 until it reaches a point at which $f_1(M, r) = r$ and slope 1 thereafter. In other words,

$$R(M \otimes F_\rho) = \begin{cases} p^{-1/(p-1)}|\lambda|^{-1/2}\rho^{1/2} & \rho > p^{-2/(p-1)}|\lambda|^{-1}, \\ \rho & \rho \leq p^{-2/(p-1)}|\lambda|^{-1}. \end{cases}$$

In particular, the fundamental solution matrix of M in the given basis converges on the open disc of radius $p^{-2/(p-1)}|\lambda|^{-1} = \lim_{\rho \to 1^-} R(M \otimes F_\rho)^2$ but not on any larger disc.

Remark 13.7.3. The final assertion of Theorem 13.7.1 (if M is solvable at 1 then the fundamental solution matrix converges in the open unit disc) can also be proved without relying on the p-adic Fuchs theorem for annuli (Theorem 13.6.1); see the notes. This in turn can be used to give a proof of Corollary 13.6.4 without using Theorem 13.6.1, as follows.

Let M be a finite differential module, on the open annulus $\alpha < |t| < \beta$ for the derivation $t\,d/dt$, satisfying the Robba condition. Suppose that the restriction M' of M to some smaller open annulus $\gamma < |t| < \delta$ is unipotent. Then, on this smaller annulus, by Lemma 9.2.3 we obtain a basis of M on which td/dt acts via a nilpotent matrix over K. This defines a differential module on the disc $|t| < \delta$ with a nilpotent regular singularity at $t = 0$; by gluing the module to M over the annulus $\gamma < |t| < \delta$ we obtain a differential module on the disc $|t| < \beta$ with a nilpotent regular singularity at $t = 0$, which is solvable at the boundary of the disc. By the final assertion of Theorem 13.7.1 this module admits a basis on which td/dt acts via a nilpotent matrix over K. This gives us such a basis of M over the annulus $\gamma < |t| < \beta$; by a similar argument, we get a similar basis over the annulus $\alpha < |t| < \delta$. These bases can only differ by a K-linear transformation (as in the proof of Theorem 13.7.1), so each gives a basis of M itself on which td/dt acts via a nilpotent matrix over K. Hence M is unipotent.

Notes

The definition of a p-adic Liouville number was introduced by Clark [**54**]; our presentation follows [**80**, §VI.1].

The cited theorem of Clark [**54**, Theorem 3] is actually somewhat stronger than Theorem 13.2.3 as it allows differential operators of possibly infinite order.

The example in Remark 13.2.4 was loosely inspired by an example of Monsky (a counterexample against a slightly different assertion); see [**81**, §7] or [**80**, §IV.8] for discussion.

Proposition 13.3.2 was originally due to Christol; compare [**80**, Proposition IV.7.7].

The theory of exponents for differential modules on a p-adic annulus satisfying the Robba condition was originally developed by Christol and Mebkhout [**50**, §4–5]; in particular, Theorem 13.6.1 appears therein as [**50**, Théorème 6.2–4]. A somewhat more streamlined development was later given by Dwork [**79**], in which Theorem 13.6.1 appears as [**79**, Theorem 7.1]. (Dwork notes that he did not verify the equivalence between the two constructions; we do not recommend losing any sleep over this.) Our treatment is essentially that of Dwork with a few technical simplifications. Another treatment will appears in the upcoming book of Christol [**46**]. Besides all these there is an expository article [**161**] that summarizes the proofs (using Dwork's approach) and provides some further context, including the formulation of the *p-adic index theorem*.

It is claimed in [**161**, §3.2] that the converse of Theorem 13.5.6 also holds, i.e., if A is an exponent of a differential module M and B is weakly equivalent to A then B is also an exponent of M. We do not know of a proof of this; in [**79**, Remark 4.5] it is suggested that this may not be entirely trivial.

A somewhat more elementary treatment of Theorem 13.7.1 than the one given here was given in [**80**, §6]; it does not rely on the p-adic Fuchs theorem for annuli (Theorem 13.6.1). However, it gives a weaker result: it only establishes convergence of the fundamental solution matrix in the open disc of radius $\lim_{\rho \to 1^-} R(M \otimes F_\rho)^{n^2}$. That weaker result is due to Christol, who presented it himself in [**42**, Théorème 6.4.7] and [**43**].

The weaker result just mentioned is itself sufficient to imply that if $\lim_{\rho \to 1^-} R(M \otimes F_\rho) = 1$ then the fundamental solution matrix converges in the open unit disc. In the case of a nilpotent singularity one can show this even more directly; see [**134**, Lemma 3.6.2]. As noted in Remark 13.7.3 this can be used to derive Corollary 13.6.4 without using Theorem 13.6.1.

An intriguing archimedean analogue to the theory of p-adic exponents appears in the local analytic theory of q-difference equations in the case $|q| = 1$. This seems to extend to an analogy between q-difference equations with $|q| \neq 1$ and p-adic differential modules with intrinsic subsidiary radii strictly less than 1. See [**73**] for part of this story.

Exercises

(1) Prove that rational numbers are p-adic non-Liouville numbers.

(2) Give another proof of Lemma 13.1.6 (as in [**80**, Lemma VI.1.2]) by first verifying that both sides of the desired equation have the same coefficients of x^0 and x^1 and are killed by the second-order differential operator
$$\frac{d}{dx}\left(\frac{d}{dx} - \lambda - x\right).$$

(3) Show that Theorem 13.2.2 can be deduced from Theorem 13.2.3. (Hint: show that if $H^0(M) \neq 0$ then 0 must occur as an eigenvalue of N_0.)

(4) Prove that there exists $a \in \mathbb{Z}_p$ satisfying
$$\mathrm{type}(a) = 1, \qquad \mathrm{type}(-a) < 1.$$

(5) Prove Proposition 13.3.3.

(6) Verify Example 13.4.6.

(7) (a) Prove that the set of $A \in \mathbb{Z}_n^p$ which are weakly equivalent to 0 is a subgroup of $\mathbb{Z}_n^p$.

(b) Give a counterexample against the claim that if A_i is weakly equivalent to B_i for $i = 1, 2$ then $A_1 + A_2$ is weakly equivalent to $B_1 + B_2$. (Hint: give an example using rational numbers, in which case weak equivalence implies equivalence, by an earlier exercise.)

(8) Complete the proof of Proposition 13.5.3.

(9) Prove that if a differential module M over $K\langle \alpha/t, t/\beta \rangle$ admits a basis on which td/dt acts via a nilpotent matrix over K then the K-span of the basis is fixed by the Taylor series action of μ_{p^∞}, but not necessarily by the full action of $1 + \mathfrak{m}_K$. (Hint: this can be reduced to the fact that μ_{p^∞} is in the kernel of the logarithm map defined using the power series $\log(1 - z)$, which converges for $|z| < 1$.)

(10) Let M be a differential module on an open annulus satisfying the Robba condition. Use Theorem 13.6.1 to prove that $H^0(M)$ and $H^1(M)$ are both finite-dimensional and of the same dimension, i.e., that the index of M is 0.

Part IV

Difference Algebra and Frobenius Modules

14

Formalism of difference algebra

We now step away from differential modules for a little while to study the related subject of *difference algebra*. This is the theory of algebraic structures enriched not with a derivation but with an endomorphism of rings. Our treatment of difference algebra will run largely in parallel with that for differential algebra but in a somewhat abridged fashion; our goal is to be able to use difference algebra to say nontrivial things about p-adic differential equations. We will begin to do that in Part V.

In this chapter we introduce the formalism of difference rings, fields, and modules and the associated notion of twisted polynomials. Then we study briefly the analogue of algebraic closure for a difference field. Finally, we make a detailed study of difference modules over a complete nonarchimedean field culminating in a classification of difference modules for the Frobenius automorphism of a complete unramified p-adic field with algebraically closed residue field (the Dieudonné–Manin classification).

Hypothesis 14.0.1. Throughout this part of the book and the next, we retain Notation 8.0.1 with $p > 0$ (so that K is a complete nonarchimedean field of characteristic 0 and residual characteristic $p > 0$) but, unless otherwise specified, we will require K to be *discretely valued*. This is necessary to avoid a number of technical complications, some of which we will point out as we go along. (We will make almost no reference to K in this particular chapter; we only include this hypothesis here in order to place it at the beginning of Part V.)

14.1 Difference algebra

We start with the central definition. See the notes at the end of the chapter for some explanation of the term "difference" in this context.

Definition 14.1.1. A *difference ring or field* is a ring or field R equipped with an endomorphism ϕ. A *difference module* over R is an R-module M equipped with a map $\Phi : M \to M$ which is additive and ϕ-semilinear; the latter means that

$$\Phi(rm) = \phi(r)\Phi(m) \qquad (r \in R, m \in M).$$

This data determines an R-linear map $\phi^*(M) = M \otimes_{R,\phi} R \to M$ sending $m \otimes 1$ to $\Phi(m)$; conversely, any R-linear map $\phi^* M \to M$ induces the structure of a difference module on M. A free difference module is *trivial* if it admits a Φ-invariant basis; when we refer to "the" trivial module we mean the module R itself, with $\Phi = \phi$. A difference submodule of R is also called a *difference ideal*.

Morphisms of difference rings or modules are required to be ϕ-equivariant. Tensor products behave as expected: if M, N are difference modules over the same difference ring R then $M \otimes_R N$ acquires the structure of a difference module with

$$\Phi(m \otimes n) = \Phi(m) \otimes \Phi(n) \qquad (m \in M, n \in N).$$

In particular, if $R \to S$ is a morphism of difference rings then we can perform a base change from difference modules over R to difference modules over S.

As in differential algebra, difference modules can often be described in terms of matrices of action.

Definition 14.1.2. If M is a finite difference module over R freely generated by $e_1, \dots, e_n$ then we can recover the action of Φ from the $n \times n$ matrix A defined by

$$\Phi(e_j) = \sum_i A_{ij} e_i.$$

Namely, if we use the basis to identify M with the space of column vectors of length n over R then

$$\Phi(v) = A\phi(v).$$

In parallel with the differential case, we call A the *matrix of action* of Φ on the basis $e_1, \dots, e_n$.

We now record the effect of a change of basis on a matrix of action.

Remark 14.1.3. Let M be a finite free difference module over R, and let $e_1, \dots, e_n$ and $e'_1, \dots, e'_n$ be two bases of M. Recall that the change-of-basis matrix U from the first basis to the second is the $n \times n$ matrix U satisfying

$e'_j = \sum_i U_{ij} e_i$. If A is the matrix of action of Φ on the e_i then the matrix of action of Φ on the e'_i is

$$A' = U^{-1} A \phi(U).$$

It is important to consider dual modules, but the description in difference algebra is somewhat more complicated than in differential algebra. We will introduce it only in the free case. (One can extend the definition to locally free modules; compare Remark 5.3.4.)

Definition 14.1.4. Let M be a finite free difference module over R. We say that M is *dualizable* if the map $\phi^*(M) \to M$ induced by Φ is an isomorphism. Given a basis of M, it is equivalent to check that the matrix of action A of Φ on that basis is invertible (the same then holds for any other basis). However, even in this case the action of Φ on M itself is not invertible unless the action of ϕ on R is invertible (i.e., unless R is inversive in the sense of Definition 14.1.5 below).

If M is dualizable, there is a unique way to equip the module-theoretic dual $M^\vee = \mathrm{Hom}_R(M, R)$ with a Φ-action such that

$$\Phi(f)(\Phi(m)) = f(m) \qquad (f \in M^\vee, m \in M).$$

In terms of a given basis, the matrix of action on the dual basis is given by the inverse transpose A^{-T}. We call the resulting difference module the *dual* of M.

There is also a notion of an opposite difference ring but only under an extra hypothesis.

Definition 14.1.5. We say that the difference ring R is *inversive* if ϕ is an automorphism. In this case, we can define the *opposite difference ring* R^{opp} to be R again, but now equipped with the endomorphism ϕ^{-1}. If R is inversive and M is a finite free difference module over R, we define the *opposite module* M^{opp} of M to be M equipped with the inverse of Φ; that is, if A is the matrix of action of Φ on some basis then the matrix of action of Φ^{-1} is given by $\phi^{-1}(A^{-1})$.

Definition 14.1.6. For M a difference module, write

$$H^0(M) = \ker(\mathrm{id} - \Phi), \qquad H^1(M) = \mathrm{coker}(\mathrm{id} - \Phi).$$

If M_1, M_2 are difference modules with M_1 dualizable then $H^0(M_1^\vee \otimes M_2)$ computes ϕ-equivariant morphisms from M_1 to M_2 and $H^1(M_1^\vee \otimes M_2)$ computes extensions $0 \to M_2 \to M \to M_1 \to 0$ of difference modules (exercise). That is, we have natural identifications

$$H^0(M_1^\vee \otimes M_2) = \mathrm{Hom}(M_1, M_2), \qquad H^1(M_1^\vee \otimes M_2) = \mathrm{Ext}(M_1, M_2)$$

of groups (see the proof of Lemma 5.3.3 for the group structure on extensions).

14.2 Twisted polynomials

As in differential algebra, there is a relevant notion of twisted polynomials.

Definition 14.2.1. For R a difference ring, we define the *twisted polynomial ring* $R\{T\}$ as the set of finite formal sums $\sum_{i=0}^{\infty} r_i T^i$, with multiplication obeying the rule $Tr = \phi(r)T$. For any $P \in R\{T\}$, the quotient $R\{T\}/R\{T\}P$ is a difference module; if M is a difference module, we say that $m \in M$ is a *cyclic vector* if there is an isomorphism $M \cong R\{T\}/R\{T\}P$ carrying m to 1.

Definition 14.2.2. If R is inversive, we again have a formal adjoint construction: given $P \in R\{T\}$, its formal adjoint is obtained by taking the coefficients to the right-hand side of T. This may then be viewed as an element of the opposite ring of $R\{T\}$, which we may identify with $R^{\mathrm{opp}}\{T\}$.

It is not completely straightforward to analogize the cyclic vector theorem to difference modules; see the exercises for one attempt to do so. Instead, we will use only the following trivial observation.

Lemma 14.2.3. *Any irreducible finite difference module over a difference field contains a cyclic vector.*

Proof. If F is a difference field, M is a finite difference module over F, and $m \in M$ is nonzero then $m, \Phi(m), \dots$ generate a nonzero difference submodule of M. If M is irreducible, this submodule must be the whole of M. □

Definition 14.2.4. If ϕ is isometric for a norm $|\cdot|$ on F then we have the usual definition of Newton polygons and slopes for twisted polynomials, with a natural analogue of the additivity of slopes (i.e., Proposition 2.1.2). If R is inversive then a twisted polynomial and its adjoint have the same Newton polygon.

As in the differential case, we may apply the master factorization theorem (Theorem 2.2.2), as in the proof of Theorem 2.2.1, to get a factorization result. However, the result is inherently asymmetric; see Remark 14.2.6 below.

Theorem 14.2.5. *Let F be a difference field complete for a norm $|\cdot|$ under which ϕ is isometric. Then any monic twisted polynomial $P \in F\{T\}$ admits a unique factorization*

$$P = P_{r_1} \cdots P_{r_m}$$

for some $r_1 < \cdots < r_m$, where each P_{r_i} is monic with all slopes equal to r_i. (The same holds with the factors in the opposite order if F is inversive, but not always otherwise; see Remark 14.2.6 and the exercises.)

Remark 14.2.6. It is worth clarifying why the conditions of Theorem 2.2.2 can be satisfied by $F\{T\}$ but not necessarily by its opposite ring. The key condition in this theorem is (b), which states that with

$$\begin{aligned}
U &= \{P \in F[T] : \deg(P) \leq \deg(S) - m\}, \\
V &= \{P \in F[T] : \deg(P) \leq m - 1\}, \\
W &= \{P \in F[T] : \deg(P) \leq \deg(S)\}, \\
a &= 1, \\
b &= T^m,
\end{aligned}$$

the map $(u, v) \mapsto av + ub$ surjects $U \times V$ onto W. This holds because, on the one hand, any element of $F\{T\}$ whose coefficients of $1, T, \dots, T^{m-1}$ vanish is divisible by T^m on the right. On the other hand, if $r \notin \phi(F)$ then rT^m is not divisible on the left by T, let alone by T^m.

14.3 Difference-closed fields

Now we study briefly the difference-theoretic analogue of the algebraic closure property of ordinary fields.

Definition 14.3.1. We will say that a difference field F is *weakly difference-closed* if every dualizable finite difference module over F is trivial. We say F is *strongly difference-closed* if F is inversive and weakly difference-closed.

Remark 14.3.2. Note that the property that F is weakly difference-closed includes the fact that short exact sequences of dualizable finite difference modules over F always split. By contrast, if for instance ϕ is the identity map then this is never true even if F is algebraically closed, because linear transformations need not be semisimple.

Here is a useful criterion for checking for difference closure.

Lemma 14.3.3. *The difference field F is weakly difference-closed if and only if the following conditions all hold.*

(a) *Every nonconstant monic twisted polynomial $P \in F\{T\}$ factors as a product of linear factors.*
(b) *For every $c \in F^\times$, there exists $x \in F^\times$ with $\phi(x) = cx$.*
(c) *For every $c \in F^\times$, there exists $x \in F^\times$ with $\phi(x) - x = c$.*

Proof. We first suppose that F is weakly difference-closed. To prove (a), it suffices to check that if $P \in F\{T\}$ is nonconstant monic with nonzero constant term then P factors as $P_1 P_2$ with P_2 linear. The nonzero constant term implies that $M = F\{T\}/F\{T\}P$ is a dualizable finite difference module over F and so must be trivial by the hypothesis that F must be weakly difference-closed. In particular, there exists a short exact sequence $0 \to M_1 \to M \to M_2 \to 0$ with M_2 trivial of dimension 1; this corresponds to a factorization $P = P_1 P_2$ with P_2 linear.

To prove (b), note that $F\{T\}/F\{T\}(T - c^{-1})$ must be trivial. Since each element of this difference module is represented by some $x \in F$, we must have $x \in F^\times$ such that $Tx - x$ is divisible on the right by $T - c^{-1}$. By comparing degrees, we see that $Tx - x = y(T - c^{-1})$ for some $y \in F$. Then $y = \phi(x)$ and $yc^{-1} = x$, proving the claim.

To prove (c), form the ϕ-module V corresponding to the matrix $\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$. By construction, we have a short exact sequence $0 \to V_1 \to V \to V_2 \to 0$ with V_1, V_2 trivial; since V must also be trivial, this extension must split. In other words we can find $x \in F$ such that

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\phi(x) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

that is, $\phi(x) - x = c$.

Conversely, suppose that (a)–(c) hold. Every nonzero dualizable finite difference module over F admits an irreducible quotient. This quotient admits a cyclic vector, by Lemma 14.2.3, and so admits a quotient of dimension 1 by (a). That quotient in turn is trivial by (b). By induction, we deduce that every dualizable finite difference module over F admits a filtration whose successive quotients are trivial of dimension 1. This filtration splits, by (c). □

The notion of difference closure is particularly simple for the absolute Frobenius endomorphism (the p-power map) on a field of characteristic $p > 0$.

Proposition 14.3.4. *Let F be a separably (resp. algebraically) closed field of characteristic $p > 0$ equipped with a power of the absolute Frobenius endomorphism. Then F is weakly (resp. strongly) difference-closed.*

Proof. For $P = \sum_{i=0}^m P_i T^m \in F\{T\}$, with $m > 0$, $P_m = 1$, and $P_0 \neq 0$, the polynomial $Q(x) = \sum_{i=0}^m P_i x^{q^i}$ has degree $q^m \geq 2$, and $x = 0$ occurs as a root only with multiplicity 1. Moreover, the formal derivative of P is a constant polynomial and so has no common roots with P; hence P is a separable polynomial. Since F is separably closed, there must exist a nonzero root x of Q; this implies criteria (a) and (b) of Lemma 14.3.3. To deduce (c) note that, for $c \in F^\times$, the polynomial $x^q - x - c$ is again separable and so has a root in F. □

14.4 Difference algebra over a complete field

Hypothesis 14.4.1. For the rest of this chapter, let F be a difference field complete for a nonarchimedean norm $|\cdot|$, with respect to which ϕ is isometric. (For short, we will say that F is an *isometric* complete nonarchimedean difference field.) We do not assume that F is inversive; if it is not then we can embed F into an inversive difference field by forming the completion F' of the direct limit of the system

$$F \xrightarrow{\phi} F \xrightarrow{\phi} \cdots .$$

We sometimes call F' the *perfection*, or more precisely the *ϕ-perfection*, of F.

As in the differential case, we would like to classify finite difference modules over F by the spectral radius of Φ; this turns out to be somewhat easier in the difference case because there is now no analogue of the distinction between the visible and full spectra. A related fact is that the use of matrix inequalities here is much closer to that in Chapter 4 than that in Chapter 6; the main difference is that we do not begin with a satisfactory theory of eigenvalues or eigenvectors.

We first note the following analogue of Lemma 6.2.5.

Lemma 14.4.2. *Let V be a nonzero finite difference module over F. Let $e_1, \dots, e_n$ be a basis of V. For each nonnegative integer s, let A_s be the matrix of action of Φ^s on $e_1, \dots, e_n$. Then*

$$|\Phi|_{\mathrm{sp},V} = \lim_{s\to\infty} |A_s|^{1/s}.$$

Proof. It suffices to observe that, using the supremum norm defined by $e_1, \dots, e_n$, the operator norm of Φ^s is precisely $|A_s|$. Thus the limit on the right-hand side exists and matches the definition of the left-hand side. $\square$

The following basic properties will help with our classification, as long as we are mindful of the discrepancies between the differential and difference cases. For instance, the difference case has a simpler rule for tensor products but a less simple rule for duals. (It may be helpful to keep in mind the case where ϕ is the identity map, and relate the following observations to eigenvalues of linear transformations.)

Lemma 14.4.3. *Let V, V_1, V_2 be nonzero finite difference modules over F.*

(a) *For a short exact sequence $0 \to V_1 \to V \to V_2 \to 0$,*

$$|\Phi|_{\mathrm{sp},V} = \max\{|\Phi|_{\mathrm{sp},V_1}, |\Phi|_{\mathrm{sp},V_2}\}.$$

(b) *We have*

$$|\Phi|_{\mathrm{sp},V_1\otimes V_2} = |\Phi|_{\mathrm{sp},V_1}|\Phi|_{\mathrm{sp},V_2}.$$

(c) *For any complete extension E of F to which ϕ extends isometrically,*

$$|\Phi|_{\mathrm{sp},V} = |\Phi|_{\mathrm{sp},V\otimes E}.$$

(d) *For V dualizable,*

$$|\Phi|_{\mathrm{sp},V}|\Phi|_{\mathrm{sp},V^\vee} \geq 1.$$

Proof. Exercise. □

Lemma 14.4.4. *If $V \cong F\{T\}/F\{T\}P$ and P has only one slope $r < +\infty$ in its Newton polygon, then V admits a basis on which*

$$|\Phi|_V = e^{-r}, \qquad |\Phi^{-1}|_{V\otimes F'} = e^r.$$

Consequently

$$|\Phi|_{\mathrm{sp},V} = e^{-r}, \qquad |\Phi^\vee|_{\mathrm{sp},V} = e^r$$

and, if F is inversive, then also

$$|\Phi^{-1}|_{\mathrm{sp},V} = e^r.$$

Proof. Put $n = \deg(P)$, and define a norm on V by

$$|a_0 + \cdots + a_{n-1}T^{n-1}| = \max_i\{|a_i|e^{-ri}\};$$

then clearly

$$|\Phi|_V = e^{-r}, \qquad |\Phi^{-1}|_{V\otimes F'} = |\Phi|_{V^\vee} = e^r.$$

We deduce that

$$|\Phi|_{\mathrm{sp},V} \leq e^{-r}, \qquad |\Phi^{-1}|_{\mathrm{sp},V\otimes F'}, |\Phi|_{V^\vee} \leq e^r;$$

since

$$1 \leq |\Phi|_{\mathrm{sp},V}|\Phi^{-1}|_{\mathrm{sp},V\otimes F'} \leq e^{-r}e^r, \qquad 1 \leq |\Phi|_{\mathrm{sp},V}|\Phi|_{\mathrm{sp},V^\vee} \leq e^{-r}e^r,$$

we obtain the desired equalities. □

Corollary 14.4.5. *For any nonzero finite difference module V over F, either $|\Phi|_{\mathrm{sp},V} = 0$ or there exists an integer $m \in \{1,\dots,\dim_F V\}$ such that $|\Phi|^m_{\mathrm{sp},V} \in |F^\times|$.*

Definition 14.4.6. Let V be a nonzero finite difference module over F. We say that V is *pure (and) of norm s* if all the Jordan-Hölder constituents of V have spectral radius s. Note that V is pure of norm 0 if and only if $\Phi^{\dim_F V} = 0$. If V is pure of norm 1, we also say that V is *étale* or *unit-root*.

Remark 14.4.7. It is more common to classify pure modules using additive notation, i.e., in terms of the *slope* $(-\log s)$ instead of the norm s. (Correspondingly, pure modules are also called *isoclinic* modules.) For better or worse, we have decided here to keep the notation consistent with the multiplicative terminology used in the differential case. We will switch to the additive language only in the next section, in order to talk about Hodge and Newton polygons.

Proposition 14.4.8. *Let V be a nonzero finite difference module over F. Then V is pure of norm $s > 0$ if and only if*

$$|\Phi|_{\mathrm{sp},V\otimes F'} = s, \qquad |\Phi^{-1}|_{\mathrm{sp},V\otimes F'} = s^{-1}; \tag{14.4.8.1}$$

in this case, $V^\vee$ is pure of norm s^{-1}.

Proof. If V is pure of norm s then (14.4.8.1) holds by Lemmas 14.4.3(a), (c) and 14.4.4. Conversely, if (14.4.8.1) holds and W is an irreducible subquotient of V then, by Lemma 14.4.3(a),

$$|\Phi|_{\mathrm{sp},W\otimes F'} \leq |\Phi|_{\mathrm{sp},V\otimes F'}, \qquad |\Phi^{-1}|_{\mathrm{sp},W\otimes F'} \leq |\Phi^{-1}|_{\mathrm{sp},V\otimes F'}.$$

We thus have

$$1 \leq |\Phi|_{\mathrm{sp},W\otimes F'} |\Phi^{-1}|_{\mathrm{sp},W\otimes F'} \leq ss^{-1} = 1,$$

which forces the equalities $|\Phi|_{\mathrm{sp},W} = |\Phi|_{\mathrm{sp},W\otimes F'} = s$ and $|\Phi^{-1}|_{\mathrm{sp},W\otimes F'} = s^{-1}$. In particular, W is pure of norm s. Moreover, by Lemma 14.2.3 there is an isomorphism $W \cong F\{T\}/F\{T\}P$ for some twisted polynomial P; by Theorem 14.2.5, P has only one slope in its Newton polygon. By Lemma 14.4.4 that slope must equal $-\log s$, and so $W^\vee$ must be pure of norm s^{-1}, as then is $V^\vee$. □

Corollary 14.4.9. *Let V_1, V_2 be nonzero finite difference modules over F which are pure and of respective norms s_1, s_2. Then $V_1 \otimes V_2$ is pure of norm $s_1 s_2$.*

Note that this does not follow immediately from Lemma 14.4.3(b) because the tensor product of two irreducibles need not be irreducible.

Proof. If $s_1 s_2 = 0$ then it is easy to check that $V_1 \otimes V_2$ is pure of norm 0. Otherwise, Lemma 14.4.3(b) plus Proposition 14.4.8 yields

$$|\Phi|_{\mathrm{sp},V_1\otimes V_2\otimes F'} = |\Phi|_{\mathrm{sp},V_1\otimes F'} |\Phi|_{\mathrm{sp},V_2\otimes F'} = s_1 s_2,$$
$$|\Phi^{-1}|_{\mathrm{sp},V_1\otimes V_2\otimes F'} = |\Phi^{-1}|_{\mathrm{sp},V_1\otimes F'} |\Phi^{-1}|_{\mathrm{sp},V_2\otimes F'} = s_1^{-1} s_2^{-1};$$

thus Proposition 14.4.8 again implies that $V_1 \otimes V_2$ is pure of norm $s_1 s_2$. □

Corollary 14.4.10. *Let V be a nonzero finite difference module over F. Then, for any positive integer d, V is pure of norm s if and only if V becomes pure of norm s^d when viewed as a difference module over (F, ϕ^d).*

Proposition 14.4.11. *Let V be a nonzero finite difference module over F. Suppose that either*

(a) $|\Phi|_{\mathrm{sp},V} < 1$ *or*

(b) *F is inversive and* $|\Phi^{-1}|_{\mathrm{sp},V} < 1$.

Then $H^1(V) = 0$.

Proof. In case (a), given $v \in V$, the series

$$w = \sum_{i=0}^{\infty} \Phi^i(v)$$

converges to a solution of $w - \Phi(w) = v$. In case (b), the series

$$w = -\sum_{i=0}^{\infty} \Phi^{-i-1}(v)$$

does likewise. □

Corollary 14.4.12. *If V_1, V_2 are nonzero finite differential modules over F which are pure of respective norms s_1, s_2 and if either*

(a) $s_1 < s_2$ *or*

(b) *F is inversive and $s_1 > s_2$,*

then any exact sequence $0 \to V_1 \to V \to V_2 \to 0$ splits.

Proof. If $s_2 > 0$ then, by Proposition 14.4.8 and Corollary 14.4.9, $V_2^\vee \otimes V_1$ is pure of norm s_1/s_2, so Proposition 14.4.11 gives the desired splitting. Otherwise we must be in case (b), so we can pass to the opposite ring to make the same conclusion. □

If F is inversive, we again get a decomposition theorem.

Theorem 14.4.13. *Suppose that F is inversive. Let V be a finite difference module over F. Then there exists a unique direct sum decomposition*

$$V = \bigoplus_{s \geq 0} V_s$$

of difference modules, in which each nonzero V_s is pure of norm s. (Note that V is dualizable if and only if $V_0 = 0$.)

Proof. This follows at once from Corollary 14.4.12. □

Remark 14.4.14. Note that if ϕ is the identity map on F, Theorem 14.4.13 simply reproduces the decomposition of V in which V_s consists of the generalized eigenspaces for all eigenvalues of norm s.

If F is not inversive, we get a filtration instead of a decomposition.

Theorem 14.4.15. *Let V be a finite difference module over F. Then there exists a unique filtration*

$$0 = V_0 \subset V_1 \subset \cdots \subset V_l = V$$

of difference modules, such that each successive quotient V_i/V_{i-1} is pure of some norm s_i and $s_1 > \cdots > s_l$. (Note that V is dualizable if and only if $V = 0$ or $s_l > 0$.)

Proof. Start with any filtration $0 = V_0 \subset V_1 \subset \cdots \subset V_l = V$ with irreducible successive quotients. Let s_i be the radius of V_i/V_{i-1}. In the case $s_i < s_{i+1}$ we may apply Corollary 14.4.12 to choose another difference module V_i', containing V_{i-1} and contained in V_{i+1}, such that V_i'/V_{i-1}, V_{i+1}/V_i' are pure and have slopes s_{i+1}, s_i, respectively.

By repeating this process, we eventually reach the case where $s_1 \geq \cdots \geq s_l$. We obtain the existence of a suitable filtration from the given filtration by simply omitting V_i whenever $s_i = s_{i-1}$. Uniqueness follows by tensoring with F' and invoking the uniqueness in Theorem 14.4.13. $\square$

The following alternative characterization of pureness may be useful in some situations.

Proposition 14.4.16. *Let V be a finite difference module over F, and choose $\lambda \in F^\times$. Then V is pure of norm $|\lambda|$ if and only if there exists a basis of V on which Φ acts via λ times an element of* $\mathrm{GL}_n(\mathfrak{o}_F)$.

Proof. If such a basis exists then Proposition 14.4.8 implies that V is pure of norm $|\lambda|$. Conversely, if V is irreducible of spectral radius $|\lambda|$ then Lemma 14.4.4 provides a basis of the desired form. Otherwise we proceed by induction on $\dim_F V$. Suppose that we are given a short exact sequence $0 \to V_1 \to V \to V_2 \to 0$ in which V_1, V_2 admit bases of the desired form. Let $e_1, \dots, e_m \in V$ form such a basis for V_1, and let $e_{m+1}, \dots, e_n \in V$ lift such a basis for V_2. Then, for $\mu \in F$ of sufficiently small norm,

$$e_1, \dots, e_m, \mu e_{m+1}, \dots, \mu e_n$$

will form a basis of V of the desired form. $\square$

Remark 14.4.17. Note that, whenever V is pure and of positive norm, we can apply Proposition 14.4.16 after replacing Φ by some power of itself, thanks to Corollary 14.4.5.

14.5 Hodge and Newton polygons

The theory of Hodge and Newton polygons, which we introduced when studying nonarchimedean matrix inequalities in Chapter 4, admits a close analogue when considering a difference algebra over a complete nonarchimedean field. Throughout this section, we continue to retain Hypothesis 14.4.1.

Definition 14.5.1. Let V be a finite difference module over F equipped with the supremum norm with respect to some basis. Let A be the matrix of action of Φ on this basis; define the *Hodge polygon* of V as the Hodge polygon of the matrix A (see Definition 4.3.3). Given the choice of the norm on V, this definition is independent of the choice of the basis: we can change basis only by using a matrix $U \in \mathrm{GL}_n(\mathfrak{o}_F)$ which replaces A by $U^{-1}A\phi(U)$; since ϕ is an isometry, this ensures that $\phi(U) \in \mathrm{GL}_n(\mathfrak{o}_F)$ also. As in the linear case we list the Hodge slopes $s_{H,i}, \dots, s_{H,n}$ in increasing order.

Definition 14.5.2. Let V be a finite difference module over F. Define the *Newton polygon* of V to have slopes $s_{N,1}, \dots, s_{N,n}$ such that r appears with multiplicity equal to the dimension of the quotient of the norm e^{-r} in Theorem 14.4.15.

Lemma 14.5.3. *Let V be a finite difference module over F. We have*

$$s_{H,1} + \cdots + s_{H,i} = -\log |\Phi|_{\wedge^i V} \qquad (i = 1, \dots, n),$$
$$s_{N,1} + \cdots + s_{N,i} = -\log |\Phi|_{\mathrm{sp},\wedge^i V} \qquad (i = 1, \dots, n).$$

Proof. The first assertion follows from the corresponding fact in the linear case (which is analogous to Lemma 4.1.9). The second assertion reduces to the fact that if V is irreducible of dimension n and spectral radius s then $\wedge^i V$ has spectral radius s^i for $i = 1, \dots, n$; this follows from the fact that, in the basis given by Lemma 14.4.4, $\left|\wedge^i \Phi\right|_{\wedge^i V} = s^i$. $\square$

Corollary 14.5.4 (Newton above Hodge). *We have*

$$s_{N,1} + \cdots + s_{N,i} \geq s_{H,1} + \cdots + s_{H,i} \qquad (i = 1, \dots, n)$$

with equality for $i = n$.

As in the linear case (Theorem 4.3.11), we have a Hodge–Newton decomposition theorem.

Theorem 14.5.5. *Let V be a finite difference module over F equipped with a basis. If for some $i \in \{1, \dots, n-1\}$ we have*

$$s_{N,i} < s_{N,i+1}, \qquad s_{N,1} + \cdots + s_{N,i} = s_{H,1} + \cdots + s_{H,i}$$

then we can change basis, using a matrix in $\mathrm{GL}_n(\mathfrak{o}_F)$, so that the matrix of action of Φ becomes block upper triangular; the top left block accounts for the first i Hodge and Newton slopes of V. Moreover, if F is inversive and $s_{H,i} > s_{H,i+1}$, we can ensure that the matrix of action of Φ becomes block diagonal.

Proof. First we change basis by a matrix in $\mathrm{GL}_n(\mathfrak{o}_F)$ such that the F-span of the first i basis vectors equals the step of the filtration of Theorem 14.4.15 consisting of norms greater than or equal to $e^{-s_{N,i}}$. We may then proceed as in Theorem 4.3.11. □

Remark 14.5.6. Be aware that the Newton polygon, unlike the Hodge polygon, cannot be read off directly from the matrix of action of Φ; see the exercises for an example. However, this can be done if the matrix of Φ is a companion matrix. A restatement follows.

Proposition 14.5.7. *If $V \cong F\{T\}/F\{T\}P$ then the Newton polygon of V coincides with that of P.*

Proof. This reduces to Lemma 14.4.4 using Theorem 14.2.5. □

We also obtain the following analogue of Proposition 4.4.10.

Proposition 14.5.8. *Let V be a finite difference module over F, equipped with the supremum norm for some basis. For $i = 1, 2, \dots,$ let $s_{H,1,i}, \dots, s_{H,n,i}$ be the Hodge slopes of V as a difference module over (F, ϕ^i). Then, as $i \to \infty$, the quantities $i^{-1}s_{H,1,i}, \dots, i^{-1}s_{H,n,i}$ converge to the Newton slopes of V.*

Proof. The claim is independent of the choice of basis, so by Theorem 14.4.15 we may reduce to considering a pure module. In that case Lemma 14.4.2 implies that $i^{-1}s_{H,1,i} \to s_{N,1}$ as $i \to \infty$.

Note that $s_{N,1} = \cdots = s_{N,n}$ by purity and that $i^{-1}s_{H,1,i} \geq i^{-1}s_{H,j,i}$ for $j \geq 1$. Also, the Newton slopes of V as a difference module over (F, ϕ^i) are $is_{N,1}, \dots, is_{N,n}$; by the equality case of Corollary 14.5.4 we obtain $i^{-1}s_{H,1,i} + \cdots + i^{-1}s_{H,n,i} = s_{N,1} + \cdots + s_{N,n}$. All this plus the fact that $i^{-1}s_{H,1,i} \to s_{N,1}$ as $i \to \infty$ yields the desired convergence. □

Proposition 14.5.9. *Suppose that $F = K$ (i.e., F is discretely valued) and that $R = F[\![t]\!]_0$ carries the structure of an isometric complete nonarchimedean difference ring for the norm $|\cdot|_1$. Let M be a finite free difference module over*

R with least Newton slope c. Then there exists a basis of M with respect to which, for each positive integer m, the least Hodge slope of Φ^m is the least element of $v(F^\times)$ greater than or equal to cm.

Proof. Recall that if K is discrete then $\mathcal{E}$ is the completed fraction field of R (see Definition 9.4.3). Construct a suitable basis of $M \otimes \mathcal{E}$ by imitating the proof of Lemma 4.3.13 and then apply Lemma 8.6.1. □

14.6 The Dieudonné–Manin classification theorem

We also have an analogue in difference algebra of the fact that linear transformations over an algebraically closed field can be put into Jordan normal form. In fact, the situation is even better: in this setting all objects are semisimple. We continue to retain Hypothesis 14.4.1.

We first define some standard difference modules of a particularly simple form.

