# THE DE RHAM PROPERTY FOR REPRESENTATIONS OF PRODUCTS OF GALOIS GROUPS

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## 1. Introduction

Let $\Delta$ be a finite set. Let $K$ be a complete discretely valued field of characteristic $p$ with perfect residue field $k$. For each $\alpha \in \Delta$, let $G_{K,\alpha}$ be a copy of the absolute Galois group $G_K$ of $K$, and put $G_{K,\Delta} := \prod_{\alpha \in \Delta} G_{K,\alpha}$. Let $\mathbf{Rep}_{\mathbb{Q}_p}(G_{K,\Delta})$ be the category of continuous representations of $G_{K,\Delta}$ on finite-dimensional $\mathbb{Q}_p$-vector spaces.

The purpose of this note is to compare various possible definitions of the *de Rham* property for objects of $\mathbf{Rep}_{\mathbb{Q}_p}(G_{K,\Delta})$.

**Theorem 1.1.** *For $V \in \mathbf{Rep}_{\mathbb{Q}_p}(G_{K,\Delta})$, the following conditions are equivalent.*

(a) *For each $\alpha \in \Delta$, the restriction of $V$ along $G_{K,\alpha} \to G_{K,\Delta}$ is de Rham in the sense of Fontaine, i.e., it is admissible with respect to the period ring $\mathbf{B}_{\mathrm{dR},\alpha}$.*
(b) *The representation $V$ is admissible with respect to the uncompleted tensor product of the rings $\mathbf{B}_{\mathrm{dR},\alpha}$ over $\mathbb{Q}_p$.*
(c) *The representation $V$ is de Rham in the sense of* [1, Definition above Proposition 4.18]*, i.e., it is admissible with respect to the period ring $\mathbf{B}_{\mathrm{dR},\Delta}$ (a certain completed tensor product of the rings $\mathbf{B}_{\mathrm{dR},\alpha}$ over $\mathbb{Q}_p$).*
(d) *The module with integrable connection $D_{\mathrm{dif}}(V)$* [1, Definition 5.17] *is trivial.*

The implication from (a) to (b) is essentially formal (see Proposition 2.3), as is the implication from (b) to (c). The equivalence of (c) and (d) is [1, Proposition 5.18]. The main content is therefore the implication from (d) to (a), which we deduce by observing that the construction of $D_{\mathrm{dif}}(V)$ is compatible with restriction along $G_{K,\alpha} \to G_{K,\Delta}$ (see Lemma 4.3).

From Theorem 1.1, it should be possible to deduce analogues of the equivalence among (a), (b), (c) for the crystalline and semistable properties. We also expect a corresponding statement for the Hodge–Tate property, including a modified form of (d): the connection on $D_{\mathrm{dif}}(V)$ should be unipotent, not necessarily trivial.

We also expect similar results about multivariate $(\varphi, \Gamma)$-modules which need not be étale. At present, this is obstructed by the fact that it is not obvious how to define the analogue of restriction along $G_{K,\alpha} \to G_{K,\Delta}$; this is closely related to the fact that we do not have an analogue of Drinfeld's lemma for general multivariate $(\varphi, \Gamma)$-modules; see Conjecture 5.2.

We conclude this introduction by mentioning one potential improvement to Theorem 1.1. By [1, Proposition 4.19], if the equivalent conditions of Theorem 1.1 hold (in particular (c)), then the $K_\Delta$-module $D_{\mathrm{dR}}(V)$ is free of rank $\dim_{\mathbb{Q}_p} V$. We do not know if the converse implication holds.

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## 2. Restriction of representations

We start with a purely group-theoretical lemma.

**Definition 2.1.** For $\Delta' \subseteq \Delta$, let $\mathrm{Res}_{\Delta,\Delta'} : \mathbf{Rep}_{\mathbb{Q}_p}(G_{K,\Delta}) \to \mathbf{Rep}_{\mathbb{Q}_p}(G_{K,\Delta'})$ be the functor given by restriction along the inclusion $\iota_{\Delta',\Delta} : G_{K,\Delta'} \to G_{K,\Delta}$. When $\Delta'$ is the singleton set $\alpha$, we substitute $\alpha$ for $\Delta'$ in this notation.

