VECTOR BUNDLES ON PRODUCTS OF FARGUES–FARGUES CURVES

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Abstract. We construct products of Fargues–Fontaine curves associated to a perfectoid field in the categories of schemes and adic spaces. Our main result is a "GAGA" theorem: the natural morphism from the adic to the schematic construction induces an equivalence of categories of vector bundles and isomorphisms of sheaf cohomology groups. We also discuss the theory of slope filtrations for vector bundles.

Throughout this paper, fix a local field E of residue characteristic p, with residue field of cardinality q, and fix a uniformizer ϖ of E. Over many years and multiple changes in perspective, it has emerged that continuous representations of the absolute Galois group G_E on finite-dimensional E-vector spaces is closely linked with the geometry of a certain space, the *Fargues–Fontaine curve* with coefficients in E [1]. This name in fact applies to two different but closely related objects in the category of locally ringed spaces: the *schematic Fargues–Fontaine curve*, which is a scheme over E, and the *adic Fargues–Fontaine curve*, which belongs to Huber's category of adic spaces over E. The relationship between these two spaces can be summarized as follows: the adic curve carries a particular ample line bundle $\mathcal{O}(1)$, and the schematic curve is defined by applying the Proj construction to the graded ring of sections of tensor powers of $\mathcal{O}(1)$. Notably, these two spaces are related by a theorem analogous to Serre's GAGA theorem in complex analytic geometry [11]: there is a natural morphism from the adic space to the scheme, and pullback along this morphism induces an equivalence of categories of coherent sheaves and isomorphisms of sheaf cohomology groups [1, §11.3].

In this note, we demonstrate similar results for the product of two or more copies of the Fargues–Fontaine curve with coefficients in E. Such products appear naturally in the local Langlands correspondence [2] by analogy with the use of "Drinfeld's lemma" to describe the global Langlands correspondence for function fields [10]. Our main result is a GAGA statement, but limited to vector bundles due to the failure of these products to be locally noetherian (Theorem 3.8).

One lingering question is to what extent the classification of vector bundles on the Fargues–Fontaine curve [1, Chapter 9] admits an analogue for products of curves. We plan to address this issue in a subsequent version of this document.

1. Relative FF curves

In order to study vector bundles on products of Fargues–Fontaine curves, we need to consider somewhat more general sheaves on relative Fargues–Fontaine curves. However, to

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avoid having to worry about sheaf-theoretic issues, we formulate the relevant statements in algebraic language.

Hypothesis 1.1. Throughout §1, let I be a closed subinterval of $(0, +\infty)$ such that $I \cap I^{1/q} \neq \emptyset$; let R be a perfectoid Banach ring of characteristic p (i.e., a perfect Banach ring over $\mathbb{F}_p$ carrying a power-multiplicative norm); and let S be a Banach ring over E. We assume further that R is Tate, i.e., it contains a topologically nilpotent unit.

Definition 1.2. Write

$$W_E(R) := W(R) \widehat{\otimes}_{\mathbb{Z}_p} \mathfrak{o}_E = \varprojlim_n (W(R)/p^n \otimes_{\mathbb{Z}_p} \mathfrak{o}_E).$$

For $\alpha > 0$, let $|\bullet|_\alpha$ denote the α-Gauss norm on $W_E(R)[p^{-1}]$:

$$\left| \sum_m p^n [\overline{x}_n] \right| = \sup_n \{\alpha^{-n} |\overline{x}_n|\}.$$

This defines a power-multiplicative norm. Let $B^I_{R,E}$ be the completion of $W_E(R)[p^{-1}]$ with respect to the supremum of the $|\bullet|_\alpha$ over all $\alpha \in I$. Define also $B^I_{R,S} := B^I_{R,E} \widehat{\otimes}_E S$.

The q-power Frobenius map on $W_E(R)$ induces a map $\varphi : B^I_{R,S} \to B^{I^{1/q}}_{R,S}$. By a *Banach φ-module* over $B^I_{R,S}$, we will mean a Banach module M over $B^I_{R,S}$ equipped with an isomorphism

$$\Phi : \varphi^* M \widehat{\otimes}_{B^{I^{1/q}}_{R,S}} B^{I \cap I^{1/q}}_{R,S} \cong M \widehat{\otimes}_{B^I_{R,S}} B^{I \cap I^{1/q}}_{R,S}.$$

We define $H^0_\varphi(M)$ and $H^1_\varphi(M)$ as the kernel and cokernel of the induced φ-semilinear map

$$M \to M \widehat{\otimes}_{B^I_{R,S}} B^{I \cap I^{1/q}}_{R,S}, \qquad m \mapsto \Phi(m \otimes 1).$$

For $n \in \mathbb{Z}$, define the *twist* $M(n)$ of M by multiplying the action of Φ by ϖ^{-n}.