Definition 14.6.1. For $\lambda \in F$ and d a positive integer, let $V_{\lambda,d}$ be the difference module over F with basis $e_1, \dots, e_d$ such that

$$\Phi(e_1) = e_2, \quad \dots, \quad \Phi(e_{d-1}) = e_d, \quad \Phi(e_d) = \lambda e_1.$$

Lemma 14.6.2. *Suppose that $\lambda \in F^\times$ and the positive integer d are such that there is no $i \in \{1, \dots, d-1\}$ such that $|\lambda|^{i/d} \in |F^\times|$. Then $V_{\lambda,d}$ is irreducible.*

Proof. Note that

$$\Phi^d(e_i) = \phi^{i-1}(\lambda)e_i \qquad (i = 1, \dots, d).$$

Hence, by Proposition 14.4.16, $V_{\lambda,d}$ is pure of norm $\lambda^{1/d}$, as then is any submodule. But if the submodule were proper and nonzero then we would have a violation of Corollary 14.4.5. □

Next we show that, if F has a sufficiently large residue field, one can classify all dualizable finite difference modules in terms of the standard modules $V_{\lambda,d}$. One can always enlarge F to reach this case; see the exercises.

Theorem 14.6.3. *Let F be a complete discretely valued field, equipped with an isometric endomorphism ϕ, such that κ_F is strongly difference-closed. Then every dualizable finite difference module over F can be split (non-uniquely) as a direct sum of submodules, each of the form $V_{\lambda,d}$ for some λ, d. Moreover, given any generator π of $\mathfrak{m}_F$, we can force each λ to be a power of π.*

Proof. We first check that if V is pure of norm 1 then V is trivial; for this step, we only need κ_F to be weakly difference-closed. We must show that, for any $A \in \mathrm{GL}_n(\mathfrak{o}_F)$, there exists a convergent sequence $U_1, U_2, \dots \in \mathrm{GL}_n(\mathfrak{o}_F)$ such that

$$U_m^{-1} A\phi(U_m) \equiv I_n \pmod{\pi^m}.$$

Specifically, we will insist that $U_{m+1} \equiv U_m \pmod{\pi^m}$. Finding U_1 amounts to trivializing a dualizable difference module of dimension n over κ_F, which is possible because κ_F is assumed to be weakly difference-closed. For $m > 1$, given U_m we wish to set $U_{m+1} = U_m(I_n + \pi^m X_m)$, for some X_m, in such a way that

$$(I_n + \pi^m X_m)^{-1}(U_m^{-1} A\phi(U_m))(I_n + \phi(\pi)^m \phi(X_m)) \equiv I_n \pmod{\pi^{m+1}}.$$

Since we already have $U_m^{-1} A\phi(U_m) \equiv I_n \pmod{\pi^m}$, this amounts to solving

$$-X_m + \pi^{-m}(U_m^{-1} A\phi(U_m) - I_n) + \phi(\pi)^m \pi^{-m} \phi(X_m) \equiv 0 \pmod{\pi},$$

which we can do by applying criterion (c) from Lemma 14.3.3.

Next we check that ϕ is surjective on $\mathfrak{o}_F$, which implies that F is inversive. Given $x \in \mathfrak{o}_F$, it suffices to exhibit a sequence $y_0, y_1, y_2, \dots$, with $y_m \equiv y_{m+1} \pmod{\pi^m}$, such that $\phi(y_m) \equiv x \pmod{\pi^m}$. We start with $y_0 = 0$; given y_m, solving the equation

$$\phi(\pi^{-m}(y_{m+1} - y_m)) \equiv \phi(\pi)^{-m}(x - \phi(y_m)) \pmod{\pi}$$

is possible because κ_F is inversive.

We now check that if V is trivial then $H^1(V) = 0$. Given $x \in \mathfrak{o}_F$, it suffices to exhibit a sequence $y_0, y_1, y_2, \dots$, with $y_m \equiv y_{m+1} \pmod{\pi^m}$, such that $\phi(y_m) - y_m \equiv x \pmod{\pi^m}$. Again, we start with $y_0 = 0$; given y_m, solving the equation

$$\pi^{-m}\phi(\pi^m)\phi(\pi^{-m}(y_{m+1} - y_m)) - \pi^{-m}(y_{m+1} - y_m) \equiv \pi^{-m}(x - \phi(y_m) + y_m) \pmod{\pi}$$

is possibly by criteria (b) and (c) from Lemma 14.3.3. (We must use (b) to remove the leading coefficient in the first term before applying (c).)

At this point, we may apply Theorem 14.4.13 to reduce the desired result to the case where V is pure of norm $s > 0$. Let d be the smallest positive integer such that $s^d = |\pi^m|$ for some integer m. Then the first paragraph of the proof implies that $\pi^{-m}\Phi^d$ fixes some nonzero element of V; this gives us a nonzero map from $V_{\pi^m,d}$ to V. By Lemma 14.6.2 this map must be injective. Repeating this argument, we can write V as a successive extension of copies of $V_{\pi^m,d}$. However, $V_{\pi^m,d}^\vee \otimes V_{\pi^m,d}$ is pure of norm 1 and so has trivial H^1 as above. Thus V splits as a direct sum of copies of $V_{\pi^m,d}$, as desired. □

By Proposition 14.3.4, Theorem 14.6.3 has the following immediate corollary.

Corollary 14.6.4. *Let F be a complete discretely valued field such that κ_F is algebraically closed and of characteristic $p > 0$. Let $\phi : F \to F$ be an isometric automorphism lifting a power of the absolute Frobenius on κ_F. Then every dualizable finite difference module over F can be split (non-uniquely) as a direct sum of difference submodules, each of the form $V_{\lambda,d}$ for some $\lambda \in F^\times$ and some positive integer d coprime to the valuation of λ. Moreover, given any generator π of $\mathfrak{m}_F$, we can force each λ to be a power of π.*

Remark 14.6.5. Let k be an algebraically closed field of characteristic $p > 0$, let $W(k)$ be the ring of p-typical Witt vectors (i.e., the unique complete discrete valuation ring with residue field k and maximal ideal generated by p), and put $F = \text{Frac}(W(k))$. Then, for each power q of p, there is a unique Frobenius lift ϕ on $W(k)$, namely the Witt vector Frobenius lift. For this data, Corollary 14.6.4 with $\pi = p$ is precisely the *Dieudonné–Manin theorem*, i.e., the classification theorem of rational Dieudonné modules over k. (For more on Witt vectors, see the notes and the exercises.)

Corollary 14.6.6. *Let F be a complete discretely valued field, equipped with an isometric endomorphism ϕ, such that κ_F is strongly difference-closed. Then, for any finite difference module V over F, $H^1(V) = 0$.*

Proof. By Theorem 14.4.13 this reduces to the case where V is pure and of some norm $s \geq 0$. If $s > 0$ then V is dualizable and Theorem 14.6.3 implies the claim. If $s = 0$ then the action of Φ on V is nilpotent, so we may check the claim for V by checking the kernel and image of V. We may thus reduce to the case where the action of Φ on V is zero, when F becomes inversive; this was also checked in the proof of Theorem 14.6.3. □

Notes

The parallels between difference algebra and differential algebra are close enough that a survey of references for difference algebra strongly resembles its differential counterpart. A well-established, perhaps rather dry, reference is [**56**]; a somewhat more lively and more modern reference, which develops difference Galois theory under somewhat restrictive conditions, is [**201**]. We again mention [**4**] as a useful unifying framework for difference and differential algebra.

The choice of the modifier "difference" in the phrase "difference algebra" is motivated by the following basic example. Consider the automorphism on the ring of entire functions $f : \mathbb{C} \to \mathbb{C}$ given by $\phi(f)(z) = f(z + q)$ for some q. The difference operator

$$\Delta(f) = \frac{\phi(f) - f}{q}$$

obeys the twisted Leibniz rule

$$\Delta(fg) = \phi(f)\Delta(g) + \Delta(f)(g),$$

and its limit as $q \to 1$ is the usual derivation with respect to z.

Proposition 14.3.4 can be found in SGA7 [**70**, Exposé XXII, Corollaire 1.1.10], wherein Katz attributes it to Lang. Indeed, it is a special case of the nonabelian Artin–Schreier theory associated with an algebraic group over a field of positive characteristic (in our case GL_n) via the *Lang torsor*; see [**153**]. It is also the basis for the theory of (ϕ, Γ)-modules, which we will mention briefly in Chapter 24. (See also Section 19.1.)

Suppose that k is a perfect field of characteristic $p > 0$ and that C_k is a complete discretely valued field with residue field k and maximal ideal generated by p (e.g., $k = \mathbb{F}_p$ and $C_k = \mathbb{Z}_p$). It was originally noticed by Teichmüller that each element $x \in k$ has a unique lift $[x] \in C_k$ admitting p^nth roots for all n; this is called the *Teichmüller lift* of x. (These were first considered for unramified extensions of $\mathbb{Z}_p$, in which context we have already seen them, in Chapter 0.) One can use Teichmüller lifts as digits to form a canonical base-p expansion of any element of C_k; the arithmetic operations can be expressed using certain polynomials in the p-power roots of these digits. These polynomials were used by Witt to prove that C_k exists, is unique, and is functorial in k. More generally, Witt gave a functor for each p accepting arbitrary rings (the functor of *p-typical Witt vectors*) and associating k as above to C_k; he also gave a natural functor through which all the functors, for different choices of p, factorize (the functor of *big Witt vectors*). These have applications far beyond their origins, in fields as diverse as arithmetic geometry, algebraic topology, and combinatorics. See [**110**] for a comprehensive summary.

In the special case of the difference field $\mathrm{Frac}(W(k))$, with k perfect and of characteristic $p > 0$ and ϕ equal to the unique lift of the absolute p-power Frobenius morphism, a number of the results in this chapter appear (in marginally less generality) in [**120**], such as the following.

- Corollary 14.5.4 reproduces Mazur's [**120**, Theorem 1.4.1].
- Theorem 14.5.5 is [**120**, Theorem 1.6.1].
- Proposition 14.5.8 reproduces [**120**, Corollary 1.4.4] without requiring nonnegative Hodge slopes (as Katz does in his "basic slope estimate" [**120**, 1.4.3]).
- Proposition 14.5.9 reproduces (and slightly generalizes) one case of [**120**, Theorem 2.6.1].

For the original classification of rational Dieudonné modules over an algebraically closed field, see Manin's original paper [**166**] or the book of Demazure [**71**]. We do not have a prior reference for Theorem 14.6.3, but nevertheless we do not believe it to be original.

An equal-characteristic analogue of Remark 14.6.5 is to take $F = k((z))$ for k an algebraically closed field of characteristic $p > 0$, with ϕ acting as the Frobenius morphism on k and trivially on z. This special case of Corollary 14.6.4 is due to Laumon [**155**, Theorem 2.4.5].

As in Chapter 4, one can interpret what has been done here as the special case for GL_n of a construction for any reductive algebraic group. This point of view was originally introduced by Kottwitz [**148, 149**], but a full development of the analogy is the subject of ongoing work of Kottwitz [**150**] and Csima [**64**].

Exercises

(1) For M_1, M_2 difference modules over a difference ring R, with M_1 dualizable, give a canonical identification of $H^1(M_1^\vee \otimes M_2)$ with the Yoneda extension group $\mathrm{Ext}(M_1, M_2)$. (Hint: as in Lemma 5.3.3 you may wish to first reduce to the case $M_1 = R$.)

(2) Prove Lemma 14.4.3. (Hint: Lemma 14.4.2 may be helpful, particularly for (c) and (d).)

(3) Let F be a difference field (of arbitrary characteristic) containing an element x such that $\phi(x) = \lambda x$ for some λ fixed by ϕ which is not a root of unity. Prove that every finite difference module for M admits a cyclic vector. (Hint: imitate the proof of Theorem 5.4.2. At the key step, instead of getting a polynomial in s you should get a polynomial in λ^s; the root-of-unity condition forces λ^s to take infinitely many values.)

(4) Let F be the completion of $\mathbb{Q}_p(t)$ for the 1-Gauss norm, viewed as a difference field for ϕ equal to the substitution $t \mapsto t^p$. Let V be the difference module corresponding to the matrix

$$A = \begin{pmatrix} 1 & t \\ 0 & p \end{pmatrix}.$$

Prove that there is a nonsplit short exact sequence $0 \to V_1 \to V \to V_2 \to 0$ with V_1, V_2 pure and of norms s_1, s_2 with $s_1 < s_2$.

(5) Here is a beautiful example from [**120**, §1.3] (attributed to B. Gross). Let p be a prime congruent to 3 modulo 4, put $F = \mathbb{Q}_p(i)$ with $i^2 = -1$, and let ϕ be the automorphism $i \mapsto -i$ of F over $\mathbb{Q}_p$. Define a difference module V of rank 2 over F, using the matrix

$$A = \begin{pmatrix} 1-p & (p+1)i \\ (p+1)i & p-1 \end{pmatrix}.$$

Compute the Newton polygons of A and V and verify that they do not coincide. (Hint: find another basis of V, on which Φ acts diagonally.)

(6) (a) Prove that every difference field can be embedded into a strongly difference-closed field. (Hint: use your favorite equivalent of the axiom of choice, e.g., Zorn's lemma or transfinite induction.)

(b) Prove that every complete isometric discretely valued difference field can be embedded into a complete isometric difference field with the same value group but having a strongly difference-closed residue field.

(7) Prove that Theorem 14.6.3 continues to hold if the hypothesis that F is discretely valued is replaced by one of the following hypotheses.

(a) There exists a constant $\epsilon > 0$ with the following property: for $c \in \{0, 1\}$ and $x \in \mathfrak{o}_F$ there exists $y \in \mathfrak{o}_F$ with $|\phi(y) - cy - x| < \epsilon|x|$. (Hint: use ϵ to replace the π-adically convergent sequences in the proof of Theorem 14.6.3.)

(b) The field F is spherically complete. (Hint: attempt to form convergent sequences as in (a) and then invoke spherical completeness if the sequences fail to converge.)

(8) This exercise is related to the construction of Witt vectors. Let k, ℓ be perfect fields of characteristic $p > 0$. Suppose that C_k, C_ℓ are complete discrete valuation rings, having residue fields identified with k, ℓ, respectively, whose maximal ideals are generated by p. Prove that any homomorphism $k \to \ell$ lifts uniquely to a homomorphism $C_k \to C_\ell$. (Hint: read about Teichmüller lifts in the notes.)

15
Frobenius modules

Having introduced the general formalism of difference algebra and made a more careful study over a complete nonarchimedean field, we specialize to the sort of power series rings over which we studied differential algebra. We have seen most of these rings before, but we will encounter a couple of new variations, notably the *Robba ring*. This ring consists of power series convergent on some annulus of outer radius 1 but with unspecified inner radius, which may vary with the choice of power series. This may seem to be a strange construction at first, but it is rather natural from the point of view of difference algebra: the endomorphisms that we will consider (Frobenius lifts) do not preserve the region of convergence of an individual series but do act on the Robba ring as a whole.

This chapter serves mostly to set the definitions and notation for what follows. One nontrivial result here is the behavior of the Newton polygon under specialization. Remember that Hypothesis 14.0.1 is still in force, so the field K will always be discretely valued.

15.1 A multitude of rings

One can talk about Frobenius structures on a variety of rings; for convenience, we review here the definitions of some rings introduced in Chapter 8 and then add some special notation that will be useful later.

Remark 15.1.1. The following rings were defined in Chapter 8:

$$K\langle \alpha/t, t/\beta\rangle = \left\{ \sum_{i\in\mathbb{Z}} c_i t^i : c_i \in K, \lim_{i\to-\infty} |c_i|\alpha^i = 0, \lim_{i\to+\infty} |c_i|\beta^i = 0 \right\},$$

$$K[\![t]\!]_0 = \left\{ \sum_{i=0}^{\infty} c_i t^i : c_i \in K, \sup_i\{|c_i|\} < \infty \right\},$$

$$K\{t\} = \left\{ \sum_{i=0}^{\infty} c_i t^i : c_i \in K, \lim_{i \to \infty} |c_i| \rho^i = 0 \quad (\rho \in (0,1)) \right\},$$

$$K\langle \alpha/t, t \rrbracket_0 = \left\{ \sum_{i \in \mathbb{Z}} c_i t^i : c_i \in K, \lim_{i \to -\infty} |c_i| \alpha^i = 0, \sup_i \{|c_i|\} < \infty \right\},$$

$$K\langle \alpha/t, t\} = \left\{ \sum_{i \in \mathbb{Z}} c_i t^i : c_i \in K, \lim_{i \to -\infty} |c_i| \alpha^i = 0, \lim_{i \to +\infty} |c_i| \rho^i = 0 \quad (\rho \in (0,1)) \right\}.$$

Definition 15.1.2. For later use, we give a special notation to certain rings appearing in this framework. We have already defined $\mathcal{E}$ to be the completion of $\mathfrak{o}_K((t)) \otimes_{\mathfrak{o}_K} K = K\llbracket t \rrbracket_0[t^{-1}]$ for the 1-Gauss norm; that is, $\mathcal{E}$ consists of formal sums $\sum_i c_i t^i$ which have bounded coefficients and satisfy $|c_i| \to 0$ as $i \to -\infty$. Since we are assuming that K is discretely valued, $\mathcal{E}$ is a complete nonarchimedean field with residue field $\kappa_K((t))$. We next set

$$\mathcal{E}^\dagger = \bigcup_{\alpha \in (0,1)} K\langle \alpha/t, t \rrbracket_0;$$

that is, $\mathcal{E}^\dagger$ consists of formal sums $\sum c_i t^i$ which have bounded coefficients and converge in some range $\alpha \leq |t| < 1$. This ring is sometimes called the *bounded Robba ring*, since it consists of the bounded elements of the Robba ring (see Definition 15.1.4 below). Note that it can also be written as

$$\mathcal{E}^\dagger = \bigcup_{\alpha \in (0,1)} K\llbracket \alpha/t, t \rrbracket_0.$$

Lemma 15.1.3.

(a) *The ring $\mathcal{E}^\dagger$ is a field.*

(b) *Under the 1-Gauss norm $|\cdot|_1$, the valuation ring $\mathfrak{o}_{\mathcal{E}^\dagger}$ is a local ring with maximal ideal $\mathfrak{m}_K \mathfrak{o}_{\mathcal{E}^\dagger}$.*

(c) *The field $\mathcal{E}^\dagger$, equipped with $|\cdot|_1$, is henselian (see Remark 3.0.2).*

This last property implies that finite separable extensions of $\kappa_{\mathcal{E}^\dagger} = \kappa_K((t))$ lift functorially to finite étale extensions of $\mathfrak{o}_{\mathcal{E}^\dagger}$ (and to unramified extensions of $\mathcal{E}^\dagger$). In particular, the maximal unramified extension $\mathcal{E}^{\dagger,\mathrm{unr}}$ carries an action of the absolute Galois group of $\kappa_K((t))$.

Proof. To check (a), note that the Newton polygon of any nonzero $x \in \mathcal{E}^\dagger$ has finite width (since K is discretely valued). We can thus choose α such that $x \in K\langle \alpha/t, t \rrbracket_0$ has no slope between 0 and $-\log \alpha$. In this case x is a unit in $K\langle \alpha/t, t \rrbracket_0$ by Lemma 8.2.6(c), yielding (a). We may deduce (b) as an immediate corollary of (a).

To prove (c) it suffices to check the criterion of Remark 3.0.2: for any monic polynomial $P(x) \in \mathfrak{o}_{\mathcal{E}^\dagger}[x]$ and any simple root $\overline{r} \in \kappa_{\mathcal{E}^\dagger}$ of $\overline{P} \in \kappa_{\mathcal{E}^\dagger}[x]$, there exists a unique root $r \in \mathfrak{o}_{\mathcal{E}^\dagger}$ of P lifting $\overline{r}$. In fact, there exists a unique such root $r \in \mathfrak{o}_{\mathcal{E}}$ by Hensel's lemma (see Remark 2.2.3), so it suffices to produce such a root in $\mathfrak{o}_{\mathcal{E}^\dagger}$.

We may rescale P by a unit in $\mathfrak{o}_{\mathcal{E}^\dagger}$ to force $\overline{P}'(\overline{r}) = 1$. Choose $r_0 \in \mathfrak{o}_{\mathcal{E}^\dagger}$ lifting $\overline{r}$. We can then choose $\alpha \in (0,1)$ such that $K\langle \alpha/t, t]\!]_0$ contains both r_0 and the coefficients of P and, in addition, $|P(r_0)|_\alpha, |1 - P'(r_0)|_\alpha < 1$. Since $|P(r_0)|_1, |1 - P'(r_0)|_1 < 1$, this implies that $|P(r_0)|_\rho, |1 - P'(r_0)|_\rho < 1$ for $\rho \in [\alpha, 1)$ by Proposition 8.2.3(c).

We may now finish using a standard application of Newton's method. Given r_0 as defined above, for $i \geq 0$ set $r_{i+1} = r_i - P(r_i)/P'(r_i)$. We find by induction on i (exercise) that $|P(r_{i+1})|_\rho \leq |P(r_i)|_\rho^2$ and $|1 - P'(r_i)|_\rho < 1$ for $\rho \in [\alpha, 1)$. Consequently, the r_i form a Cauchy sequence under $|\cdot|_\rho$ for each $\rho \in [\alpha, 1)$; since $K\langle \alpha/t, t]\!]_0$ is Fréchet complete for these norms (Proposition 8.2.5), the sequence has a limit r of the desired form. This yields (c). □

We next confect another novel but important ring.

Definition 15.1.4. Put

$$\mathcal{R} = \bigcup_{\alpha \in (0,1)} K\langle \alpha/t, t\};$$

that is, $\mathcal{R}$ consists of formal sums $\sum_i c_i t^i$ which converge in some range $\alpha \leq |t| < 1$ but need not be bounded. The ring $\mathcal{R}$ is commonly known as the *Robba ring* having coefficients in K.

Remark 15.1.5. Be aware that, since $\mathcal{R}$ consists of series with unbounded coefficients, the 1-Gauss norm $|\cdot|_1$ is not defined on the whole of $\mathcal{R}$. We will, conventionally, write $|x|_1 = +\infty$ if $x \in \mathcal{R}$ has unbounded coefficients. A related issue is that $\mathcal{R}$ is not a field; indeed, it is not even noetherian, by a similar argument as that for $K\{t\}$ (see the exercises for Chapter 8). We will return to this and related ring-theoretic issues concerning $\mathcal{R}$ later (Section 16.2).

15.2 Frobenius lifts

Next, we equip these rings with a particular sort of endomorphism.

Definition 15.2.1. Let q be a power of p. Let R be one of the following rings:

- $K\langle t\rangle$, $K[\![t]\!]_0$, or $K\{t\}$;
- the union of $K\langle \alpha/t, t\rangle$, $K\langle \alpha/t, t]\!]_0$, or $K\langle \alpha/t, t\}$ over all $\alpha \in (0,1)$;
- F_1, the completion of $K(t)$ for the 1-Gauss norm;
- $\mathcal{E}$, the completion of $K[\![t]\!]_0[t^{-1}]$ for the 1-Gauss norm.

By a *q-power Frobenius lift* on R we will mean a map $\phi : R \to R$ of the form

$$\sum_i c_i t^i \mapsto \sum_i \phi_K(c_i) u^i,$$

where:

- the map $\phi_K : K \to K$ is an isometry;
- the element $u \in R$ satisfies $|u - t^q|_1 < 1$. (If $R = \mathcal{R}$, this forces $u \in \mathcal{E}^\dagger$.)

(These are not the most inclusive conditions possible; see the notes.) The most important case is when ϕ is *absolute*, i.e., when ϕ_K provides the q-power Frobenius lift on κ_K.

Remark 15.2.2. Note that, unless ϕ is absolute, the property that it is a Frobenius lift depends on the implicit choice of the series parameter t; that is, ϕ is not invariant under isometric automorphisms of R.

Remark 15.2.3. Note that, in Definition 15.2.1, one cannot define a Frobenius lift on an individual ring like $K\langle \alpha/t, t]\!]_0$; for instance, the simple substitution $t \to t^q$ carries $K\langle \alpha/t, t]\!]_0$ to $K\langle \alpha^{1/q}/t, t]\!]_0$. One can make this remark more quantitative, as in the following lemma.

Lemma 15.2.4. *Let ϕ be a Frobenius lift on $\mathcal{E}^\dagger$. Then there exists $\epsilon \in (0,1)$ such that, for $\beta, \gamma \in [\epsilon, 1)$ with $\beta \leq \gamma$, ϕ carries $K\langle \beta/t, t/\gamma\rangle$ to $K\langle \beta^{1/q}/t, t/\gamma^{1/q}\rangle$ and*

$$|f|_\beta = |\phi(f)|_{\beta^{1/q}}.$$

Proof. Since $|\phi(t)t^{-q} - 1|_1 < 1$, by continuity we can choose $\epsilon \in (0,1)$ and $\eta \in (0,1)$ so that $|\phi(t)t^{-q} - 1|_{\rho^{1/q}} \leq \eta$ for $\rho \in [\epsilon, 1]$. This inequality implies that $|\phi(t^i)t^{-qi} - 1|_{\rho^{1/q}} \leq \eta$ for all $i \in \mathbb{Z}$. For such an ϵ, the claim is easily verified: if $f = \sum_i f_i t^i$ then, for $\beta \in [\epsilon, 1)$,

$$|f|_\beta = \left|\sum_i \phi_K(f_i) t^{qi}\right|_{\beta^{1/q}} \geq \eta \left|\sum_i \phi_K(f_i)(t^{qi} - \phi(t^i))\right|_{\beta^{1/q}}.$$

□

Definition 15.2.5. Let ϕ be a Frobenius lift on $K[\![t]\!]_0$. We say that ϕ is *centered* if there exists $\lambda \in \mathfrak{m}_K$ such that

$$\phi(t - \lambda) \equiv 0 \pmod{t - \lambda}.$$

We call such a λ a *center* of ϕ; it follows from Lemma 15.2.6 below that λ is unique if it exists. Furthermore, it always exists if ϕ_K is inversive (but not always otherwise; see the exercises). We say that ϕ is *zero-centered* if its center is equal to 0, i.e., if $\phi(t) \equiv 0 \pmod t$.

Lemma 15.2.6. *Suppose that ϕ is a Frobenius lift on $K[\![t]\!]_0$ and that $\phi_K : K \to K$ is inversive. Then ϕ is centered and its center is unique.*

Proof. Exercise. □

15.3 Generic versus special Frobenius lifts

For difference modules over $K[\![t]\!]_0$ there are two natural Newton polygons, and there is an important relationship between them.

Definition 15.3.1. Let M be a finite free difference module over $K[\![t]\!]_0$, for a centered Frobenius lift ϕ. Define the *generic Newton polygon* of M to be the Newton polygon of $M \otimes \mathcal{E}$. Define the *special Newton polygon* of M to be the Newton polygon of $M/(t-\lambda)M$, for λ the center of ϕ.

The following result is sometimes called the *semicontinuity theorem* for Newton polygons.

Theorem 15.3.2 (Grothendieck, Katz). *Let M be a finite free difference module over $K[\![t]\!]_0$, for a centered Frobenius lift ϕ. Then the special Newton polygon lies on or above the generic Newton polygon with the same endpoints.*

Proof. Choose a basis of M, and use it to define supremum norms on $M \otimes \mathcal{E}$ and $M/(t-\lambda)M$. Then it is evident that, for any positive integer n, the Hodge polygon of Φ^n acting on $M/(t-\lambda)M$ lies on or above the Hodge polygon of Φ^n acting on $M \otimes \mathcal{E}$ with the same endpoints. If we divide all slopes by n and take limits as $n \to \infty$ then Proposition 14.5.8 implies that the generic or special Hodge slopes converge to the generic or special Newton slopes. □

As in the comparison of Hodge and Newton polygons, one obtains a decomposition result in the case when the special and generic Newton polygons touch somewhere.

Theorem 15.3.3. *Let M be a finite free difference module of rank n over $K[\![t]\!]_0$, for a centered Frobenius lift ϕ. Let $s_{g,1} \leq \cdots \leq s_{g,n}$ and $s_{s,1} \leq \cdots \leq s_{s,n}$ be the generic and special Newton slopes, respectively. Suppose that, for some $i \in \{1, \dots, n-1\}$, we have*

$$s_{s,i} < s_{s,i+1}, \qquad s_{g,1} + \cdots + s_{g,i} = s_{s,1} + \cdots + s_{s,i}.$$

Then there exists a difference submodule N of M, with M/N free, whose generic and special Newton slopes are $s_{g,1}, \dots, s_{g,i}$ and $s_{s,1}, \dots, s_{s,i}$, respectively. Moreover, if $s_{g,i} < s_{g,i+1}$ then N is unique.

We will relax the hypothesis of uniqueness later (Corollary 16.4.7).

Proof. We may assume that ϕ is zero-centered. Uniqueness in the case $s_{g,i} < s_{g,i+1}$ follows from the uniqueness in $M \otimes \mathcal{E}$, as in Theorem 14.4.15. For the existence, we first replace ϕ by a suitable power to ensure that all the slopes are in the additive value group of K; we then apply Proposition 14.4.16 followed by Lemma 8.6.1 to change basis in M, in order to ensure that the generic Hodge slopes of M are also equal to $s_{g,1}, \dots, s_{g,n}$.

If $s_{s,H,1}, \dots, s_{s,H,n}$ denote the special Hodge slopes in this basis then we have

$$s_{s,1} + \cdots + s_{s,i} \geq s_{s,H,1} + \cdots + s_{s,H,i}$$

by Corollary 14.5.4, but also

$$s_{s,H,1} + \cdots + s_{s,H,i} \geq s_{g,1} + \cdots + s_{g,i}$$

as in the proof of Theorem 15.3.2 (since $s_{g,1}, \dots, s_{g,i}$ match the generic Hodge slopes of M). Consequently, $s_{s,1} + \cdots + s_{s,i} = s_{s,H,1} + \cdots + s_{s,H,i}$; that is, for this basis, the condition of Theorem 14.5.5 is also satisfied by M/tM.

We can thus change the basis over $\mathfrak{o}_K[\![t]\!]$ to obtain a new basis of M on which the action of Φ is via the block matrix

$$A_0 = \begin{pmatrix} B_0 & C_0 \\ D_0 & E_0 \end{pmatrix},$$

in which, modulo t, B_0 accounts for the first i Hodge and Newton slopes of M/tM, E_0 accounts for the remaining Hodge and Newton slopes of M/tM, and D_0 vanishes. In particular, $\det(B_0 \pmod{t})$ has the valuation $s_{s,1} + \cdots + s_{s,i}$. The valuation of $\det(B_0)$ itself cannot be any greater, but it must be at least the sum of the first i generic Hodge slopes of M, which we also know to be $s_{s,1} + \cdots + s_{s,i}$. Consequently $\det(B_0)$ is a unit in $K[\![t]\!]_0$, and it has minimal valuation among all $i \times i$ minors of A_0. Then, by Cramer's rule (as in the proof of Theorem 4.3.11), $D_0 B_0^{-1}$ and $B_0^{-1} C_0$ have entries in $\mathfrak{o}_K[\![t]\!]$. Note also that the least Hodge slope of B_0^{-1} is $-s_{g,i}$, since $\det(B_0)$ is a unit and the entries of $\det(B_0) B_0^{-1}$ are the cofactors of B_0.

Conjugating by the block lower triangular unipotent matrix U_0 with off-diagonal block $D_0 B_0^{-1}$, we obtain

$$A_1 = U_0^{-1} A_0 \phi(U_0) = \begin{pmatrix} B_0 + C_0\phi(D_0 B_0^{-1}) & C_0 \\ E_0\phi(D_0 B_0^{-1}) - D_0 B_0^{-1} C_0 \phi(D_0 B_0^{-1}) & E_0 - D_0 B_0^{-1} C_0 \end{pmatrix}.$$

Since $D_0 B_0^{-1}$ has entries in $t\mathfrak{o}_K[\![t]\!]$, we have $A_1 \equiv A_0 \pmod{t}$. We can thus iterate the construction to obtain a sequence of block matrices

$$A_l = \begin{pmatrix} B_l & C_l \\ D_l & E_l \end{pmatrix} \qquad (l = 0, 1, \dots)$$

for which $A_{l+1} = U_l^{-1} A_l \phi(U_l)$, with $U_l \in \mathrm{GL}_n(\mathfrak{o}_K[\![t]\!])$ equal to the block lower triangular unipotent matrix with off-diagonal block $D_l B_l^{-1}$. Note that $B_{l+1} = B_l(I + B_l^{-1} C_l \phi(D_l B_l^{-1}))$, in which $B_l^{-1} C_l \phi(D_l B_l^{-1})$ has entries in $(t, \mathfrak{m}_K)$, so the Hodge slopes of B_l are preserved. In particular, the least Hodge slope of B_l^{-1} equals $s_{g,i}$.

To complete the proof we must show that the U_l converge to the identity for the $(t, \mathfrak{m}_K)$-adic topology. Since the least Hodge slope of B_l^{-1} equals $s_{g,i}$, it is enough to check that the D_l converge to zero for the same topology. Moreover, it suffices to check that, for each positive integer m, $D_l \pmod{t^m}$ converges to zero for the $\mathfrak{m}_K$-adic topology.

We do this by induction on m, the case $m = 1$ being clear because $D_l \equiv 0 \pmod{t}$. Assume that the convergence is known modulo t^m. Write

$$D_{l+1} = E_l \phi(D_l) \phi(B_l^{-1}) - D_l (B_l^{-1} C_l) \phi(D_l B_l^{-1}).$$

Setting $D_l = \sum_h D_{l,h} t^h$, we then have

$$D_{l+1,m} t^m \equiv E_l \phi(D_{l,m} t^m) \phi(B_l^{-1}) - D_{l,m} t^m (B_l^{-1} C_l) \phi(D_l B_l^{-1}) \\ + \cdots \pmod{t^{m+1}}, \tag{15.3.3.1}$$

where the ellipses represent terms depending on $D_{l,0}, \dots, D_{l,m-1}$ which are already known to converge to 0 as $l \to \infty$.

On the one hand the sum of the least Hodge slopes of the reductions of E_l and $\phi(B_l^{-1})$ modulo t equals $s_{g,i+1} - s_{g,i} \geq 0$. Thus the reduction of $E_l \phi(D_{l,m} t^m) \phi(B_l^{-1})$ modulo t^{m+1} has p-adic valuation no less than that of the reduction of $\phi(D_{l,m} t^m)$, which in turn has p-adic valuation strictly greater than that of the reduction of $D_{l,m} t^m$. On the other hand, $B_l^{-1} C_l$ has entries in $\mathfrak{o}_K[\![t]\!]$, and $\phi(D_l B_l^{-1})$ has entries in the ideal $(t, \mathfrak{m}_K)$ since D_l is divisible by t. Thus the reduction of $D_{l,m} t^m (B_l^{-1} C_l) \phi(D_l B_l^{-1})$ modulo t^{m+1} has valuation strictly greater than that of the reduction of $D_{l,m} t^m$. We may conclude that, for any $c > 0$, for l sufficiently large (so that the terms represented by the ellipses in (15.3.3.1) all have norm less than c) we have $|D_{l+1,m}| < \max\{|D_{l,m}|, c\}$. This yields the desired convergence. $\square$

Theorem 15.3.4. *Let M be a finite free difference module of rank n over $K[\![t]\!]_0$, for ϕ a zero-centered Frobenius lift. Suppose that the generic and special Frobenius slopes of M are all equal to a single value r. Then there is a canonical isomorphism $M \cong (M/tM) \otimes_K K[\![t]\!]_0$ of differential modules.*

Proof. First suppose that $r = 0$. By Lemma 8.6.1 we can choose a basis for which the generic Hodge slopes are all equal to 0. Let A be the matrix of action of Φ on this basis. We wish to construct an $n \times n$ matrix $U = \sum_{i=0}^{\infty} U_i t^i$ over $\mathfrak{o}_K[\![t]\!]$, with $U_0 = I_n$, such that $U^{-1} A \phi(U) = A_0$ or, equivalently, $U = A\phi(U)A_0^{-1}$. Since the map $U \mapsto A\phi(U)A_0^{-1}$ is contractive for the $(t, \mathfrak{m}_K)$-adic topology on $I_n + tM_{n\times n}(\mathfrak{o}_K[\![t]\!])$, it has a unique fixed point, by the contraction mapping theorem, which gives the desired isomorphism.

If $r \in v(K^\times)$, we may apply the above argument after twisting by a scalar. Otherwise we may replace ϕ by a power and then twist and apply the above argument. □

15.4 A reverse filtration

Since $\mathcal{E}^\dagger$ is not complete, we cannot apply Theorem 14.4.15 to filter a finite difference module over $\mathcal{E}^\dagger$ with pure quotients of decreasing norms. It was originally observed by de Jong that one can get a filtration with pure quotients of *increasing* norms, but at the expense of replacing $\mathcal{E}^\dagger$ with its ϕ-perfection.

Definition 15.4.1. Let ϕ be a Frobenius lift on $\mathcal{E}^\dagger$. Let $\mathcal{E}_\phi$ denote the ϕ-perfection of $\mathcal{E}$, that is, the completion of the direct limit $R_0 \to R_1 \to \cdots$, with $R_i = \mathcal{E}$ and the transition map $R_i \to R_{i+1}$ being ϕ. Choose ϵ as in Lemma 15.2.4. For $\alpha \in [\epsilon, 1)$ we may define $|\cdot|_\alpha$ on $\mathcal{E}_\phi$ as the function $|\cdot|_{\alpha^{1/q^i}}$ on R_i; this is consistent because Lemma 15.2.4 guarantees that the transition maps are isometries. Let $\mathcal{E}_\phi^\dagger$ be the subring of $x \in \mathcal{E}_\phi$ such that, for some $\alpha \in [\epsilon, 1)$ depending on x, $|x|_\beta < +\infty$ for $\beta \in [\alpha, 1]$. Note that, within $\mathcal{E}_\phi$,

$$\mathcal{E} \cap \mathcal{E}_\phi^\dagger = \mathcal{E}^\dagger.$$

We obtain the following analogue of Lemma 15.1.3.

Lemma 15.4.2.

(a) *The ring $\mathcal{E}_\phi^\dagger$ is a field.*

(b) *Under the 1-Gauss norm $|\cdot|_1$, the valuation ring $\mathfrak{o}_{\mathcal{E}_\phi^\dagger}$ is a local ring with maximal ideal $\mathfrak{m}_K \mathfrak{o}_{\mathcal{E}_\phi^\dagger}$.*

(c) *The field $\mathcal{E}_\phi^\dagger$ equipped with $|\cdot|_1$ is henselian.*

Proof. Exercise. □

Example 15.4.3. Suppose that ϕ_K is the identity and $\phi(t) = t^q$. Then $\mathcal{E}_\phi$ may be identified with the set of formal sums $\sum_{i \in \mathbb{Z}[1/p]} c_i t^i$ having the property that, for each $\epsilon > 0$, the set of $i \in \mathbb{Z}[1/p]$ for which $|c_i| \geq \epsilon$ is both bounded below and has bounded denominators. In this interpretation, $\mathcal{E}_\phi^\dagger$ consists of those sums $\sum_i c_i t^i$ for which there exists $\alpha \in (0, 1)$ for which $|c_i|\alpha^i \to 0$ as $i \to -\infty$. (Equivalently, there exists $\alpha \in (0, 1)$ for which $|c_i|\alpha^i$ remains bounded as $i \to -\infty$.)