**Lemma 2.2.** *Write $\Delta$ as a disjoint union $\Delta_1 \sqcup \Delta_2$. Define the functor $\boxtimes : \mathbf{Rep}_{\mathbb{Q}_p}(G_{K,\Delta_1}) \times \mathbf{Rep}_{\mathbb{Q}_p}(G_{K,\Delta_2}) \to \mathbf{Rep}_{\mathbb{Q}_p}(G_{K,\Delta})$ taking the pair $V_1, V_2$ to the tensor product of the objects obtained from each $V_i$ by restriction along the projection $\pi_{\Delta,\Delta_i} : G_{K,\Delta} \to G_{K,\Delta_i}$. Then every object of $\mathbf{Rep}_{\mathbb{Q}_p}(G_{K,\Delta})$ is a quotient of some object in the essential image of $\boxtimes$.*

*Proof.* Set $V_1 := \mathrm{Res}_{\Delta,\Delta_1}(V)$ and let $W$ be the $G_{K,\Delta_1}$-invariant subspace of $(V_1 \boxtimes 1)^\vee \otimes V$. Since $W$ is evidently also stable under $G_{K,\Delta_2}$, it can be interpreted as $1 \boxtimes V_2$ for $V_2 := \mathrm{Res}_{\Delta,\Delta_2}(W)$.

Now consider the following composition of morphisms in $\mathbf{Rep}_{\mathbb{Q}_p}(G_{K,\Delta})$:

$$(V_1 \boxtimes 1) \otimes W \to (V_1 \boxtimes 1) \otimes (V_1 \boxtimes 1)^\vee \otimes V \to V, \tag{1}$$

where the last map is the natural contraction of the first and second factors in the triple tensor product. After applying $\mathrm{Res}_{\Delta,\Delta_1}$, we obtain maps in $\mathbf{Rep}_{\mathbb{Q}_p}(G_{K,\Delta_1})$ of the form

$$V_1 \otimes \mathrm{Res}_{\Delta,\Delta_1}(W) \to V_1 \otimes V_1^\vee \otimes V_1 \to V_1.$$

Now let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be a basis of $V_1$ and let $\mathbf{e}_1^\vee, \dots, \mathbf{e}_n^\vee$ denote the dual basis of $V_1^\vee$. Then $\mathrm{Res}_{\Delta,\Delta_1}(W)$ contains the element $\sum_i \mathbf{e}_i^\vee \otimes \mathbf{e}_i$ of $V_1^\vee \otimes V_1$ corresponding to the identity map; hence for $j = 1, \dots, n$, $V_1 \otimes \mathrm{Res}_{\Delta,\Delta_1}(W)$ contains the element $\mathbf{e}_j \otimes \sum_i (\mathbf{e}_i^\vee \otimes \mathbf{e}_i)$ which contracts to $\mathbf{e}_j$. We conclude that (1) is surjective, and hence $V$ is a quotient of $V_1 \boxtimes W$. $\square$

From this lemma, we deduce the implication from (a) to (b) in Theorem 1.1, as well as comparable results for other period rings besides $\mathbf{B}_{\mathrm{dR}}$.

**Proposition 2.3.** *Suppose that $V \in \mathbf{Rep}_{\mathbb{Q}_p}(G_{K,\Delta})$ has the property that for each $\alpha \in \Delta$, $\mathrm{Res}_{\Delta,\alpha}(V)$ is admissible with respect to the period ring $\mathbf{B}_{*,\alpha}$ for some $* \in \{\mathrm{dR}, \mathrm{HT}, \mathrm{crys}, \mathrm{ss}\}$. Then the representation $V$ is admissible with respect to the uncompleted tensor product $\mathbf{B}$ of the rings $\mathbf{B}_{*,\alpha}$ over $\mathbb{Q}_p$.*