Remark 1.3. We will use frequently the fact that because R is Tate, it admits at least one untilt $R^\sharp$. For example, we can find a topologically nilpotent unit $\pi \in R$ such that $\log_{|\varpi|} |\pi| \in I$ (e.g., by starting with any topologically nilpotent unit and applying some power of Frobenius or its inverse), and then the quotient of $B^I_{R,S}$ by the ideal $(\varpi - [\pi])$ is an untilt of R. This yields an exact sequence of φ-modules of the form

$$0 \to B^I_{R,S}(-1) \to B^I_{R,S} \to (R^\sharp)^{\mathbb{Z}} \to 0. \tag{1}$$

Lemma 1.4. *We have $H^0_\varphi(B^I_{R,E}) = E$.*

Proof. See [8, Lemma 4.2.10] in the case $E = \mathbb{Q}_p$, the general case being similar. □

Definition 1.5. For A a Banach ring and T a set, let $A^{\widehat{T}}$ be the completion of the free A-module A^T for the supremum norm.

Lemma 1.6. *Let T be any set and view $F = (B^I_{R,E})^{\widehat{T}}$ as a Banach φ-module via the coordinatewise action of Frobenius. Then for all $n > 0$:*

- (a) *$H^0_\varphi(F(n))$ generates $F(n)$;*
- (b) *$H^1_\varphi(F(n)) = 0$;*

(c) *for any affinoid algebra S over E, the maps*

$$H^0_\varphi(F(n))\widehat{\otimes}_E S \to H^0_\varphi(F(n)\widehat{\otimes}_E S), \qquad H^1_\varphi(F(-n))\widehat{\otimes}_E S \to H^1_\varphi(F(-n)\widehat{\otimes}_E S)$$

are isomorphisms.

Proof. Parts (a) and (b) follow as in [8, Proposition 6.2.2, Proposition 6.2.4]. To check (c), note that if S is topologically countably generated over E, then S admits a Schauder basis (that is, S is a topologically free E-module) and so the computation of H^0_φ and H^1_φ both commute with completed base extension. In general S is the *uncompleted* direct limit of its topologically countably generated affinoid subalgebras and so the same conclusion holds. □

Lemma 1.7. *With notation as in Lemma 1.6, for all $n \leq 0$:*

(a) $H^0_\varphi(F(n)) = E^{\widehat{T}}$ *if* $n = 0$ *and* $H^0_\varphi(F(n)) = 0$ *if* $n < 0$;
(b) $H^1_\varphi(F(n))$ *is a Banach module over E;*
(c) *for any affinoid algebra S over E, the maps*

$$H^1_\varphi(F(n))\widehat{\otimes}_E S \to H^1_\varphi(F(n)\widehat{\otimes}_E S)$$

are isomorphisms.

Proof. We proceed by descending induction on n. By twisting the sequence (1) and taking cohomology, we obtain an exact sequence

$$0 \to H^0_\varphi(F(n)) \to H^0_\varphi(F(n+1)) \to (R^\sharp)^{\widehat{T}} \to H^1_\varphi(F(n)) \to H^1_\varphi(F(n+1)) \to 0.$$

For the base case $n = 0$, (a) follows from Lemma 1.6; this implies that the map $H^0_\varphi(F(n+1)) \to R^\sharp$ is strict. Meanwhile, the term $H^1_\varphi(F(1))$ vanishes by Lemma 1.6, so $H^1_\varphi(F)$ is an extension of two Banach modules over E. For the induction step $n < 0$, we have $H^0_\varphi(F(n+1)) = 0$ unless $n = -1$, in which case $H^0_\varphi(F) \to \tilde{F}$ is injective; in either case we obtain (a). We then obtain (b) as in the base case. Given (a) and (b), we check (c) as in Lemma 1.6. □

Remark 1.8. In general, $H^1_\varphi(B^I_{R,S})$ is nonzero and its formation does *not* compute with completed base change on S. It can be interpreted as the first continuous Galois cohomology group of the trivial E-local system on $\operatorname{Spec} R$ [8, Theorem 8.6.4]; in particular, it vanishes when R is an algebraically closed perfectoid field, or more generally an *absolutely integrally closed* (AIC) perfectoid ring.