Theorem 15.4.4. *Let V be a finite free dualizable difference module over $\mathcal{E}_\phi^\dagger$. Then there exists a unique filtration*

$$0 = V_0 \subset V_1 \subset \cdots \subset V_l = V$$

of difference modules, such that each successive quotient V_i/V_{i-1} is pure and of some norm s_i and $s_1 < \cdots < s_l$.

Proof. By Theorem 14.4.15 we obtain a decomposition of $V' = V \otimes_{\mathcal{E}_\phi^\dagger} \mathcal{E}_\phi$ as a direct sum $\oplus_s V'_s$ of difference modules in which each nonzero V'_s is pure of norm s. By replacing ϕ with a power, we can ensure that each s for which $V'_s \neq 0$ appears in $|F^\times|$. It suffices to check that the summand of V' of least norm descends to a submodule of V, as we may then repeat the argument after quotienting by this submodule.

By twisting and then approximating a suitable basis of V', we obtain a basis of V on which the matrix of action of Φ^{-1} is a block matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

in which A is invertible over $\mathfrak{o}_{\mathcal{E}_\phi^\dagger}$ and B, C, D have entries in $\mathfrak{m}_K \mathfrak{o}_{\mathcal{E}_\phi^\dagger}$. Put $X = CA^{-1}$ and then change basis using the block lower triangular unipotent matrix with lower left block X, to obtain the new matrix of action

$$\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} = \begin{pmatrix} A + B\phi^{-1}(X) & B \\ (D - XB)\phi^{-1}(X) & D - XB \end{pmatrix}.$$

Repeating this operation yields a p-adically convergent sequence of changes of basis; in the limit, we get a block upper triangular matrix in which the first block corresponds to the summand of V' of least norm. To prove that this summand descends to a submodule of V, it suffices to check that the change-of-basis matrices are bounded in norm by 1 under $|\cdot|_\alpha$ for some $\alpha \in (0, 1)$.

At the first stage, for α sufficiently close to 1 we have

$$|A^{-1}|_\alpha \max\{|B|_\alpha, |C|_\alpha, |D|_\alpha\} < 1,$$

which implies that $|X|_\alpha < 1$. Since we also have $|X|_1 < 1$, by Proposition 8.2.3(c) we have $|X|_{\alpha^{1/q}} < 1$; by Lemma 15.2.4, $|\phi^{-1}(X)|_\alpha = |X|_{\alpha^{1/q}} < 1$. Hence

$$|A^{-1}A' - I_n|_\alpha < 1,$$
$$\max\{|B'|_\alpha, |C'|_\alpha, |D|_\alpha\} \le \max\{|B|_\alpha, |C|_\alpha, |D|_\alpha\}.$$

We may thus use the same α for the next stage, so $|X|_\alpha < 1$ at all stages. □

In order to use the reverse filtration later, we will need the following projection construction.

Lemma 15.4.5. *There exists a $\mathcal{E}$-linear map $\lambda : \mathcal{E}_\phi \to \mathcal{E}$ sending $\mathcal{E}_\phi^\dagger$ to $\mathcal{E}^\dagger$, such that $\lambda(x) = x$ for all $x \in \mathcal{E}$.*

Proof. Suppose first that κ_K is inversive. In this case, the residue field of $\mathcal{E}_\phi$ can be written as $\cup_{i=0}^\infty \kappa_K((t^{1/q^i}))$. Each element of this union can be written uniquely as the finite sum $\sum_{i=0}^\infty \phi^{-1}(x_i)$, with $x_i \in \kappa_K((t))$ such that the coefficient of t^j in x_i vanishes whenever $i > 0$ and j is divisible by q.

This implies by an easy induction that each $x \in \mathcal{E}_\phi$ can be written uniquely as a convergent (for the $\mathfrak{m}_K$-adic topology) sum $\sum_{i=0}^\infty \phi^{-i}(x_i)$, with $x_i \in \mathcal{E}$ such that the coefficient of t^j in x_i vanishes whenever $i > 0$ and j is divisible by q. In this presentation, for ϵ as in Lemma 15.2.4 we have $|\phi^{-i}(x_i)|_\alpha \le |x|_\alpha$ for $\alpha \in [\epsilon, 1)$ (exercise). We may thus take $\lambda(x) = x_0$.

In the general case, let K' be the ϕ-perfection of K and put $\tilde{\mathcal{E}} = K'\langle 1/t, t]\!]_0$. (That is, $\tilde{\mathcal{E}}$ is the analogue of $\mathcal{E}$ with base field K' instead of K.) Argue as above to construct a map $\mathcal{E}_\phi \to \tilde{\mathcal{E}}$. Then choose any continuous linear map $K' \to K$ whose composition with the inclusion $K \to K'$ is the identity, and use it to define a projection $\tilde{\mathcal{E}} \to \mathcal{E}$. The composition $\mathcal{E}_\phi \to \tilde{\mathcal{E}} \to \mathcal{E}$ has the desired effect. □

Corollary 15.4.6. *The multiplication map $\mu : \mathcal{E} \otimes_{\mathcal{E}^\dagger} \mathcal{E}_\phi^\dagger \to \mathcal{E}_\phi$ is injective.*

Proof. Exercise. □

Notes

Much existing literature makes the restriction that Frobenius lifts must be absolute. The generalization that we have considered here is relevant for some applications (e.g., to families of Galois representations in p-adic Hodge theory), so it is prudent to allow it as much as possible.

However, we have not opted to consider cases where ϕ does not carry K into K. (At that level of generality, one might consider that our definition restricts to *scalar-preserving* Frobenius lifts.) Also, we have not allowed K to be nondiscrete; this avoids having to worry about whether to allow Frobenius lifts ϕ for which $\phi(t) - t^q$ has 1-Gauss norm 1 but each of its coefficients has norm less than 1. (This cannot occur for K discrete.)

As in the previous chapter, a number of the results are based on [**120**], at least when restricted to an absolute Frobenius lift.

- Theorem 15.3.2 is a local formulation of a geometric result of Grothendieck. The proof given is from [**120**, Theorem 2.3.1].
- Theorem 15.3.3 is a variant of [**120**, Theorem 2.4.2].
- Theorem 15.3.4 is an adaptation of [**120**, Theorem 2.7.1].

For stronger global results along these lines (in the theory of *F-crystals*), see [**67**], [**226**], and [**176**].

Theorem 15.4.4 in the case of an absolute Frobenius lift is due to de Jong [**65**, Proposition 5.8]. Corollary 15.4.6 and its proof are a variant on [**65**, Proposition 8.1].

Exercises

(1) Verify the claims about Newton iteration from the proof of Lemma 15.1.3(c).

(2) Suppose that κ_K is perfect and that $\phi : \mathcal{E} \to \mathcal{E}$ is a continuous isometric endomorphism inducing the absolute q-power Frobenius lift on $\kappa_K((t))$. Prove that ϕ is an absolute Frobenius lift in the sense of Definition 15.2.1. (Hint: use Witt vector functoriality to show that ϕ carries K into K. In other words, for $x \in K$, show that $\phi(x^{p^n}) = \phi(x)^{p^n}$ is p-adically close to an element of K.)

(3) Prove Lemma 15.2.6. (Hint: show that the map $\lambda \mapsto \phi_K^{-1}(\phi(t)|_{t=\lambda})$ on $\mathfrak{m}_K$ is contractive; here $\phi(t)|_{t=\lambda}$ denotes the substitution $t \mapsto \lambda$.)

(4) Let K be the completion of $\mathbb{Q}_p(x)$ for the 1-Gauss norm. Prove that the Frobenius lift ϕ defined by $\phi(t) = t^p + px$ does not have a center.

(5) Prove Lemma 15.4.2.

(6) Check the omitted details in the proof of Lemma 15.4.5. (Hint: you can in fact show that $|x|_\alpha = \sup_i\{|\phi^{-1}(x_i)|_\alpha\}$. To check this, it suffices to consider x in the dense subset $\cup_{m=0}^{\infty}\phi^{-m}(\mathcal{E}^\dagger)$ of $\mathcal{E}^\dagger_\phi$.)

(7) Prove Corollary 15.4.6. (Hint: if the claim fails, we can choose a nonzero element $z = \sum_{i=1}^n x_i \otimes y_i$ of the kernel with n as small as possible. Now use Lemma 15.4.5 to construct an element with an even shorter representation, as in the proof of Lemma 1.3.11. A related argument in Galois theory is Artin's proof of the linear independence of characters.)

16

Frobenius modules over the Robba ring

In Chapter 14 we discussed some structure theory for finite difference modules over a complete isometric nonarchimedean difference field. This theory can be applied to the field $\mathcal{E}$, which is the p-adic completion of the bounded Robba ring $\mathcal{E}^\dagger$; however, the information it gives is somewhat limited.

For the purposes of studying Frobenius structures on differential modules (see Part V), it would be useful to have a structure theory over $\mathcal{E}^\dagger$ itself. This is a bit too much to ask for; what we can provide is a structure theory that applies over the Robba ring $\mathcal{R}$, which is somewhat analogous to what we obtain over $\mathcal{E}$. In particular, with an appropriate definition of pure modules, we obtain a slope filtration theorem analogous to Theorem 14.4.15 but valid over $\mathcal{R}$.

Given a difference module over $\mathcal{E}^\dagger$, one obtains slope filtrations and Newton polygons over both $\mathcal{E}$ and $\mathcal{R}$. For a module over $K[\![t]\!]_0$ these turn out to match the generic and special Newton polygons, and so in particular they need not coincide. However, they do admit a specialization property analogous to Theorem 15.3.2.

Unfortunately, a proof of the slope filtration theorem over $\mathcal{R}$ would take us rather far afield, so we do not include one here. Instead, we limit ourselves to a brief overview of the proof and consign further discussion and references to the notes.

Hypothesis 16.0.1. Throughout this chapter, let ϕ be a Frobenius lift on the Robba ring $\mathcal{R}$. (For a possible relaxation of this hypothesis, see the notes.) All difference modules over $\mathcal{R}$ will be taken with respect to ϕ unless otherwise specified.

16.1 Frobenius modules on open discs

We start with the fact that, over the ring $K\{t\}$, the classification of finite free difference modules reduces (mostly) to classification over K.

Theorem 16.1.1. *Suppose that κ_K is strongly difference-closed (e.g., ϕ is an absolute Frobenius lift and κ_K is algebraically closed). Let M be a finite free dualizable difference module over $K\{t\}$ for a zero-centered Frobenius lift. Then there exists a noncanonical isomorphism of difference modules $M \cong (M/tM) \otimes_K K\{t\}$.*

Proof. Let c be the reduction of $\phi(t)/t$ modulo t, so that $c \in \mathfrak{m}_K$. Let $e_1, \dots, e_n$ be a basis for M. Let A be the matrix of action of Φ on this basis, which is invertible because M is dualizable. Let A_0 be the constant term of A, which is an invertible matrix over K. We wish to find a matrix U over $K\{t\}$ with $U \equiv I_n \pmod{t}$ such that $U^{-1}A\phi(U) = A_0$.

We first construct a sequence of matrices $U_0, U_1, U_2, \dots$ over $K[\![t]\!]$, with $U_0 = I_n$ and $U_{i+1} = U_i(I_n + X_i t^i)$ for some matrix X_i over K such that $U_i^{-1}A\phi(U_i)A_0^{-1} \equiv I_n \pmod{t^{i+1}}$. Namely, given U_i, we find X_i by solving the equation

$$c^i A_0\phi(X_i)A_0^{-1} - X_i + t^{-i}(U_i^{-1}A\phi(U_i)A_0^{-1} - I_n) \equiv 0 \pmod{t}, \quad (16.1.1.1)$$

which amounts to trivializing a class in $H^1(V)$ for some finite difference module V over K. This is possible by Corollary 14.6.6, although it can not necessarily be done uniquely. However, let us choose $h \geq 0$ such that $|c^h||A_0||A_0^{-1}| < 1$. Then, for $i \geq h$ and for any choice of X_i, $|c^i A_0\phi(X_i)A_0^{-1}| < |X_i|$, so that X_i is uniquely determined by (16.1.1.1); moreover, it has the same norm as the reduction of $t^{-i}U_i^{-1}A\phi(U_i)A_0^{-1} - I_n$ modulo t.

Choose α so that, for $i = h$,

$$|U_i - I_n|_\alpha, |U_i^{-1} - I_n|_\alpha, |U_i^{-1}A\phi(U_i)A_0^{-1} - I_n|_\alpha < 1. \qquad (16.1.1.2)$$

Then, by the previous paragraph, $|X_i t^i|_\alpha \leq |U_i^{-1}A\phi(U_i)A_0^{-1} - I_n|_\alpha$. By our choice of h, $|A_0\phi(X_i t^i)A_0^{-1}|_\alpha < |X_i t^i|_\alpha$; since

$$U_{i+1}^{-1}A\phi(U_{i+1})A_0^{-1} = (I_n + X_i t^i)^{-1}U_i^{-1}A\phi(U_i)A_0^{-1}(I_n + A_0\phi(X_i t^i)A_0^{-1}),$$

we may conclude that (16.1.1.2) holds also with i replaced by $i + 1$. By induction, we see that (16.1.1.2) holds for all $i \geq h$.

This means that, for any $\beta < \alpha$, the matrices U and U^{-1} have entries in $K\langle t/\beta \rangle$. Using the relation $U = A\phi(U)A_0^{-1}$, we may conclude that U and U^{-1} also have entries in $K\langle t/\beta' \rangle$, for β' equal to the minimum norm of a root of $\phi(t) - c$ over all $c \in K^{\mathrm{alg}}$ of norm β. Since ϕ is zero-centered, by considering the Newton polygon of $\phi(t) - c$ we see that $\beta' \geq \min\{\beta^{1/q}, \beta/|\pi|\}$ for π a generator of $\mathfrak{m}_K$. Iterating the function $\beta \mapsto \min\{\beta^{1/q}, \beta/|\pi|\}$, we see that U and U^{-1} have entries in $K\langle t/\beta \rangle$ for all $\beta \in (0, 1)$, proving the claim. $\square$

Remark 16.1.2. We will see a similar result later in the presence of a differential structure. This is introduced as Dwork's trick (Corollary 17.2.2). The main differences are that in the setting of Dwork's trick there will be a *canonical* isomorphism $M \cong (M/tM) \otimes_K K\{t\}$, and no restriction on K is needed.

Remark 16.1.3. Note that the function $\beta \mapsto \min\{\beta^{1/q}, \beta/|\pi|\}$ becomes exactly the function $\lambda \mapsto \min\{\lambda^{1/p}, p\lambda\}$ from Chapter 10 in the case where $q = p$ and K is absolutely unramified. See also Theorem 17.2.1.

One can apply Theorem 16.1.1 to coherent locally free modules on the open unit disc, by virtue of the following observation. (Compare Definition 8.4.3, remembering that in this part of the book K is assumed to be discretely valued.)

Proposition 16.1.4. *Any coherent locally free module on the open unit disc over K is freely generated by finitely many global sections.*

Proof. Let M be such a module. Choose a sequence $0 < \beta_1 < \beta_2 < \cdots$ with limit 1, and put $M_i = M \otimes K\langle t/\beta_i\rangle$. Choose any basis B_1 of M_1. Given a basis B_i of M_i, there must exist $X_i \in \mathrm{GL}_n(K\langle t/\beta_i\rangle)$ such that changing basis from B_i via X_i produces a basis of M_{i+1}. By Lemma 8.3.4 we can factor X_i as $V_i W_i$, with $V_i \in \mathrm{GL}_n(K\langle t/\beta_i\rangle)$ such that $|V_i - I_n|_{\beta_i} < 1$ and $W_i \in \mathrm{GL}_n(K[t])$. Write $V_i = \sum_{j=0}^{\infty} V_{i,j} t^j$, let Y_i be the sum of $V_{i,j} t^j$ over only those j for which $|V_{i,j}|_1 \leq 1$, and put $U_i = V_i Y_i^{-1}$. Changing basis from B_i via U_i also produces a basis B_{i+1} of M_{i+1}.

Since K is discretely valued, for each positive integer j and for i sufficiently large (depending on j), we have $|V_{i,j} t^j|_1 \leq 1$ if and only if $|V_{i,j} t^j|_{\beta_i} \leq 1$. Consequently, the U_i converge t-adically to the identity matrix. Given any $\beta \in (0, 1)$, if we choose $\gamma \in (\beta, 1)$ and combine the previous observation with the fact that the U_i are bounded under $|\cdot|_\gamma$, we may deduce that the U_i converge under $|\cdot|_\beta$ to the identity matrix. Hence the product $U_1 U_2 \cdots$ is the change-of-basis matrix from B_1 to a simultaneous basis of each M_i, as desired. $\square$

16.2 More on the Robba ring

To discuss the classification of difference modules on annuli, we must recall some properties of the Robba ring beyond simply its definition.

Remark 16.2.1. Recall (Definition 15.1.4) that we have defined the Robba ring to be

$$\mathcal{R} = \cup_{\alpha \in (0,1)} K\langle \alpha/t, t\};$$

that is, $\mathcal{R}$ consists of formal sums $\sum_i c_i t^i$ that converge in some range $\alpha \leq |t| < 1$ but need not have bounded coefficients. Unlike its subring $\mathcal{E}^\dagger$, $\mathcal{R}$ is not a field; for instance the element

$$\log(1+t) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} t^i$$

is not invertible (because its Newton polygon has infinitely many slopes). More generally, we have the following easy fact.

Lemma 16.2.2. *We have* $\mathcal{R}^\times = (\mathcal{E}^\dagger)^\times = \mathcal{E}^\dagger \setminus \{0\}$.

Proof. On the one hand, a unit in $K\langle \alpha/t, t\}$ must have empty Newton polygon, so it must belong to $\mathcal{E}^\dagger$. On the other hand, $\mathcal{E}^\dagger$ is a field by Lemma 15.1.3(a). This proves the claim. □

Definition 16.2.3. Because $\mathcal{R}$ consists of series with possibly unbounded coefficients, it does not carry a natural p-adic norm or topology. The most useful topology on $\mathcal{R}$ is the *LF topology*, which is the direct limit of the Fréchet topology on each $K\langle \alpha/t, t\}$ defined by the $|\cdot|_\rho$ for $\rho \in [\alpha, 1)$. That is, a sequence converges if on the one hand it is contained in some $K\langle \alpha/t, t\}$ and on the other hand it converges for the Fréchet topology on that ring. Note that it does not matter which α is chosen since, for $\gamma > \alpha$, the inclusion $K\langle \alpha/t, t\} \to K\langle \gamma/t, t\}$ is strict (i.e., it is a homeomorphism to its image equipped with the subspace topology).

In fact, the ring $\mathcal{R}$ is not even noetherian (by an argument similar to that for $K\{t\}$; see the exercises for Chapter 8), but the following useful facts, mentioned in the notes, are true.

Proposition 16.2.4. *For an ideal I of $\mathcal{R}$, the following are equivalent.*

(a) *The ideal I is closed in the LF topology.*
(b) *The ideal I is finitely generated.*
(c) *The ideal I is principal.*

Remark 16.2.5. The equivalence of (b) and (c) in Proposition 16.2.4 implies that $\mathcal{R}$ is a *Bézout domain*, i.e., an integral domain in which every finitely generated ideal is principal. Such rings enjoy many properties analogous to principal ideal domains; see the exercises.

We also have the following analogue of Proposition 16.1.4.

Proposition 16.2.6. *Any coherent locally free module on the half-open annulus with closed inner radius α and open outer radius* 1 *is represented by a* *finite free module over $K\langle \alpha/t, t\}$ and so corresponds to a finite free module over $\mathcal{R}$.*

Proof. Let M be such a module. Then $M \otimes K\langle \alpha/t, t/\alpha\rangle$ is finite free since $K\langle \alpha/t, t/\alpha\rangle$ is a principal ideal domain by Proposition 8.3.2. By choosing a basis of $M \otimes K\langle \alpha/t, t/\alpha\rangle$ and then invoking Lemma 8.3.6, we may extend M to a coherent locally free module on the whole unit disc. We may then deduce the claim from Proposition 16.1.4. □

16.3 Pure difference modules

Over a complete nonarchimedean difference field such as $\mathcal{E}^\dagger$, we already have a notion of a pure difference module. Since $\mathcal{R}$ does not come with a Frobenius-invariant norm we cannot use the same definition. The appropriate definition in this case is as follows.

Definition 16.3.1. A finite free difference module M over $\mathcal{R}$ is *pure of norm s* if there exists a finite free difference module $M^\dagger$ over $\mathcal{E}^\dagger$ which is pure of norm s (in the sense of Definition 14.4.6) and such that $M \cong M^\dagger \otimes_{\mathcal{E}^\dagger} \mathcal{R}$. We will see shortly (Corollary 16.3.7) that the module $M^\dagger$ is uniquely determined by this requirement.

Lemma 16.3.2. *Let A be an $n \times n$ matrix over $\mathfrak{o}_{\mathcal{E}^\dagger}$. Then the map $v \mapsto v - A\phi(v)$ induces a bijection on $(\mathcal{R}/\mathcal{E}^\dagger)^n$.*

Proof. Exercise, or see [**136**, Proposition 1.2.6]. □

Corollary 16.3.3. *Let M be a finite free difference module over $\mathcal{E}^\dagger$ such that $|\Phi|_{\mathrm{sp},M\otimes\mathcal{E}} \leq 1$. Then the map $H^1(M) \to H^1(M \otimes_{\mathcal{E}^\dagger} \mathcal{R})$ is bijective.*

Proof. This will follow from Lemma 16.3.2 once we produce a basis of M on which the matrix of action of Φ has entries in $\mathfrak{o}_{\mathcal{E}^\dagger}$. Such a basis exists, as explained in the proof of Proposition 14.5.9. □

Corollary 16.3.4. *Let $M_1^\dagger, M_2^\dagger$ be two finite free dualizable difference modules over $\mathcal{E}^\dagger$. Suppose that every Newton slope of $M_1^\dagger \otimes_{\mathcal{E}^\dagger} \mathcal{E}$ is less than or equal to every Newton slope of $M_2^\dagger \otimes_{\mathcal{E}^\dagger} \mathcal{E}$. Then*

$$\mathrm{Hom}(M_1^\dagger, M_2^\dagger) = \mathrm{Hom}(M_1^\dagger \otimes_{\mathcal{E}^\dagger} \mathcal{R},\ M_2^\dagger \otimes_{\mathcal{E}^\dagger} \mathcal{R}).$$

Proof. Note that $(M_1^\dagger)^\vee \otimes M_2^\dagger$ has norm less than or equal to 1 (because its Newton slopes are all nonnegative). It thus suffices to check that, for $M^\dagger$ of norm $s \leq 1$, $H^0(M^\dagger) = H^0(M^\dagger \otimes_{\mathcal{E}^\dagger} \mathcal{R})$. As in the previous proof, we can find a basis $e_1, \ldots, e_n$ of $M^\dagger$ such that the matrix A of action of Φ has entries in $\mathfrak{o}_{\mathcal{E}^\dagger}$. Any element of $H^0(M^\dagger \otimes_{\mathcal{E}^\dagger} \mathcal{R})$, when written in terms of the basis, corresponds to a column vector $v \in \mathcal{R}^n$ satisfying $A\phi(v) = v$. By Lemma 16.3.2 this forces $v \in (\mathcal{E}^\dagger)^n$, so v corresponds to an element of M itself. □

Corollary 16.3.5. *Let M_1, M_2 be finite free difference modules over $\mathcal{R}$ that are pure of norms s_1, s_2, respectively. If $s_1 > s_2 > 0$ then* $\mathrm{Hom}(M_1, M_2) = 0$.

Proof. By Corollary 16.3.4 it suffices to check that if $M_1^\dagger, M_2^\dagger$ are finite free difference modules over $\mathcal{E}^\dagger$ that are pure of norms s_1, s_2, respectively, and $s_1 > s_2$ then $\mathrm{Hom}(M_1, M_2) = 0$. This holds because the image of any morphism $M_1^\dagger \to M_2^\dagger$, if nonzero, would have to be both pure of norm s_1 and pure of norm s_2, which is impossible. □

Remark 16.3.6. In the proof of Corollary 16.3.5 we used the fact that, when working over a difference field, purity of the middle term in a short exact sequence $0 \to M_1 \to M \to M_2 \to 0$ implies purity (of the same norm) at the extremes. This is not true over $\mathcal{R}$, which means that one cannot deduce Corollary 16.3.5 in the case $s_1 < s_2$. For example, if ϕ is a Frobenius lift for which $\phi(1+t) = (1+t)^p$, and $M = \mathcal{R}v$ with $\Phi(v) = p^{-1}v$, then

$$\log(1+t)v \in H^0(M);$$

this constitutes a key example in p-adic Hodge theory (see Chapter 24).

Corollary 16.3.7. *Let M be a finite free difference module over $\mathcal{R}$ that is pure of norm $s > 0$. Then there is a unique finite free difference module $M^\dagger$ over $\mathcal{E}^\dagger$, which is pure of norm s, such that $M \cong M^\dagger \otimes_{\mathcal{E}^\dagger} \mathcal{R}$.*

Remark 16.3.8. Even if M is pure of norm s, the uniqueness in Corollary 16.3.7 can fail if we do not require $M^\dagger$ to be pure.

Proposition 16.3.9. *Let $M_1^\dagger, M_2^\dagger$ be finite free difference modules over $\mathcal{E}^\dagger$ that are pure of the same norm $s > 0$. Then any finite free difference module M over $\mathcal{R}$ fitting into a short exact sequence of the form*

$$0 \to M_1^\dagger \otimes_{\mathcal{E}^\dagger} \mathcal{R} \to M \to M_2^\dagger \otimes_{\mathcal{E}^\dagger} \mathcal{R} \to 0$$

is also pure of norm s.

Proof. It is equivalent to check that

$$\mathrm{Ext}(M_1^\dagger, M_2^\dagger) = \mathrm{Ext}(M_1^\dagger \otimes_{\mathcal{E}^\dagger} \mathcal{R},\ M_2^\dagger \otimes_{\mathcal{E}^\dagger} \mathcal{R}).$$

If we set $M^\dagger = (M_1^\dagger)^\vee \otimes M_2^\dagger$, it is also equivalent to check that the natural map

$$H^1(M^\dagger) \to H^1(M^\dagger \otimes_{\mathcal{E}^\dagger} \mathcal{R})$$

is a bijection (see Definition 14.1.6); this holds by Corollary 16.3.3. □

16.4 The slope filtration theorem

The fundamental theorem in the theory of difference modules over the Robba ring is the following result, which is analogous to Theorem 14.4.15. We do not give a complete proof in this book, but we will sketch the argument in the next section. (As noted in Remark 14.4.7, this theorem is usually stated in additive rather than multiplicative terms; this explains our choice of the name "slope filtration theorem" rather than "norm filtration theorem".)

Theorem 16.4.1 (Slope filtration theorem). *Let M be a finite free dualizable difference module over $\mathcal{R}$. Then there exists a unique filtration $0 = M_0 \subset \cdots \subset M_l = M$ by dualizable difference submodules with the following properties.*

(a) *Each successive quotient M_i/M_{i-1} is finite free and is pure of some norm s_i (in the sense of Definition 16.3.1).*

(b) *We have $s_1 > \cdots > s_l$.*

In this book the main application of the slope filtration will be to the p-adic local monodromy theorem for differential modules over $\mathcal{R}$ (Theorem 20.1.4). For the moment, let us record one or two additional corollaries. (For further applications, see the notes.)

Definition 16.4.2. Let M be a finite free dualizable difference module over $\mathcal{R}$. Set the notation as in Theorem 16.4.1. Define the *Newton polygon* of M to be the polygon associated with the multiset containing $-\log s_i$ with multiplicity $\mathrm{rank}(M_i/M_{i-1})$.

By the compatibility of purity of difference modules over $\mathcal{E}^\dagger$ with tensor products (Corollary 14.4.9) and duals (Proposition 14.4.8), we obtain the usual behavior of Newton slopes under tensor products, exterior powers, and duals.

Lemma 16.4.3. *Let M, N be finite free dualizable difference modules over $\mathcal{R}$. Let $s_{M,1}, \ldots, s_{M,m}$ and $s_{N,1}, \ldots, s_{N,n}$ be the Newton slopes of M and N, respectively.*

(a) *The Newton slopes of $M \otimes N$ are $s_{M,i} + s_{N,j}$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$.*

(b) *For $j = 1, \ldots, \mathrm{rank}(M)$, the Newton slopes of $\wedge^j M$ are $s_{M,i_1} + \cdots + s_{M,i_j}$ for $1 \leq i_1 < \cdots < i_j \leq m$.*

(c) *The Newton slopes of $M^\vee$ are $-s_{M,1}, \ldots, -s_{M,m}$.*

We have the following generalization of Corollary 16.3.5.

Proposition 16.4.4. *Let M_1, M_2 be finite free dualizable difference modules over $\mathcal{R}$. Suppose that every Newton slope of M_1 is less than every Newton slope of M_2. Then* $\mathrm{Hom}(M_1, M_2) = 0$.

Proof. We induct on $\mathrm{rank}(M_1) + \mathrm{rank}(M_2)$. If M_1 and M_2 are both pure then Corollary 16.3.5 yields the claim. If M_1 is not pure, by Theorem 16.4.1 there exists a proper nonzero difference submodule N of M_1 such that N and M_1/N are both finite free, and we deduce that $\mathrm{Hom}(M_1, M_2) = 0$ from the fact that $\mathrm{Hom}(N, M_2) = \mathrm{Hom}(M_1/N, M_2) = 0$. (Namely, any morphism from M_1 to M_2 must vanish on N, yielding a morphism from M_1/N to M_2 that also vanishes.) We argue similarly if M_2 fails to be pure. $\square$

We now mention a result that extends the semicontinuity theorem for Newton polygons (Theorem 15.3.2). See Example 20.2.1 for an explicit example.

Definition 16.4.5. For M a finite free dualizable difference module over $\mathcal{E}^\dagger$, we can construct two different Newton polygons, namely those associated with $M \otimes_{\mathcal{E}^\dagger} \mathcal{E}$ and $M \otimes_{\mathcal{E}^\dagger} \mathcal{R}$. We call these the *generic Newton polygon* and the *special Newton polygon*, respectively. The terminology is justified by the fact that if κ_K is strongly difference-closed, ϕ is induced by a zero-centered Frobenius lift on $K[\![t]\!]_0$, and M is obtained by base extension from a finite free dualizable difference module M_0 over $K[\![t]\!]_0$, then by Theorem 16.1.1 the special Newton polygon of M coincides with the special Newton polygon of M_0.

Theorem 16.4.6. *Let M be a finite free dualizable difference module over $\mathcal{E}^\dagger$. Then the special Newton polygon lies on or above the generic Newton polygon with the same endpoints.*

Proof. It suffices to show that the least special slope of M is greater than or equal to the least generic slope, as then applying this inequality to the exterior powers of M yields the comparison of Newton polygons by Lemmas 14.5.3 and 16.4.3.

Suppose on the contrary that the least special slope is less than the least generic slope. Let M_1 be the first step in the filtration of $M \otimes_{\mathcal{E}^\dagger} \mathcal{R}$ given by Theorem 14.4.15. Since M_1 is pure, it descends to a difference module N_1 over $\mathcal{E}^\dagger$ of the same norm. By hypothesis, the generic slopes of $N_1^\vee \otimes M$ are all positive. It follows that $H^0(N_1^\vee \otimes M) = 0$ are and then by Corollary 16.3.4 that $H^0((N_1^\vee \otimes M) \otimes_{\mathcal{E}^\dagger} \mathcal{R}) = 0$ also. However, the latter contradicts the fact that N_1 is a submodule of $M \otimes_{\mathcal{E}^\dagger} \mathcal{R}$. We thus deduce the claim. $\square$

We can similarly use slope filtrations to finish the uniqueness part of Theorem 15.3.3.

Corollary 16.4.7. *With the notation of Theorem 15.3.3, the submodule N is always unique.*

Proof. It suffices to check that $N \otimes_{K[\![t]\!]_0} \mathcal{R}$ is uniquely determined; this follows because it coincides with one step of the slope filtration of $M \otimes_{K[\![t]\!]_0} \mathcal{R}$. $\square$

Remark 16.4.8. Note that in Corollary 16.4.7 we need only the uniqueness of the slope filtration in Theorem 16.4.1. The existence, which lies much deeper, can be supplied using Theorem 16.1.1 because we start with a difference module over $K[\![t]\!]_0$.

16.5 Proof of the slope filtration theorem

The proof of Theorem 16.4.1 is unfortunately in some respects orthogonal to much of the material presented in this book. We will thus limit ourselves to an indication of the key steps. See the notes for further discussion.

Definition 16.5.1. Let M be a finite free dualizable difference module over $\mathcal{R}$ of rank $n > 0$. Define the *average norm* of M to be

$$\mu(M) = |\Phi|^{1/n}_{\mathrm{sp}, \wedge^n M};$$

this quantity is multiplicative in the sense that $\mu(M_1 \otimes M_2) = \mu(M_1)\mu(M_2)$. We say M is *semistable* if we have $\mu(M) \geq \mu(N)$ for any nonzero difference submodule N of M.

Lemma 16.5.2. *Let M be a nonzero finite free dualizable difference module over $\mathcal{R}$ that is pure of some norm. Then M is semistable.*

Proof. Suppose on the contrary that there exists a difference submodule N of M with $\mu(N) > \mu(M)$. Put $e = \mathrm{rank}(N)$; then $\wedge^e N$ is of rank 1 and hence pure. Consequently, the inclusion $\wedge^e N \to \wedge^e M$ violates Corollary 16.3.5; this contradiction yields the claim. □

Lemma 16.5.3. *Let M be a nonzero finite free dualizable difference module over $\mathcal{R}$. There exists a unique filtration $0 = M_0 \subset \cdots \subset M_l = M$ by difference submodules, such that each quotient M_i/M_{i-1} is free and semistable and*

$$\mu(M_1/M_0) > \cdots > \mu(M_l/M_{l-1}).$$

Proof. First note that the claim holds for any module of rank 1 by Lemma 16.5.2. From this point, the rest of the proof is formal; we leave it as an exercise. □

Definition 16.5.4. The filtration in Lemma 16.5.3 is called the *Harder–Narasimhan* or *HN filtration* of the module M. To prove Theorem 16.4.1 it will suffice to check that semistable modules over $\mathcal{R}$ are pure.

Remark 16.5.5. The reader may wonder why it is necessary to prove Theorem 16.4.1 if defining the HN filtration is a mere formality. The answer is that one cannot prove much about the norms occurring in the HN filtration; for instance, it has not been formally proved that the tensor product of semistable modules is semistable but this is indeed a consequence of Theorem 16.4.1. A similar situation arises in the theory of vector bundles, on which the usage of the word "semistable" is modeled; see the discussion preceding [**136**, Theorem 1.7.1]. Another loosely analogous situation is Deligne's use of *determinantal weights* in his second proof of the Weil conjectures [**69**]; there one has matrices over a p-adic field whose eigenvalues are supposed to be algebraic numbers with certain archimedean norms, but one has direct control only over the determinants.

We need to introduce something like a residual difference closure of the ring $\mathcal{R}$. (This characterization is justified by Corollary 16.5.8.) This construction, based on Mal'cev-Neumann series (Example 1.5.8), is called the *extended Robba ring* in [**136**].

Definition 16.5.6. Let $\tilde{\mathcal{R}}$ be the ring of formal sums $x = \sum_{i \in \mathbb{Q}} x_i t^i$ with $x_i \in K$ that satisfy the following properties. (Here α is a number in $(0, 1)$ which may vary with x.)

(a) For $\rho \in (\alpha, 1)$ and we have $\lim_{i \to \pm\infty} |x_i|\rho^i = 0$.

(b) For $\rho \in (\alpha, 1)$ and for $c > 0$, the set of indices i for which $|x_i|\rho^i \geq c$ is a well-ordered subset of $\mathbb{R}$ (i.e., it contains no infinite decreasing subsequence).

We may also construct $\tilde{\mathcal{R}}$ as follows. Recall that the field $K((t^{\mathbb{Q}}))$ of Mal'cev-Neumann series over K consists of formal sums $\sum_{i \in \mathbb{Q}} x_i t^i$ for which $\{i \in \mathbb{Q} : x_i \neq 0\}$ is well-ordered. On the subring of $K((t^{\mathbb{Q}}))$ consisting of series with bounded coefficients, for $\rho \in (0, 1)$ define the Gauss norm

$$\left| \sum_{i \in \mathbb{Q}} x_i t^i \right|_\rho = \sup_i \{|x_i|\rho^i\}.$$

For each $\alpha \in (0, 1)$ take the Fréchet completion for the norms $|\cdot|_\rho$, for $\rho \in (\alpha, 1)$; the union of these completions over all α gives $\tilde{\mathcal{R}}$.

Like the usual Robba ring, the ring $\tilde{\mathcal{R}}$ turns out to be a Bézout domain (compare Proposition 16.2.4). Let $\tilde{\mathcal{E}}^\dagger$ be the subring of $\tilde{\mathcal{R}}$ consisting of formal sums with bounded coefficients. By imitating the proofs for $\mathcal{E}^\dagger$ (Lemma 15.1.3) it can be shown that $\tilde{\mathcal{E}}^\dagger$ is a henselian discretely valued field with residue field $\kappa_K((t^{\mathbb{Q}}))$. Various definitions (e.g., purity, semistability) carry over from $\mathcal{R}$ to $\tilde{\mathcal{R}}$; we will not write these out explicitly.

We view $\tilde{\mathcal{R}}$ as a difference ring with the Frobenius morphism given by

$$\phi\left(\sum_{i\in\mathbb{Q}} x_i t^i\right) = \sum_{i\in\mathbb{Q}} \phi_K(x_i)t^{qi}.$$

The following lemma is where most of the hard work is concentrated.

Lemma 16.5.7. *Suppose that κ_K is strongly difference-closed. Then every semistable dualizable finite difference module over $\tilde{\mathcal{R}}$ is pure of some norm.*

Sketch of proof. One first constructs (using an explicit calculation) a filtration in which the successive quotients are each pure of some norm, but these norms do not necessarily increase in the right direction as we proceed up the filtration. One then argues (using a second explicit calculation) that if there are two steps in the wrong order that cannot be switched, because the extension between them is not split, then one can change the filtration to move the norms of these steps closer to each other. (Note the similarity to Grothendieck's classification of vector bundles on the projective line.) See [**136**, Theorem 2.1.8] for full details. □

Although we will not need it explicitly, we mention the following analogue of the generalized Dieudonné–Manin classification (Theorem 14.6.3). It is not stated explicitly in [**136**] but is an easy consequence of results from there.