*Proof.* We proceed by induction on $\#\Delta$, the case where this quantity is 1 being a vacuous base case. Suppose $\#\Delta > 1$ and write $\Delta$ as a disjoint union $\Delta_1 \sqcup \Delta_2$ of two nonempty subsets. Define $V_1, W, V_2$ as in the proof of Lemma 2.2. Then $\mathrm{Res}_{\Delta_1,\alpha}(V_1) \cong \mathrm{Res}_{\Delta,\alpha}(V)$ is admissible with respect to $\mathbf{B}_{*,\alpha}$ for each $\alpha \in \Delta_1$. Meanwhile, $\mathrm{Res}_{\Delta_2,\alpha}(V_1 \boxtimes 1)$ is trivially admissible with respect to $\mathbf{B}_{*,\alpha}$ for each $\alpha \in \Delta_2$, as then is $\mathrm{Res}_{\Delta_2,\alpha}((V_1 \boxtimes 1) \otimes V)$. Since admissibility passes to subobjects, we deduce that $\mathrm{Res}_{\Delta_2,\alpha}(W)$ is admissible with respect to $\mathbf{B}_{*,\alpha}$ for each $\alpha \in \Delta_2$.

Applying the induction hypothesis, we deduce that $V_1 \boxtimes V_2$ is admissible with respect to the uncompleted tensor product $\mathbf{B}$ of the rings $\mathbf{B}_{*,\alpha}$ over $\mathbb{Q}_p$. Since this property passes to quotients, the same holds for $V$. $\square$

## 3. Multivariate $(\varphi, \Gamma)$-modules

We recall briefly how the standard framework of $(\varphi, \Gamma)$-modules adapts to the multivariate context.

**Definition 3.1.** Let $F$ be a field of the form $W(k_0)[p^{-1}]$ for some perfect field $k_0$ of characteristic $p$. Let $\mathbf{A}_F$ be the completion of $W(k_0)((\varpi))$ for the $p$-adic topology. Elements of $\mathbf{A}_F$ are naturally represented as formal Laurent series $\sum_{i \in \mathbb{Z}} c_i \varpi^i$ with $c_i \in W(k_0)$. This ring admits a continuous action of the monoid $\mathbb{Z}_p \setminus \{0\}$ given by

$$\gamma \left( \sum_{i \in \mathbb{Z}} c_i \varpi^i \right) = \sum_{i \in \mathbb{Z}} c_i ((\varpi + 1)^{\gamma(i)} - 1);$$

let $\varphi$ denote the endomorphism given by the action of $p \in \mathbb{Z}_p \setminus \{0\}$.

Each finite étale algebra over $F$ lifts uniquely to a finite étale extension of $\mathbf{A}_F$ equipped with an extension of the action of $\mathbb{Z}_p \setminus \{0\}$ (need reference). Let $\mathbf{A}_K$ denote the extension of $\mathbf{A}_F$ corresponding to $K$; the group $\mathbb{Z}_p^\times$ acts transitively on the connected components of $\mathbf{A}_K$ with stabilizer $\Gamma_K$.

For each $\alpha \in \Delta$, let $\mathbf{A}_{K,\alpha}$ be a copy of $\mathbf{A}_K$ with the variable $\varpi$ replaced by $\varpi_\alpha$, and let $\mathbf{A}_{K,\Delta}$ be the $p$-adic completion of the tensor product of the rings $\mathbf{A}_{K,\alpha}$ over $\mathbb{Z}_p$. This ring carries an action of the product of $\mathbb{Z}_p \setminus \{0\}$ indexed by $\Delta$, which for uniformity we denote by $(\mathbb{Z}_p \setminus \{0\})_\Delta$.

Put $\mathbf{B}_K := \mathbf{A}_K[p^{-1}]$, $\mathbf{B}_{K,\Delta} := \mathbf{A}_{K,\Delta}[p^{-1}]$.