Theorem 1.9. *For any Banach φ-module M over $B^I_{R,S}$, there exists an integer N such that for $n \geq N$,*

(a) $H^0_\varphi(M(n))$ *generates* $M(n)$;
(b) $H^1_\varphi(M(n)) = 0$;
(c) *for any Banach ring S' over S, the map*

$$H^0_\varphi(M(n))\widehat{\otimes}_S S' \to H^0_\varphi(M(n)\widehat{\otimes}_S S')$$

is an isomorphism.

Proof. The proof of [8, Proposition 6.2.2] directly generalizes to yield (b). Given (b), we may then deduce (a) by following [8, Proposition 6.2.4]. To check (c), using (a) we may (after twisting) reduce to the case where there exists an exact sequence

$$0 \to P \to F \to M \to 0$$

of Banach φ-modules with $F = (B^I_{R,E})^{\widehat{T}}$ for some set T. From the exact sequence

$$0 \to P(n) \to F(n) \to M(n) \to 0$$

and (a), we obtain an exact sequence

$$\begin{tikzcd}
 & H^0_\varphi(P(n))\widehat{\otimes}_S S' \arrow[r] \arrow[d] & H^0_\varphi(F(n))\widehat{\otimes}_S S' \arrow[r] \arrow[d] & H^0_\varphi(M(n))\widehat{\otimes}_S S' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^0_\varphi(P(n)\widehat{\otimes}_S S') \arrow[r] & H^0_\varphi(F(n)\widehat{\otimes}_S S') \arrow[r] & H^0_\varphi(M(n)\widehat{\otimes}_S S') \arrow[r] & 0
\end{tikzcd}$$

Lemma 1.6 shows that the middle vertical arrow is an isomorphism; this implies that the right vertical arrow is surjective. By the same token, the left vertical arrow is surjective; by the five lemma, the right vertical arrow is injective. ∎

2. Multiple FF curves: algebraic description

We next formally promote the previous result to the case of multiple Frobenius actions. This will then specialize to tell us about vector bundles on products of Fargues–Fontaine curves.

Hypothesis 2.1. Throughout §2, continue to retain Hypothesis 1.1. In addition, fix a positive integer m and let $I_1, \dots, I_m$ be closed subintervals of $(0, +\infty)$ such that $I_i \cap I_i^{1/q} \neq \emptyset$ for $i = 1, \dots, m$.

Definition 2.2. For m a positive integer, let $B^{I_1,\dots,I_m}_{R,S}$ denote the m-fold completed tensor product $B^{I_1}_{R,S}\widehat{\otimes}_S \cdots \widehat{\otimes}_S B^{I_m}_{R,S}$. We again have $B^{I_1,\dots,I_m}_{R,S} \cong B^{I_1,\dots,I_m}_{R,E}\widehat{\otimes}_E S$. To simplify notation, when $I_1 = \cdots = I_m = I$ we denote $B^{I_1,\dots,I_m}_{R,S}$ also as $B^{I,m}_{R,S}$.

For $i = 1, \dots, m$, let $\varphi_i : B^{I_1,\dots,I_m}_{R,S} \to B^{I_1,\dots,I_i^{1/q},\dots,I_m}_{R,S}$ be the map induced by φ on the i-th factor of the completed tensor products. By a *Banach* $(\varphi_1, \dots, \varphi_m)$*-module* over $B^{I_1,\dots,I_m}_{R,S}$, we will mean a Banach module M over $B^{I_1,\dots,I_m}_{R,S}$ equipped with isomorphisms

$$\Phi_i : \varphi_i^* M \widehat{\otimes}_{B^{I_1,\dots,I_i^{1/q},\dots,I_m}_{R,S}} B^{I_1,\dots,I_i\cap I_i^{1/q},\dots,I_m}_{R,S} \cong M \widehat{\otimes}_{B^{I_1,\dots,I_m}_{R,S}} B^{I_1,\dots,I_i\cap I_i^{1/q},\dots,I_m}_{R,S} \qquad (i = 1, \dots, m)$$

which "commute" in the following sense. For $P \subseteq \{1, \dots, m\}$, let C_P be the module

$$M \widehat{\otimes}_{B^{I_1,\dots,I_i,\dots,I_m}_{R,S}} B^{J_1,\dots,J_m}_{R,S}, \qquad J_i = \begin{cases} I_i & i \notin P \\ I_i \cap I_i^{1/q} & i \in P. \end{cases}$$

For $i \notin P$, consider the map $C_P \to C_{P \cup \{i\}}$ given by $m \mapsto \Phi_i(m \otimes 1)$. Then these maps form a commuting diagram.