Corollary 16.5.8. *Suppose that κ_K is strongly difference-closed. Then every dualizable finite difference module over $\tilde{\mathcal{R}}$ can be split (non-uniquely) as a direct sum of difference submodules, each of the form $V_{\lambda,d}$ for some λ, d (as in Definition 14.6.1).*

Proof. By [**136**, Proposition 2.1.6], the categories of pure dualizable difference modules of a given norm over K and over $\tilde{\mathcal{R}}$ are equivalent. Thus, by Lemma 16.5.7 and Theorem 14.6.3, it suffices to split any short exact sequence $0 \to M_1 \to M \to M_2 \to 0$ in which M_1, M_2 are base extensions of pure difference modules over K with $\mu(M_1) > \mu(M_2)$. This can be deduced from the proof of [**136**, Proposition 2.1.5] or by a simple direct calculation. □

It remains to descend in our understanding of semistability and purity from $\tilde{\mathcal{R}}$ to $\mathcal{R}$. We must first make the following observation.

Remark 16.5.9. Let L be a complete extension of K as a difference field, such that κ_L is strongly difference-closed. (Such an L can always be constructed; see the exercises for Chapter 14 or [**136**, Proposition 3.2.4].) Let $\tilde{\mathcal{R}}_L$ denote the extended Robba ring constructed using L as the coefficient field. Then there is always an embedding of difference rings $\psi : \mathcal{R} \to \tilde{\mathcal{R}}_L$ that is isometric;

i.e., for some $\alpha \in (0,1)$ and for $\rho \in (\alpha, 1)$ we have $|\psi(x)|_\rho = |x|_\rho$ for all $x \in \mathcal{R}$ (in particular, the two quantities are either both finite or both infinite). This is clear if for instance $\phi(t) = t^q$, as we can then set $\psi(t) = t$; it was verified in general by [**136**, Proposition 2.2.6].

We then obtain the following results; the key tool used is descent for modules along a faithfully flat ring homomorphism. (We also make considerable use of an analogue of the projection $\mathcal{E}_\phi \to \mathcal{E}$ of Lemma 15.4.5.)

Lemma 16.5.10. *Let M be a semistable dualizable finite difference module over $\mathcal{R}$. Then $M \otimes_{\mathcal{R}} \tilde{\mathcal{R}}_L$ is semistable.*

Proof. See [**136**, Theorem 3.1.2]. □

Lemma 16.5.11. *Let M be a dualizable finite difference module over $\mathcal{R}$ such that $M \otimes_{\mathcal{R}} \tilde{\mathcal{R}}_L$ is pure. Then M is pure.*

Proof. See [**136**, Theorem 3.1.3]. □

Putting everything together, we may prove Theorem 16.4.1 as follows.

Proof of Theorem 16.4.1. We wish to show that the HN filtration of M is pure. By Lemma 16.5.10, if we start with the HN filtration of M and tensor with $\tilde{\mathcal{R}}_L$ then the resulting filtration still has semistable successive quotients. Consequently, it must be the HN filtration of $M \otimes_{\mathcal{R}} \tilde{\mathcal{R}}_L$. By Lemma 16.5.7, each successive quotient of the HN filtration of M becomes pure when tensored with $\tilde{\mathcal{R}}_L$. By Lemma 16.5.11 each successive quotient is itself pure. □

Notes

The restriction that ϕ must be a Frobenius lift is probably stronger than necessary. In fact, as suggested in [**136**], if the action of ϕ on $\kappa_K((t))$ satisfies $v(\phi(x)) = mv(x)$ for some integer $m > 1$ this should suffice.

Our proof of Theorem 16.1.1 is essentially that of [**130**, Proposition 4.3], except that in the latter only the absolute case is treated, and it is further assumed that M descends to $K[\![t_0]\!]$.

Proposition 16.2.4 is the essential content of a paper of Lazard [**156**]. Note that it depends on K being spherically complete and is false otherwise; however, we have assumed in this part that K is discretely valued, so we are safe.

The treatment of pure modules over $\mathcal{R}$ is an abridged version of [**136**, §1].

Corollary 16.3.4 has appeared in several guises previously. It figures in the work of Cherbonnier and Colmez [**38**], which we will discuss in

Chapter 24 (see Remark 24.2.6); it also appears in work of Tsuzuki [**206**, Proposition 4.1.1].

The existence of slope filtration for Frobenius modules over the Robba ring was anticipated by Tsuzuki [**209**], who introduced the definition in the case of a Frobenius structure on a differential module [**209**, Definition 5.1.1]. For more on why he did this, see the notes to Chapter 20.

The slope filtration theorem itself (Theorem 16.4.1), in the case of an absolute Frobenius lift with K of characteristic 0, was originally proved by Kedlaya in [**125**, Theorem 6.10]. That proof was significantly more complicated than the one given here. A second proof in the absolute case (again with K of characteristic 0), which also gives an important generalization (see below), was given in [**129**]; this proof introduced the formalism of semistability, inspired by some parallel work of Hartl and Pink [**104**] (on which more below). Our sketch was modeled on a third proof, this time for an arbitrary Frobenius lift, given in [**136**, Theorem 1.7.1]; the latter proof adds several technical simplifications to the previous proofs. One is the direct characterization of pure modules; the previous proofs used a characterization in terms of an appropriate analogue of the Dieudonné–Manin classification (Corollary 16.5.8). Another is the use of faithful flat descent, replacing a more complicated Galois descent argument used in the previous proofs. A third simplification is the use of the extended Robba ring $\tilde{\mathcal{R}}$ as described here. In the previous proofs the role of $\tilde{\mathcal{R}}$ was played by a somewhat smaller ring: its bounded elements (plus 0) form a field whose residue field is the algebraic closure of $\kappa_K((t))$ rather than the much larger field of generalized power series.

The proof of Theorem 16.4.1 in [**129**] includes a nontrivial generalization of the original slope filtration theorem that was not covered by [**136**]. In this generalization, the ring $\mathcal{E}$ is replaced by a Cohen ring (or a ramified extension thereof), not for a field of power series but for a more general complete nonarchimedean field of characteristic p; the rings $\mathcal{E}^\dagger$ and $\mathcal{R}$ are replaced by appropriate analogues. The resulting theorem plays an important role in the study of isocrystals; see the notes for Chapter 23. It also is important for applications to Rapoport–Zink spaces; see below.

In the case of an absolute Frobenius lift, Theorem 16.4.6 becomes [**129**, Proposition 5.5.1]. (This reference is also the source of the terminology of special and generic Newton polygons over $\mathcal{E}^\dagger$). The proof in [**129**] uses the reverse filtration of de Jong (see Theorem 15.4.4).

Besides the p-adic local monodromy theorem (Theorem 20.1.4), several additional applications of the slope filtration theorem have been found. One of these is in p-adic Hodge theory; this is discussed further in the notes to Chapter 24. Another is a q-difference analogue of the p-adic local monodromy theorem, due to André and Di Vizio [**8**]. A third application has been pursued by Hartl in the context of period morphisms for Rapoport–Zink spaces. These are moduli spaces for deformations of certain p-divisible groups into mixed characteristic; the main conjecture of Rapoport and Zink is that a period morphism between this space and a space of linear-algebraic objects can be constructed, via which the Galois representations for to the p-divisible groups correspond to some coherent data. In equal positive characteristic, Hartl [**101**] established the appropriate analogue of this conjecture, using a form of Theorem 16.4.1 in which K is of positive characteristic. (Hartl prove that case of Theorem 16.4.1 directly, building on work of Hartl and Pink [**104**] giving a form of Corollary 16.5.8 for K of characteristic p.) In the original mixed-characteristic setting, Hartl has given some partial results [**102, 103**]; a complementary but related partial result has been established by Faltings (though only the brief research announcement [**86**] was available at the time of writing), and the combination of ideas may lead to a proof of the full Rapoport–Zink conjecture.

Exercises

(1) Show that if ϕ is a Frobenius lift such that $\phi(t) \equiv 0 \pmod{t^2}$ then the conclusion of Theorem 16.1.1 holds without any restriction on K.

(2) Let R be a Bézout domain (an integral domain in which every finitely generated ideal is principal).
   (a) Prove that any $x_1, \dots, x_n \in R$ that generate the unit ideal appear as the first row in some $n \times n$ invertible matrix over R.
   (b) Prove that every finitely generated torsion-free R-module is free.

(3) Prove Lemma 16.3.2. (Hint: reduce to the case where $|A|_\rho \leq 1$ for $\rho \in [\alpha, 1)$. For the injectivity, given that $v - A\phi(v) \in (\mathcal{E}^\dagger)^n$ show that $|v|_\rho$ is bounded for $\rho \in [\alpha, 1)$ by comparing $|v|_\rho$ with $|v|_{\rho^{1/q}}$, using Lemma 15.2.4. For the surjectivity, to find the class of $w \in \mathcal{R}^n$ in the image of the map, separate off the positive terms w_+ of w, replace w with $A\phi(w_+) + (w - w_+)$, and then repeat this procedure.)

(4) Prove that any $A \in \mathrm{GL}_n(\mathcal{R})$ can be factored as UV with $U \in \mathrm{GL}_n(K\{t\})$ and $V \in \mathrm{GL}_n(\mathcal{E}^\dagger)$. (Hint: imitate the proof of Proposition 8.3.5.)

(5) Let M be a finite free difference module over $K\{t\}$ such that $M \otimes_{K\{t\}} \mathcal{R}$ is pure of some norm s. Let $M^\dagger$ be the pure module over $\mathcal{E}^\dagger$ satisfying $M \otimes_{K\{t\}} \mathcal{R} \cong M^\dagger \otimes_{\mathcal{E}^\dagger} \mathcal{R}$. Prove that there exists a basis $e_1, \dots, e_n$ of M that is also a basis of $M^\dagger$. (Hint: imitate the proof of Lemma 8.3.6 using the previous exercise.)

(6) Prove Lemma 16.5.3. (Hint: first check that $\mu(N) \leq \mu(M)$ whenever N is a difference submodule of M of full rank, by comparing the top exterior powers of M and N. See also [**136**, Proposition 1.4.15].)

(7) Let M be a nonzero finite dualizable difference module over $\mathcal{R}$. Let $0 = M_0 \subset \cdots \subset M_l = M$ be any filtration of M by difference submodules, such that each quotient M_i/M_{i-1} is free and semistable. Form the convex polygon of length n associated (as in Definition 4.3.2) with the multiset of slopes given by $-\log \mu(M_i/M_{i-1})$, with multiplicity $\operatorname{rank}(M_i/M_{i-1})$, for $i = 1, \ldots, l$. Prove that this polygon lies on or below the corresponding polygon for the HN filtration with the same endpoint. (Again, this is a purely formal consequence of the definitions of semistability and the HN filtration, so the proof is as in the theory of vector bundles.)

Part V

Frobenius Structures

17

Frobenius structures on differential modules

In this part of the book, we bring together the streams of differential algebra (from Part III) and difference algebra (from Part IV), realizing Dwork's fundamental insight that the study of differential modules on discs and annuli is greatly enhanced by the introduction of Frobenius structures.

This chapter sets the foundations for this study. First, we introduce the notion of a Frobenius structure on a differential module, with some examples. Then we consider the effect of Frobenius structures on the generic radius of convergence and obtain the fact that a differential module on a disc has a full basis of horizontal sections ("Dwork's trick"). We also show that the existence of a Frobenius structure does not depend on the particular choice of Frobenius lift; this independence plays an important role in rigid cohomology (Chapter 23).

Throughout Part V, Hypothesis 14.0.1 remains in force unless explicitly contravened. In particular, K will by default be a *discretely valued* complete nonarchimedean field.

17.1 Frobenius structures

We start with the basic compatibility between differential and difference structures.

Definition 17.1.1. Let R be a ring as in Definition 15.2.1. For M a finite free differential module over R, a *Frobenius structure* on M with respect to a Frobenius lift ϕ on R is an isomorphism $\Phi : \phi^* M \cong M$ of differential modules. In more explicit terms, we must equip M with the structure of a dualizable difference module over (R, ϕ), such that

$$D(\Phi(m)) = \frac{d\phi(t)}{dt}\Phi(D(m)) \qquad (m \in M).$$

In even more explicit terms, if A, N are the matrices of action of Φ, D on some basis then A is invertible, and we have the compatibility

$$NA + \frac{dA}{dt} = \frac{d\phi(t)}{dt} A\phi(N). \qquad (17.1.1.1)$$

Remark 17.1.2. We may also speak of Frobenius structures on finite free differential modules for the derivation td/dt; the analogue of (17.1.1.1) is

$$NA + t\frac{dA}{dt} = \frac{t}{\phi(t)} \frac{d\phi(t)}{dt} A\phi(N). \qquad (17.1.2.1)$$

However, if R is a subring of $K[\![t]\!]$ then (17.1.2.1) only makes sense if $\phi(t) = t^q u$ for $u \in R^\times$, in which case taking constant terms in (17.1.2.1) yields $N_0 A_0 = q u_0 A_0 \phi(N_0)$. This gives $|N_0|_{\mathrm{sp}} = |A_0^{-1} N_0 A_0|_{\mathrm{sp}} = q^{-1}|\phi(N_0)|_{\mathrm{sp}} = q^{-1}|N_0|_{\mathrm{sp}}$, so N_0 must have spectral radius 0 and hence must be nilpotent.

We now describe some examples where Frobenius structures can be constructed explicitly.

Definition 17.1.3. Suppose that $\pi \in K$ satisfies $\pi^{p-1} = -p$. Then the power series $E(t) = \exp(\pi t - \pi t^p)$ (sometimes called the *Dwork exponential series*) has radius of convergence $p^{(p-1)/p^2}$ (exercise), even though the series $\exp(\pi t)$ has radius of convergence 1.

Example 17.1.4 (Dwork). Assume that K contains an element π with $\pi^{p-1} = -p$. Pick any $f \in \mathfrak{o}_{\mathcal{E}^\dagger}$, and let M_f be the differential module over $\mathcal{E}^\dagger$ of rank 1 with a generator v satisfying $D(v) = \pi(df/dt)v$. (Note that this is the pullback along f of Example 9.3.5.) This module is typically nontrivial because the exponential series $\exp(-\pi f)$ does not represent an element of $\mathcal{E}^\dagger$ (e.g., consider $f = t^{-1}$). However, the exponential series gives an important clue about how to associate a Frobenius structure with M_f; namely, for ϕ an absolute Frobenius lift we would like to define

$$\Phi(v) = \exp(\pi f - \pi\phi(f))v.$$

To verify this formula, note that it suffices to do so in the case $\phi(t) = t^q$, since $\exp(\pi(f^q - \phi(f)))$ is already well-defined in $\mathcal{E}^\dagger$. We can then write

$$\exp(\pi f - \pi f^{p^a}) = E(f)E(f^p)\cdots E(f^{p^{a-1}}),$$

where E is the Dwork exponential series from Definition 17.1.3.

Example 17.1.5. We can similarly construct a Frobenius structure for Example 9.9.3. (The case $h = 0$ will reproduce Example 17.1.4; see Remark 17.1.7 below.) For h a nonnegative integer and $\zeta \in K$ a primitive p^{h+1}th root of unity, let M be the differential module of rank 1 over $\mathcal{E}^\dagger$ with the action of D on a generator v given by

$$D(v) = -\sum_{i=0}^{h} (\zeta^{p^i} - 1) t^{-p^i - 1} v.$$

(Note that we have replaced t by t^{-1} in the formula from Example 9.9.3.) Let $\phi : \mathcal{E}^\dagger \to \mathcal{E}^\dagger$ be an absolute Frobenius lift fixing ζ. On the disc $|t^{-1}| < 1$ we have already constructed the horizontal section

$$\frac{E_p(t^{-1})}{E_p(\zeta t^{-1})} v = \exp\left(\sum_{i=0}^{h} \frac{1 - \zeta^{p^i}}{p^i} t^{-p^i} \right) v,$$

where

$$E_p(t) = \exp\left(\sum_{i=0}^{\infty} \frac{t^{p^i}}{p^i} \right)$$

is the Artin–Hasse exponential. This suggests that one should define a Frobenius action fixing this horizontal section, which would then be given by the formula

$$\Phi(v) = \frac{E_p(t^{-1}) E_p(\zeta t^{-p})}{E_p(t^{-p}) E_p(\zeta t^{-1})} v. \tag{17.1.5.1}$$

In fact, this gives a Frobenius structure defined over $\mathcal{E}^\dagger$, because the coefficient of v in (17.1.5.1), as a power series in t^{-1}, has radius of convergence strictly greater than 1. We will not verify this here; it was shown for $p > 2$ by Matsuda [**169**] by an explicit calculation, and for all p by Pulita [**185**] as part of a much broader result. We note here that the existence of this Frobenius structure implies the following strengthening of Theorem 12.7.2.

Theorem 17.1.6. *Let b be a positive integer. Let K be a complete nonarchimedean field (not necessarily discretely valued) containing the p^hth roots of unity for all $h \leq \log_p b$ and having a perfect residue field. Let M denote a finite differential module of rank 1 on a half-open annulus with open outer radius 1, which is solvable at 1 with differential slope b. Then there exist $c_1, \dots, c_b \in \{0\} \cup \mathfrak{o}_K^\times$ and nonnegative integers $j_1, \dots, j_b$ such that*

$$M \otimes M_{1,c_1} \otimes \cdots \otimes M_{b,c_b}$$

has differential slope 0, for M_{i,c_i} defined as in Theorem 12.7.2.

Proof. Since κ_K is perfect, K contains a subfield K_0 isomorphic to the fraction field of the Witt vectors of κ_K. Let ϕ_0 be the Witt vector Frobenius morphism on K_0. Let ψ be the substitution $t \mapsto t^p$, and set $\phi = \psi \circ \phi_0$ as an endomorphism of $\mathcal{E}_0^\dagger = \cup_{\alpha \in (0,1)} K_0\langle \alpha/t, t]\!]_0$.

On the one hand, by Example 17.1.5, for any $c_b \in \mathfrak{o}_{K_0}$ we have $M_{b,c_b} \cong \phi^*(M_{b,c_b}) = \psi^*(M_{b,\phi_0(c_b)})$. On the other hand, for $c_b, d_b \in \mathfrak{o}_K$ with $|c_b - d_b| < 1$, $M_{b,c_b}^\vee \otimes M_{b,d_b}$ is trivial because we can use (9.9.3.1) to write down a horizontal section. Since κ_K is perfect, given $c_b \in \mathfrak{o}_K$ we can choose $c_b' \in \mathfrak{o}_{K_0}$ with $|(c_b')^p - c_b| < 1$, and so $|\phi_0(c_b') - c_b| < 1$. We thus deduce that

$$M_{b,c_b'} \cong \psi^*(M_{b,c_b}).$$

With this fact, we may deduce the claim from Theorem 12.7.2. □

Remark 17.1.7. We will have special need later for the case of Theorem 17.1.6, in which $M^{\otimes p}$ has differential slope 0. In this case, we only deal with objects of the form M_{i,c_i} with i not divisible by p, thanks to Corollary 12.7.3.

This implies first that we only need K to contain the pth roots of unity, not the p^hth roots of unity, for all $h \leq \log_p b$. It also implies that we can use the Frobenius structure from Example 17.1.4 instead of that from Example 17.1.5. Namely, for ζ a primitive pth root of unity, there exists a unique $\pi \in \mathbb{Q}_p(\zeta)$ with $\pi^{p-1} = -p$ and $|\pi - (\zeta_p - 1)| < p^{-1/(p-1)}$ (exercise) and, for this choice, if $c_i \in \mathfrak{o}_K$ and $f = c_i t^{-i}$ then $M_f^\vee \otimes M_{i,c_i}$ has differential slope 0 by an explicit calculation (exercise).

Remark 17.1.8. In addition to the above examples, Dwork managed to construct explicit Frobenius structures in several classical cases, using explicit formal solutions of the corresponding differential equations; see for instance Example 20.2.1. These examples are uniformly explained by the fact that Picard–Fuchs modules carry Frobenius structures in rather broad generality. See Chapter 22 for further discussion.

17.2 Frobenius structures and the generic radius of convergence

One of Dwork's early discoveries was that the presence of a Frobenius structure forces solvability at the boundary. (There is also a converse for modules of rank 1; see the notes.)

Theorem 17.2.1. *Let M be a finite differential module on the half-open annulus with closed inner radius α and open outer radius 1 equipped with a Frobenius structure. Then*

$$\lim_{\rho \to 1^-} IR(M \otimes F_\rho) = 1,$$

that is, M is solvable at 1*. More precisely, for $\rho \in (0, 1)$ sufficiently close to* 1*,*

$$IR(M \otimes F_{\rho^{1/q}}) \geq IR(M \otimes F_\rho)^{1/q}.$$

Proof. By imitating the proof of Lemma 10.3.2 (using Lemma 15.2.4), we may show that, for $\rho \in (0, 1)$ sufficiently close to 1,

$$IR(M \otimes F_{\rho^{1/q}}) \geq \min\{IR(M \otimes F_\rho)^{1/q},\ qIR(M \otimes F_\rho)\}.$$

The function $f(s) = \min\{s^{1/q}, qs\}$ on $(0, 1]$ is strictly increasing, and any sequence of the form $s, f(s), f(f(s)), \dots$ converges to 1 (as in the proof of Theorem 16.1.1). This proves the first claim; for the second claim, note that $f(s) = s^{1/q}$ when s is sufficiently close to 1. □

The following corollary is sometimes called *Dwork's trick*. It may be viewed as a true nonarchimedean analogue of the fundamental theorem of ordinary differential equations.

Corollary 17.2.2 (Dwork). *Let M be a finite differential module on the open unit disc, such that the restriction of M to some half-open annulus with open outer radius* 1 *admits a Frobenius structure. Then M admits a basis of horizontal sections.*

Proof. By Theorem 17.2.1, for each $\lambda < 1$ there exists $\rho \in (\lambda, 1)$ such that $R(M \otimes F_\rho) > \lambda$. By Dwork's transfer theorem (Theorem 9.6.1), $M \otimes K\langle t/\lambda \rangle$ admits a basis of horizontal sections. Taking λ arbitrarily close to 1 yields the claim. □

Remark 17.2.3. The proof of Corollary 17.2.2 admits the following geometric interpretation. By Proposition 9.3.3 the horizontal sections converge on some disc of positive radius ρ. Pulling back by a Frobenius lift gives a new space of horizontal sections on the disc of radius $\min\{\rho^{1/q}, q\rho\}$, but this space must coincide with the original space. Repeating the construction, we can eventually stretch the horizontal sections over the entire open unit disc.

One also has a nilpotent analogue of Dwork's trick by using Theorem 13.7.1 in place of Theorem 9.6.1.

Corollary 17.2.4. *Let M be a finite differential module, on the open unit disc for the derivation $t\,d/dt$ with a nilpotent singularity at $t = 0$, such that the restriction of M to some annulus with open outer radius* 1 *admits a Frobenius structure. Then M has radius of convergence* 1*; that is, for any $\beta < 1$ and for any basis of $M \otimes K\langle t/\beta \rangle$, the fundamental solution matrix for that basis* *has entries in* $K\langle t/\beta\rangle$. *(Note that the nilpotency of the singularity is automatic if the Frobenius lift* ϕ *is of the form described in Remark 17.1.2 and if the Frobenius structure is defined on the entire disc.)*

A nice application of Dwork's trick is the following.

Proposition 17.2.5. *Let* M *be a finite differential module over* $K[\![t]\!]_0$ *with* $R(M) = 1$. *(For instance, this holds if* M *admits a Frobenius structure, by Dwork's trick.) Then* $H^0(M) = H^0(M \otimes \mathcal{E}^\dagger)$.

Proof. By Theorem 9.6.1 there exists a horizontal basis $e_1, \dots, e_n$ of $M \otimes_{K[\![t]\!]_0} K\{t\}$. If $v \in H^0(M \otimes_{K[\![t]\!]_0} \mathcal{E}^\dagger)$ then when we write $v = \sum_{i=1}^n v_i e_i$ with $v_i \in \mathcal{R}$ we must have $d(v_i) = 0$ for $i = 1, \dots, n$. This forces $v_i \in K$ for $i = 1, \dots, n$, and so

$$v \in (M \otimes_{K[\![t]\!]_0} \mathcal{E}^\dagger) \cap (M \otimes_{K[\![t]\!]_0} K\{t\}) = M \otimes_{K[\![t]\!]_0} (\mathcal{E}^\dagger \cap K\{t\}) = M. \quad \square$$

17.3 Independence from the Frobenius lift

Another key property of Frobenius structures is that their existence does not depend on the exact shape of the Frobenius lift.

Proposition 17.3.1. *Let* ϕ_1, ϕ_2 *be two Frobenius lifts on* R *that agree on* K. *Let* M *be a finite free differential module over* R *equipped with a Frobenius structure for* ϕ_1. *Then there is a functorial way to equip* M *with a Frobenius structure for* ϕ_2.

Proof. The Frobenius structure for ϕ_2 is defined by the following Taylor series:

$$\Phi_2(m) = \sum_{i=0}^{\infty} \frac{(\phi_2(t) - \phi_1(t))^i}{i!} \Phi_1(D^i(m)). \tag{17.3.1.1}$$

By Theorem 17.2.1 and the fact that $|\phi_2(t) - \phi_1(t)|_1 < 1$, this series converges under $|\cdot|_\rho$ for $\rho \in (0, 1)$ sufficiently close to 1 (if this is well-defined for R), and also under $|\cdot|_1$ (if this is well-defined for R). $\square$

Corollary 17.3.2. *Let* ϕ_1, ϕ_2 *be two Frobenius lifts on* R *that agree on* K. *Then there is a canonical equivalence between the categories of finite free differential modules over* R *equipped with Frobenius structures with respect to* ϕ_i *for* $i = 1, 2$*; this equivalence is the identity functor on the underlying difference modules.*

Definition 17.3.3. Corollary 17.3.2 allows us to switch from one Frobenius lift on R to another that may be more convenient. One useful choice is what we call the *standard q-power Frobenius lift* for a given choice of ϕ_K, namely the Frobenius lift ϕ for which $\phi(t) = t^q$.

We will also see that changing Frobenius lifts preserves purity.

Lemma 17.3.4. *Let M be a finite free differential module over $\mathcal{E}^\dagger$ equipped with a unit-root Frobenius structure. Let M_0 be a finite free dualizable difference module over $\mathfrak{o}_{\mathcal{E}^\dagger}$ such that M is isomorphic as a difference module to $M_0 \otimes_{\mathfrak{o}_{\mathcal{E}^\dagger}} \mathcal{E}^\dagger$. Then, under any such isomorphism, M_0 is stable under $D^i/i!$ for each nonnegative integer i.*

Proof. We proceed by induction on i. Note that $\mathfrak{o}_{\mathcal{E}^\dagger}$ is stable under $d^i/i!$ for each i; hence, by the Leibniz rule, given the claim for all $j < i$ it suffices to exhibit a single basis of M_0 on which $D^i/i!$ acts via a basis over $\mathfrak{o}_{\mathcal{E}^\dagger}$.

Choose a matrix of M_0, and let A, N be the matrices of action of Φ, D on this basis. If we apply Φ to this basis, the matrices of action of Φ, D on the resulting basis are $\Phi(A)$, $(d\phi(t)/dt)\phi(N)$, respectively. By repeating this construction, for any $\epsilon < 0$ we can find a basis of M_0 on which the matrix of action of D has norm at most ϵ.

In particular, we will deduce the base case $i = 1$. For $i > 1$, use the previous paragraph to choose a basis $e_1, \dots, e_n$ of M_0 such that $(D/i)(e_j) \in M_0$ for $j = 1, \dots, n$ and then invoke the induction hypothesis to deduce that $(D^i/i!)(e_j) = (D^{i-1}/(i-1)!)((D/i)(e_j)) \in M_0$. □

Proposition 17.3.5. *Suppose that $R = \mathcal{E}^\dagger$ or $R = \mathcal{R}$. Let ϕ_1, ϕ_2 be two Frobenius lifts on R that agree on K. Let M be a finite free differential module over R equipped with a Frobenius structure for ϕ_1 that is pure of some norm. Then M is also pure of the same norm with respect to the induced Frobenius structure for ϕ_2.*

Proof. It suffices to check the case $R = \mathcal{E}^\dagger$, because if M is pure over $\mathcal{R}$, and $M^\dagger$ is the unique pure module over $\mathcal{E}^\dagger$ with $M \cong M^\dagger \otimes_{\mathcal{E}^\dagger} \mathcal{R}$, then Φ acts on M by Corollary 16.3.4. We may also reduce to the case where M is pure of norm 1, by replacing ϕ_1, ϕ_2 by powers of themselves and invoking Corollary 14.4.5. By Proposition 14.4.16 we can write $M = M_0 \otimes_{\mathfrak{o}_{\mathcal{E}^\dagger}} \mathcal{E}^\dagger$ for some finite free $\mathfrak{o}_{\mathcal{E}^\dagger}$-module M_0 such that M_0 and $M_0^\vee$ are stable under the action of ϕ_1. By Lemma 17.3.4, M_0 is stable under $D^i/i!$ for each nonnegative integer i. By (17.3.1.1) the Frobenius structure with respect to ϕ_2 carries M_0 into itself, and similarly for $M_0^\vee$. This yields the claim. □

17.4 Slope filtrations and differential structures

In order to apply the slope filtration theorem (Theorem 16.4.1) to differential modules with a Frobenius structure, we must check some compatibilities between slope filtrations and differential modules.

Lemma 17.4.1. *Let M be a finite differential module over $\mathcal{R}$ equipped with a Frobenius structure. Then the steps of the filtration of Theorem 16.4.1 are differential modules, not just difference modules.*

Proof. It suffices to check this for M_1, as then we may quotient by M_1 and repeat the argument. Note that the composition $M_1 \stackrel{D}{\to} M \to M/M_1$ is $\mathcal{R}$-linear, since $D(rv) = rD(v) + d(r)v$ and the second term becomes zero in the quotient. Hence it suffices to check that $M_1 \to M/M_1$ is the zero map. However, M_1 is pure of some norm that is greater than every norm appearing in the slope filtration of M/M_1. Consequently, $\mathrm{Hom}(M_1, M/M_1) = 0$ by Proposition 16.4.4. □

Lemma 17.4.2. *Let M be a finite free unit-root difference module over $\mathcal{E}^\dagger$ such that $M \otimes_{\mathcal{E}^\dagger} \mathcal{R}$ admits a compatible differential structure. Then this structure is induced by a corresponding differential structure on M itself.*

Proof. Let N, A be the matrices via which D and Φ act on a basis of M. Write the commutation relation (17.1.2.1) between N and A in the form $N - (t/\phi(t))(d\phi(t)/dt)A\phi(N)A^{-1} = (d/dt)(A)A^{-1}$. We deduce from Lemma 16.3.2 that N has entries in $\mathcal{E}^\dagger$. □

From Lemmas 17.4.1 and 17.4.2, we deduce the following refinement of the slope filtration theorem in the presence of a differential structure.

Theorem 17.4.3. *Let M be a differential module over $\mathcal{R}$ equipped with a Frobenius structure. Then there exists a unique filtration $0 = M_0 \subset \cdots \subset M_l = M$, by differential submodules preserved by the Frobenius structure, with the following properties.*

(a) *Each successive quotient M_i/M_{i-1} is finite free and descends uniquely to a differential module over $\mathcal{E}^\dagger$ with an induced Frobenius structure that is pure of some norm s_i (in the sense of Definition 16.3.1).*
(b) *We have $s_1 > \cdots > s_l$.*

17.5 Extension of Frobenius structures

The following result allows us to extend certain Frobenius actions. An important application will be to Picard–Fuchs modules (Chapter 22).

Proposition 17.5.1. *Let M be a finite differential module on the open unit disc for the derivation td/dt with a nilpotent singularity at $t = 0$. Assume that either:*

(a) *the Frobenius lift ϕ on $K[\![t]\!]_0$ is arbitrary, and M has no singularity at $t = 0$; or*

(b) *the Frobenius lift ϕ on $K[\![t]\!]_0$ satisfies $\phi(t) = t^q u$ for some $u \in \mathfrak{o}_K[\![t]\!]^\times$.*

Then any Frobenius structure with respect to ϕ on the restriction of M to some annulus with open outer radius 1 is induced by a Frobenius structure on M itself.

Proof. It suffices to check this for a standard Frobenius lift, as then under either (a) or (b) we may switch back and forth between the given Frobenius lift and a standard Frobenius lift using Proposition 17.3.1. Thus we assume hereafter that $\phi(t) = t^q$.

Let Φ be a Frobenius structure with respect to ϕ on the restriction of M to some annulus with open outer radius 1. By Corollary 17.2.4, M admits a basis on which the matrix of action of D is a nilpotent matrix N over K. Let $A = \sum_{i \in \mathbb{Z}} A_i t^i$ be the matrix of action of Φ on the same basis; then the commutation relation (17.1.2.1) between ϕ and D states that $NA + t dA/dt = qA\phi_K(N)$. Consequently, for each $i \in \mathbb{Z}$ we have $NA_i + iA_i = qA_i\phi_K(N)$. By Lemma 7.3.5 the operator $X \mapsto NX + iX - qX\phi_K(N)$ on $n \times n$ matrices over K has all its eigenvalues equal to i, because N and $\phi_K(N)$ are both nilpotent. Hence $A_i = 0$ for $i \neq 0$, so $A \in \mathrm{GL}_n(K)$. In particular the Frobenius structure Φ can be defined on the entire open unit disc. □

Notes

The statement of Theorem 17.1.6 (whose proof includes that of Theorem 12.7.2) is a slight weakening of [**171**, Corollaire 2.0–2], with a similar proof; the technique goes back to Robba [**190**]. A complete classification of rank 1 solvable modules on an open annulus with outer radius 1 (and unspecified inner radius) has been given by Pulita [**185**] and can be formulated in terms of any Lubin–Tate group; using the formal multiplicative group returns Theorem 17.1.6.

Theorem 17.2.1 admits the following partial converse: if M is a finite free differential module over $\mathcal{R}$ of rank 1 that is solvable at 1 then there exists $\lambda \in \mathbb{Z}_p$ such that $M \otimes V_\lambda$ admits a Frobenius structure for some absolute Frobenius lift. (Here V_λ is defined as in Example 9.5.2.) For M defined over F_1 the field of analytic elements, this was shown by Chiarellotto and Christol

[**39**]; it is straightforward to extend their argument to $\mathcal{E}^\dagger$, but not to $\mathcal{R}$. The general case was shown by Pulita [**184**]; it constitutes a refinement of our Theorem 17.1.6.

We cannot resist viewing Dwork's trick (Corollary 17.2.2) as an instance of a general principle articulated beautifully by Coleman [**57**, §III]:

> Rigid analysis was created to provide some coherence in an otherwise totally disconnected p-adic realm. Still, it is often left to Frobenius to quell the rebellious outer provinces.

A direct proof of Corollary 17.2.4, which does not employ the transfer theorem for a nilpotent regular singularity (Theorem 13.7.1), appeared in [**134**, §3.6].

The proof of Proposition 17.3.1 was taken from [**209**, Theorem 3.4.10].

Theorem 17.4.3, in the case of an absolute Frobenius lift, is the original form of the slope filtration theorem suggested, though not explicitly conjectured, by Tsuzuki in [**209**]. Its derivation from Theorem 16.4.1 is the same as the derivation of [**125**, Theorem 6.12] from [**125**, Theorem 6.10]. See also [**129**, §7.1].

Exercises

(1) Verify the unproved assertion in Definition 17.1.3. (Hint: see [**191**, §7.2]).
(2) Verify the unproved assertions in Remark 17.1.7.

18

Effective convergence bounds

In this chapter, we discuss some effective bounds on the solutions of p-adic differential equations with nilpotent singularities. These come in two forms. We start by discussing bounds that make no reference to a Frobenius structure; these are due to Christol, Dwork, and Robba. They could have been presented earlier, and indeed one was invoked in Chapter 13; we chose to postpone them until this point so that we could better contrast them with the bounds available in the presence of a Frobenius structure. The latter are original, though strongly inspired by some recent results of Chiarellotto and Tsuzuki.

These results carry both theoretical and practical interest. Besides their application in the study of p-adic exponents mentioned above (and in the proof of the unit-root p-adic local monodromy theorem to follow; see Theorem 19.3.1), another theoretical point of interest is their use in the study of the logarithmic growth of horizontal sections at a boundary. We will discuss some recent advances in this subject due to André, Chiarellotto, and Tsuzuki. (An area of application that we will not discuss is the theory of G-functions, as found in [**80**].)

A point of practical interest is that effective convergence bounds are useful for carrying out rigorous numerical calculations, e.g., in the machine computation of zeta functions of varieties over finite fields. See the notes for Chapter 23 for further discussion.

Hypothesis 18.0.1. In this chapter, we will drop the running restriction that K is discretely valued, imposing it only when we discuss Frobenius structures. We retain the condition $p > 0$, however.

18.1 A first bound

We now go back to the p-adic Fuchs theorem for discs (Theorem 13.2.2) and extract an effective convergence bound in the case of a nilpotent singularity.

One can also give effective bounds for more general regular singularities (as long as one takes into account the condition of p-adic non-Liouville differences), but the nilpotent case is sufficient for most applications in algebraic geometry and the bounds are much easier to describe in this case. (The case of no singularity reproduces the p-adic Cauchy theorem, Proposition 9.3.3.)

Proposition 18.1.1. *Let $N = \sum_{i=0}^{\infty} N_i t^i$ be an $n \times n$ matrix over $K[\![t/\beta]\!]_0$ corresponding to the differential system $D(v) = Nv + d(v)$, where $d = t\,d/dt$. Assume that N_0 is nilpotent with nilpotency index m, that is, $N_0^m = 0$ but $N_0^{m-1} \neq 0$. Assume also that $|N|_\beta \leq 1$. Then the fundamental solution matrix $U = \sum_{i=0}^{\infty} U_i t^i$ over $K[\![t]\!]$ (as in Proposition 7.3.6) satisfies*

$$|U_i|\beta^i \leq |i!|^{-2m+1} \qquad (i = 1, 2, \dots). \tag{18.1.1.1}$$

Consequently, U has entries in $K[\![t/(p^{-(2m-1)/(p-1)}\beta)]\!]_0$, as does its inverse.