**Theorem 3.2.** *Let $\mathbf{A}_\Delta$ denote the $p$-adic completion of $\bigcup_{K'/K} \mathbf{A}_{K',\Delta}$ (where $K'$ runs over finite extensions of $K$). Then for $T \in \mathbf{Rep}_{\mathbb{Z}_p}(G_{K,\Delta})$,*

$$D(T) := \left(T \otimes_{\mathbb{Z}_p} \mathbf{A}_\Delta\right)^{G_K}$$

*is a finite projective $\mathbf{A}_{K,\Delta}$-module of rank $\operatorname{rank}_{\mathbb{Z}_p}(T)$, which inherits an action of $(\mathbb{Z}_p \setminus \{0\})_\Delta$ from the action on $\mathbf{A}_\Delta$, and the natural map*

$$D(T) \otimes_{\mathbf{A}_{K,\Delta}} \mathbf{A}_\Delta \to T \otimes_{\mathbb{Z}_p} \mathbf{A}_\Delta$$

*is an isomorphism. (In particular, every $T$ is admissible for the period ring $\mathbf{A}_\Delta$.)*

*Proof.* In the case $\Delta = \{\alpha\}$ this dates back to Fontaine. For the general case, apply [2, Theorem 1.1]. □

**Corollary 3.3.** *Set $\mathbf{B}_\Delta := \mathbf{A}_\Delta[p^{-1}]$. Then for $V \in \mathbf{Rep}_{\mathbb{Q}_p}(G_{K,\Delta})$,*

$$D(V) := \left(V \otimes_{\mathbb{Q}_p} \mathbf{B}_\Delta\right)^{G_K}$$

*is a finite projective $\mathbf{B}_{K,\Delta}$-module of rank $\dim_{\mathbb{Q}_p}(V)$, which inherits an action of $(\mathbb{Z}_p \setminus \{0\})_\Delta$ from the action on $\mathbf{B}_\Delta$, and the natural map*

$$D(V) \otimes_{\mathbf{B}_{K,\Delta}} \mathbf{B}_\Delta \to V \otimes_{\mathbb{Q}_p} \mathbf{B}_\Delta$$

*is an isomorphism. (In particular, every $V$ is admissible for the period ring $\mathbf{B}_\Delta$.)*

*Proof.* Since $G_{K,\Delta}$ is a compact group, its action on $V$ must stabilize some lattice $T$. We may ths deduce the claim from Theorem 3.2. □

**Definition 3.4.** Let $\mathbf{A}^r_{F,\Delta}$ be the subring of $\mathbf{A}_{F,\Delta}$ consisting of formal Laurent series which converge for $p^{-r} \leq |\varpi_\alpha| < 1$. This ring is stable under the action of $(\mathbb{Z}_p \setminus \{0\})_\Delta$ on $\mathbf{A}_{F,\Delta}$.

The pair $(\mathbf{A}^\dagger_{F,\Delta}, (p))$ is henselian, so we have a canonical identification of $\mathbf{A}_{K,\Delta}$ with $\mathbf{A}^\dagger_{K,\Delta} \otimes_{\mathbf{A}^\dagger_{F,\Delta}} \mathbf{A}_{K,\Delta}$ for some finite étale $\mathbf{A}^\dagger_{F,\Delta}$-algebra $\mathbf{A}^\dagger_{K,\Delta}$ equipped with an extension of the action of $(\mathbb{Z}_p \setminus \{0\})_\Delta$.

Put $\mathbf{B}^\dagger_K = \mathbf{A}^\dagger_K[p^{-1}]$, $\mathbf{B}^\dagger_{K,\Delta} := \mathbf{A}^\dagger_{K,\Delta}[p^{-1}]$.

**Theorem 3.5.** *For $T \in \mathbf{Rep}_{\mathbb{Z}_p}(G_{K,\Delta})$, there is a unique finite projective $\mathbf{A}^\dagger_{K,\Delta}$-submodule $D^\dagger(T)$ of $D(T)$ which is stable under the action of $(\mathbb{Z}_p \setminus \{0\})_\Delta$ and for which the natural map*

$$D^\dagger(T) \otimes_{\mathbf{A}^\dagger_{K,\Delta}} \mathbf{A}_{K,\Delta} \to D(T)$$

*is an isomorphism.*

*Proof.* For $\Delta = \{\alpha\}$, this is a formulation of the theorem of Cherbonnier–Colmez [3]. For the general case, apply [2, Theorem 1.1]. □