With notation as above, define the groups $H^i_{\varphi_1,\dots,\varphi_m}(M)$ as the cohomology groups of the totalization of the m-fold complex C_P. By definition, these groups vanish for $i > m$.

For $n_1, \dots, n_m \in \mathbb{Z}$, define the *twist* $M(n_1, \dots, n_m)$ of M by multiplying the action of Φ_i by ϖ^{-n_i}.

Theorem 2.3. *Let M be a Banach $(\varphi_1, \dots, \varphi_m)$-module over $B^{I,m}_{R,S}$. Then there exists an integer N such that for $n_1, \dots, n_m \geq N$,*

(a) $H^0_{\varphi_1,\dots,\varphi_m}(M(n_1,\dots,n_m))$ *generates* $M(n_1,\dots,n_m)$*;*
(b) $H^i_{\varphi_1,\dots,\varphi_m}(M(n_1,\dots,n_m)) = 0$ *for all* $i > 0$*;*
(c) *for any Banach algebra* S' *over* S*, the map*

$$H^0_{\varphi_1,\dots,\varphi_m}(M(n_1,\dots,n_m))\widehat{\otimes}_S S' \to H^0_{\varphi_1,\dots,\varphi_m}(M(n_1,\dots,n_m)\widehat{\otimes}_S S')$$

is an isomorphism.

Proof. We proceed by induction on m, the case $m = 1$ being Theorem 1.9. For $m > 1$, by applying Theorem 1.9 repeatedly, we obtain the corresponding statements for n_1 sufficiently large, n_2 sufficiently large as a function of n_1, and so on. Applying (a) in this weaker form suffices to produce (after a single twist) an exact sequence

$$0 \to P \to F \to M \to 0$$

in which F is the completion (for the supremum norm) of a free module equipped with the coordinatewise action of the partial Frobenius maps.

It remains to show that after replacing M with some twist, statements (a), (b), (c) become true for all $n_1,\dots,n_m > 0$. By Lemma 1.6 this is already true for F itself. Consequently, (b) holds trivially for $i > m$, and then by descending induction (using the fact that anything we can conclude about M is also true about P, but for a different twist) we may deduce it for all $i > 0$. We may then similarly deduce (a) and (c). $\square$

3. GAGA for multiple FF curves

We now recast the previous discussion into geometric language, following [8, Chapter 8].

Hypothesis 3.1. Throughout §3, retain Hypothesis 2.1 except for the choice of the intervals I and $I_1,\dots,I_m$ (which we will allow to vary).

Lemma 3.2. *The underlying Huber ring of* $B^{I,m}_{R,E}$ *is sheafy.*

Proof. When E is of characteristic p, the perfect closure of $B^{I,m}_{R,E}$ is the completed direct limit of $B^{I,m}_{R,E} \otimes_E E^{p^{-n}}$ over all n, and the embedding of $B^{I,m}_{R,E}$ splits in the category of Banach modules over $B^{I,m}_{R,E}$. This splitting then transfers to any rational localization of $B^{I,m}_{R,E}$, so $B^{I,m}_{R,E}$ is stably uniform and hence sheafy [4, Theorem 1.2.13].

Similarly, when E is of characteristic 0, the completed direct limit of $B^{I,m}_{R,E} \otimes_E E(\varpi^{p^{-n}})$ over all n is a perfectoid ring, and the embedding of $B^{I,m}_{R,E}$ splits in the category of Banach modules over $B^{I,m}_{R,E}$. This splitting then transfers to any rational localization of $B^{I,m}_{R,E}$, so $B^{I,m}_{R,E}$ is stably uniform and hence sheafy. In other words, $B^{I,m}_{R,E}$ is a *sousperfectoid ring* in the sense of [3]. $\square$

Definition 3.3. Let Y be the union of the spaces $\mathrm{Spa}(B^{I,m}_{R,E}, (B^{I,m}_{R,E})^\circ)$ over all closed subintervals $I \subset (0,+\infty)$; by Lemma 3.2, Y carries the structure of an adic space. The multiple Frobenius maps define a properly continuous action of $\mathbb{Z}^m$ on Y; let $\mathrm{FF}_{R,q,m}$ be the quotient $Y/\mathbb{Z}^m$. We call this the *adic* m*-fold (*q*-power) Fargues–Fontaine curve* over R (even though it is not an adic space over R).