Proof. Recall that U is determined by the recursion (7.3.6.1):

$$N_0 U_i - U_i N_0 + iU_i = -\sum_{j=1}^{i} N_j U_{i-j} \qquad (i > 0).$$

By Lemma 7.3.5 the map $f(X) = N_0 X - X N_0$ on $n \times n$ matrices is nilpotent, with nilpotency index $2m - 1$. Hence the map $X \mapsto iX + f(X)$ has inverse

$$X \mapsto \sum_{j=0}^{2m-2} (-1)^j i^{-j-1} f^j(X).$$

This gives the claim by induction on i. □

18.2 Effective bounds for solvable modules

We now give an improved version of Proposition 18.1.1 under the hypothesis that U has entries in $K\{t/\beta]\!]_0\}$. The hypothesis is only qualitative, in that it implies that $|U_i|\beta^i \to 0$ as $i \to \infty$, but it does not give a specific bound on $|U_i|$ for any particular i. Somewhat surprisingly, this hypothesis plus an explicit bound on N together imply a rather strong explicit bound on $|U_i|$. (We will continue to restrict to the case of nilpotent singularities; see Section 18.5 for what happens when the exponents are allowed to range over $\mathbb{Z}_p$.)

Theorem 18.2.1. *Let $N = \sum_{i=0}^{\infty} N_i t^i$ and $U = \sum_{i=0}^{\infty} U_i t^i$ be $n \times n$ matrices over $K[\![t]\!]$ satisfying the following conditions.*

(a) *The matrix N has entries in $K[\![t/\beta]\!]_0$.*
(b) *We have $U_0 = I_n$.*

(c) *We have* $U^{-1}NU + U^{-1}t(d/dt)(U) = N_0$.
(d) *The matrix* N_0 *is nilpotent.*
(e) *The matrices* U *and* U^{-1} *have entries in* $K\{t/\beta\}$.
Then, for every positive integer i,

$$|U_i|\beta^i \leq p^{(n-1)\lfloor \log_p i \rfloor} \max\{1, |N|_\beta^{n-1}\}.$$

The first step in the proof of Theorem 18.2.1 is to change basis to reduce $|N|_\beta$; this comes at the expense of enlarging K slightly and decreasing β slightly.

Lemma 18.2.2. *With notation as in Theorem 18.2.1, suppose that* K *has value group* $\mathbb{R}$. *Then, for any* $\lambda < 1$, *there exists an invertible* $n \times n$ *matrix* X *over* $K[\![t/(\lambda\beta)]\!]_0$ *such that*

$$\left| X^{-1}NX + X^{-1}t\frac{d}{dt}(X) \right|_{\lambda\beta} \leq 1$$
$$|X^{-1}|_{\lambda\beta} \leq 1$$
$$|X|_{\lambda\beta} \leq |N|_{\lambda\beta}^{n-1}.$$

Proof. Let M be the differential module over $K[\![t/\beta]\!]_0$ for the operator td/dt, with a basis on which D acts via N, and let $|\cdot|$ be the supremum norm defined by this basis. Over the closed disc of radius $\lambda\beta$, M becomes isomorphic to a successive extension of trivial differential modules. Consequently the generic radius of convergence of $M \otimes F_{\lambda\beta}$ is equal to $\lambda\beta$. In particular,

$$p^{-1/(p-1)}\lambda\beta = |t^{-1}D|_{\mathrm{sp}, M\otimes F_{\lambda\beta}} \leq \left|\frac{d}{dt}\right|_{F_{\lambda\beta}} = \lambda\beta.$$

By Proposition 6.5.6 plus Lemma 8.6.1, we obtain the desired matrix X. □

Using Lemma 18.2.2, we wish to prove Theorem 18.2.1 by using Frobenius antecedents to reduce the index from i to $\lfloor i/p \rfloor$. One can improve upon this argument if one has a Frobenius structure on the differential module; see Lemma 18.3.2 below.

Lemma 18.2.3. *With notation as in Theorem 18.2.1, suppose that* $|N|_\beta \leq 1$. *Then there exist* $n \times n$ *matrices* N', U' *over* $K[\![t/\beta^p]\!]$, *satisfying the hypotheses of Theorem 18.2.1, such that*

$$|N'|_{\beta^p} \leq p,$$
$$\max\{|U_j|\beta^j : 0 \leq j \leq i\} \leq \max\{|U'_j|\beta^{pj} : 0 \leq j \leq i/p\}.$$

Proof. Define the invertible $n \times n$ matrix $V = \sum_{i=0}^{\infty} V_i t^i$ over $K[\![t/\beta]\!]$ as follows. Start with $V_0 = I_n$. Given $V_0, \dots, V_{i-1}$, if $i \equiv 0 \pmod p$ then put $V_i = 0$. Otherwise, put $W = \sum_{j=0}^{i-1} V_j t^j$ and $N_W = W^{-1}NW + W^{-1}t(d/dt)(W)$, and let V_i be the unique solution of the matrix equation

$$N_0 V_i - V_i N_0 + i V_i = -(N_W)_i.$$

By induction on i we have $|V_i|\beta^i \leq 1$ for all i, so V is invertible over $K[\![t/\beta]\!]_0$. Let $\phi : K[\![t]\!] \to K[\![t]\!]$ denote the substitution $t \mapsto t^p$. Put $N'' = V^{-1}NV + V^{-1}t(d/dt)(V)$; then N'' has entries in $K[\![t^p]\!]$ and $|\phi^{-1}(N'')|_{\beta^p} \leq 1$. Put $U'' = V^{-1}U$; then

$$(U'')^{-1}N''U'' + (U'')^{-1}t\frac{d}{dt}(U'') = N_0'' = N_0,$$

which forces U'' also to have entries in $K[\![t^p]\!]$. We may then take $N' = p^{-1}\phi^{-1}(N'')$ and $U' = \phi^{-1}(U'')$. $\square$

We now put everything together.

Proof of Theorem 18.2.1. There is no harm in enlarging K, so we may assume that K has value group $\mathbb{R}$. We will then prove the claim by induction on i, in three stages. First, if $i < p$ and $|N|_\beta \leq 1$ then the desired estimate follows from Proposition 18.1.1. Second, for any given i, the desired estimate for general N follows from the estimate for the same i in the case $|N|_\beta \leq 1$, by Lemma 18.2.2. (More precisely, for any $\lambda < 1$, replace the pair N, U by $X^{-1}NX + X^{-1}t(d/dt)(X), X^{-1}UX_0$ and then take the limit as $\lambda \to 1$.) Third, if $|N|_\beta \leq 1$ then the desired estimate for any given i follows from the corresponding estimate for general N with i replaced by $\lfloor i/p \rfloor$, by Lemma 18.2.3. $\square$

Example 18.2.4. It is easy to give an example that shows that one cannot significantly improve the bound of Theorem 18.2.1 without extra hypotheses. (There is a tiny improvement possible; see the notes.) For instance, the functions

$$f_i = \frac{1}{i!}(\log(1+t))^i \qquad (i = 0, \dots, n-1)$$

satisfy the differential system

$$\frac{d}{dt}f_0 = 0, \quad \frac{d}{dt}f_i = \frac{1}{1+t}f_{i-1} \qquad (i = 1, \dots, n-1),$$

in which the coefficients have 1-Gauss norm at most 1.

One important special case of these results is that of a generic disc. Considering this case will fulfill an earlier promise to prove Lemma 13.5.4.

Corollary 18.2.5. *Let V be a finite differential module over F_β (for the derivation d/dt) for some $\beta > 0$ such that $IR(V) = 1$. Choose a basis of V and, for $i \geq 0$, let D_i be the matrix of action of D^i on this basis. Then*

$$\left|\frac{D_i}{i!}\right|_\beta \beta^i \leq p^{(n-1)\lfloor \log_p i \rfloor} \max\{1, |D_1|_\beta^{n-1}\} \qquad (i > 0).$$

Proof. Let L be the completion of $K(t_\beta)$ for the β-Gauss norm, so that $t_\beta \in L$ is a generic point of norm β. As in Section 9.4, we may base-extend V to the open disc of radius β with series parameter $t - t_\beta$. The fundamental solution matrix at t_β can be computed using Remark 5.8.4; it is

$$\sum_{i=0}^{\infty} \frac{(t_\beta - t)^i}{i!} D_i.$$

If we write this matrix as $\sum_{i=0}^{\infty} T_i (t - t_\beta)^i$, where T_i has entries in L, we obtain from Theorem 18.2.1 the bound

$$|T_i|\beta^i \leq p^{(n-1)\lfloor \log_p i \rfloor} \max\{1, |D_1|_\beta^{n-1}\}.$$

We now deduce the claim by induction on i. Write D_i as a power series $\sum_{j=0}^{\infty} D_{i,j}(t - t_\beta)^j$ whose coefficients have entries in L, so that $|D_{i,j}|\beta^j \leq |D_i|_\beta$ for $j \geq 0$ with equality for $j = 0$. (See Section 9.4 to recall where these inequalities come from.) We then have

$$T_i = \sum_{j=0}^{i} (-1)^j \frac{D_{j,i-j}}{j!}.$$

By the induction hypothesis, for $j < i$ we have

$$\begin{aligned} |D_{j,i-j}/j!|\beta^i &\leq |D_j/j!|_\beta \beta^j \\ &\leq p^{(n-1)\lfloor \log_p j \rfloor} \max\{1, |D_1|_\beta^{n-1}\} \\ &\leq p^{(n-1)\lfloor \log_p i \rfloor} \max\{1, |D_1|_\beta^{n-1}\}. \end{aligned}$$

Combined with the bound on T_i, this yields

$$|D_i/i!|_\beta \beta^i = |D_{i,0}/i!|\beta^i \leq p^{(n-1)\lfloor \log_p i \rfloor} \max\{1, |D_1|_\beta^{n-1}\},$$

as desired. ∎

18.3 Better bounds using Frobenius structures

Although Theorem 18.2.1 is close to optimal under its hypotheses, it can be improved if the differential module in question admits a Frobenius structure. For simplicity, we restrict to standard Frobenius structures.

Hypothesis 18.3.1. In this section, we restore the hypothesis that K is discretely valued. Fix a power q of p, and let ϕ be the standard qth-power Frobenius lift on $K[\![t]\!]_0$ with respect to some isometry $\phi_K : K \to K$.

The key here is to imitate the proof of Theorem 18.2.1 but with the differential equation replaced by a certain difference equation.

Lemma 18.3.2. *Let $U = \sum_{i=0}^{\infty} U_i t^i$, $A = \sum_{i=0}^{\infty} A_i t^i$ be $n \times n$ matrices over $K[\![t]\!]$ satisfying the following conditions.*

(a) *The matrix A has entries in $K[\![t]\!]_0$.*
(b) *We have $U_0 = I_n$, and the matrix A_0 is invertible.*
(c) *We have $U^{-1} A \phi(U) = A_0$.*

Then

$$\max\{|U_j| : 0 \leq j \leq i\} \leq |A_0^{-1}||A|_1 \max\{|U_j| : 0 \leq j \leq \lfloor i/q \rfloor\}.$$

Consequently, for every positive integer i,

$$|U_i| \leq (|A_0^{-1}||A|_1)^{\lceil \log_q i \rceil}.$$

Proof. Note that (c) can be rewritten as

$$U = A\phi(U)A_0^{-1}.$$

This gives the first inequality. To deduce the second inequality, we proceed as in the proof of Theorem 18.2.1 except that we iterate $\lceil \log_q i \rceil$ times to reach the case $i = 0$ (rather than iterating $\lfloor \log_q i \rfloor$ times to reach the case $0 < i < p$). □

Theorem 18.3.3. *Let $N = \sum_{i=0}^{\infty} N_i t^i$, $U = \sum_{i=0}^{\infty} U_i t^i$, and $A = \sum_{i=0}^{\infty} A_i t^i$ be $n \times n$ matrices over $K[\![t]\!]$ satisfying the following conditions.*

(a) *The matrix A has entries in $K[\![t]\!]_0$.*
(b) *We have $U_0 = I_n$, and the matrix A_0 is invertible.*
(c) *We have $U^{-1} N U + U^{-1} t(d/dt)(U) = N_0$.*
(d) *We have $NA + t(d/dt)(A) = qA\phi(N)$.*

Then $U^{-1} A \phi(U) = A_0$ and, for every positive integer i,

$$|U_i| \leq (|A_0^{-1}||A|_1)^{\lceil \log_q i \rceil}.$$

Proof. As noted in Remark 17.1.2, the commutation relation (d) implies that $N_0A_0 = qA_0\phi(N_0)$, which forces N_0 to be nilpotent. Put $B = U^{-1}A\phi(U) = \sum_{i=0}^{\infty} B_i t^i$. Then $B_0 = A_0$ and $N_0B + t(d/dt)(B) = qB\phi(N_0)$. Hence

$$N_0B_i + iB_i = qB_i\phi(N_0) = B_iA_0^{-1}N_0A_0$$

or

$$N_0(B_iA_0^{-1}) + i(B_iA_0^{-1}) = (B_iA_0^{-1})N_0. \tag{18.3.3.1}$$

By Lemma 7.3.5 the operator $X \mapsto N_0X - XN_0 + iX$ on $n \times n$ matrices is invertible for $i \neq 0$, so (18.3.3.1) implies that $B_i = 0$ for $i > 0$. Hence $U^{-1}A\phi(U) = A_0$ does hold, so we may conclude by applying Lemma 18.3.2, to reduce to the case $i < q$, Theorem 18.2.1. □

Remark 18.3.4. By combining Theorem 18.3.3 with Theorem 18.2.1 (applying the latter for $i < q$), we can obtain the bound

$$|U_i| \leq |N|_1^{n-1} p^{(n-1)\lfloor \log_p i - (\log_p q)\lfloor \log_q i \rfloor \rfloor} (|A_0^{-1}||A|_1)^{\lfloor \log_q i \rfloor}.$$

Remark 18.3.5. In applications to Picard–Fuchs modules, the difference between the bounds given by Theorems 18.2.1 and 18.3.3 can be quite significant. For instance, given a Picard–Fuchs module arising from a family of curves of genus g, the bound of Theorem 18.2.1 contains the factor $p^{(2g-1)\lfloor \log_p i \rfloor}$ but the bound of Theorem 18.3.3 replaces the factor $2g - 1$ by 1. In general, it should be possible to use Theorem 18.3.3 (and perhaps also Theorem 18.3.6) to explain various instances in which a calculation of n terms of a power series involves a precision loss $p^{O(\log(n))}$ even though the accumulated factors p by which one divides throughout the calculation amount to $p^{O(n)}$. (A typical example of this is given in [**124**, Lemma 3].)

We record also a sharper form of Theorem 18.3.3 for use in the discussion of logarithmic growth in the next section.

Theorem 18.3.6. *Let v be a column vector of length n over $K[\![t]\!]$, let $A = \sum_{i=0}^{\infty} A_i t^i$ be an $n \times n$ matrix over $K[\![t]\!]$, and let $\lambda \in K$ be chosen to satisfy the following conditions.*

(a) *The matrix A has entries in $K[\![t]\!]_0$.*
(b) *The matrix A_0 is invertible.*
(c) *We have $A\phi(v) = \lambda v$.*

Then

$$\max\{|v_j| : 0 \leq j \leq i\} \leq |\lambda^{-1}||A|_1 \max\{|v_j| : 0 \leq j \leq \lfloor i/q \rfloor\}.$$

Consequently, for every positive integer i*,*

$$|v_i| \leq |v_0|(|\lambda^{-1}||A|_1)^{\lceil \log_q i \rceil}.$$

Proof. Rewrite (c) as $v = \lambda^{-1} A\sigma(v)$ and proceed as in Lemma 18.3.2. $\square$

18.4 Logarithmic growth

In general, the fundamental solution matrix of a differential system with a nilpotent regular singularity at 0 need not be bounded on a closed disc, even if the matrix defining the system is bounded and the solution matrix converges on the open disc. However, one gets a fairly mild growth condition at the boundary; better still, one can extract some interesting information by distinguishing different solutions on the basis of their order of growth. This is loosely inspired by a comparable archimedean situation, for which see the notes.

Definition 18.4.1. For $\delta \geq 0$, let $K[\![t]\!]_\delta$ be the subset of $K[\![t]\!]$ consisting of those $f = \sum_{i=0}^{\infty} f_i t^i$ for which

$$|f|_\delta = \sup_i \left\{ \frac{|f_i|}{(i+1)^\delta} \right\} < +\infty;$$

note that $K[\![t]\!]_\delta$ is complete under the norm $|\cdot|_\delta$. For $\delta = 0$ we recover the ring $K[\![t]\!]_0$ of bounded power series. However, $K[\![t]\!]_\delta$ is not a ring for $\delta > 0$; rather, we have

$$K[\![t]\!]_{\delta_1} K[\![t]\!]_{\delta_2} \subset K[\![t]\!]_{\delta_1+\delta_2}.$$

Also, $K[\![t]\!]_\delta$ is stable under d/dt but antidifferentiation carries it into $K[\![t]\!]_{\delta+1}$. Put

$$K[\![t]\!]_{\delta+} = \bigcap_{\delta' > \delta} K[\![t]\!]_{\delta'}.$$

For another useful characterization of $K[\![t]\!]_\delta$, see the exercises.

Definition 18.4.2. For $f \in K[\![t]\!]$, we say that f has *order of log-growth* δ if $f \in K[\![t]\!]_\delta$ but $f \notin K[\![t]\!]_{\delta'}$ for any $\delta' < \delta$. We say f has *order of log-growth* $\delta+$ if $f \notin K[\![t]\!]_\delta$ but $f \in K[\![t]\!]_{\delta'}$ for any $\delta' > \delta$. We have similar definitions for vectors or matrices over $K[\![t]\!]$ and for elements of $M \otimes_{K[\![t]\!]_0} K[\![t]\!]$ if M is a finite free module over $K[\![t]\!]_0$ (by computing in terms of a basis, the choice of which will not affect the answer).

We then deduce the following from Theorem 18.2.1.

Proposition 18.4.3. *Let M be a differential module of rank n over $K[\![t]\!]_0$ for the operator td/dt, such that the action of D on M/tM is nilpotent. Then any element of $H^0(M \otimes_{K[\![t]\!]_0} K[\![t]\!])$ has order of log-growth at most $n-1$.*

Remark 18.4.4. One can say something slightly stronger than Proposition 18.4.3, by observing that $M \otimes_{K[\![t]\!]_0} K[\![t]\!][\log t]$ admits a basis of horizontal sections, each of which has degree at most $n-1$ in $\log t$ (exercise). If we write some $m \in H^0(M \otimes_{K[\![t]\!]_0} K[\![t]\!][\log t])$ as a formal sum $\sum_{i=0}^{n-1} m_i (\log t)^i$, with $m_i \in M \otimes_{K[\![t]\!]_0} K[\![t]\!]$, then Theorem 18.2.1 implies that, for $i = 0, \dots, n-1$, m_i has order of log-growth at most $n-1$. However, we suspect that this can be improved to $n-1-i$.

In the presence of a Frobenius structure, one obtains a much sharper bound, due to Chiarellotto and Tsuzuki [**41**, Theorem 6.17]. (One can also formulate a refinement in the manner of Remark 18.4.4.)

Theorem 18.4.5. *Assume that K is discretely valued. Let M be a differential module of rank n over $K[\![t]\!]_0$ for the operator td/dt, equipped with a Frobenius structure for a qth-power Frobenius lift as in Remark 17.1.2. Then any element $v \in H^0(M \otimes_{K[\![t]\!]_0} K[\![t]\!])$ satisfying $\Phi(v) = \lambda v$ for some $\lambda \in K$ has order of log-growth at most $(-\log|\lambda| - s_0)/(\log q)$, where s_0 is the least generic Newton slope of M.*

Proof. We may assume that the Frobenius lift is standard, thanks to Proposition 17.3.1. By then replacing the Frobenius lift by some power, we can reduce to the case where s_0 is a multiple of $-\log p$. We can then twist into the case $s_0 = 0$. By Proposition 14.5.9, we can choose a basis of M such that the least generic Hodge slope of M is also 0. Then the claim follows immediately from Theorem 18.3.6. □

Remark 18.4.6. Refining a conjecture of Dwork, it was conjectured by Chiarellotto and Tsuzuki [**41**] that if M is indecomposable then Theorem 18.4.5 is optimal. That is, in the notation of Theorem 18.4.5, v should have order of log-growth exactly $(-\log|\lambda| - s_0)/(\log q)$; Chiarellotto and Tsuzuki have proved this for $\mathrm{rank}(M) \leq 2$ [**41**, Theorem 7.2]. It should be possible to extend their proof to all cases where $-\log|\lambda|$ is less than or equal to the least Newton slope of M strictly greater than s_0, but it is less clear how to extend to the general case.

Remark 18.4.7. By contrast, if M does not carry a Frobenius structure then the order of log-growth of a horizontal section behaves much less predictably.

For instance, it need not be rational, and it could have the form $\delta+$ instead of δ [**41**, §5.2].

18.5 Nonzero exponents

So far, we have considered only regular differential systems with all exponents equal to zero. We now allow for arbitrary exponents in $\mathbb{Z}_p$; the bound we get is very slightly weaker in this case.

Theorem 18.5.1. *Let $N = \sum_{i=0}^{\infty} N_i t^i$ and $U = \sum_{i=0}^{\infty} U_i t^i$ be $n \times n$ matrices over $K[\![t]\!]$ satisfying the following conditions for some $\beta > 0$.*

(a) *The matrix N has entries in $K[\![t/\beta]\!]_0$.*
(b) *We have $U_0 = I_n$.*
(c) *We have $U^{-1}NU + U^{-1}t(d/dt)(U) = N_0$.*
(d) *The matrix N_0 has prepared eigenvalues in $\mathbb{Z}_p$.*
(e) *The matrices U and U^{-1} have entries in $K\{t/\beta\}$.*

Then, for every positive integer i,

$$|U_i|\beta^i \leq p^{n+(n-1)\lceil \log_p i \rceil} \max\{1, |N|_\beta^{n-1}\}.$$

Proof. The statement of Lemma 18.2.2 carries over to this situation without change. However, we can carry out the construction in the proof of Lemma 18.2.3 only if the exponents are divisible by p. Otherwise we must first enforce this condition by performing up to $p-1$ shearing transformations, yielding the weaker bound

$$\max\{|U_j|\beta^j : 0 \leq j \leq i\} \leq \max\{|U'_j|\beta^{pj} : 0 \leq j \leq (i+p-1)/p\}.$$

In the case $i \leq p$ (and $|N|_\beta = 1$), we instead perform the shearing transformations and then calculate explicitly as in Proposition 18.1.1 to obtain the bound

$$|U_i|\beta^i \leq p^{2n-1}.$$

If $i \leq p^{h+1}$ for some $h \geq 0$ then, after h iterations of the map $i \mapsto \lfloor (i+p-1)/p \rfloor$, we end up with a quantity that is at most p. We thus obtain the claimed bound. □

Notes

In the case of no singularities ($N_0 = 0$), the effective bound of Theorem 18.2.1 is due to Dwork and Robba [**82**], but is slightly stronger: one may replace $p^{(n-1)\lfloor \log_p i \rfloor}$ with the maximum of $|j_1 \cdots j_{n-1}|^{-1}$ over $j_1, \dots, j_{n-1} \in \mathbb{Z}$ with $1 \leq j_1 < \cdots < j_{n-1} \leq i$. See also [**80**, Theorem IV.3.1].

The general case of Theorem 18.2.1 is due to Christol and Dwork [**47**], except that their bound is significantly weaker: it is roughly $p^{c(n-1)\lfloor \log_p i \rfloor}$ with $c = 2 + 1/(p-1)$. The discrepancy comes from the fact that the role of Proposition 6.5.6 is played in [**47**] by an effective version of the cyclic vector theorem, which does not give optimal bounds. As usual, the use of cyclic vectors also introduces singularities, which must then be removed, leading to some technical difficulties. See also [**80**, Theorem V.2.1].

Theorems 18.3.3 and 18.3.6 are original, but they owe a great debt to the proof of [**41**, Theorem 7.2]. The main difference is that we prefer to argue in terms of matrices rather than cyclic vectors.

In the case of no singularities, Proposition 18.4.3 was first proved by Dwork; it appears in [**75**] and [**76**]. (See also [**42**].) The extension suggested by Remark 18.4.4 is original; as noted above, the effective bounds in [**47**] are not strong enough to imply this.

The archimedean motivation for the study of logarithmic growth comes from Deligne's study of regular singularities [**68**]. He showed that a differential module over the ring of germs of meromorphic functions at a point has a regular singularity if and only if the horizontal sections have at worst logarithmic growth at the singular point.

In the p-adic setting, the theory of logarithmic growth emerged from some close analysis made by Dwork [**75, 76**] of the finer convergence behavior of solutions of certain p-adic differential equations. The subject languished until the recent work of Chiarellotto and Tsuzuki [**41**]; inspired by this, André [**6**] proved a conjecture of Dwork [**76**, Conjecture 2] analogizing the semicontinuity theorem for Newton polygons (Theorem 15.3.2) to logarithmic growth.

Theorem 18.5.1 is an improvement on [**80**, Theorem V.9.1], in which the bound takes the form $p^{(n^2+cn)\lfloor \log_p i \rfloor}$ for some constant c. Again, part (though not all) of the improvement is due to the avoidance of cyclic vectors.

Exercises

(1) Prove that, for $\delta \geq 0$,

$$K\llbracket t \rrbracket_\delta = \{f \in K\llbracket t \rrbracket : \limsup_{\rho \to 1^-} \frac{|f|_\rho}{(-\log \rho)^\delta} < \infty\}.$$

(Hint: the inequality

$$\sup_i \{(i+1)^\delta \rho^i\} \leq \rho^{-1} \left(\frac{\delta}{e}\right)^\delta (-\log \rho)^{-\delta}$$

may be helpful.)

(2) Let M be a differential module of rank n over $K[\![t]\!]$ for the operator td/dt, such that the action of D on M/tM is nilpotent. Show that if we extend the action of td/dt to $K[\![t]\!][\log t]$, by setting $(td/dt)(\log t) = 1$, then $M \otimes_{K[\![t]\!]} K[\![t]\!][\log t]$ admits a basis of horizontal sections, and each horizontal section has degree in $\log t$ bounded by $n - 1$.

19

Galois representations and differential modules

In this chapter we construct a class of examples of differential modules on open annuli which carry Frobenius structures and hence are solvable at a boundary. These modules are derived from continuous linear representations of the absolute Galois group of a positive-characteristic local field.

We first construct a correspondence between Galois representations and differential modules over $\mathcal{E}$ carrying a unit-root Frobenius structure. The basic mechanism for producing these modules is to tensor with a large ring carrying a Galois action and then take Galois invariants. This mechanism will reappear when we turn to p-adic Hodge theory, at which point we will attempt to simulate this situation using the Galois group of a mixed-characteristic local field. See Chapter 24.

Then we refine the construction to compare Galois representations having finite image of inertia with differential modules over $\mathcal{E}^\dagger$ carrying a unit-root Frobenius structure; the main result here is an equivalence of categories due to Tsuzuki. It is generalized by the absolute case of the p-adic local monodromy theorem (Theorem 20.1.4 below) and indeed can be used together with the slope filtration theorem (Theorem 17.4.3) to prove the monodromy theorem in the absolute case. This result also has an analogue in p-adic Hodge theory; see Theorem 24.2.5.

We finally describe (without proof) a numerical relationship between the wild ramification of a Galois representation and the convergence of solutions of p-adic differential equations. Besides making explicit the analogy between the wild ramification of Galois representations and the irregularity of meromorphic differential systems, it also suggests an approach to higher-dimensional ramification theory. We reserve most discussion of the latter to the notes.

Remark 19.0.1. The reader encountering this material for the first time is strongly encouraged to assume that κ_K is perfect. For an explanation of how things get complicated otherwise, see Section 20.5.

Notation 19.0.2. In this chapter we write d and ϕ instead of D and Φ for the actions on a differential or difference module. This is to avoid confusion with another standard usage of the letter D, which will first appear in Definition 19.1.2.

19.1 Representations and differential modules

We first describe a simple correspondence between Galois representations and differential modules due to Fontaine. This will serve as a model later, when we introduce (ϕ, Γ)-modules associated with Galois representations of local fields in mixed characteristic (Chapter 24).

Definition 19.1.1. For L a finite separable extension of $\kappa_K((t))$, let $\mathcal{E}_L$ be the finite unramified extension of $\mathcal{E}$ with residue field L (see Corollary 3.2.4). Let $\tilde{\mathcal{E}}$ be the completion of the maximal unramified extension of $\mathcal{E}$, which in particular contains the completion $\tilde{K}$ of the maximal unramified extension of K. Then $G_{\kappa_K((t))}$ acts on $\tilde{\mathcal{E}}$ with fixed field $\mathcal{E}$.

Definition 19.1.2. Let V be a finite-dimensional vector space over K, and let $\tau : G_{\kappa_K((t))} \to \mathrm{GL}(V)$ be a continuous homomorphism for the p-adic topology on $\mathrm{GL}(V)$. Let us view $V \otimes_K \tilde{\mathcal{E}}$ as a left $G_{\kappa_K((t))}$-module, with the action on the first factor coming from τ and the natural action on the second factor. Put

$$D(V) = (V \otimes_K \tilde{\mathcal{E}})^{G_{\kappa_K((t))}}.$$

Lemma 19.1.3. *The space $D(V)$ is an $\mathcal{E}$-vector space of dimension* $\dim_K(V)$. *Equivalently, the natural map $D(V) \otimes_{\mathcal{E}} \tilde{\mathcal{E}} \to V \otimes_K \tilde{\mathcal{E}}$ is an isomorphism.*

Proof. We first check this in the case where τ has finite image; then τ factors through $G_{E/\kappa_K((t))}$ for some finite separable extension E of $\kappa_K((t))$. In this case the claim is a consequence of Noether's nonabelian version of Hilbert's Theorem 90: for any finite Galois extension E/F of fields, the nonabelian cohomology set $H^1(G_{E/F}, \mathrm{GL}_n(E))$ is trivial.

In the general case we must argue a bit more carefully. Since $G_{\kappa_K((t))}$ is compact as a topological group, its image under τ is also compact. This implies that we can find an $\mathfrak{o}_{\mathcal{E}}$-lattice T in V that is stable under the Galois action, e.g., by starting with any lattice and taking the span of its images under the

Galois action. (The compactness ensures that the resulting $\mathfrak{o}_{\mathcal{E}}$-submodule of V is indeed finitely generated.) For any positive integer i the induced topology on $T/\mathfrak{m}_K^i T$ is discrete, so the Galois action factors through the Galois group of a finite separable extension of $\kappa_K((t))$. We may thus argue for each i as above and then take the inverse limit. □

Definition 19.1.4. Note that d/dt extends uniquely to $\mathcal{E}_L$, and hence to $D(V)$ by taking the action on V to be trivial. Since the action of d/dt commutes with the Galois action, we also obtain an action on $D(V)$. That is, $D(V)$ is a differential module over $\mathcal{E}$. By the same token, if we equip V with the structure of a difference module with respect to ϕ_K, then $D(V)$ inherits a Frobenius structure for any Frobenius lift ϕ on $\mathcal{E}$ acting on K via ϕ_K. Since we can always start with a unit-root Frobenius structure on V (e.g., by forcing a basis of V to be fixed), we deduce that $D(V)$ admits a unit-root Frobenius structure.

We obtain an instance of nonabelian Artin–Schreier theory. See the notes to Chapter 14 for some background.

Proposition 19.1.5. *Suppose that ϕ is an absolute qth-power Frobenius lift and that the fixed field K_0 of ϕ on K has residue field $\mathbb{F}_q$ and the same value group as K. Given a continuous representation of $G_{\kappa_K((t))}$ on a finite-dimensional K_0-vector space V_0, equip $V = V_0 \otimes_{K_0} K$ with the Frobenius action induced by the trivial action on V_0. Then $V_0 \mapsto D(V)$ is an equivalence of categories between the category of continuous representations of $G_{\kappa_K((t))}$ on finite-dimensional K_0-vector spaces and the category of finite differential modules over $\mathcal{E}$ equipped with a unit-root Frobenius structure.*

Proof. We will show that the reverse equivalence is provided by the functor V_0 on finite differential modules over $\mathcal{E}$ equipped with unit-root Frobenius structures and defined by

$$V_0(D) = (D \otimes_{\mathcal{E}} \tilde{\mathcal{E}})^{\phi=1}.$$

Since $D(V_0) \otimes_{\mathcal{E}} \tilde{\mathcal{E}} \to V_0 \otimes_{K_0} \tilde{\mathcal{E}}$ is an isomorphism by Lemma 19.1.3, we may naturally identify $V_0(D(V_0))$ with $(V \otimes_K \tilde{\mathcal{E}})^{\phi=1} = V_0$. (Here the conditions on K_0 are needed to ensure that $\tilde{\mathcal{E}}^{\phi=1} = K_0$. See the exercises at the end of the chapter.)

It remains to give a canonical isomorphism of $D(V_0(D))$ with D; by the same token, it is sufficient to check that the natural map $V_0(D) \otimes_{K_0} \tilde{\mathcal{E}} \to D \otimes_{\mathcal{E}} \tilde{\mathcal{E}}$ is a bijection. This is exactly the statement that $D \otimes_{\mathcal{E}} \tilde{\mathcal{E}}$ is a trivial difference module. This holds by the first part of the proof of Theorem 14.6.3, since the residue field of $\tilde{\mathcal{E}}$ is separably closed and hence weakly difference-closed for absolute Frobenius lifts. □

Corollary 19.1.6. *Suppose that ϕ is an absolute qth-power Frobenius lift and that $K = \mathbb{Q}_q$ is the unramified extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$. Then the functor D, from continuous representations of $G_{\mathbb{F}_q((t))}$ on finite-dimensional $\mathbb{Q}_q$-vector spaces to finite differential modules over $\mathcal{E}$ equipped with unit-root Frobenius structure, is an equivalence of categories.*

Remark 19.1.7. In Proposition 19.1.5 the differential structure is not necessary; see the exercises. What we do need is the symmetry between two actions on $\tilde{\mathcal{E}}$, that of the Galois group and that of the monoid given by the nonnegative powers of ϕ. The proof works because $\tilde{\mathcal{E}}$ is large enough to trivialize both the Galois action (by the Hilbert–Noether theorem) and the ϕ-action (by the Dieudonné–Manin theorem). Starting from this point, Fontaine realized that one could set up an analogous situation when $G_{\mathbb{F}_q((t))}$ is replaced by the Galois group of a finite extension of $\mathbb{Q}_p$; the result is a central construction in p-adic Hodge theory. See Chapter 24.

Remark 19.1.8. If ϕ is a Frobenius lift that is not absolute, then Definition 19.1.4 remains valid. The reason is that even though an individual finite separable extension of $\kappa_K((t))$ may not carry an action of ϕ, the separable closure of $\kappa_K((t))$ does carry such an action. However, Proposition 19.1.5 does not remain valid, because the residue field of $\tilde{\mathcal{E}}$ need not be weakly difference-closed.

19.2 Finite representations and overconvergent differential modules

Following an idea of Crew, we give a refinement of the previous construction for some special representations.

Definition 19.2.1. Since $\mathcal{E}^\dagger$ is henselian (Lemma 15.1.3), for each finite separable extension L of $\kappa_K((t))$ there exists a unique finite unramified extension $\mathcal{E}^\dagger_L$ with residue field L. In fact

$$\mathcal{E}_L \cong \mathcal{E} \otimes_{\mathcal{E}^\dagger} \mathcal{E}^\dagger_L,$$

and $\mathcal{E}^\dagger_L$ is the integral closure of $\mathcal{E}^\dagger$ in $\mathcal{E}_L$. In particular, $G_{\kappa_K((t))}$ acts on $\mathcal{E}^\dagger_L$ with fixed field $\mathcal{E}^\dagger$.

Definition 19.2.2. Let V be a finite-dimensional vector space over K, and let $\tau : G_{\kappa_K((t))} \to \mathrm{GL}(V)$ be a continuous homomorphism for the *discrete* topology on $\mathrm{GL}(V)$. That is, τ factors through $G_{L/\kappa_K((t))}$ for some finite separable extension L of $\kappa_K((t))$. Let us view $V \otimes_K \mathcal{E}_L^\dagger$ as a $G_{\kappa_K((t))}$-module in which the action on the first factor comes from τ and the action on the second factor is as above. Put

$$D^\dagger(V) = (V \otimes_K \mathcal{E}_L^\dagger)^{G_{\kappa_K((t))}}.$$

Lemma 19.2.3. *The space $D^\dagger(V)$ is an $\mathcal{E}^\dagger$-vector space of dimension $\dim_K(V)$. Equivalently, the natural map $D^\dagger(V) \otimes_{\mathcal{E}^\dagger} \mathcal{E}_L^\dagger \to V \otimes_K \mathcal{E}_L^\dagger$ is an isomorphism. (In particular, $D^\dagger(V)$ is canonically independent of the choice of L.)*

Proof. This again follows from the Hilbert–Noether theorem. □

Definition 19.2.4. As in Definition 19.1.4, $D^\dagger(V)$ is a differential module over $\mathcal{E}^\dagger$ admitting a unit-root Frobenius structure for any Frobenius lift ϕ on $\mathcal{E}^\dagger$. However, now that we have a module over $\mathcal{E}^\dagger$ it makes sense to compute the subsidiary radii of $D^\dagger(V) \otimes F_\rho$ for $\rho \in (0, 1)$ sufficiently close to 1. Namely, realize $D^\dagger(V)$ as a differential module over $K\langle \alpha/t, t]\!]_0$ for some α and compute there. Be aware that any two such realizations for a given α need only become isomorphic over $K\langle \beta/t, t]\!]_0$ for some $\beta \in [\alpha, 1)$. However, statements about the germ at 1 of the function $\rho \mapsto R(D^\dagger(V) \otimes F_\rho)$ are unambiguous.

Proposition 19.2.5. *The generic radius of convergence of $D(V)$ is equal to* 1. *Consequently (by the continuity of the generic radius of convergence, as in Theorem 11.3.2(a)), $D^\dagger(V)$ is solvable at* 1.

Proof. This follows from the existence of a Frobenius structure on $D^\dagger(V)$, using Theorem 17.2.1. □

Example 19.2.6. Assume that K contains an element π with $\pi^{p-1} = -p$; then K contains a unique pth root of unity ζ_p satisfying $1 - \zeta_p \equiv \pi \pmod{\pi^2}$ (see Remark 17.1.7). Let $L = \kappa_K((t))[z]/(z^p - z - \overline{f})$ be an Artin–Schreier extension, and let V be the Galois representation corresponding to the character of $G_{L/\kappa_K((t))}$ taking the automorphism $z \mapsto z + 1$ to ζ_p. We can then explicitly describe $D^\dagger(V)$: it is the module M_f of Example 17.1.4 (exercise).

Similarly, one can realize the construction of Example 9.9.3 as $D^\dagger(V)$ for a certain explicit character of order p^h. This observation has been thoroughly generalized by Pulita [**185**].

Remark 19.2.7. Note that the kernel of d on $\mathcal{E}_L^\dagger$ is the integral closure K' of K in $\mathcal{E}_L^\dagger$ (exercise). Consequently, the space of horizontal sections of $D^\dagger(V) \otimes_{\mathcal{E}^\dagger} \mathcal{E}_L^\dagger$ is equal to $V \otimes_K K'$. This suggests that we cannot recover the whole of V from $D^\dagger(V)$, at least if we use only the differential structure; instead, we recover the restriction of V to the inertia subgroup of $G_{\kappa_K((t))}$, which we can identify with $G_{\kappa_K^{\mathrm{sep}}((t))}$.