**Corollary 3.6.** *For $V \in \mathbf{Rep}_{\mathbb{Q}_p}(G_{K,\Delta})$, there is a unique finite projective $\mathbf{B}^\dagger_{K,\Delta}$-submodule $D^\dagger(V)$ of $D(V)$ which is stable under the action of $(\mathbb{Z}_p \setminus \{0\})_\Delta$ and for which the natural map*

$$D^\dagger(V) \otimes_{\mathbf{B}^\dagger_{K,\Delta}} \mathbf{B}_{K,\Delta} \to D(V)$$

*is an isomorphism.*

*Proof.* This again follows from Theorem 3.5 and the existence of stable lattices. □

**Remark 3.7.** We sketch an alternate proof of Corollary 3.6 using only Corollary 3.3 and the Cherbonnier–Colmez theorem. This short-circuits some arguments from [2, §5] which adapt the approach of [4, §2] to the Cherbonnier–Colmez theorem. A similar approach can be used to recover Theorem 3.5.

We proceed by induction on $\#\Delta$, using Cherbonnier–Colmez for the base case. In the induction step, choose a partition $\Delta = \Delta_1 \sqcup \Delta_2$ into nonempty subsets, then apply Lemma 2.2 to write $V$ as a quotient of $V_1 \boxtimes V_2$ for some $V_i \in \mathbf{Rep}_{\mathbb{Q}_p}(G_{K,\Delta_i})$. By the induction hypothesis, $D^\dagger(V_i)$ is a finite projective $\mathbf{B}^\dagger_{K,\Delta_i}$-module and the natural map $D^\dagger(V_i) \otimes_{\mathbf{B}^\dagger_{K,\Delta_i}} \mathbf{B}_{K,\Delta} \to D(V_i)$ is an isomorphism. This implies at once that $D^\dagger(V) \otimes_{\mathbf{B}^\dagger_{K,\Delta}} \mathbf{B}_{K,\Delta} \to D(V)$ is surjective. By repeating the argument with $V$ replaced by the kernel of $V_1 \boxtimes V_2 \to V$, we may then apply the five lemma to deduce that in fact $D^\dagger(V) \otimes_{\mathbf{B}^\dagger_{K,\Delta}} \mathbf{B}_{K,\Delta} \to D(V)$ is also injective, and hence an isomorphism. Since $\mathbf{A}^\dagger_{K,\Delta} \to \mathbf{A}_{K,\Delta}$ is faithfully flat (being a completion with respect to an ideal in the Jacobson radical), so is $\mathbf{B}^\dagger_{K,\Delta} \to \mathbf{B}_{K,\Delta}$; we may thus conclude that $D^\dagger(V)$ is a finite projective $\mathbf{B}^\dagger_{K,\Delta}$-module.

## 4. Multivariate $(\varphi, \Gamma)$-modules and Berger's construction

We follow [1, §5].

**Definition 4.1.** Let $\mathbf{B}^r_{\mathrm{rig},F,\Delta}$ denote the ring of rigid analytic functions on the product of the annuli $p^{-r} \leq |\varpi_\alpha| < 1$ over $\alpha \in \Delta$. Put $\mathbf{B}^\dagger_{\mathrm{rig},F,\Delta} := \bigcup_{r>0} \mathbf{B}^r_{\mathrm{rig},F,\Delta}$; when $\Delta = \{\varpi\}$ this is commonly called the *Robba ring* over $F$ in the variable $\varpi_\alpha$. Set $\mathbf{B}^\dagger_{\mathrm{rig},K,\Delta} := \mathbf{B}^\dagger_{\mathrm{rig},F,\Delta} \otimes_{\mathbf{A}^\dagger_{F,\Delta}} \mathbf{A}^\dagger_{K,\Delta}$.

For $\alpha \in \Delta$, define

$$t_\alpha = \log(1+\varpi_\alpha) \in \mathbf{B}^\dagger_{\mathrm{rig},F,\Delta};$$

it has the property that $\gamma(t_\alpha) = \gamma \cdot t_\alpha$ for $\gamma \in (\mathbb{Z}_p \setminus \{0\})_\alpha$.