We may identify vector bundles with $\mathrm{FF}_{R,q,m}$ with $\mathbb{Z}^m$-equivariant vector bundles on Y. For $n_1,\dots,n_m \in \mathbb{Z}$, let $\mathcal{O}(n_1,\dots,n_m)$ be the line bundle on $\mathrm{FF}_{R,q,m}$ corresponding to the trivial line bundle on Y for the action of $\mathbb{Z}^m$ in which the i-th generator acts as multiplication by ϖ^{-n_i}. We may then form the multigraded ring

$$P_{R,q,m} = \bigoplus_{n_1,\dots,n_m \geq 0} \Gamma(\mathrm{FF}_{R,q,m}, \mathcal{O}(n_1,\dots,n_m))$$

and take its Proj; this yields the *schematic m-fold (q-power) Fargues–Fontaine curve* over R.

By construction, there is a natural morphism $\mathrm{FF}_{R,q,m} \to \operatorname{Proj} P_{R,q,m}$ of locally ringed spaces. We will view pullback along this morphism as an *analytification* functor.

Remark 3.4. It is natural but slightly confusing terminology to say that the previous constructions give spaces "over R", as neither of them gives rise to a locally ringed space mapping to $\operatorname{Spec} R$. The adic construction does give rise to a morphism to R in the larger category of *diamonds* whose underlying map of sets is the topological projection onto $\operatorname{Spa}(R, R^\circ)$.

Remark 3.5. Note that both $\operatorname{Proj} P_{R,q,m}$ and $\mathrm{FF}_{R,q,m}$ are quasicompact and separated locally ringed spaces, and so have finite cohomological dimension.

Definition 3.6. In the category of diamonds, we may interpret $\mathrm{FF}_{R,q,1}$ as the quotient $(\operatorname{Spd} R \times \operatorname{Spd} E)/\langle \varphi \rangle$, with φ acting only on the first factor, equipped with the structure morphism to $\operatorname{Spd} E$ given by the second projection. By the same token, $\mathrm{FF}_{R,q,m}$ can be viewed as the quotient

$$\frac{\operatorname{Spd} R \times \cdots \times \operatorname{Spd} R \times \operatorname{Spd} E}{\langle \varphi_1, \dots, \varphi_m \rangle} \cong \frac{\operatorname{Spd} R \times \cdots \times \operatorname{Spd} R}{\langle \varphi_1, \dots, \varphi_m \rangle} \times \operatorname{Spd} E. \tag{2}$$

In particular, we have canonical isomorphisms

$$\begin{aligned}\mathrm{FF}_{R,q,m} &\cong ((\operatorname{Spd} R)/\varphi \times \operatorname{Spd} E) \times_{\operatorname{Spd} E} \cdots \times_{\operatorname{Spd} E} ((\operatorname{Spd} R)/\varphi \times \operatorname{Spd} E) \\ &\cong \mathrm{FF}_{R,q,1} \times_E \cdots \times_E \mathrm{FF}_{R,q,1};\end{aligned}$$

the outer isomorphism can also be interpreted directly in the category of adic spaces.

Lemma 3.7. *For $n_1, \dots, n_m \in \mathbb{Z}$, the natural maps*

$$H^i(\operatorname{Proj} P_{R,q,m}, \mathcal{O}(n_1,\dots,n_m)) \to H^i(\mathrm{FF}_{R,q,m}, \mathcal{O}(n_1,\dots,n_m))$$

are isomorphisms.