The previous remark suggests the following construction.

Definition 19.2.8. Let V be a finite-dimensional vector space over K, and let $\tau : G_{\kappa_K((t))} \to \mathrm{GL}(V)$ be a continuous homomorphism for the p-adic topology. We say that τ has *finite local monodromy* if the image of the inertia subgroup of $G_{\kappa_K((t))}$ is finite. (That inertia subgroup is isomorphic to $G_{\kappa_K^{\mathrm{sep}}((t))}$.) In this case, let $\mathcal{E}^\dagger_{\kappa_K^{\mathrm{sep}}((t))}$ be the ring defined in the same fashion as $\mathcal{E}^\dagger$ but using $\widehat{K^{\mathrm{unr}}}$, the completion of the maximal unramified extension of K, for the coefficients; let $G_{\kappa_K((t))}$ act on this ring via the quotient by its inertia subgroup. We can then define

$$D^\dagger(V) = (V \otimes_K (\mathcal{E}^\dagger_{\kappa_K^{\mathrm{sep}}((t))})^{\mathrm{unr}})^{G_{\kappa_K((t))}}$$

and this will be a differential module over $\mathcal{E}^\dagger$ of the correct dimension, again admitting a unit-root Frobenius structure for any Frobenius lift.

19.3 The unit-root p-adic local monodromy theorem

We have the following refinement of Proposition 19.1.5. See the notes for a detailed attribution. (For an analogous fact in p-adic Hodge theory, see Theorem 24.2.5.)

Theorem 19.3.1 (Tsuzuki). *Suppose that ϕ is an absolute qth-power Frobenius lift, and that the fixed field K_0 of ϕ on K has residue field $\mathbb{F}_q$ and the same value group as K. Given a continuous representation of $G_{\kappa_K((t))}$ with finite local monodromy on a finite-dimensional K_0-vector space V_0, equip $V = V_0 \otimes_{K_0} K$ with the Frobenius action induced by the trivial action on V_0. Then $V_0 \mapsto D^\dagger(V)$ is an equivalence of categories with the category of finite differential modules over $\mathcal{E}^\dagger$ equipped with unit-root Frobenius structure.*

Although it is possible to deduce this theorem from the p-adic local monodromy theorem (see Remark 20.1.5 below), it is both instructive and historically appropriate to give a proof using the tools we have available at this point. We proceed to this task now.

Definition 19.3.2. Let M be a finite differential module over $\mathcal{E}^\dagger$ equipped with a unit-root Frobenius structure. For $c \in (0, 1)$, we say that M is *c-constant* if there exists a basis of M on which Φ acts via a matrix A with $|A - I_n|_1 \leq c$; we call such a basis a *c-constant basis*.

If ϕ is absolute, we can see directly that the property of M of being c-constant is invariant under a change in Frobenius lift (Corollary 17.3.2). Namely, by Proposition 19.1.5, M is c-constant if and only if it occurs as $D(V)$ for some representation $\tau : G_{\kappa_K((t))} \to \mathrm{GL}(V)$ such that V admits a supremum norm $|\cdot|$ for which $|\tau(g)(x) - x| \leq c|x|$ for all $g \in G_{\kappa_K((t))}$ and $x \in V$.

For arbitrary ϕ, we may reach the same conclusion by imitating the proof of Proposition 17.3.5. See the exercises.

Lemma 19.3.3. *Let M be a finite differential module over $\mathcal{E}^\dagger$ admitting a unit-root Frobenius structure for a standard (but not necessarily absolute) Frobenius lift. Then there exists a positive integer m coprime to p such that $M \otimes_{\mathcal{E}^\dagger} \mathcal{E}^\dagger[t^{1/m}]$ admits a basis on which the matrices A, N of action of Φ, tD have entries in $\mathcal{E}^\dagger[t^{1/m}] \cap \mathfrak{o}_K[\![t^{-1/m}]\!]$, as does A^{-1}. Moreover, if M is c-constant for some $c < 1$ then we can ensure that $m = 1$ and $|A - I_n|_1 \leq c$.*

Proof. Since M admits a unit-root Frobenius structure, we can choose a basis on which the matrix of action A of Φ belongs to $\mathrm{GL}_n(\mathfrak{o}_{\mathcal{E}^\dagger})$. By reordering the basis vectors, we can ensure that the minimum t-adic valuation of the reduction of $\overline{A}_{ij}$ occurs for $i = j = 1$. By adjoining $t^{1/(q-1)}$ and then rescaling, we can force this minimum valuation to equal 0. We can then conjugate to ensure that $\overline{A}_{i1} = 0$ for $i = 2, \dots, n$. Proceeding in this manner, we can force $\overline{A}$ to be upper triangular and invertible over $\kappa_K[\![t^{1/m}]\!]$.

We next put A into the desired form, by repeating the following operation. (If M is c-constant for some $c < 1$, we may start the argument here.) Write $A = \sum_i A_i t^{i/m}$, so that A_0 is upper triangular and invertible modulo $\mathfrak{m}_K$ and A_i vanishes modulo $\mathfrak{m}_K$ for $i < 0$. Then replace A by $U^{-1}A\phi(U)$, with

$$U = I_n + A_0^{-1}\left(\sum_{i>0} A_i t^{i/m}\right).$$

The matrices U then converge to the identity under the $(t^{1/m}, \mathfrak{m}_K)$-adic topology on $\mathfrak{o}_K[\![t^{1/m}]\!]$.

Finally, we note that the compatibility $N - qA\phi(N)A^{-1} = t(d/dt)(A)A^{-1}$ from (17.1.2.1) implies that having A in the desired form forces N to be in the desired form as well. Namely, the compatibility first implies that N

modulo q involves only nonnegative powers of $t^{-1/m}$. We then derive the same conclusion modulo q^2, q^3, and so on. $\square$

The crux of the proof is the following lemma.

Lemma 19.3.4. *Let M be a finite differential module over $\mathcal{E}^\dagger$ admitting a unit-root Frobenius structure. Suppose that M is c-constant for some $c < p^{-1/(p-1)}$. Then M is trivial as a differential module.*

Proof. We may assume that the Frobenius lift is standard. By Lemma 19.3.3 we may assume that there is a c-constant basis on which Φ, tD act via matrices A, N over $\mathcal{E}^\dagger \cap \mathfrak{o}_K[\![t^{-1}]\!]$. Then A and N together represent a finite differential module over $\cup_{\alpha \in (0,1)} K\langle \alpha/t \rangle$ equipped with a unit-root Frobenius structure. Moreover, the commutation relation $NA + t(d/dt)(A) = qA\phi(N)$ from (17.1.2.1) and the fact that $|A_0| = |A_0^{-1}| = |A|_1 = |A^{-1}|_1 = 1$ together force $N_0 = 0$ and $|N|_1 \le c$; consequently, there exists $\alpha \in (0, 1)$ for which $|N|_\alpha \le 1$.

We now proceed as in the proof of Lemma 18.2.3. As in that proof, we construct a matrix V, with entries in $\mathcal{E}^\dagger \cap K[\![t^{-1}]\!]$ and with $|V - I_n|_1 \le c$ and $|V - I_n|_\alpha \le 1$, for which

$$A' = V^{-1} A \phi(V), \qquad N' = V^{-1} N V + V^{-1} t \frac{d}{dt}(V)$$

have entries in $\mathcal{E}^\dagger \cap K[\![t^{-p}]\!]$. Let $\psi : K[\![t^{-1}]\!] \to K[\![t^{-1}]\!]$ be the K-linear substitution $t^{-1} \mapsto t^{-p}$. Since ϕ is standard, ϕ commutes with ψ; consequently, $A'' = \psi^{-1}(A')$, $N'' = p^{-1}\psi^{-1}(N')$ again satisfy the commutation relation $N''A'' + t(d/dt)(A'') = qA''\phi(N'')$. Since $|A'' - I_n|_1 = |A' - I_n|_1 \le c$, we deduce that $|N''|_1 \le c$; we also have $|N''|_{\alpha^p} = p|N'|_\alpha \le p$. To choose $u \in [0, 1]$ such that $p^u c^{1-u} = 1$ or, in other words, $u \log p + (1-u)\log c = 0$, we take

$$u = \frac{\log c}{\log c - \log p}.$$

Using Proposition 8.2.3(b) we obtain $|N''|_\beta \le 1$ for

$$\log \beta = up \log \alpha = \frac{p \log c}{\log c - \log p} \log \alpha.$$

Since $p \log c < \log c - \log p < 0$, we have $-\log \beta \ge (1 + \epsilon)(-\log \alpha)$ for some fixed $\epsilon > 0$.

Consequently, we can replace A, N by another pair for which the desired result can be derived equivalently, with α arbitrarily small. In particular we can force $\alpha < p^{-1/(p-1)}$. Now let M' be the differential module on the disc of radius α^{-1} in the coordinate t^{-1} with the action of $t^{-1}d/dt^{-1} = -td/dt$ given by $-N$. The fact $|N|_\alpha \leq 1$ implies by Theorem 6.5.3 that the spectral radius of d/dt^{-1} on M' is at most α. Hence the generic radius of convergence of M' at radius α^{-1} is at least $p^{-1/(p-1)}\alpha^{-1} > 1$, so, using Theorem 9.6.1 it can be proved that the local horizontal sections at infinity converge on a disc of radius greater than 1 in the parameter t^{-1}. We may then restrict these to a basis of horizontal sections of M. □

Proof of Theorem 19.3.1. It follows from Proposition 19.1.5 that $D^\dagger$ is fully faithful, so the key point is to show that $D^\dagger$ is essentially surjective. That is, for any finite differential module M over $\mathcal{E}^\dagger$ equipped with a unit-root Frobenius structure, there exists a finite separable extension L of $\kappa_K((t))$ such that $M \otimes_{\mathcal{E}^\dagger} \mathcal{E}_L^\dagger$ is a trivial differential module (and so corresponds to an unramified representation of $G_{\kappa_K((t))}$). Using Proposition 19.1.5, we may choose L so that $M \otimes_{\mathcal{E}^\dagger} \mathcal{E}_L^\dagger$ is c-constant for some $c < p^{-1/(p-1)}$. Then Lemma 19.3.4 implies the desired result. □

19.4 Ramification and differential slopes

In the equal-characteristic case, we can relate the upper numbering ramification filtration (Definition 3.4.3) of a Galois representation with finite local monodromy to the generic radius of convergence of an associated differential module, as follows. We will not give a proof here; see the notes for attributions and references plus some speculations about a mixed-characteristic analogue.

Theorem 19.4.1. *Assume that κ_K is perfect. Let V be a finite-dimensional vector space over K, and let $\tau : G_{\kappa_K((t))} \to \mathrm{GL}(V)$ be a continuous homomorphism for the p-adic topology on $\mathrm{GL}(V)$, with finite local monodromy. Then, for $\rho \in (0, 1)$ sufficiently close to 1,*

$$R(D^\dagger(V) \otimes F_\rho) = \rho^b, \qquad b = \max\{i \geq 1 : G^i_{\kappa_K((t))} \not\subseteq \ker(\tau)\}.$$

Corollary 19.4.2. *With the notation of Theorem 19.4.1, let $V_1, \dots, V_m$ be the constituents of V, and let $\tau_j : G_{\kappa_K((t))} \to \mathrm{GL}(V_j)$ be the corresponding homomorphisms for $j = 1, \dots, m$. For $\rho \in (0, 1)$ sufficiently close to 1, the multiset of subsidiary radii of $D^\dagger(V) \otimes F_\rho$ consists of $\rho^{b_1}, \dots, \rho^{b_n}$, where the multiset $\{b_1, \dots, b_n\}$ consists of $\max\{i \geq 1 : G_{\kappa_K((t)),i} \not\subseteq \ker(\tau_j)\}$ with multiplicity $\dim(V_j)$ for $j = 1, \dots, m$.*

Remark 19.4.3. One interpretation of Theorem 19.4.1 is that the decomposition of V by ramification numbers matches up with the Christol–Mebkhout decomposition of $D^\dagger(V) \otimes \mathcal{R}$ provided by Theorem 12.6.4. While the latter was inspired by analogues for meromorphic connections in the complex analytic setting, the analogy with wild ramification was anticipated somewhat before it was realized in Theorem 19.4.1.

Remark 19.4.4. Using the integrality properties of subsidiary radii (Theorem 11.3.2(b)), we may deduce that, for $\rho \in (0,1)$ sufficiently close to 1, the product of the subsidiary radii is an integral power of ρ; this amounts to verifying the *Hasse–Arf theorem* for V (the integrality of the Artin conductor).

An interesting corollary of Theorem 19.4.1 is the following.

Proposition 19.4.5. *Let M be a finite free differential module, on the open annulus with inner radius α and outer radius β, satisfying the Robba condition. Suppose that for some closed interval $[\gamma, \delta]$ with $\alpha < \gamma < \delta < \beta$ there exists a finite étale extension R of $K\langle \gamma/t, t/\delta\rangle$ such that the differential module $M \otimes_{K\langle \gamma/t, t/\delta\rangle} R$ is unipotent (i.e., a successive extension of trivial modules). Then there exists a positive integer m coprime to p such that the pullback of M along the map $t \mapsto t^m$ is unipotent.*

Proof. Note that the conclusion may be checked by replacing K by a finite Galois extension K'; this follows from the fact that

$$H^0(M \otimes_K K')^{G_{K'/K}} = H^0(M).$$

We may thus enlarge K and then rescale to ensure that $1 \in (\alpha, \beta)$. We may also assume that R is Galois over $K\langle \gamma/t, t/\delta\rangle$ and (possibly after enlarging K again) geometrically connected. That is, R is connected and remains so after any further finite enlargement of K.

Put $G = R \otimes_{K\langle \gamma/t, t/\delta\rangle} F_1$. Our hypotheses so far ensure that G is a field so, by Theorem 1.4.9, G carries a unique multiplicative norm extending $|\cdot|_1$. Since that norm is computed using Newton polygons, we can replace K by a finite extension to ensure that the multiplicative value group of G is the same as that of K.

By Lemma 8.6.1 the norm on G can be defined as the supremum norm for some basis of $R \otimes_{K\langle \gamma/t, t/\delta\rangle} K\langle t, t^{-1}\rangle$. After moving γ and δ closer to 1, we can define the same norm for a basis $e_1, \dots, e_n$ of R itself.

Let A be the matrix of the trace pairing of R in terms of this basis, i.e., A_{ij} is the trace of multiplication by $e_i e_j$ as an endomorphism of R over $K\langle \gamma/t, t/\delta\rangle$. On the one hand, because G has the same multiplicative value group as K, we must have $|\det(A)|_1 = 1$. On the other hand, since R is étale over $K\langle \gamma/t, t/\delta\rangle$, $\det(A)$ must be a unit in R. We conclude that $\det(A) = ct^m$ for some $c \in \mathfrak{o}_K^\times$

and some $m \in \mathbb{Z}$. In particular, R induces a finite étale extension $\overline{R}$ of $\kappa_K[t, t^{-1}]$.

Let $0 = M_0 \subset M_1 \subset \cdots \subset M_l = M$ be a filtration of M such that, for $i = 1, \dots, l$, $M_i/M_{i-1} \otimes_{K\langle \gamma/t, t/\delta \rangle} R$ is a trivial differential module. View the K-vector space

$$\bigoplus_i H^0((M_i/M_{i-1}) \otimes_{K\langle \gamma/t, t/\delta \rangle} R)$$

as a representation of $\mathrm{Aut}(\overline{R}/\kappa_K[t, t^{-1}])$. By restricting to the inertia groups at $t = 0$ and $t = \infty$, applying Theorem 19.4.1, and possibly making a different choice of $[\gamma, \delta]$ (but still with 1 in its interior), we can construct a subextension R' of R which induces a tamely ramified extension of $\kappa_K[t, t^{-1}]$ such that $M \otimes_{K\langle \gamma/t, t/\delta \rangle} R'$ is unipotent. However, a finite étale extension of $\kappa_K[t, t^{-1}]$ that is tamely ramified at $t = 0$ and $t = \infty$ must be contained in an extension of the form $\kappa'_K[t^{1/m}, t^{-1/m}]$ for some positive integer m and some finite separable extension κ'_K of κ_K. (This is usually deduced as a consequence of Grothendieck's theory of the tame quotient of the étale fundamental group of a scheme; see [**99**, Exposé XIII].)

This proves the claim for the restriction of M to the open annulus with inner radius γ and outer radius δ. The claim for M itself follows by Corollary 13.6.4. □

Notes

A more detailed survey of most material in this chapter (excluding the unit-root p-adic local monodromy theorem) is given in the article [**128**].

Proposition 19.1.5 was originally formulated by Fontaine [**89**, A1.2.6], in a slightly less general form and with no reference to differential modules. Our presentation more closely follows [**207**, Theorem 4.1.3]; a related result in the language of F-isocrystals is the theorem of Crew [**61**, Theorem 2.1].

For κ_K perfect, the rank 1 case of Theorem 19.3.1 is due to Crew [**61**, Theorem 4.12], while the general case is due to Tsuzuki [**207**, Theorem 5.1.1]; however, both arguments can be extended easily to the general case. An alternate exposition was given by Christol [**44**] in the case of a standard Frobenius lift, although without discussion of the fact that standardness is not stable under the replacement of $\kappa_K((t))$ by a finite separable extension. (In our argument this is treated by the invariance of the c-constant property under a change in Frobenius lift, as in Definition 19.3.2. The representation-theoretic approach we used is that taken in [**207**].) Yet another exposition may be inferred from [**135**, Theorem 4.5.2], where a stronger result is proved. (The stronger result is used in the study of semistable reduction for isocrystals; see the notes for Chapter 23.) All these proofs are similar in form to the proof given here; by contrast, one may infer a rather different proof by specializing Theorem 20.1.4 below to the unit-root case (Remark 20.1.5).

The fact that one can give a direct quantitative relationship between wild ramification and the spectral properties of differential modules fits nicely into a well-developed analogy between the structures of irregular formal meromorphic connections and those of wildly ramified ℓ-adic étale sheaves. An early suggestion along these lines was given by Katz [**121**] and pursued further by Terasoma [**204**]; some further work in this direction is that of Beilinson, Bloch, and Esnault [**13**].

Theorem 19.4.1 was originally stated in its present form by Matsuda [**169**, Corollary 8.8]; a reformulation in the formalism of Tannakian categories was given by André [**5**, Complement 7.1.2] as part of his formulation and proof of the p-adic local monodromy theorem. However, thanks to the p-adic global index theorem of Christol and Mebkhout [**51**, Theorem 8.4–1], [**52**, Corollaire 5.0–12], this could have been deduced from the Grothendieck–Ogg–Shafarevich formula for unit-root overconvergent F-isocrystals in rigid cohomology; such a formula was proved by Tsuzuki [**208**, Theorem 7.2.2] (by Brauer induction, as is possible in the ℓ-adic case) and Crew [**63**, Theorem 5.4] (using the Katz–Gabber theory of canonical extensions, as also is possible in the ℓ-adic case). For a proof by direct computation and Brauer induction (not using the Christol–Mebkhout index theory), see [**128**, Theorem 5.23].

In the case of an imperfect residue field, it was originally suggested by Matsuda [**170**] that one should formulate an analogue of Theorem 19.4.1 relating the Abbes–Saito conductor (discussed in the notes for Chapter 3) to a suitable differential analogue. That differential analogue was described by Kedlaya [**133**]; a comparison with the Abbes–Saito conductor was established by Chiarellotto and Pulita [**40**] for one-dimensional representations, and by Xiao [**219**] in the general case. This has the side effect of establishing the integrality of the Abbes–Saito conductor in the case of equal characteristic, which is not evident from the original construction.

In mixed characteristic the appropriate analogue of the functor $V \mapsto D(V)$ is provided by the theory of (ϕ, Γ)-modules; see Chapter 24. The sort of analogue of Theorem 19.4.1 that should exist in mixed characteristic is distinctly less clear. Even in the case of a perfect residue field, where one is asking for a differential interpretation of the usual conductor, only a partial answer is known, by a result of Marmora [**167**]: one has a differential interpretation of the conductor once one passes to a suitably large cyclotomic extension. In general, one does at least have integrality of the Abbes–Saito conductor when the residue characteristic is odd; see Xiao [**220**].

Exercises

(1) Suppose that the fixed field K_0 of ϕ on K has residue field $\mathbb{F}_q$ and the same value group as K. Prove that $\tilde{\mathcal{E}}^{\phi=1} = K_0$. (Hint: reduce to a corresponding equality of residue fields.)

(2) Prove that any finite free unit-root difference module over $\mathcal{E}$ admits a unique compatible differential structure.

(3) Prove that the kernel of d/dt on $\tilde{\mathcal{E}}$ equals $\tilde{K}$. In particular, for any finite separable extension L of $\kappa_K((t))$, prove that the kernel of d on $\mathcal{E}_L^\dagger$ is the integral closure of K in $\mathcal{E}_L^\dagger$.

(4) Write out explicitly the isomorphism $D^\dagger(V) \cong M$ implied by Example 19.2.6. (Hint: this module is generated by $(1 + p\pi f)^{1/p}$ for any lift $f \in \mathcal{E}^\dagger$ of $\overline{f}$.)

(5) Prove that, for an arbitrary Frobenius lift, the property whereby a finite differential module over $\mathcal{E}$ or $\mathcal{E}^\dagger$ equipped with a unit-root Frobenius structure is c-constant remains invariant under a change in Frobenius lift. (Hint: in Lemma 17.3.4, if the matrix of action of D/i has norm at most ϵ then so does the matrix of action of $D^i/i!$. Apply this observation to (17.3.1.1).)

20

The p-adic local monodromy theorem

We are now ready to state the capstone theorem of this book, the p-adic local monodromy theorem. This asserts that a finite differential module over an annulus carrying a Frobenius structure has "finite local monodromy", in the sense that it becomes unipotent after making a suitable finite étale cover of the annulus. In this chapter, we give the precise statement of the theorem, illustrate it with an example and a couple of basic applications, and discuss some technical points that arise if the field K has imperfect residue field. We will postpone discussion of the proof(s) of the theorem to the next chapter.

We will discuss two broad areas of application of the p-adic local monodromy theorem in later chapters. One of these is in the subject of rigid cohomology, where the theorem plays a role analogous to the ℓ-adic local monodromy theorem of Grothendieck in the subject of étale cohomology (hence the name); see Chapter 23. The other is in p-adic Hodge theory, where the theorem clarifies the structure of certain p-adic Galois representations; see Chapter 24.

Hypothesis 20.0.1. Throughout this chapter and the next, fix a homomorphism $\overline{\phi} : \kappa_K((t)) \to \kappa_K((t))$ preserving (but not necessarily fixing) κ_K and carrying t to t^q for some power q of p. We will assume that all the Frobenius lifts considered are in fact lifts of this particular $\overline{\phi}$.

20.1 Statement of the theorem

Definition 20.1.1. Let L be a finite separable extension of $\kappa_K((t))$ to which $\overline{\phi}$ extends. Put

$$\mathcal{R}_L = \mathcal{R} \otimes_{\mathcal{E}^\dagger} \mathcal{E}_L^\dagger.$$

We say that a finite free differential module M over $\mathcal{R}$ is *quasiconstant* if there exists L such that $M \otimes_{\mathcal{R}} \mathcal{R}_L$ is trivial. We say M is *quasiunipotent* if it is a successive extension of quasiconstant modules; this holds if and only if $M \otimes_{\mathcal{R}} \mathcal{R}_L$ is unipotent for some L (exercise).

Remark 20.1.2. It is more typical to make Definition 20.1.1 without reference to $\overline{\phi}$, so that L may be any finite separable extension of $\kappa_K((t))$. This corresponds precisely to the case where $\overline{\phi}$ is an absolute Frobenius morphism, as then $\overline{\phi}$ extends to any finite separable extension of $\kappa_K((t))$.

The condition of quasiunipotence implies solvability.

Proposition 20.1.3. *Let M be a finite free quasiunipotent differential module over $\mathcal{R}$. Then M is solvable at* 1.

Proof. We may reduce to the case where M is irreducible and hence quasiconstant. We may thus pick an L for which $M \otimes_{\mathcal{R}} \mathcal{R}_L$ is trivial. It is then clear that $M \otimes_{\mathcal{R}} \mathcal{R}_L$ is solvable at 1, and one can argue using direct calculation that this implies the same for M. Here, we will give a slightly different argument.

There is no harm in enlarging K, so we may assume that K is integrally closed in $\mathcal{R}_L$. In this case, $H^0(M \otimes_{\mathcal{R}} \mathcal{R}_L)$ is a vector space over K equipped with a linear action of $G_{L/\kappa_K((t))}$; that is, we have a representation of a finite group on a finite-dimensional vector space over a field of characteristic 0. (Without the enlargement of K, we might have only a semilinear group action.) It is a basic fact of the representation theory of finite groups that this representation can also be defined over a subfield of K that is finite over $\mathbb{Q}$; in particular, we can pick a finite extension K_0 of $\mathbb{Q}_p$ contained in K and a K_0-lattice T in $H^0(M \otimes_{\mathcal{R}} \mathcal{R}_L)$ that is stable under the Galois action.

Since the desired result makes no reference to $\overline{\phi}$, we can choose ϕ to be an absolute Frobenius lift fixing K_0. We then obtain a Frobenius structure on $M \otimes_{\mathcal{R}} \mathcal{R}_L$ by declaring T to be fixed under the Frobenius action. Since T is stable under $G_{L/\kappa_K((t))}$, this action descends to a Frobenius structure on M itself. We may thus deduce that M is solvable at 1, by applying Theorem 17.2.1. (The reader may recognize this construction from the proof of Theorem 19.3.1.) □

Conversely, the following important theorem asserts that many naturally occurring solvable differential modules, including Picard–Fuchs modules, are quasiunipotent. In the case of an absolute Frobenius lift, the theorem is due independently to André [**5**], Kedlaya [**125**], and Mebkhout [**171**]; the generalization to nonabsolute Frobenius lifts is original to this book. See Chapter 21 for a proof and see the notes for further discussion.

Theorem 20.1.4 (**p-adic local monodromy theorem**). *Let M be a finite free differential module over $\mathcal{R}$ admitting a Frobenius structure for some Frobenius lift. Then M is quasiunipotent.*

Remark 20.1.5. In the special case where ϕ is absolute and $M = M^\dagger \otimes_{\mathcal{E}^\dagger} \mathcal{R}$ for $M^\dagger$ a finite differential module over $\mathcal{E}^\dagger$ equipped with a unit-root Frobenius structure, Theorem 20.1.4 almost recovers Tsuzuki's theorem (Theorem 19.3.1). The missing ingredients are as follows.

(a) We must show that M is quasiconstant, not just quasiunipotent. This follows from the fact that a unipotent differential module over $\mathcal{R}$ equipped with a unit-root Frobenius structure must be trivial (exercise).

(b) We must show that $M^\dagger \otimes_{\mathcal{E}^\dagger} \mathcal{E}_L^\dagger$ is a trivial differential module. This follows from the triviality of $M \otimes_{\mathcal{R}} \mathcal{R}_L$, using Corollary 16.3.4.

20.2 An example

It may be worth seeing how Theorem 20.1.4 applies in a particular example. This example, corresponding to a Bessel equation, was originally considered by Dwork [**77**]; the analysis given here is due to Tsuzuki [**209**, Example 6.2.6] and was cited in the introduction to [**125**].

Example 20.2.1. Let p be an odd prime, put $K = \mathbb{Q}_p(\pi)$ with $\pi^{p-1} = -p$, and take ϕ to be the absolute p-power Frobenius morphism. Let M be the differential module of rank 2 over $\mathcal{R}$ with the action of D on a basis e_1, e_2 given by

$$N = \begin{pmatrix} 0 & t^{-1} \\ \pi^2 t^{-2} & 0 \end{pmatrix}.$$

Then M admits a Frobenius structure; this was shown by explicit calculation in [**77**] but can also be derived by consideration of a suitable Picard–Fuchs module. Define the tamely ramified extension L of $\kappa_K((t))$, and the corresponding extension $\mathcal{E}_L^\dagger$ of $\mathcal{E}^\dagger$, by adjoining to $\kappa_K((t))$ an element u such that $4u^2 = t$; then put

$$u_\pm = 1 + \sum_{n=1}^{\infty} (\pm 1)^n \frac{(2n)!^2}{(32\pi)^n n!^3} u^n \in K\{u\}.$$

Define the matrix

$$U = \begin{pmatrix} u_+ & u_- \\ \frac{ud}{du}(u_+) + \left(\frac{1}{2} - \pi u^{-1}\right) u_+ & \frac{ud}{du}(u_-) + \left(\frac{1}{2} + \pi u^{-1}\right) u_- \end{pmatrix}$$

and use it to change basis; then the action of d/du on the new basis e_+, e_- of $M \otimes_{\mathcal{R}} \mathcal{R}_L$ is via the matrix

$$\begin{pmatrix} -\frac{1}{2}u^{-1} + \pi u^{-2} & 0 \\ 0 & -\frac{1}{2}u^{-1} - \pi u^{-2}. \end{pmatrix}.$$

That is, $M \otimes_{\mathcal{R}} \mathcal{R}_L$ splits into two differential submodules of rank 1. To render these quasiconstant, we must adjoin to L a square root of u (to eliminate the term $-\frac{1}{2}u^{-1}$) and a root of the polynomial $z^p - z = u^{-1}$ (which, by Example 19.2.6, eliminates the terms $\pm\pi u^{-2}$).

By further analysis, carried out in [**209**, Example 6.2.6], one determines that in this example the special Newton slopes are $\frac{1}{2}\log p, \frac{1}{2}\log p$. By contrast, the generic Newton slopes (obtained by writing $M = M^\dagger \otimes_{\mathcal{E}^\dagger} \mathcal{R}$ using the chosen basis and then extending scalars to $\mathcal{E}$) are $0, \log p$. This illustrates the conclusion of the semicontinuity theorem for Newton polygons (Theorem 16.4.6).

20.3 Descent of sections

The property of quasiunipotence is rather handy because it often allows complicated-looking statements about nontrivial differential modules to be reduced to statements about trivial differential modules that can be checked by direct calculation. Over a disc, this can be achieved using Dwork's trick (Corollary 17.2.2); over an annulus, Theorem 20.1.4 often serves as a usable replacement. Here is a typical example, which builds on techniques of de Jong; see the notes for further discussion.

We start with a calculation that is in some sense dual to Lemma 16.3.2. Recall that $\mathcal{E}_\phi$ and $\mathcal{E}_\phi^\dagger$ denote the ϕ-perfections of $\mathcal{E}$ and $\mathcal{E}^\dagger$, respectively (Definition 15.4.1).

Lemma 20.3.1. *Let A be an $n \times n$ matrix over $\mathfrak{o}_{\mathcal{E}_\phi^\dagger}$, and suppose that $v \in \mathcal{E}_\phi^n$, $w \in (\mathcal{E}_\phi^\dagger)^n$ satisfy $Av - \phi(v) = w$. Then $v \in (\mathcal{E}_\phi^\dagger)^n$.*

Proof. Exercise. □

We next bring to bear de Jong's reverse filtration.

Lemma 20.3.2. *Let M be a finite dualizable difference module over $\mathcal{E}_\phi^\dagger$. Let $\psi : M \to \mathcal{E}_\phi$ be a nonzero ϕ-equivariant map (i.e., ψ commutes with ϕ using the specified ϕ-actions on M and $\mathcal{E}_\phi$). Then $\psi^{-1}(\mathcal{E}_\phi^\dagger)/\ker(\psi)$ has rank* 1 *and Newton slope* 0*, and $M/\psi^{-1}(\mathcal{E}_\phi^\dagger)$ has all its Newton slopes less than* 0*.*

Proof. We may assume that ψ is injective. Let M_1 be the first step of the filtration on M, given by Theorem 15.4.4. Then ψ induces a nonzero element of $H^0(M_1^\vee \otimes_{\mathcal{E}_\phi^\dagger} \mathcal{E}_\phi)$, which can only exist if M_1 is pure of norm 1. Moreover, by Lemma 20.3.1 (applied after using Proposition 14.4.16 to choose a suitable basis),

$$H^0(M_1^\vee) = H^0(M_1^\vee \otimes_{\mathcal{E}_\phi^\dagger} \mathcal{E}_\phi),$$

so ψ is induced by a nonzero morphism $M_1 \to \mathcal{E}_\phi^\dagger$. We deduce that $\psi^{-1}(\mathcal{E}_\phi^\dagger)$ is of rank 1; since ψ is assumed to be injective, M_1 must also be of rank 1. This proves the desired results. $\square$

To eliminate the perfection in the previous lemma, we use the projection $\mathcal{E}_\phi \to \mathcal{E}$ constructed in Lemma 15.4.5.

Proposition 20.3.3. *Let M be a finite dualizable difference module over $\mathcal{E}^\dagger$. Let $\psi : M \to \mathcal{E}$ be a nonzero ϕ-equivariant map. Then $\psi^{-1}(\mathcal{E}^\dagger)/\ker(\psi)$ has rank* 1 *and Newton slope* 0*, and $M/\psi^{-1}(\mathcal{E}^\dagger)$ has all Newton slopes less than* 0.

Proof. Put $M' = M \otimes_{\mathcal{E}^\dagger} \mathcal{E}_\phi^\dagger$, and let $\psi' : M' \to \mathcal{E}_\phi$ be the composition

$$M' \stackrel{\psi\otimes 1}{\to} \mathcal{E} \otimes_{\mathcal{E}^\dagger} \mathcal{E}_\phi^\dagger \to \mathcal{E}_\phi,$$

the last map being multiplication in $\mathcal{E}_\phi$. The map ψ' is nonzero because it restricts to ψ on M. (Note that the multiplication map is injective by Corollary 15.4.6, but we will use the proof technique of that result here rather than its statement.)

By Lemma 20.3.2 the desired results will follow from the assertion that the natural inclusions

$$\psi^{-1}(0) \otimes_{\mathcal{E}^\dagger} \mathcal{E}_\phi^\dagger \to (\psi')^{-1}(0),$$
$$\psi^{-1}(\mathcal{E}^\dagger) \otimes_{\mathcal{E}^\dagger} \mathcal{E}_\phi^\dagger \to (\psi')^{-1}(\mathcal{E}_\phi^\dagger)$$

are surjective. To check this assertion, choose a basis $m_1, \dots, m_n$ of M. Given $v \in (\psi')^{-1}(0)$ (resp. $v \in (\psi')^{-1}(\mathcal{E}_\phi^\dagger)$), write $v = \sum_{i=1}^n v_i m_i$ with $v \in \mathcal{E}_\phi^\dagger$. We induct on the largest integer j for which $v_j \neq 0$, the case where no such j exists being a trivial base case.

Define the map $\lambda : \mathcal{E}_\phi \to \mathcal{E}$ as in Lemma 15.4.5, and put $\lambda_M = \mathrm{id}_M \otimes \lambda : M' \to M$, so that $\psi \circ \lambda_M = \lambda \circ \psi'$. Then the quantity

$$\lambda_M(v/v_j) = \sum_{i=1}^{j} \lambda(v_i/v_j) m_i$$

belongs to $\psi^{-1}(0)$ (resp. to $\psi^{-1}(\mathcal{E}^\dagger)$). Moreover the coefficient of m_j in $v/v_j - \lambda_M(v/v_j)$ vanishes, so by the induction hypothesis the latter belongs to $\psi^{-1}(0) \otimes_{\mathcal{E}^\dagger} \mathcal{E}^\dagger_\phi$ (resp. to $\psi^{-1}(\mathcal{E}^\dagger) \otimes_{\mathcal{E}^\dagger} \mathcal{E}^\dagger_\phi$). Hence v does also. □

We now add the p-adic local monodromy theorem in order to split some exact sequences.

Lemma 20.3.4. *Suppose that κ_K is perfect. Let $0 \to M_1 \to M \to M_2 \to 0$ be a short exact sequence of finite differential modules over $\mathcal{E}^\dagger$ equipped with Frobenius structures. Suppose that every generic Newton slope of M_1 is strictly greater than every generic Newton slope of M_2. Then the exact sequence splits.*

Proof. As in Lemma 5.3.3, we may replace M_1, M_2 with $M_2^\vee \otimes M_1, \mathcal{E}^\dagger$; that is, we may reduce to the case where $M_2 = \mathcal{E}^\dagger$ and every generic Newton slope of M_1 is positive. By Theorem 16.4.6 every special Newton slope of M_1 is also positive. By Theorem 20.1.4, we can find a finite Galois extension L of $\kappa_K((t))$ such that $M_1 \otimes_{\mathcal{E}^\dagger} \mathcal{R}_L$ is unipotent. Since we have assumed that κ_K is perfect, $\mathcal{R}_L$ can itself be written as a Robba ring over some finite extension of K (see Remark 20.5.3 below). By a direct calculation (exercise), the exact sequence splits over $\mathcal{R}_L$. By Corollary 16.3.3, in the category of difference modules the map $H^1(M_1 \otimes_{\mathcal{E}^\dagger} \mathcal{E}^\dagger_L) \to H^1(M_1 \otimes_{\mathcal{E}^\dagger} \mathcal{R}_L)$ is injective, so the original exact sequence splits in the category of difference modules over $\mathcal{E}^\dagger_L$. That splitting is unique (as can be seen for the case of $\mathcal{E}_L$), so it descends to a splitting of the original sequence in the category of difference modules over $\mathcal{E}^\dagger$.

We now check that this splitting is also a splitting of differential modules. We start with an injective homomorphism $\mathcal{E}^\dagger \to M$ of difference modules. By comparing special slopes, we see that the resulting copy of $\mathcal{R}$ in $M \otimes_{\mathcal{E}^\dagger} \mathcal{R}$ must be the first step in the slope filtration of $M \otimes_{\mathcal{E}^\dagger} \mathcal{R}$, in the sense of Theorem 17.4.3. In particular, the copy of $\mathcal{R}$ in $M \otimes_{\mathcal{E}^\dagger} \mathcal{R}$ is a differential submodule, as is its pure descent to $\mathcal{E}^\dagger$. The latter is just the image of $\mathcal{E}^\dagger$ in M, so the claim follows. □

We now put the preceding lemmas together to get a theorem on the descent of horizontal sections.