Proof. We induct on m; within this induction, we perform a descending induction on i using Remark 3.5 as the base case. Using the hypothesis that R is Tate, we may choose an untilt of R and thus obtain a closed immersion $j : \mathrm{FF}_{R,q,m-1} \to \mathrm{FF}_{R,q,m}$ (and similarly on the schematic side). This gives us an exact sequence

$$0 \to \mathcal{O}(0,\dots,0,-1) \to \mathcal{O} \to j_* j^* \mathcal{O} \to 0;$$

by twisting this sequence, we may apply the induction hypothesis for $m-1$ to perform the descending induction on i. $\square$

Theorem 3.8. *The pullback of vector bundles along $\mathrm{FF}_{R,q,m} \to \operatorname{Proj} P_{R,q,m}$ (analytification) is an equivalence of categories. Moreover, the natural maps from the cohomology groups of any vector bundle on $\operatorname{Proj} P_{R,q,m}$ to the cohomology groups of its analytification is an isomorphism.*

Proof. We first establish the comparison of cohomology groups, which also implies full faithfulness (by taking internal Homs). Let $\mathcal{F}$ be a vector bundle on $\operatorname{Proj} P_{R,q,m}$. From the Proj construction, we obtain an exact sequence of the form

$$0 \to \mathcal{G} \to \mathcal{O}(n_1, \dots, n_m)^{\oplus d} \to \mathcal{F} \to 0$$

for some $d \geq 0$ and some $n_1, \dots, n_m \in \mathbb{Z}$. By Lemma 3.7, we have the desired isomorphism of cohomology groups at the middle of this sequence; we may thus deduce the same isomorphism for $\mathcal{F}$ by descending induction on the cohomology degree, using Remark 3.5 as the base case.

We next establish essential surjectivity. Let $\mathcal{F}$ be a vector bundle on $\mathrm{FF}_{R,q,m}$. By applying Theorem 2.3 twice, we obtain an exact sequence

$$\mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{F} \to 0$$

in which $\mathcal{E}_1, \mathcal{E}_2$ arise by pullback from $\operatorname{Proj} P_{R,q,m}$. By the previous paragraph, the morphism between them also arises by pullback from $\operatorname{Proj} P_{R,q,m}$, as then does $\mathcal{F} = \operatorname{coker}(\mathcal{E}_1 \to \mathcal{E}_2)$. $\square$

Remark 3.9. In principle, it should be possible to generalize Theorem 3.8 after taking completed tensor products over E with a Banach ring S over E. The main difficulty is in the formulation: it is not clear what conditions are needed to ensure that the underlying Huber ring of $B^{I,m}_{R,E} \widehat{\otimes}_E S$ is sheafy, so that it makes sense to form the adic relative m-fold Fargues–Fontaine curve.

One tractable case is when S is a smooth affinoid algebra over E. In this case, S locally admits an étale morphism to a torus, in which case it is again sousperfectoid. In this case, $B^{I,m}_{R,E} \widehat{\otimes}_E S$ is also sousperfectoid.

Corollary 3.10. *Let $\mathcal{F}$ be a vector bundle on* $\operatorname{Proj} P_{R,q,m}$ *of rank $r \geq m+1$. Then $\mathcal{F}$ admits a surjective morphism onto some line bundle.*

Proof. By Theorem 2.3, for $n_1, \dots, n_m \gg 0$ the bundle $\mathcal{F}^\vee(n_1, \dots, n_m)$ is generated by global sections. Since $r \geq m+1$, a generic section s of $\mathcal{F}^\vee(n_1, \dots, n_m)$ will have no zeros; it thus defines a surjective morphism $\mathcal{F} \to \mathcal{O}(n_1, \dots, n_m)$. Further details to follow. $\square$

4. A fibration construction

Definition 4.1. On the right-hand side of (2), we may refactor the first term in the product as Y_R/φ where Y_R is the quotient of $\operatorname{Spd} R \times \cdots \times \operatorname{Spd} R$ by the subgroup of $\langle \varphi_1, \dots, \varphi_m \rangle$ obtained by omitting one of the generators. That is, we may view $\mathrm{FF}_{R,q,m}$ as a relative Fargues–Fontaine curve over Y_R, which admits the untilt $\mathrm{FF}_{R,q,m-1} \times_E R^\sharp$.

For $\mathcal{F}$ a vector bundle on $\mathrm{FF}_{R,q,m}$, we may consider the *diagonal HN polygon* of $\mathcal{F}$ over Y_R. The diagonal degree of $\mathcal{F}$ is locally constant on Y_R, and hence on R because $Y_R \to \operatorname{Spd} R$ has connected fibers.

For $\mu \in \mathbb{Q}$, we say that $\mathcal{F}$ is *diagonally pure of slope* μ if the diagonal HN polygon has all slopes equal to μ; when $\mu = 0$, we also say that $\mathcal{F}$ is *diagonally étale.*

Theorem 4.2. *Suppose that R is an $\mathbb{F}_q$-algebra. For $\mu \in \mathbb{Z}$, there are canonical (depending on ϖ) equivalence of tensor categories between the following categories.*

(a) *Vector bundles on* $\mathrm{FF}_{R,q,m}$ *which are diagonally pure of slope μ.*
(b) *Pro-étale E-local systems on Y_R (i.e., finite locally free modules for the locally constant sheaf $\underline{E}$ on the pro-étale site of Y_R).*

(c) *Pro-étale E-local systems on* $\operatorname{Spd} R \times \cdots \times \operatorname{Spd} R$ *which are* $\langle \varphi_1, \ldots, \varphi_n \rangle$*-equivariant and such that the action of* $\varphi = \varphi_1 \circ \cdots \circ \varphi_n$ *becomes the endomorphism* $\varpi^{-\mu}$ *upon identifying the local system with its* φ*-pullback.*

Moreover, these equivalences are compatible with tensor products between different values of μ *(that is, the source objects belong to the categories indexed by* μ_1 *and* μ_2 *and the target to the category indexed by* $\mu_1 + \mu_2$*).*

Proof. This follows from [8, Theorem 8.5.12]. □

Definition 4.3. For $\mu_1, \ldots, \mu_m \in \mathbb{Q}$ with least common denominator s, let $\mathcal{O}(\mu_1, \ldots, \mu_m)$ be the vector bundle of rank s on $\mathrm{FF}_{R,q,m}$ corresponding to the $(\varphi_1, \ldots, \varphi_m)$-module over $B^{I,m}_{R,S}$ with free generators $\mathbf{e}_1, \ldots, \mathbf{e}_s$ with the action of φ_i given by

$$\Phi_i(\mathbf{e}_1) = \mathbf{e}_2, \quad \ldots, \quad \Phi_i(\mathbf{e}_{s-1}) = \mathbf{e}_s, \quad \Phi_i(\mathbf{e}_s) = \varpi^{-s\mu_i}\mathbf{e}_1.$$

When $s = 1$, we may apply Theorem 4.2 to obtain a pro-étale E-local system L of rank 1 on $\operatorname{Spd} R \times \cdots \times \operatorname{Spd} R$; which is $\langle \varphi_1, \ldots, \varphi_n \rangle$-equivariant; this local system admits a trivialization for which the action of φ_i is given by multiplication by $\varpi^{-\mu_i}$ for $i = 1, \ldots, n$.

Theorem 4.4. *Suppose that R is an algebraically closed perfectoid field. Then the map* $\mathbb{Z}^m \to \operatorname{Pic}(\mathrm{FF}_{R,q,m})$ *taking* $(s_1, \ldots, s_m)$ *to* $\mathcal{O}(s_1, \ldots, s_m)$ *is an isomorphism.*

Proof. It is clear that the map in question is a group homomorphism. Injectivity follows from the case $m = 1$, e.g., using the fact that $H^0(\mathrm{FF}_{R,q,1}, \mathcal{O}(s))$ is zero for $s < 0$ and nonzero for $s \geq 0$. To deduce surjectivity, using Theorem 4.2 and the decomposition $E^\times \cong \mathfrak{o}_E^\times \times \varpi^{\mathbb{Z}}$ we reduce to the assertion that every pro-étale $\mathfrak{o}_E$-local system on Y_R is trivial. This is a consequence of Drinfeld's lemma for perfectoid spaces [5]. □

Definition 4.5. Using Theorem 4.4, we may define for every vector bundle $\mathcal{F}$ on $\mathrm{FF}_{R,q,m}$ the *full degree map* as a locally constant map $\deg_{\mathcal{F}} : \pi_0(R) \to \mathbb{Z}^m$. We may then define the *full slope map* $\mu_{\mathcal{F}} = \deg_{\mathcal{F}} / \operatorname{rank} : \pi_0(R) \to \mathbb{Q}^m$.

Remark 4.6. The proof of Theorem 4.4 implicitly uses the fact that $E^\times$ injects into its profinite completion. By contrast, $\mathrm{GL}_n(E)$ does not inject into its profinite completion for $n > 1$, so étale E-local systems on Y_R of rank greater than 1 are not necessarily elements of the isogeny category of pro-étale $\mathfrak{o}_E$-local systems. See Example 4.7.