Theorem 20.3.5. *Let M be a finite differential module over $R = K[\![t]\!]_0$ or $R = \mathcal{E}^\dagger$ admitting a Frobenius structure. Then, in the category of differential modules, we have*

$$H^0(M) = H^0(M \otimes_R \mathcal{E}).$$

Proof. For the case $R = K[\![t]\!]_0$ we have $H^0(M) = H^0(M \otimes_R \mathcal{E}^\dagger)$ by Proposition 17.2.5, so we may assume that $R = \mathcal{E}^\dagger$ hereafter. We may enlarge K to force κ_K to be perfect. We first check that any nonzero Φ-invariant horizontal section v of $M \otimes_{\mathcal{E}^\dagger} \mathcal{E}$ descends to M. The section v corresponds to a nonzero ϕ-equivariant map $\psi : M^\vee \to \mathcal{E}$. By Proposition 20.3.3, $\psi^{-1}(\mathcal{E}^\dagger) \neq 0$ and the generic slopes of $M^\vee/\psi^{-1}(\mathcal{E}^\dagger)$ are all negative. By Lemma 20.3.4 the exact sequence

$$0 \to \psi^{-1}(\mathcal{E}^\dagger)/\ker(\psi) \to M^\vee/\ker(\psi) \to M^\vee/\psi^{-1}(\mathcal{E}^\dagger) \to 0$$

must split. However, again by Proposition 20.3.3, any complement of $\psi^{-1}(\mathcal{E}^\dagger)/\ker(\psi)$ in $M^\vee/\ker(\psi)$ that is in the category of difference modules must map to zero under ψ. We conclude that in fact $\psi(M^\vee) = \mathcal{E}^\dagger$, forcing $v \in H^0(M)$.

To check the original claim (still for $R = \mathcal{E}^\dagger$), we may enlarge K to have an algebraically closed residue field. In this case, Corollary 14.6.4 implies that $H^0(M \otimes_{\mathcal{E}^\dagger} \mathcal{E})$ is spanned by one-dimensional fixed subspaces for some power of the Frobenius action. The previous argument shows that any generator of one of these subspaces belongs to M, proving the claim. □

20.4 Local duality

Here is another useful property of quasiunipotent differential modules, which by Theorem 20.1.4 is present whenever one has a Frobenius structure.

Lemma 20.4.1. *Put $R = \mathcal{E}^\dagger$ or $\mathcal{R}$. Let M be a finite free differential module over R. Then, for $i = 0, 1$, the natural maps*

$$H^i(M) \to H^i(M \otimes_{\mathcal{E}^\dagger} \mathcal{E}_L^\dagger)^{G_{L/\kappa_K((t))}}$$

are bijections.

Proof. For $i = 0$, we clearly have

$$H^0(M \otimes_{\mathcal{E}^\dagger} \mathcal{E}_L^\dagger)^{G_{L/\kappa_K((t))}} = H^0((M \otimes_{\mathcal{E}^\dagger} \mathcal{E}_L^\dagger)^{G_{L/\kappa_K((t))}}) = H^0(M).$$

For $i = 1$, we have injectivity because M is a direct summand of $M \otimes_{\mathcal{E}^\dagger} \mathcal{E}_L^\dagger$. We have surjectivity, because if $x \in M \otimes_{\mathcal{E}^\dagger} \mathcal{E}_L^\dagger$ represents a Galois-invariant class in $H^1(M \otimes_{\mathcal{E}^\dagger} \mathcal{E}_L^\dagger)$ then the average of $g(x)$ over $g \in G_{L/\kappa_K((t))}$ is an element of M representing the same class. □

Proposition 20.4.2. *Suppose that κ_K is perfect. (This assumption can be removed; see Section 20.5.) Let M be a finite free quasiunipotent differential* *module over $\mathcal{R}$. Then the spaces $H^0(M)$, $H^1(M)$ are finite-dimensional, and there is a perfect pairing*

$$H^0(M) \times H^1(M^\vee) \to H^1(M \otimes M^\vee) \to H^1(\mathcal{R}) \cong K\frac{dt}{t}.$$

In particular, by Theorem 20.1.4 this holds whenever M admits a Frobenius structure.

Proof. This is straightforward to check when M is unipotent (exercise). In general, we may choose a finite Galois extension L of $\kappa_K((t))$ such that $M \otimes_{\mathcal{R}} \mathcal{R}_L$ is unipotent. Since we have assumed that κ_K is perfect, $\mathcal{R}_L$ can itself be written as a Robba ring over some finite extension of K (see Remark 20.5.3 below), so the desired assertions hold for $M \otimes_{\mathcal{R}} \mathcal{R}_L$. We may then deduce the desired results using Lemma 20.4.1. (See also [**128**, Proposition 4.26].) □

This leads to the following result of Matsuda [**169**, Theorem 7.8].

Corollary 20.4.3 (Matsuda). *Every indecomposable finite free quasiunipotent differential module over $\mathcal{R}$ has the form $M \otimes N$ with M quasiconstant and N unipotent.*

Proof. Exercise. □

20.5 When the residue field is imperfect

We mentioned earlier (Remark 19.0.1) that the reader should assume that the residue field κ_K of K is perfect. We now discuss the possible confusion arising when this hypothesis is omitted.

Remark 20.5.1. The Cohen structure theorem asserts that if F is a complete discretely valued field of characteristic $p > 0$ then there exists an isomorphism $F \cong \kappa_F((t))$. However, this isomorphism is far from unique; it depends not only on the choice of the series parameter t but also on the choice of embedding of the residue field κ_F into F. This choice is unique when κ_F is perfect (because in that case one has Teichmüller lifts; see the notes for Chapter 14), but not otherwise.

Example 20.5.2. Suppose that $F = \mathbb{F}_p(x)((t))$. Then, for any $y = \sum_{i=1}^\infty y_i t^i$, there is an embedding $\mathbb{F}_p(x) \hookrightarrow F$ given by $x \mapsto x + y$.

Remark 20.5.3. Suppose further that E is a finite separable extension of F. By the Cohen structure theorem, we can find copies of κ_E and κ_F inside E and F, respectively, and use these to present E and F as power series fields (possibly in different series parameters, if $|E^\times| \neq |F^\times|$). If κ_E is separable over κ_F, we can choose the copy of κ_E to be the integral closure in E of the chosen copy of κ_F, by Proposition 3.2.3 (or simply by applying Hensel's lemma directly). If $F = \kappa_K((t))$ and $E = L$, we can then view $\mathcal{R}_L$ as a copy of the Robba ring with coefficients in the unramified extension of $\kappa_K = \kappa_F$ with residue field κ_E. However, this may not be possible if κ_E is not separable over κ_F, as in the following example.

Example 20.5.4. Suppose again that $F = \mathbb{F}_p(x)((t))$, and put

$$E = F[z]/(z^p - z - xt^{-p}).$$

Then $\kappa_E \cong \mathbb{F}_p(x^{1/p})$, but there is no pth root of x within E itself. To write $E \cong \kappa_E((t))$, we must make a different choice of the copy of κ_E within E, e.g., $\mathbb{F}_p(zt)$.

Fortunately, in our setting there is a convenient way to skirt this issue using the Frobenius map $\overline{\phi}$.

Lemma 20.5.5. *Suppose that $\overline{\phi}$ is bijective on κ_K. Then, for any finite Galois extension E of $F = \kappa_{\mathcal{E}}$ equipped with an extension of $\overline{\phi}$, κ_E is separable over κ_F.*

Proof. Let U and T be the maximal unramified and tamely ramified subextensions, respectively, of E over F. Then the claim holds for the extensions U/F and T/U (the former by the definition of an unramified extension, the latter by Proposition 3.3.6). We may thus assume that E is totally wildly ramified over F.

By Remark 3.3.10 the extension E/F can be written as a tower of Artin–Schreier extensions $F = E_0 \subset E_1 \subset \cdots \subset E_l = E$. It may happen that E_1 is not preserved by the action of $\overline{\phi}$ on E; however, it must be carried to another $\mathbb{Z}/p\mathbb{Z}$-subextension of E over F, of which there are only finitely many. Consequently E_1 is preserved by some power of $\overline{\phi}$; similarly, we can choose a power of $\overline{\phi}$ preserving $E_2, \dots, E_l$.

Since the desired result is insensitive to the replacement of $\overline{\phi}$ by a power, we may thus reduce to considering a single Artin–Schreier extension

$$E = F[z]/(z^p - z - P), \qquad P = \sum_i c_i t^i.$$

Since E admits an extension of $\overline{\phi}$, there must exist $a \in \mathbb{F}_p^\times$ such that $P - a\overline{\phi}(P) = y^p - y$ for some $y \in F$. By replacing $\overline{\phi}$ with a suitable power, we may reduce to the case $a = 1$.

Remember that we do not change the extension by replacing P with $P + y^p - y$ for $y \in F$. We may thus choose P so that, for any two indices $i, j < 0$

such that $c_i, c_j \neq 0$, the ratio i/j is not a power of p. Now let j be the smallest integer for which $c_j \neq 0$. We are done if either $j \geq 0$, in which case E is unramified, or $j < 0$ is not divisible by p, in which case $\#(|E^\times|/|F^\times|) > 1$ and so, by Lemma 3.1.1, $\kappa_E = \kappa_F$.

Suppose that, on the contrary, j is divisible by p. By the choice of P and the fact that $P - \overline{\phi}(P) = y^p - y$ for some $y \in F$, we must have $\overline{\phi}(c_j t^j) = (c_j t^j)^q$. That is, $c_j^{1/q} = \overline{\phi}^{-1}(c_j) \in \kappa_K$ since $\overline{\phi}_K$ was assumed to be bijective. We can thus replace P by $P + (c_j t^j)^{1/p} - c_j t^j$ to increase j. This process repeats only finitely many times, after which we may deduce the claim. $\square$

Notes

The p-adic local monodromy theorem (Theorem 20.1.4) can be viewed as an archimedean analogue of the following theorem of Borel [see **194**, Theorem 6.1]. Given a vector bundle with connection over the punctured unit disc in the complex plane, equipped with a polarized variation of (rational) Hodge structures, Borel's theorem asserts that the monodromy transformation is forced to be quasiunipotent (i.e., its eigenvalues must be roots of unity). It appears that the Frobenius structure in the p-adic setting plays the role of the variation of Hodge structures in the complex analytic realm.

The p-adic local monodromy theorem was originally formulated and proved only in the absolute case. Then, it is often referred to in the literature as *Crew's conjecture* because it emerged from the work of Crew [**62**] on the finite dimensionality of rigid cohomology with coefficients in an overconvergent F-isocrystal. Crew's original conjecture was more limited still, as it concerned only modules such that the differential and Frobenius structures were both defined over $\mathcal{E}^\dagger$; it was restated in a more geometric form by de Jong [**66**]. A closer analysis of Crew's conjecture was then given by Tsuzuki [**209**], who explained (using Theorem 19.3.1, which he had proved in [**207**]) how Theorem 20.1.4 in the absolute case would follow from a slope filtration theorem [**209**, Theorem 5.2.1].

The original proofs of Theorem 20.1.4 in the absolute case can be briefly described as follows. The proof of Kedlaya uses slope filtrations (Theorem 17.4.3) to reduce everything to the unit-root case (Theorem 19.3.1). André and Mebkhout proceed by using properties of solvable differential modules to reduce everything to the p-adic Fuchs theorem of Christol and Mebkhout (Theorem 13.6.1). André's proof was phrased in the language of Tannakian categories, whereas Mebkhout's proof was more explicit; the proof in the relative case that we will give in the next chapter is closely modeled on Mebkhout's proof.

For applications to rigid cohomology, as far as we know only the absolute case of Theorem 20.1.4 is of any relevance. However, the nonabsolute case occurs in the context of relative p-adic Hodge theory. Namely, Berger and Colmez [**18**] used the full strength of Theorem 20.1.4 to prove an analogue of Fontaine's conjecture on the potential semistability of de Rham representations (Corollary 24.4.5 below) for a family of de Rham representations parametrized by an affinoid base space.

To our knowledge, no proof of the nonabsolute case of Theorem 20.1.4 has appeared prior to the one we are about to describe in Chapter 21. The proof by Kedlaya does not apply because it relies on Theorem 19.3.1, whose given proof does not extend to the relative case. The use of the nonabsolute case of Theorem 20.1.4 in [**18**] provided a strong motivation for working at this level of generality in the present book.

Lemma 20.3.1 is a mild generalization of [**206**, Proposition 2.2.2]. A weaker result in the same spirit appears in the work of Cherbonnier and Colmez [**38**].

For an absolute Frobenius lift, the case of Theorem 20.3.5 with $R = \mathcal{E}^\dagger$ was originally conjectured by Tsuzuki [**210**, Conjecture 2.3.3] and proved by Kedlaya [**126**, Theorem 1.1]. However, most of the ideas are already present in the subcase $R = K[\![t]\!]_0$, which was established by de Jong [**65**, Theorem 9.1]; the main difference is that de Jong could use Dwork's trick (Corollary 17.2.2) where we had to use the p-adic local monodromy theorem (Theorem 20.1.4). Since a weak form of Dwork's trick applies even without a differential structure (Theorem 16.1.1), it should be possible to extend the case $R = K[\![t]\!]_0$ of Theorem 20.3.5 to difference modules; this was carried out in the absolute case in [**130**], but we expect the general case to follow similarly.

The proof of Theorem 20.3.5 given here is in substance the same as the proof in the absolute case given in [**126**]. In particular, Proposition 20.3.3 is essentially [**126**, Proposition 4.2], which in turn is close to de Jong's [**65**, Corollary 8.2].

The original application of Theorem 20.3.5 in the case $R = K[\![t]\!]_0$ is de Jong's proof of the equal-characteristic analogue of Tate's extension theorem on p-divisible groups (Barsotti–Tate groups). Tate proved [**203**] that for R a complete discrete valuation ring of characteristic 0 and residual characteristic $p > 0$, given any two p-divisible groups over R, any morphism between their generic fibres extends to a full morphism. Tate's proof introduced the seeds that grew into the subject of p-adic Hodge theory (see Chapter 24); de Jong recognized that an appropriate analogue of Tate's method for R of characteristic p would proceed by means of crystalline Dieudonné theory. In this manner, de Jong [**65**, Theorem 1.1] reduced this analogue of Tate's theorem to an instance of Theorem 20.3.5.

The case $R = \mathcal{E}^\dagger$ of Theorem 20.3.5 has applications in the theory of overconvergent F-isocrystals (for more on which see Chapter 23). Namely, it implies (after some work) that, on a smooth variety over a field of characteristic $p > 0$, the restriction functor from overconvergent F-isocrystals to convergent F-isocrystals is fully faithful [**137**, Theorem 4.2.1].

The local duality for quasiunipotent differential modules (Proposition 20.4.2) is due to Matsuda [**169**]. See also the treatment in [**128**, §4].

Exercises

(1) Prove that a finite free differential module M over $\mathcal{R}$ is quasiunipotent if and only if $M \otimes_{\mathcal{R}} \mathcal{R}_L$ is unipotent for some L. (Hint: produce a nonzero quasiconstant submodule of M, e.g., by Galois descent.)

(2) Let $0 \to M_1 \to M \to M_2 \to 0$ be a nonsplit extension of differential modules over $\mathcal{R}$, where M_1 and M_2 are both trivial and of rank 1. Prove that M cannot admit a Frobenius structure that induces unit-root Frobenius structures on both M_1 and M_2. (Hint: use the fact that $H^1(\mathcal{R}) = K\,dt/t$.)

(3) Prove Lemma 20.3.1. (Hint: reduce to the case where $|A|_\rho \leq 1$ for some $\rho \in (0, 1)$ for which $|w|_\rho < \infty$. Then use $|w|_\rho$ to bound the terms of v of norm greater than some $c > 0$.)

(4) Complete the proof of Lemma 20.3.4 by proving the following assertion. Let M be a finite unipotent differential module over $\mathcal{R}$, equipped with a Frobenius structure whose Newton slopes are all positive. Prove that any exact sequence $0 \to M \to N \to \mathcal{R} \to 0$ in the category of differential modules with Frobenius structures must split. (Hint: apply Lemma 9.2.3 to N and then see how the resulting basis behaves under a standard Frobenius lift.)

(5) In the notation of Theorem 20.3.5, suppose that $\phi(t) = t^q$. Prove that the equality $H^0(M) = H^0(M \otimes_R \mathcal{E})$ also holds in the category of difference modules. (Hint: if $v \in H^0(M)$ then $t^{-1}D(v) \in H^0(M)$ also.)

(6) Prove Proposition 20.4.2 in the case where M is unipotent. (Hint: use Lemma 9.2.3.)

(7) Prove Corollary 20.4.3 using Proposition 20.4.2. (Hint: let P be an indecomposable finite free quasiunipotent differential module over $\mathcal{R}$. First, prove that P is a successive extension of copies of a single irreducible differential module M. Then construct an isomorphism $P \cong M \otimes N$, with N unipotent, by induction on the rank of P.)

21

The p-adic local monodromy theorem: proof

In this chapter we give a proof of the p-adic local monodromy theorem, at the full level of generality at which we stated it (Theorem 20.1.4). After some initial reductions, we start with the case of a module of differential slope 0, i.e., one satisfying the Robba condition. We describe how this case can be treated using either the p-adic Fuchs theorem for Christol–Mebkhout annuli (Theorem 13.6.1) or the slope filtration theorem of Kedlaya (Theorem 16.4.1). We then treat the rank 1 case using the classification of rank 1 solvable modules from Chapter 12. We then show that any module of rank greater than 1 and prime to p can be made reducible, by comparing the module with its top exterior power and using properties of refined differential modules. We finally handle the case of a module M of rank divisible by p by considering $M^\vee \otimes M$ instead.

The reader may notice some similarities to the proof of the Turrittin–Levelt–Hukuhara decomposition theorem (Theorem 7.5.1). In fact, this theorem is also known as the *p-adic Turrittin theorem* for this reason.

Besides the running hypothesis for this part of the book (Hypothesis 14.0.1) and the hypothesis from the previous chapter (Hypothesis 20.0.1), it will be convenient to set several more hypotheses.

21.1 Running hypotheses

We are going to make a number of calculations under the same hypotheses. Rather than repeat the hypotheses each time, we enunciate them once and for all here. We first explain how to deal with the case where κ_K is imperfect.

Remark 21.1.1. First, we point out that it suffices to prove quasiunipotence, after replacing K by a complete extension K' to which ϕ_K extends, either by

Proposition 6.9.1 or a more elementary argument (see [**125**, Proposition 6.11]). In particular we may take K' to be the ϕ-perfection of K, on which ϕ is bijective.

Here is a variant of the previous remark.

Remark 21.1.2. One can also proceed without ϕ-perfecting K initially but replacing K by its inverse image under ϕ at any time when needed. We end up with a finite extension L of $\phi^{-m}(\kappa_K)((t))$, for some positive integer m, such that $M \otimes_{\mathcal{R}} \mathcal{R}_L$ is quasiunipotent. In particular, $H^0(M \otimes_{\mathcal{R}} \mathcal{R}_L)$ is a nonzero space on which ϕ acts bijectively. Applying ϕ^m gives a nonzero element of $H^0(M \otimes_{\mathcal{R}} \mathcal{R}_{L'})$ for a finite separable extension L' of $\kappa_K((t))$. We may deduce that M has a nonzero Φ-stable differential submodule that is quasiunipotent; repeating this argument recovers Theorem 20.1.4 in full.

We are now ready to introduce the new running hypotheses mentioned in the introduction to this chapter.

Hypothesis 21.1.3. For the rest of the chapter, assume that ϕ_K is bijective on K; by the preceding remarks this is harmless for the purpose of proving Theorem 20.1.4. Let F be a finite Galois (but not necessarily unramified) extension of K. Put $\mathcal{R}_F = \mathcal{R} \otimes_K F$; we will not attempt to extend ϕ to $\mathcal{R}_F$. Let M be a finite differential module over $\mathcal{R}$ equipped with a Frobenius structure.

Hypothesis 21.1.4. Within each lemma in this chapter, set the notation as follows. Let N be a nonzero differential submodule (that is not a subquotient) of $M \otimes_{\mathcal{R}} \mathcal{R}_F$ for some specified F. We will use L to indicate an initially unspecified finite separable extension of $\kappa_K((t))$ to which $\overline{\phi}$ extends; since ϕ_K is bijective on K, Lemma 20.5.5 implies that $\mathcal{R}_L$ may be viewed as a Robba ring over a finite unramified extension of K. We will use F' to indicate an initially unspecified finite extension of F that is Galois over the integral closure K' of K in $\mathcal{R}_L$. (This may require the identification of a subfield of F larger than K with an isomorphic subfield of $\mathcal{R}_L$.) Write $\mathcal{R}_{L,F'} = \mathcal{R}_L \otimes_{K'} F'$.

Notation 21.1.5. We write N_0 to refer to the component of N of differential slope 0, as provided by Theorem 12.6.4.

21.2 Modules of differential slope 0

We start by describing objects of differential slope 0. This requires the use of either of two pieces of heavy machinery: the theory of p-adic exponents (Chapter 13) or the theory of slope filtrations for difference modules over the Robba ring (Theorem 17.4.3).

Lemma 21.2.1. *Suppose that N has differential slope* 0. *Then we may choose L such that $N \otimes_{\mathcal{R}_F} \mathcal{R}_{L,F}$ is unipotent.*

Before giving either proof, we insert a reduction common to both.

Remark 21.2.2. In Lemma 21.2.1 the existence of N forces M to have a nontrivial summand M_0 of differential slope 0, such that N appears in $M \otimes_{\mathcal{R}} \mathcal{R}_F$. It thus suffices to prove that if M is of differential slope 0 then we may choose L such that $M \otimes_{\mathcal{R}} \mathcal{R}_L$ is unipotent.

We first give the proof using p-adic exponents.

First proof of Lemma 21.2.1. As in Remark 21.2.2, we may assume that M itself is of differential slope 0. By Corollary 17.3.2 we may change to a standard Frobenius lift. We may then apply Corollary 13.6.2 to deduce that $M \otimes_{\mathcal{R}} \mathcal{R}[t^{1/m}]$ is unipotent for some positive integer m coprime to p. □

We next give the proof using slope filtrations.

Second proof of Lemma 21.2.1. Again as in Remark 21.2.2, we may assume that M itself is of differential slope 0. By Theorem 17.4.3 (and changing the Frobenius lift, as in Corollary 17.3.2), we may reduce to the case where M is pure of norm 1 as a difference module.

In other words, Lemma 21.2.1 is reduced to the following claim. Let M be a finite differential module over $\mathcal{E}^\dagger$ admitting a unit-root Frobenius structure for a standard Frobenius lift. Suppose that $IR(M \otimes F_\rho) = 1$ for $\rho \in (0, 1)$ sufficiently close to 1. Then there exists a positive integer m coprime to p such that $M \otimes_{\mathcal{R}} \mathcal{R}[t^{1/m}]$ is unipotent.

To prove this claim, apply Lemma 19.3.3 to construct a basis of $M \otimes_{\mathcal{R}} \mathcal{E}^\dagger[t^{1/m}]$ for some m, on which Φ, tD act via matrices A, N such that A, A^{-1}, and N have entries in $\mathcal{E}^\dagger \cap \mathfrak{o}_K[\![t^{-1/m}]\!]$. As in the proof of Lemma 19.3.4, this forces $N_0 = 0$.

For notational simplicity, assume hereafter that $m = 1$. Let $f : K[\![t^{-1}]\!] \to K[\![t]\!]$ denote the substitution $t^{-1} \mapsto t$. We may then view $f(N)$ as defining a differential module M' over $K\langle t/\beta \rangle$ for the operator d/dt for some $\beta > 1$ such that $IR(M' \otimes_{K\langle t/\beta\rangle} F_\beta) = 1$. By Theorem 9.6.1 this module is trivial on the open disc of radius β; this implies that the original module M is trivial, and in particular unipotent. □

Remark 21.2.3. It would be interesting to know whether one can prove Lemma 21.2.1 without using either p-adic exponents or slope filtrations but instead simply using the fact that the hypothesis forces M to extend across the entire punctured open unit disc (by the pasting together of Frobenius antecedents).

21.3 Modules of rank 1

We next consider the rank-1 case, using the classification of solvable rank 1 differential modules (Theorem 12.7.2). This proof is the only point in the course of the proof of Theorem 20.1.4 at which we will introduce any specific wildly ramified extensions of $\kappa_K((t))$. (We made some tamely ramified extensions in Lemma 21.2.1 and will make some unramified extensions in Lemma 21.4.1 below.)

Lemma 21.3.1. *Suppose that* $\operatorname{rank}(N) = 1$. *Then we may choose L such that $N \otimes_{\mathcal{R}_F} \mathcal{R}_{L,F}$ is trivial.*

Proof. We may assume that F contains all pth roots of 1. By Theorem 17.2.1, M is solvable at 1 as then is $M \otimes_{\mathcal{R}} \mathcal{R}_F$ and also N. By Theorem 12.7.2 there exists a nonnegative integer h such that $N^{\otimes p^h}$ has differential slope 0. If $h = 0$, we may deduce the claim from Lemma 21.2.1. It suffices to check the case $h = 1$, as we may apply this case repeatedly to deduce the general case.

In the case $h = 1$, by Theorem 17.1.6 (or the special case of the latter discussed in Remark 17.1.7), there exist $c_1, \dots, c_b \in \{0\} \cup \mathfrak{o}_F^\times$, with $c_i = 0$ whenever i is divisible by p, such that the rank 1 differential module $N_1 = M_{1,c_1} \otimes \cdots \otimes M_{b,c_b}$ over $\mathcal{R}_F$ has the property that $N_1^\vee \otimes N$ has differential slope 0. Since $N_1 \otimes_{\mathcal{R}_F} \mathcal{R}_{L_0,F}$ is unipotent for L_0 equal to the Artin–Schreier extension L_0 of $\kappa_F((t))$ defined by the parameter $\overline{c_1} t^{-1} + \cdots + \overline{c_b} t^{-b}$, so is $N \otimes_{\mathcal{R}_F} \mathcal{R}_{L_0,F}$.

Unfortunately, this construction does not guarantee that L_0 admits an action of $\overline{\phi}$. However, suppose that we run over all possible choices of N for the given M and F and define each L_0 we see, using the unique possible Artin–Schreier parameter in the additive group

$$\bigoplus_{i<0,\, p\nmid i} \kappa_K^{\mathrm{alg}} t^{-i}.$$

The resulting parameters then fill out a finite set S.

Pick an integer i with $i < 0, p \nmid i$. Let S_i be the set of coefficients of t^{-i} appearing in the elements of S. Let $\overline{P}_i(x) \in \kappa_K[x]$ be the product of the distinct minimal polynomials of the elements of S_i over κ_K. Since M admits a Frobenius structure, the roots of $\overline{\phi}(\overline{P}_i(x))$, when multiplied by t^{-qi}, must define Artin–Schreier parameters that are equivalent to the roots of $\overline{P}_i(x)$ when multiplied by t^{-i}. In other words,

$$\overline{\phi}(\overline{P}_i(x)) = \overline{P}_i(x^{1/q})^q.$$

We may thus extend $\overline{\phi}$ to the splitting field of $\overline{P}_i$ in such a way that, for each root a of $\overline{P}_i$, we have $\overline{\phi}(a) = b^q$ for some root b of $\overline{P}_i$.

This allows us to define a compositum of Artin–Schreier extensions L to which $\overline{\phi}$ extends; this includes all the choices of L_0 made above. For such an L, $N \otimes_{\mathcal{R}_F} \mathcal{R}_{L,F}$ has differential slope 0, so we can make it trivial by Lemma 21.2.1. This gives the desired result. □

21.4 Modules of rank prime to p

We next pass from rank 1 to rank prime to p, using what we know about refined differential modules over F_ρ.

Lemma 21.4.1. *Suppose that* $\mathrm{rank}(N)$ *is coprime to p. Then we may choose L, F' such that $N \otimes_{\mathcal{R}_F} \mathcal{R}_{L,F'}$ is either unipotent or reducible.*

Proof. Put $n = \mathrm{rank}(N)$. The case $n = 1$ is covered by Lemma 21.3.1, so we may assume that n is greater than 1 and coprime to p. Suppose by way of contradiction that $N \otimes_{\mathcal{R}_F} \mathcal{R}_{L,F'}$ is irreducible and nontrivial for all L, F'. By Lemmas 21.2.1 and 21.3.1, we may reduce to the case where $(M^\vee \otimes M)_0$ is unipotent and $\wedge^n N$ is trivial.

Let us realize M, M_0, N as differential modules on some annulus $\alpha < |t| < 1$. For $\rho \in (\alpha, 1)$, let $F_\rho = F[\![\rho/t, t/\rho]\!]_{\mathrm{an}}$, $F'_\rho = F'[\![\rho/t, t/\rho]\!]_{\mathrm{an}}$ denote the fields of analytic elements on the circle of radius ρ over the respective base fields F, F'. We claim that $N \otimes F_\rho$ must be refined for every $\rho \in (\alpha, 1)$. Otherwise, by Theorem 10.6.7, for a suitable choice of F' and some positive integer m coprime to p we could split N nontrivially over $F'_\rho(t^{1/m})$. Each projector for this splitting would define a horizontal section of $(N^\vee \otimes N) \otimes F'_\rho(t^{1/m})$. Since $(M^\vee \otimes M)_0$ is unipotent, so is its subquotient $(N^\vee \otimes N)_0$; consequently we would have

$$H^0(N^\vee \otimes N \otimes F'_\rho(t^{1/m})) = H^0(N^\vee \otimes N)$$

in terms of a basis of $(N^\vee \otimes N)_0$, as in Lemma 9.2.3. We would thus get a nontrivial splitting of N itself, contrary to hypothesis.

We now have that N_ρ is refined for every $\rho \in (\alpha, 1)$. Choose $\rho \in (\alpha, 1)$ such that $p^{p^{-h}/(p-1)} < IR(N \otimes F_\rho) < p^{p^{-h+1}/(p-1)}$ for some nonnegative integer h. Apply Theorem 10.4.2 to form the h-fold Frobenius antecedent P of $N \otimes F_\rho$, which is still refined. We may then apply Proposition 6.8.4(a) to deduce that $P^{\otimes n}$ has the same spectral radius as P and Proposition 6.8.4(b) to deduce that $(\wedge^n P^\vee) \otimes P^{\otimes n}$ has a strictly greater spectral radius than P. However, these two contradict each other because $\wedge^n P^\vee$ is the h-fold Frobenius antecedent of the trivial module $(\wedge^n N^\vee) \otimes F_\rho$. This contradiction yields the desired result. $\square$

21.5 The general case

We now make the step from rank prime to p to arbitrary rank. The trick used here is one familiar from elementary group theory, e.g., it is used in the proof of the Sylow theorems. See the exercises for another example of its use.

Lemma 21.5.1. *For arbitrary N, we may choose L, F' such that $N \otimes_{\mathcal{R}_F} \mathcal{R}_{L,F'}$ is either unipotent or reducible.*

Proof. By Lemma 21.4.1 it suffices to consider N of rank n divisible by p. Then the trace-0 component of $N^\vee \otimes N$ has rank $n^2 - 1$, which is not divisible by p. By repeated application of Lemma 21.4.1, we can force the trace-0 component of $(N^\vee \otimes N) \otimes_{\mathcal{R}_F} \mathcal{R}_{L,F'}$ to acquire a unipotent component. In particular, the space $V = H^0((N^\vee \otimes N) \otimes_{\mathcal{R}_F} \mathcal{R}_{L,F'})$ has F'-dimension greater than 1.

We may view V as a finite-dimensional, not necessarily commutative, F'-algebra. A standard fact about such algebras (exercise) is that, for some finite extension F'' of F', $V \otimes_{F'} F''$ fails to be a division algebra. Thus for suitable F'' we can find a pair of nonzero horizontal that endomorphisms of $N \otimes \mathcal{R}_{L,F''}$ which compose to zero. This forces N to be reducible. $\square$

Proof of Theorem 20.1.4. It suffices to consider M as irreducible. By Lemma 21.5.1, we may choose L, F' such that $M \otimes_{\mathcal{R}} \mathcal{R}_{L,F'}$ is either unipotent or reducible. In the former case we are done, as this implies that $M \otimes_{\mathcal{R}} \mathcal{R}_L$ is also unipotent. Otherwise, $M \otimes_{\mathcal{R}} \mathcal{R}_{L,F'}$ contains a proper nonzero differential submodule N. By applying Lemma 21.5.1 repeatedly, we may keep replacing N by a submodule (after changing L and F') until N becomes unipotent.

Apply Φ to $(M \otimes_{\mathcal{R}} \mathcal{R}_L) \otimes_{K'} F'$ on the left and $\mathrm{Gal}(F'/K')$ on the right. The images of N fill out a submodule of the form $N_0 \otimes_{K'} F'$, where N_0 is a nonzero differential submodule of M stable under Φ. The module N_0 is unipotent because $N_0 \otimes_{K'} F'$ is unipotent. Consequently $M \otimes_{\mathcal{R}} \mathcal{R}_L$ has a nonzero unipotent submodule N_0 stable under Φ, so we may apply the induction hypothesis to $(M \otimes_{\mathcal{R}} \mathcal{R}_L)/N_0$ to conclude. $\square$

Notes

The approach to Theorem 20.1.4 presented here is modeled on that given by Mebkhout [**171**] in the absolute case, except that Mebkhout describes only the first of our two proofs of Lemma 21.2.1. The approach of André [**5**] is substantively similar but formally different, as it is phrased in the language of Tannakian categories.

In the course of proving Lemma 21.3.1, we noticed that if $\overline{\phi}$ is not an absolute Frobenius lift, then the condition of admitting an extension of $\overline{\phi}$ is a highly nontrivial restriction on finite separable extensions of $\kappa_K((t))$. This phenomenon was noted in the context of p-adic Hodge theory by Berger and Colmez [**18**, Proposition 6.2.2].

Exercises

(1) Let G be a finite p-group and let $\tau : G \to \mathrm{GL}(V)$ be a complex linear representation of G. Prove that if τ is nontrivial then either τ or $\tau^\vee \otimes \tau$ has a nontrivial one-dimensional subrepresentation. (Hint: consider the possible dimensions of irreducible representations of G.)

(2) Let V be a finite-dimensional, not necessarily commutative, F-algebra. Prove that there exists a finite extension F' of F such that $V \otimes_{F'} F$ is not a division algebra. (Hint: we may assume that V itself is a division algebra. Pick any nonzero $x \in V$, view x as a linear transformation from V to V via left multiplication, and subtract an eigenvalue of this transformation.)

Part VI

Areas of Application

22

Picard–Fuchs modules

In this final part of the book, we touch briefly on some areas of application of the theory of p-adic differential equations. These chapters are intended more to inspire than to inform, with statements that are more illustrative than definitive.

In the present chapter we revisit the territory of Chapter 0, briefly discussing how Picard–Fuchs modules give rise to differential equations with Frobenius structures and what this has to do with zeta functions.

22.1 Origin of Picard–Fuchs modules

The original source for p-adic differential equations, which inspired the general theory, was the following construction.

Definition 22.1.1. Let K be a field of characteristic 0. Let t be a coordinate on the projective line $\mathbb{P}^1_K$. Let $f : X \to \mathbb{P}^1_K$ be a proper, flat, generically smooth morphism of algebraic varieties. Let $S \subset \mathbb{P}^1_K$ be a zero-dimensional subscheme containing ∞ (for convenience) and all points over which f is not smooth. The *Picard–Fuchs modules* on $\mathbb{P}^1_K \setminus S$ associated with f are certain finite locally free differential modules M_i for $i = 0, \dots, 2\dim(f)$ over $\Gamma(\mathbb{P}^1_K \setminus S, \mathcal{O})$ with respect to the derivation d/dt; they also have regular singularities with rational exponents at each point of S. For $\lambda \notin S$, the fibre of M_i at λ can be canonically identified with the ith de Rham cohomology of the fibre $f^{-1}(\lambda)$.

Although the classical construction of the Picard–Fuchs module is analytic (it involves viewing f as an analytically locally trivial fibration and integrating differentials against moving homology classes), there is an algebraic construction, due to Katz and Oda [**123**], involving a Leray spectral sequence for the algebraic de Rham cohomology of the total space.

Example 22.1.2. Perhaps the most fundamental example of a Picard–Fuchs module comes from the Legendre family of elliptic curves

$$y^2 = x(x-1)(x-t),$$

with $S = \{0, 1, \infty\}$. In this case, the Picard–Fuchs module is precisely the differential module derived from the hypergeometric differential equation

$$y'' + \frac{c-(a+b+1)z}{z(1-z)}y' - \frac{ab}{z(1-z)}y = 0,$$

with parameters $(a, b, c) = (1/2, 1/2, 1)$, as considered in Chapter 0. See [**213**, §7] for an algebraic derivation of this fact, in the manner of Katz and Oda.

22.2 Frobenius structures on Picard–Fuchs modules

The example considered in Chapter 0 was just one of a number of explicit examples studied by Dwork and others, in which there seemed to be some strong relationship between the Picard–Fuchs equations derived in characteristic 0 and the zeta functions observed over finite fields. Dwork was able to give a somewhat systematic explanation, in some cases, in terms of Frobenius structures; nowadays, the technology of p-adic cohomology (about which more in Chapter 23) can be used to give a fairly general explanation.

We first give an explicit statement to the effect that Picard–Fuchs modules always carry Frobenius structures. See the notes for a more detailed discussion.

Theorem 22.2.1. *With notation as above, assume that the field K is complete for a discrete valuation. Also suppose that f extends to a proper morphism $\mathfrak{X} \to \mathbb{P}^1_{\mathfrak{o}_K}$ such that the intersection of $\mathbb{P}^1_k$ with the nonsmooth locus is contained in the intersection of $\mathbb{P}^1_k$ with the Zariski closure of S (i.e., the morphism is smooth over all points of $\mathbb{P}^1_k$ that are not the reductions of points in S). Let M_i be the ith Picard–Fuchs module for f, and let $\phi : \mathbb{P}^1_{\mathfrak{o}_K} \to \mathbb{P}^1_{\mathfrak{o}_K}$ be a Frobenius lift (e.g., $t \mapsto t^p$) that acts on $\mathfrak{o}_K$ as a lift of the absolute Frobenius morphism. Then, for some $\alpha \in (0, 1)$, there exists an isomorphism $\phi^*(M_i) \cong M_i$ over the Fréchet completion R of $\Gamma(\mathbb{P}^1_K \setminus S, \mathcal{O})$ for the ρ^{-1}-Gauss norm and the Gauss norms $|t - \lambda| = \rho$, $\rho \in [\alpha, 1)$ and $\lambda \in S$.*

Remark 22.2.2. Geometrically, the Frobenius structure is defined on the complement in $\mathbb{P}^1_K$ of a union of discs around the points of S, each of radius less than 1. (This includes a disc of radius less than 1 around ∞, by which we mean the complement of a disc of radius greater than 1 around 0.) In particular, by working in a unit disc not containing any points of S, we obtain a differential module with Frobenius structure over $K[\![t]\!]_0$. In a unit disc containing one or more points of S, we obtain a differential module with Frobenius structure only over $\cup_{\alpha>0} K\langle \alpha/t, t]\!]_0$. If the disc contains exactly one point of S and the exponents at that point are all 0, we can also obtain a differential module with Frobenius structure over $K[\![t]\!]_0$ for the derivation $t\,d/dt$, provided that ϕ fixes that point.