Example 4.7. Let R be an algebraically closed perfectoid field and take $m = 2$ and $E = \mathbb{Q}_p$. By choosing an untilt $R^\sharp$ of R containing $\mathbb{Q}_p$, we may identify Y_R with the diamond associated to $\mathrm{FF}_{R,q,1} \times_E R^\sharp$.

The Lubin–Tate moduli space gives rise to an irreducible $\mathbb{Q}_p$-local system L of rank 2 on $\mathbf{P}^1_{\mathbb{Q}_p}$. By choosing two sections of $\mathcal{O}(1)$ on $\mathrm{FF}_{R,q,1}$ with distinct zeros, we may pull L back to an irreducible $\mathbb{Q}_p$-local system on $\mathrm{FF}_{R,q,1} \times_E R^\sharp$ with monodromy group $\mathrm{PSL}_2(\mathbb{Q}_p)$. Via the tilting correspondence, L can also be viewed as an irreducible $\mathbb{Q}_p$-local system of rank 2 on Y_R.

Example 4.8. With notation as in Remark 1.3, choose two E-linearly independent elements $s_1, s_2 \in (B^I_{R,E})^{\varphi=\varpi}$; these can also be viewed as elements of $H^0(\mathrm{FF}_{R,q,1}, \mathcal{O}(1))$. Let $t_1, t_2 \in R^\sharp$ be the images of s_1, s_2 via the projection in (1) (projecting to the summand indexed by 0); these can also be viewed as elements of $H^0(\mathrm{FF}_{R,q,1}, \mathcal{O}(1)/\mathcal{O})$, or of

$H^0(\mathrm{FF}_{R,q,1}, \mathcal{O}(-1)/\mathcal{O}(-2))$ using the fact that $\mathcal{O}(1)/\mathcal{O}$, being supported at a point, is isomorphic to all of its twists.

Now view $f := s_1 \otimes t_2 - s_2 \otimes t_1$ as an element of

$$H^0(\mathrm{FF}_{R,q,1}, \mathcal{O}(1)) \widehat{\otimes} H^0(\mathrm{FF}_{R,q,1}, \mathcal{O}(-1)/\mathcal{O}(-2)) \cong H^0(\mathrm{FF}_{R,q,2}, \mathcal{O}(1,-1)/\mathcal{O}(1,-2)).$$

Using the exact sequence

$$0 \to \mathcal{O}(1,-2) \to \mathcal{O}(1,-1) \to \mathcal{O}(1,-1)/\mathcal{O}(1,-2) \to 0$$

derived from $R^\sharp$ as in Remark 1.3, we may push f along the connecting homomorphism to a class in $H^1(\mathrm{FF}_{R,q,2}, \mathcal{O}(1,-2))$. We thus obtain an exact sequence

$$0 \to \mathcal{O}(1,-2) \to \mathcal{F} \to \mathcal{O} \to 0$$

For any given point x of Y_R, the pullback of $\mathcal{F}$ to the fiber of $\mathrm{FF}_{R,q,2}$ above y can similarly be obtained by restricting f to the fiber and then applying the connecting homomorphism. For a generic choice of x, this results in a nonzero extension class, in which case the diagonal polygon of $\mathcal{F}$ at x must lie strictly above the polygon with slopes $0, -1$; the only remaining option is the polygon with slopes $-1/2, -1/2$.

Now take x to be the $\operatorname{Spd} R$-valued point of Y_R given by composing by the diagonal map $\Delta : \operatorname{Spd} R \to \operatorname{Spd} R \times \operatorname{Spd} R$ with the tautological quotient map $\operatorname{Spd} R \times \operatorname{Spd} R \to Y_R$. Then by design the extension class vanishes, so the diagonal polygon of $\mathcal{F}$ at x has slopes $0, -1$.

Remark 4.9. By virtue of the analogy between F-isocrystals at a geometric point and vector bundles on a Fargues–Fontaine curve, one might hope that *Drinfeld's lemma* for F-isocrystals [6, 9] admits an analogue for vector bundles on a product of Fargues–Fontaine curves. For example, a direct analogue of [6, Corollary 7.4] would be that for R an algebraically closed perfectoid field and $\mathcal{F}$ a vector bundle on $\mathrm{FF}_{R,q,2}$, the fibers of $\mathcal{F}$ along one of the projection maps all have the same HN polygon; this is refuted by Example 4.8.

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