Example 22.2.3. In Example 22.1.2, Theorem 22.2.1 applies directly except when $p = 2$. In that case the reduction modulo p fails to be generically smooth; one must change the defining equation to get a usable description mod 2.

22.3 Relationship to zeta functions

We next give an explicit statement to the effect that the Frobenius structure on a Picard–Fuchs module can be used to compute zeta functions. (The condition on λ allows for a unique choice in each residue disc, by Lemma 15.2.6.)

Theorem 22.3.1. *Retain the notation of Theorem 22.2.1, but now assume that $\kappa_K = \mathbb{F}_q$ with $q = p^a$ and that ϕ is a qth-power Frobenius lift on $\mathbb{P}^1_{\mathfrak{o}_K}$. Suppose that $\lambda \in \mathfrak{o}_K$ satisfies $\phi(t-\lambda) \equiv 0 \pmod{t-\lambda}$ and that f extends smoothly over the residue disc containing λ. Then*

$$\zeta(f^{-1}(\overline{\lambda}), T) = \prod_{i=0}^{2\dim(f)} \det(1 - T\Phi, (M_i)_\lambda)^{(-1)^{i+1}}.$$

This suggests an interesting strategy for computing zeta functions, which was described by Alan Lauder.

Remark 22.3.2. Suppose that we have in hand the differential module M_i, plus the action of Φ on some individual $(M_i)_\lambda$. (This data would ordinarily be specified by a basis of M_i, the matrix of action of D, and the matrix of action of Φ modulo $t - \lambda$.) View the equation

$$NA + \frac{dA}{dt} = \frac{d\phi(t)}{dt} A\phi(N)$$

as a differential equation with initial condition provided by $(M_i)_\lambda$; we may then solve for A and evaluate at another λ.

More explicitly, let us suppose for simplicity that $\lambda = 0$ is the starting value and $\lambda = 1$ is the target value. In the open unit disc around 0, we can compute U such that

$$U^{-1}NU + U^{-1}\frac{dU}{dt} = 0$$

and then write down

$$A = UA_0\phi(U^{-1}).$$

This gives only a power series representation around $t = 0$ with radius of convergence 1; it does not give any way in which we can specialize to $\lambda = 1$.

However, Theorem 22.2.1 implies that the entries of A can be written as uniform limits of rational functions with limited denominators. For the purposes of computing the zeta function, we can limit how much p-adic accuracy is needed in the computations by giving some bounds on the degrees of the polynomials that appear and the sizes of the coefficients (e.g., using the Weil conjectures, Theorem 0.2.5). To obtain this much accuracy, we must compute A modulo some particular power of p. This means that we must determine some rational function whose degree we can (in principle) control, so it suffices to determine suitably many terms in the power series expansion around 0. We can then reconstruct a sufficiently good rational function approximation to A and evaluate at $\lambda = 1$.

Remark 22.3.3. One can recover from Theorem 22.3.1 the example of Dwork discussed in Chapter 0. In that example one is separating the Picard–Fuchs module, which has rank 2, into a unit-root component and a component of slope $\log p$. For this to be possible, one must be in the situation of Theorem 15.3.4; this fails precisely at the residue discs at which the Igusa polynomial vanishes, which is why one must invert the Igusa polynomial in the course of the computation.

Notes

The notion of algebraic de Rham cohomology was introduced by Grothendieck in [**98**], where he gave an algebraic construction of the topological de Rham cohomology of a complex algebraic variety. The construction involves commingling the straightforward cohomology of the de Rham complex (constructed from a module of Kähler differentials) with the cohomology of coherent algebraic sheaves. The subject was further developed by Hartshorne [**105, 106**]. A thorough treatment of Grothendieck's theorem can be found in [**97**].

In the final footnote to [**98**], Grothendieck gave what we believe was the first suggestion that one should consider in general what we call Picard–Fuchs modules. His suggestion was based on Manin's work on the Mordell conjecture over function fields; partly for this reason, Picard–Fuchs modules are also commonly known as *Gauss–Manin connections*. (This terminology is apparently due to Grothendieck. The name presumably refers to Gauss's study of hypergeometric differential equations, which incorporated the original Picard–Fuchs equation governing the periods of elliptic curves.) A good collection of onward references can be found in [**109**].

The fact that a Picard–Fuchs module has regular singularities with rational exponents can be proved in several ways, both geometric (Griffiths, Landman) and arithmetic (Katz). One can also formulate a more abstract version, concerning polarized variations of Hodge structures; this was proved by Borel and extended by Schmid (see the notes to Chapter 20 for further discussion of this last point). Again, see [**109**] for references.

The construction of Frobenius structures on Picard–Fuchs modules is a consequence of general results in the theory of p-adic cohomology, which we discuss further in the next chapter. For the moment we point out that Theorem 22.2.1 in its stated form can be found in [**211**, Theorem 3.3.1].

Lauder's strategy for computing zeta functions (also called the *deformation method*) was introduced in [**154**]; it has been worked out in detail for hyperelliptic curves by Hubrechts [**116**]. Hubrechts implemented the resulting algorithm in version 2.14 of the computer algebra system `Magma`. A version for hypersurfaces was described by Gerkmann [**96**].

23

Rigid cohomology

It has been suggested several times in this book that the study of p-adic differential equations is deeply connected with the theory of *p-adic cohomology* for varieties over finite fields. In particular, the Frobenius structures arising on Picard–Fuchs modules, discussed in the previous chapter, appear within this theory.

In this chapter, we introduce a little of the theory of rigid p-adic cohomology, as developed by Berthelot and others. In particular, we illustrate the role played by the p-adic local monodromy theorem in a fundamental finiteness problem in the theory.

23.1 Isocrystals on the affine line

We start with a concrete description of p-adic cohomology in a very special case, namely the cohomology of "locally constant" coefficient systems on the affine line over a finite field. This is due to Crew [**62**].

Definition 23.1.1. Let k be a perfect (for simplicity) field of characteristic $p > 0$. Let K be a complete discrete (again for simplicity) nonarchimedean field of characteristic 0 with $\kappa_K = k$. An *overconvergent F-isocrystal* on the affine line over k (with coefficients in K) is a finite differential module with Frobenius structure on the ring $\mathcal{A} = \cup_{\beta>1} K\langle t/\beta \rangle$, for some absolute Frobenius lift ϕ; as in Proposition 17.3.1 the resulting category is independent of the choice of Frobenius lift.

Definition 23.1.2. Let M be an overconvergent F-isocrystal on the affine line over k. Let $\mathcal{R}$ be a copy of the Robba ring with series parameter t^{-1}, so that we can identify $\mathcal{A}$ as a subring of $\mathcal{R}$. (We can write explicitly $\mathcal{R} = \cup_{\beta>1} K\langle \beta^{-1}/t^{-1}, t^{-1}\}$.) Define

$$\begin{aligned}
H^0(\mathbb{A}^1_k, M) &= \ker(D, M), \\
H^1(\mathbb{A}^1_k, M) &= \operatorname{coker}(D, M), \\
H^0_{\mathrm{loc}}(\mathbb{A}^1_k, M) &= \ker(D, M \otimes_{\mathcal{A}} \mathcal{R}), \\
H^1_{\mathrm{loc}}(\mathbb{A}^1_k, M) &= \operatorname{coker}(D, M \otimes_{\mathcal{A}} \mathcal{R}), \\
H^1_c(\mathbb{A}^1_K, M) &= \ker(D, M \otimes_{\mathcal{A}} (\mathcal{R}/\mathcal{A})), \\
H^2_c(\mathbb{A}^1_K, M) &= \operatorname{coker}(D, M \otimes_{\mathcal{A}} (\mathcal{R}/\mathcal{A})).
\end{aligned}$$

By taking kernels and cokernels in the short exact sequence

$$0 \to M \to M \otimes_{\mathcal{A}} \mathcal{R} \to M \otimes_{\mathcal{A}} (\mathcal{R}/\mathcal{A}) \to 0$$

and applying the snake lemma, we get an exact sequence

$$\begin{aligned}
0 \to H^0(\mathbb{A}^1_k, M) \to H^0_{\mathrm{loc}}(\mathbb{A}^1_k, M) \to H^1_c(\mathbb{A}^1_k, M) \\
\to H^1(\mathbb{A}^1_k, M) \to H^1_{\mathrm{loc}}(\mathbb{A}^1_k, M) \to H^2_c(\mathbb{A}^1_k, M) \to 0.
\end{aligned}$$

Remark 23.1.3. Crew [**62**] showed that in this construction H^i computes the *rigid cohomology* of M, H^i_c computes the *rigid cohomology with compact supports*, and H^i_{loc} computes some sort of local cohomology at ∞.

Crew's main result in this setting was the following.

Theorem 23.1.4 (Crew). *The spaces $H^i(\mathbb{A}^1_k, M)$, $H^i_c(\mathbb{A}^1_k, M)$, $H^i_{\mathrm{loc}}(\mathbb{A}^1_K, M)$ are all finite-dimensional over K. Moreover, the Poincaré pairings*

$$\begin{aligned}
H^i(\mathbb{A}^1_k, M) \times H^{2-i}_c(\mathbb{A}^1_k, M^\vee) &\to H^2_c(\mathbb{A}^1_k, \mathcal{A}) \cong K, \\
H^i_{\mathrm{loc}}(\mathbb{A}^1_k, M) \times H^{1-i}_{\mathrm{loc}}(\mathbb{A}^1_k, M^\vee) &\to H^1_{\mathrm{loc}}(\mathbb{A}^1_K, \mathcal{A}) \cong K
\end{aligned}$$

are perfect.

The key ingredient is the fact that, by the p-adic local monodromy theorem (Theorem 20.1.4), $M \otimes_{\mathcal{A}} \mathcal{R}$ is quasiunipotent which implies the finiteness of $H^i_{\mathrm{loc}}(\mathbb{A}^1_k, M)$. This further implies the finite dimensionalities, except for $H^1_c(\mathbb{A}^1_k, M)$ and $H^1(\mathbb{A}^1_k, M)$; however, these are related by a map with finite-dimensional kernel and cokernel. Moreover, they carry incompatible topologies: the former is a Fréchet space while the latter is dual to a Fréchet space. This incompatibility can be resolved only if both spaces are finite-dimensional.

23.2 Crystalline and rigid cohomology

Next, we briefly discuss how the previous example fits into a broader theory of p-adic cohomology, deferring most references to the notes.

Motivated by the work of Dwork and also by related work of Monsky and Washnitzer, Grothendieck proposed a method of constructing an analogue of algebraic de Rham cohomology in positive characteristic. This was developed by Berthelot and Ogus into the theory of *crystalline cohomology*. An important example of this is the fact that if X is a smooth proper scheme over $\mathbb{Z}_p$ then the algebraic de Rham cohomology of X carries a Frobenius action that computes the zeta function of the special fibre. (This generalizes to explain the results on Frobenius structures and zeta functions discussed in the previous chapter.)

One defect of crystalline cohomology is that it gives a coherent cohomology theory only for schemes over a finite field that are smooth and proper. By contrast, the work of Monsky and Washnitzer had given a good theory for smooth *affine* schemes. To fuse these theories, Berthelot introduced the theory of *rigid cohomology* as well as a theory of locally constant coefficient objects, *overconvergent F-isocrystals*. The example in the previous paragraph demonstrates the computation of the cohomology of coefficients on the affine line; a theory of cohomology with compact supports, and a local cohomology theory, are also demonstrated by Crew's construction.

For trivial coefficients, it was shown by Berthelot that rigid cohomology has all the desired properties of a Weil cohomology: finite dimensionality, Poincaré duality, Künneth formula, cycle class maps, the Lefschetz trace formula for Frobenius morphisms, etc. These were extended to nonconstant coefficients by Kedlaya, using a relative version of Theorem 23.1.4.

Berthelot also suggested a theory of constructible coefficients, based on ideas from the theory of algebraic $\mathcal{D}$-modules. (These are modules over a ring of differential operators; they are the natural coefficient objects in algebraic de Rham cohomology.) Recent work of Caro, Kedlaya, and Tsuzuki has shown that they form a good theory of coefficients, enjoying the same formal properties as their ℓ-adic étale counterparts. For example, one can execute a proof of the Weil conjectures entirely using p-adic cohomology [**132**].

23.3 Machine computations

In recent years, interest has emerged in explicit computation of the zeta functions of algebraic varieties defined over finite fields. Some of this interest has come from cryptography, in particular the use of the Jacobians of elliptic (and later hyperelliptic) curves over finite fields as "black box abelian groups" for certain public-key cryptography schemes (Diffie-Hellman, ElGamal). For elliptic curves, a good method for doing this was proposed by Schoof [**196**]. It amounts to computing the trace of the Frobenius morphism on the ℓ-torsion points, the set of which is otherwise known as the étale cohomology with

$\mathbb{F}_\ell$-coefficients, for values of ℓ small enough to determine uniquely the one unknown coefficient of the zeta function within the range prescribed by the Hasse–Weil bound.

It turns out to be somewhat more difficult to execute Schoof's scheme for curves of higher genus, as discovered by Pila [**180**]. One is forced to work with higher-division polynomials in order to compute the torsion of the Jacobian of the curve; an interpretation in terms of étale cohomology is of little value because the definition of étale cohomology is not intrinsically computable. (It is easy to write down cohomology classes, but it is difficult to test two such classes for equality.)

It has been noticed by several authors that rigid cohomology is intrinsically more computable and so lends itself better to this sort of task. Specifically, Kedlaya [**124**] proposed an algorithm using rigid cohomology to compute the zeta function of a hyperelliptic curve over a finite field of small odd characteristic. The limitation to odd characteristic was lifted by Denef and Vercauteren [**72**]; the limitation to small characteristic was somewhat remedied by Harvey [**108**], who improved the dependence on the characteristic p from $O(p)$ to $O(p^{1/2+\epsilon})$.

More recently, interest has emerged in considering also higher-dimensional varieties; this has come partly from potential applications in the study of mirror symmetry for Calabi–Yau varieties. In this case étale cohomology is of no help at all, since there is no geometric interpretation of H^i_{et} for $i > 1$ analogous to the interpretation for $i = 1$ in terms of the Jacobian. Rigid cohomology should still be computable, but relatively little progress has been made in making these computations practical (one exception being the treatment of smooth surfaces in projective 3-space in [**3**]). It may be necessary to combine these techniques with Lauder's deformation method (see Remark 22.3.2) to obtain the best results.

Notes

Crew's work, and subsequent work that builds on it (e.g., [**131**]), makes essential use of nonarchimedean functional analysis, as was evident in the discussion of Theorem 23.1.4. We recommend Schneider's book [**195**] as a user-friendly introduction to this topic.

For general surveys of the subject of p-adic cohomology, we recommend [**117**] (for crystalline cohomology only) and [**141**] (broader but more advanced). Also useful at this level of generality is Berthelot's original article outlining the theory of rigid cohomology [**22**].

For the basics of crystalline cohomology, see [**27**] and references within (largely to Berthelot's thesis [**21**]). For the work of Monsky and Washnitzer which motivated it, in addition to the original papers [**172, 173, 174**] there is also a useful survey by van der Put [**213**].

Until recently no comprehensive introductory text on rigid cohomology was available. That state of affairs was remedied by the appearance of the book of Le Stum [**157**]. Berthelot's own attempt at a foundational treatise remains incomplete, but what does exist [**23**] may also be helpful. Some of the function of a foundational text has been served by Berthelot's articles concerning finite dimensionality [**24**] and Poincaré duality [**25**].

For coefficients in an overconvergent F-isocrystal, the basic properties (finite dimensionality, Poincaré duality, the Künneth formula) are proved in [**131**] using techniques similar to those of Crew in [**62**]. However, these techniques do not appear to suffice for the construction of a full coefficient theory. For this, one needs a higher-dimensional analogue of Theorem 20.1.4, called the *semistable reduction theorem* for overconvergent F-isocrystals. This asserts that an overconvergent F-isocrystal on a noncomplete variety can always be extended to a logarithmic isocrystal on some compactification, after pulling back along a generically finite cover; such a result allows reductions to arguments about (logarithmic) crystalline cohomology. (Except for the part of the theorem that concerns the cover, this is analogous to Deligne's theory of canonical extensions in [**68**].) See [**134, 137, 140, 143**] for the proof; the original results from the present book feature prominently in [**143**].

Berthelot wrote a useful survey of his theory of arithmetic $\mathcal{D}$-modules [**26**]. It is this theory that has been developed by Caro (in part jointly with Tsuzuki) into a full theory of p-adic coefficients. This work is quite expansive and requires a summary more detailed than we can provide here; see [**35, 36**] for a representative sample.

On the subject of machine calculations, as a companion to our original paper on hyperelliptic curves [**124**] we recommend Edixhoven's course notes [**83**]. Some discussion is also included in [**93**, Chapter 7]. We gave a condensed summary of the general approach in [**127**]. For more on the role of elliptic and hyperelliptic curves in cryptography (including the relevance of the problem of the machine computation of zeta functions), the standard first reference is [**55**] even though it is some years behind the frontiers of this fast-moving subject. (For instance, it predates the growing area of *pairing-based cryptography*.)

24

p-adic Hodge theory

For our last application we turn to the subject of p-adic Hodge theory. Recall that in Chapter 19 we described a "nonabelian Artin–Schreier" construction, giving an equivalence of categories between continuous representations of the absolute Galois group of a positive characteristic local field on a p-adic vector space and certain differential modules with Frobenius structures. In this chapter, we describe an analogous construction for the Galois group of a mixed-characteristic local field. We also mention a couple of applications of this construction.

Hypothesis 24.0.1. Throughout this chapter, let K be a finite extension of $\mathbb{Q}_p$, let V be a finite-dimensional $\mathbb{Q}_p$-vector space, and let $\tau : G_K \to \mathrm{GL}(V)$ be a continuous homomorphism for the p-adic topology on V.

24.1 A few rings

We begin with the "field of norms" construction of Fontaine and Wintenberger.

Definition 24.1.1. Put $K_n = K(\zeta_{p^n})$ and $K_\infty = \cup_n K_n$. Let $F = \operatorname{Frac} W(\kappa_K)$ and F' be the maximal subfields of K and K_∞, respectively, that are unramified over $\mathbb{Q}_p$. Put $H_K = G_{K_\infty}$ and $\Gamma_K = G_{K_\infty/K} = G_K/H_K$.

Definition 24.1.2. Put $\mathfrak{o} = \mathfrak{o}_{\mathbb{C}_p}$. Let $\tilde{\mathbf{E}}^+$ be the inverse limit of the system

$$\cdots \to \mathfrak{o}/p\mathfrak{o} \to \mathfrak{o}/p\mathfrak{o},$$

in which each map is the p-power Frobenius morphism (which is a ring homomorphism). More explicitly, the elements of $\tilde{\mathbf{E}}^+$ are sequences $(x_0, x_1, \dots)$ of elements of $\mathfrak{o}/p\mathfrak{o}$ for which $x_{n+1}^p = x_n$ for all n. In particular, for any nonzero $x \in \tilde{\mathbf{E}}^+$, the quantity $p^n v_p(x_n)$ is the same for all n for which

$x_n \neq 0$; we call this quantity $v(x)$ and conventionally put $v(0) = +\infty$. Choose $\epsilon = (\epsilon_0, \epsilon_1, \dots) \in \tilde{\mathbf{E}}^+$ with $\epsilon_0 = 1$ and $\epsilon_1 \neq 1$. (This choice is somewhat analogous to the choice of a square root of -1 in $\mathbb{C}$.)

The following observations were made by Fontaine and Wintenberger [**92**].

Proposition 24.1.3. *The following are true.*

(a) *The ring $\tilde{\mathbf{E}}^+$ is a domain in which $p = 0$, with fraction field $\tilde{\mathbf{E}} = \tilde{\mathbf{E}}^+[(\epsilon - 1)^{-1}]$.*

(b) *The function $v : \tilde{\mathbf{E}}^+ \to [0, +\infty]$ extends to a valuation on $\tilde{\mathbf{E}}$ under which $\tilde{\mathbf{E}}$ is complete and $\mathfrak{o}_{\tilde{\mathbf{E}}} = \tilde{\mathbf{E}}^+$.*

(c) *The field $\tilde{\mathbf{E}}$ is the algebraic closure of $\kappa_K((\epsilon - 1))$. (The embedding of $\kappa_K((\epsilon - 1))$ into $\tilde{\mathbf{E}}$ exists because $v(\epsilon - 1) = p/(p-1) > 0$.)*

Definition 24.1.4. Let $\tilde{\mathbf{A}}$ be the ring of Witt vectors of $\tilde{\mathbf{E}}$, i.e., the unique complete discrete valuation ring with maximal ideal p and residue field $\tilde{\mathbf{E}}$. The uniqueness follows from the fact that $\tilde{\mathbf{E}}$ is algebraically closed, hence perfect. In particular, the p-power Frobenius morphism on $\tilde{\mathbf{E}}$ lifts to an automorphism ϕ of $\tilde{\mathbf{A}}$. (See the notes for Chapter 14 for further discussion of Witt vectors.)

Definition 24.1.5. Each element of $\tilde{\mathbf{A}}$ can be uniquely written as the sum $\sum_{n=0}^{\infty} p^n [x_n]$, where $x_n \in \tilde{\mathbf{E}}$ and $[x_n]$ denotes the *Teichmüller lift* of x_n (the unique lift of x_n that has a p^mth root in $\tilde{\mathbf{A}}$ for all positive integers m); note that $\phi([x]) = [x^p] = [x]^p$. We may thus equip $\tilde{\mathbf{A}}$ with a *weak topology*, in which a sequence $x_m = \sum_{n=0}^{\infty} p^n [x_{m,n}]$ converges to zero if, for each n, $v(x_{m,n}) \to +\infty$ as $m \to +\infty$. Let $\mathbf{A}_{\mathbb{Q}_p}$ be the completion of $\mathbb{Z}_p[([\epsilon] - 1)^{\pm}]$ in $\tilde{\mathbf{A}}$ for the weak topology; as a topological ring, it is isomorphic to the ring $\mathfrak{o}_{\mathcal{E}}$ defined over the base field $\mathbb{Q}_p$ with its own weak topology. It is also ϕ-stable because $\phi([\epsilon]) = [\epsilon]^p$.

Definition 24.1.6. Let $\mathbf{A}$ be the completion of the maximal unramified extension of $\mathbf{A}_{\mathbb{Q}_p}$, viewed as a subring of $\tilde{\mathbf{A}}$. Put

$$\mathbf{A}_K = \mathbf{A}^{H_K},$$

where the right-hand side makes sense because we have chosen all the rings so far in a functorial fashion, so that they indeed carry a G_K-action. Note that $\mathbf{A}_K$ can be written as a ring of the form $\mathfrak{o}_{\mathcal{E}}$, but with coefficients in F' rather than in $\mathbb{Q}_p$.

Definition 24.1.7. For any ring denoted with a boldface $\mathbf{A}$ so far, define the corresponding ring with $\mathbf{A}$ replaced by $\mathbf{B}$ by tensoring over $\mathbb{Z}_p$ with $\mathbb{Q}_p$. For instance, $\tilde{\mathbf{B}} = \tilde{\mathbf{A}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is the fraction field of $\tilde{\mathbf{A}}$.

24.2 (ϕ, Γ)-modules

We are now ready to describe the mechanism, introduced by Fontaine, for converting Galois representations into modules over various rings equipped with much simpler group actions.

Definition 24.2.1. Recall that V is a finite-dimensional vector space equipped with a continuous G_K-action. Put

$$D(V) = (V \otimes_{\mathbb{Q}_p} \mathbf{B})^{H_K};$$

by the Hilbert–Noether theorem, $D(V)$ is a finite-dimensional $\mathbf{B}_K$-vector space, and the natural map $D(V) \otimes_{\mathbf{B}_K} \mathbf{B} \to V \otimes_{\mathbb{Q}_p} \mathbf{B}$ is an isomorphism. Since we have taken only H_K-invariants, $D(V)$ retains a semilinear action of $G_K/H_K = \Gamma_K$; it also inherits an action of ϕ from $\mathbf{B}$. That is, $D(V)$ is a (ϕ, Γ)-*module* over $\mathbf{B}_K$, i.e., a finite free $\mathbf{B}_K$-module equipped with semilinear ϕ and Γ_K-actions that commute with each other. It is also *étale*, which is to say the ϕ-action is étale (unit-root), because, as in Definition 19.2.4, one can find a G_K-invariant lattice in V.

Theorem 24.2.2 (Fontaine). *The functor D, from the category of continuous representations of G_K on finite-dimensional $\mathbb{Q}_p$-vector spaces to the category of étale (ϕ, Γ)-modules over $\mathbf{B}_K$, is an equivalence of categories.*

Proof. The inverse functor is

$$V = (D(V) \otimes_{\mathbf{B}_K} \mathbf{B})^{\phi=1}.$$

The argument is similar to that in the proof of Proposition 19.1.5; see [**89**]. □

In much the same way that Proposition 19.1.5 was refined by Theorem 19.3.1, Theorem 24.2.2 was refined by Cherbonnier and Colmez as follows [**38**]. The big difference is that no additional restriction on the Galois representations is imposed.

Definition 24.2.3. Let $\mathbf{B}^{\dagger}_{\mathbb{Q}_p}$ be the image of the bounded Robba ring $\mathcal{E}^{\dagger}$, under the identification of $\mathcal{E}$ (having coefficients in $\mathbb{Q}_p$) with $\mathbf{B}_{\mathbb{Q}_p}$, sending t to $[\epsilon] - 1$. Let $\mathbf{B}^{\dagger}_K$ be the integral closure of $\mathbf{B}^{\dagger}_{\mathbb{Q}_p}$ in $\mathbf{B}_K$. Again, $\mathbf{B}^{\dagger}_K$ carries the actions of ϕ and Γ_K.

Definition 24.2.4. Let $\mathbf{A}^{\dagger}$ be the set of $x = \sum_{n=0}^{\infty} p^n [x_n] \in \mathbf{A}$ such that

$$\liminf_{n\to\infty} \{v(x_n)/n\} > -\infty.$$

Define

$$D^{\dagger}(V) = (V \otimes_{\mathbb{Q}_p} \mathbf{B}^{\dagger})^{H_K};$$

it is an étale (ϕ, Γ)-module over $\mathbf{B}^{\dagger}_K$.

The following is the main result of [**38**].

Theorem 24.2.5 (Cherbonnier–Colmez). *The functor $D^\dagger$, from the category of continuous representations of G_K on finite-dimensional $\mathbb{Q}_p$-vector spaces to the category of étale (ϕ, Γ)-modules over $\mathbf{B}_K^\dagger$, is an equivalence of categories.*

Remark 24.2.6. By Theorem 24.2.2, to check Theorem 24.2.5 it suffices to check that the base extension functor from étale (ϕ, Γ)-modules over $\mathbf{B}_K^\dagger$ to étale (ϕ, Γ)-modules over $\mathbf{B}_K$ is an equivalence. The full faithfulness of this functor is elementary and it follows from Lemma 16.3.2. The essential surjectivity is much deeper; it amounts to the fact that the natural map

$$D^\dagger(V) \otimes_{\mathbf{B}_K^\dagger} \mathbf{B}^\dagger \to V \otimes_{\mathbb{Q}_p} \mathbf{B}^\dagger$$

is an isomorphism. Verifying this requires the development of an appropriate analogy to Sen's theory of decompletion; this analogy has been developed into a full abstract Sen theory by Berger and Colmez [**18**].

A further variant was proposed by Berger [**14**].

Definition 24.2.7. Using the identification $\mathbf{B}_{\mathbb{Q}_p}^\dagger \cong \mathcal{E}^\dagger$, put

$$\mathbf{B}_{\mathrm{rig},K}^\dagger = \mathbf{B}_K^\dagger \otimes_{\mathbf{B}_{\mathbb{Q}_p}^\dagger} \mathcal{R},$$

where $\mathcal{R}$ is the Robba ring over $\mathbb{Q}_p$; the subscript "rig" indicates "rigid" in the sense of rigid analytical geometry. Note that $\mathbf{B}_{\mathrm{rig},K}^\dagger$ admits continuous extensions (for the LF-topology) of the actions of ϕ and Γ_K. Define

$$D_{\mathrm{rig}}^\dagger(V) = D^\dagger(V) \otimes_{\mathbf{B}_K^\dagger} \mathbf{B}_{\mathrm{rig},K}^\dagger;$$

it is an étale (ϕ, Γ)-module over $\mathbf{B}_{\mathrm{rig},K}^\dagger$.

Theorem 24.2.8 (Berger). *The functor $D_{\mathrm{rig}}^\dagger$, from the category of continuous representations of G_K on finite-dimensional $\mathbb{Q}_p$-vector spaces to the category of étale (ϕ, Γ)-modules over $\mathbf{B}_{\mathrm{rig},K}^\dagger$, is an equivalence of categories.*

Remark 24.2.9. The principal new content in Theorem 24.2.8 is that the base extension functor from étale ϕ-modules over $\mathcal{E}^\dagger$ to étale ϕ-modules over $\mathcal{R}$ is fully faithful; this follows from Corollary 16.3.4. The essential surjectivity of the functor is almost trivial, since the étaleness of the ϕ-action is defined over the Robba ring by base extension from $\mathcal{E}^\dagger$ (Definition 16.3.1). One need only check that the Γ_K-action also descends to any étale lattice, but this is easy: the proof is similar to that of Lemma 17.4.2.

24.3 Galois cohomology

Since the functor D and its variants lose no information about Galois representations, it is unsurprising that they can be used to recover basic invariants of a representation, such as its Galois cohomology.

Definition 24.3.1. Assume for simplicity that Γ_K is procyclic; this eliminates only the case where $p = 2$ and $\{\pm 1\} \subset \Gamma$, for which see Remark 24.3.2 below. Let γ be a topological generator of Γ. Define the *Herr complex* over $\mathbf{B}_K$ associated with V as the complex (with the first nonzero term placed in degree 0)

$$0 \to D(V) \to D(V) \oplus D(V) \to D(V) \to 0;$$

here the first map is $m \mapsto ((\phi - 1)m, (\gamma - 1)m)$ and the second is $(m_1, m_2) \to (\gamma - 1)m_1 - (\phi - 1)m_2$. (The fact that this is a complex follows from the commutativity of ϕ and γ.) Similarly, define the Herr complex over $\mathbf{B}_K^\dagger$ or $\mathbf{B}_{\mathrm{rig},K}^\dagger$ by replacing $D(V)$ by $D^\dagger(V)$ or $D_{\mathrm{rig}}^\dagger(V)$, respectively.

Remark 24.3.2. In a more conceptual description, which also covers the case where Γ_K need not be profinite, one takes the total complex associated with

$$0 \to C^{\cdot}(\Gamma_K, D(V)) \stackrel{\phi-1}{\to} C^{\cdot}(\Gamma_K, D(V)) \to 0,$$

where $C^{\cdot}(\Gamma_K, D(V))$ is the usual cochain complex for computing Galois cohomology (as in [**199**, §I.2.2]). One might think of this as the "monoid cohomology" of $\Gamma_K \times \phi^{\mathbb{Z}_{\geq 0}}$ acting on $D(V)$.

Theorem 24.3.3. *The cohomology of the Herr complex computes the Galois cohomology of V.*

Proof. For $\mathbf{B}_K$, the desired result was established by Herr [**111**]. The argument proceeds in two steps. One takes first the cohomology of the Artin–Schreier sequence

$$0 \to \mathbb{Q}_p \to \mathbf{B} \stackrel{\phi-1}{\to} \mathbf{B} \to 0,$$

after tensoring with V. This reduces the claim to the fact that the inflation homomorphisms

$$H^i(\Gamma_K, D(V)) \to H^i(G_K, V \otimes_{\mathbb{Q}_p} \mathbf{B})$$

are bijections; this is proved by adapting a technique introduced by Sen.

For $\mathbf{B}_K^\dagger$ and $\mathbf{B}_{\mathrm{rig},K}^\dagger$, the desired result was established by Liu [**160**]; his argument proceeds by comparison with the original Herr complex rather than by imitation of the above argument, though one could probably do that also. $\square$

Remark 24.3.4. As in [**111, 160**], one can make Theorem 24.3.3 more precise. For instance, the construction of Galois cohomology is functorial; there is also an interpretation in the Herr complex of the cup product in cohomology.

Remark 24.3.5. One can also use the Herr complex to recover Tate's fundamental theorems on Galois cohomology (finite dimensionality, the Euler–Poincaré characteristic formula, local duality). This was done by Herr in [**112**].

24.4 Differential equations from (ϕ, Γ)-modules

One of the original goals of p-adic Hodge theory was to associate finer invariants with p-adic Galois representations, so as, for instance, to distinguish those representations which arise in geometry (i.e., from the étale cohomology of varieties over K). This was originally done using a collection of "period rings" introduced by Fontaine; more recently, Berger's work has demonstrated that one can reproduce these constructions using (ϕ, Γ)-modules. Here is a brief description of an example that shows the relevance of p-adic differential equations to this study. We will make reference to Fontaine's rings $\mathbf{B}_{\mathrm{dR}}, \mathbf{B}_{\mathrm{st}}$ without giving definitions of them, for which see [**15**].

Definition 24.4.1. Let $\chi : \Gamma_K \to \mathbb{Z}_p^\times$ denote the cyclotomic character; that is, for all nonnegative integers m and all $\gamma \in \Gamma_K$,

$$\gamma(\zeta_{p^m}) = \zeta_{p^m}^{\chi(\gamma)}.$$

For $\gamma \in \Gamma_K$ sufficiently close to 1, we may compute

$$\nabla = \frac{\log \gamma}{\log \chi(\gamma)}$$

as an endomorphism of $D(V)$, using the power series for $\log(1+x)$. The result does not depend on γ.

Remark 24.4.2. If one views Γ_K as a one-dimensional p-adic Lie group over $\mathbb{Z}_p$ then ∇ is the action of the corresponding Lie algebra.

Definition 24.4.3. Note that ∇ acts on $\mathbf{B}^\dagger_{\mathrm{rig},K}$ with respect to

$$f \mapsto [\epsilon] \log[\epsilon] \frac{df}{d[\epsilon]}.$$

As a result, it does not induce a differential module structure with respect to d/dt on $D(V)$, but only on $D(V) \otimes \mathbf{B}^\dagger_{\mathrm{rig},K}[(\log[\epsilon])^{-1}]$. We say that V is *de Rham* if there exists a differential module with Frobenius structure M over $\mathbf{B}^\dagger_{\mathrm{rig},K}$ and an isomorphism

$$D(V) \otimes \mathbf{B}^{\dagger}_{\mathrm{rig},K}[(\log[\epsilon])^{-1}] \to M \otimes \mathbf{B}^{\dagger}_{\mathrm{rig},K}[(\log[\epsilon])^{-1}]$$

of differential modules with Frobenius structure.

One then has the following results of Berger [**14**].

Theorem 24.4.4 (Berger).

(a) *The representation V is de Rham if and only if it is de Rham in Fontaine's sense, i.e., if*

$$D_{\mathrm{dR}}(V) = (V \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{dR}})^{G_K}$$

satisfies

$$D_{\mathrm{dR}}(V) \otimes_K \mathbf{B}_{\mathrm{dR}} \cong V \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{dR}}.$$

(b) *Suppose that V is de Rham. If V is semistable in Fontaine's sense, i.e., if*

$$D_{\mathrm{st}}(V) = (V \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{st}})^{G_K}$$

satisfies

$$D_{\mathrm{st}}(V) \otimes_F \mathbf{B}_{\mathrm{st}} \cong V \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{st}}$$

then there exists an M as in Definition 24.4.3 that is unipotent. Conversely, if such an M exists then V is potentially semistable, i.e., it becomes semistable upon restriction to G_L for some finite extension L of K.

Applying Theorem 20.1.4 then yields the following corollary, which was previously a conjecture of Fontaine [**90**, 6.2].

Corollary 24.4.5 (Berger). *Every de Rham representation is potentially semistable.*

Remark 24.4.6. The descriptor "de Rham" is meant to convey the fact that if $V = H^i_{\mathrm{et}}(X \times_K K^{\mathrm{alg}}, \mathbb{Q}_p)$ for X a smooth proper variety over K then V is de Rham and one can use the aforementioned constructions to recover $H^i_{\mathrm{dR}}(X, K)$ functorially from V (solving Grothendieck's "problem of the mysterious functor"). See [**15**] for more of this story.

24.5 Beyond Galois representations

The category of arbitrary (ϕ, Γ)-modules over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ turns out to have its own representation-theoretic interpretation: it is equivalent to the category of B-*pairs* introduced by Berger [**16**]. One can associate "Galois cohomology" with such objects using the Herr complex; it has been shown by Liu [**160**] that the analogues of Tate's theorems (see Remark 24.3.5) still hold. These functors can be interpreted as the derived functors of $\mathrm{Hom}(D^{\dagger}_{\mathrm{rig}}(V_0), \cdot)$ when V_0 is the trivial representation [**142**, Appendix].

It may be wondered why we should be interested in (ϕ, Γ)-modules over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ if ultimately we have in mind an application concerning only Galois representations. One answer is that converting Galois representations into (ϕ, Γ)-modules exposes extra structure that is not visible without such a conversion.

Definition 24.5.1 (Colmez). We say that V is *trianguline* if $D^{\dagger}_{\mathrm{rig}}(V)$ is a successive extension of (ϕ, Γ)-modules of rank 1 over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$. The point is that these need not be étale, so V need not be a successive extension of representations of dimension 1.

Trianguline representations have the dual benefits of being relatively easy to classify and also somewhat commonplace. On one hand, Colmez [**58**] classified the two-dimensional trianguline representations of $G_{\mathbb{Q}_p}$; the classification included a parameter (the $\mathcal{L}$-invariant) relevant to p-adic L-functions. On the other hand, a result of Kisin [**146**] shows that the Galois representations associated with many classical modular forms are in fact trianguline.

Notes

Our presentation here was largely a summary of Berger's [**15**], which we highly recommend. For detailed study, the recent notes by Brinon and Conrad for the 2009 Clay Mathematics Institute summer school [**32**] may be of considerable value.

A variant of the theory of (ϕ, Γ)-modules was introduced by Kisin [**147**], using the Kummer tower $K(p^{1/p^n})$ instead of the cyclotomic tower $K(\zeta_{p^n})$. This leads to certain advantages, particularly when studying crystalline representations. For instance, Kisin was able to use his construction to establish some classification theorems for finite flat group schemes and for p-divisible groups, as conjectured by Breuil. Kisin's work was based on an earlier paper of Berger [**17**]; both of these use slope filtrations (as in Theorem 16.4.1) to recover a theorem of Colmez and Fontaine classifying semistable Galois representations in terms of certain linear algebraic data.

After [**14**] had appeared, Fontaine succeeded in giving a direct proof of Corollary 24.4.5 that did not involve p-adic differential equations; see [**91**].

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Notation

$|\cdot|_\rho$ (ρ-Gauss (semi)norm), 26

$|A|$ (norm of a matrix), 56, 63

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