**Definition 4.2.** For $V \in \mathbf{Rep}_{\mathbb{Q}_p}(G_{K,\Delta})$, define $D^\dagger_{\mathrm{rig}}(V) := D^\dagger(V) \otimes_{\mathbf{B}^\dagger_{K,\Delta}} \mathbf{B}^\dagger_{\mathrm{rig},K,\Delta}$. Note that for any given $V$, we can fix $r > 0$ so that $D^\dagger_{\mathrm{rig}}(V)$ is the base extension of a finite projective $\mathbf{B}^r_{F,\Delta}$-module $M$ on which $\Gamma_{K,\Delta}$ acts (but not $\varphi_\alpha$).

The action of $\Gamma_{K,\Delta}$ on $M$ then induces an action of its Lie algebra $\operatorname{Lie}\Gamma_{K,\Delta}$ with logarithmic singularities at the zeroes of $\prod_{\alpha\in\Delta} t_\alpha$. For zeroes in the region $p^{-r} \leq |\varpi_\alpha| < 1$ for $r$ sufficiently small, the residue of the action of $\operatorname{Lie}\Gamma_{K,\alpha}$ will be the same on all components of the singular locus. Crucially, since $\operatorname{Lie}\Gamma_{K,\Delta}$ is commutative, this residue will be defined over $K$.

**Lemma 4.3.** *In Theorem 1.1, condition (d) implies condition (a).*

*Proof.* Condition (d) implies that the residue of the action of $\operatorname{Lie}\Gamma_{K,\alpha}$ is zero for every $\alpha \in \Delta$. From the last point of Definition 4.2, we see that this condition persists on passage from $V$ to $\mathrm{Res}_{\Delta,\alpha}(V)$. This proves the claim. □

## 5. Restriction for multivariate $(\varphi, \Gamma)$-modules

**Definition 5.1.** A $(\varphi_\Delta, \Gamma_\Delta)$-*module* over $\mathbf{B}^\dagger_{\mathrm{rig},K,\Delta}$ is a finite projective module over this ring equipped with a semilinear action of $(\mathbb{Z} \setminus \{0\})_\Delta$ which is continuous for the LF (limit of Fréchet) topology on the base ring.

While we do not know how to define a restriction functor on Galois representations for not necessarily étale $(\varphi_\Delta, \Gamma_\Delta)$-modules, we can formulate an analogue of the conclusion of Lemma 2.2 as a conjecture.

**Conjecture 5.2.** *For any partition $\Delta = \Delta_1 \sqcup \Delta_2$, every $(\varphi_\Delta, \Gamma_\Delta)$-module over $\mathbf{B}^\dagger_{\mathrm{rig},K,\Delta}$ is the quotient of some object in the essential image of $\boxtimes$.*

**Remark 5.3.** It follows from [2, Theorem 1.1] that Conjecture 5.2 holds in the case of an étale $(\varphi_\Delta, \Gamma_\Delta)$-module. It is also possible to prove it for rank 1 modules and for de Rham objects; the latter uses the results of [5, §3] to obtain a form of Drinfeld's lemma for multivariate connections with partial Frobenius structures.

One subtlety inherent in Conjecture 5.2 is that the natural analogue for vector bundles on a product of Fargues–Fontaine curves is known to be false, even for bundles of rank 2. In particular, one cannot directly reduce to the étale case using slope filtrations. See [6] for further discussion.

## References


[1] O. Brinon, B. Chiarellotto, and N. Mazzari, Multivariable de Rham representations, Sen theory and $p$-adic differential equations, arXiv:2111.11563v1 (2021).

[2] A. Carter, K.S. Kedlaya, and G. Zábrádi, Drinfeld's lemma for perfectoid spaces and overconvergence of multivariate $(\varphi, \Gamma)$-modules, *Doc. Math.* **26** (2021), 1329–1393.

[3] F. Cherbonnier and P. Colmez, Réprésentations $p$-adiques surconvergentes, *Invent. Math.* **133** (1998), 581–611.
[4] K.S. Kedlaya, New methods for $(\varphi, \Gamma)$-modules, *Res. Math. Sci.* **2** (2015) (Robert Coleman memorial issue).
[5] K.S. Kedlaya, Monodromy representations of $p$-adic differential equations in families, arXiv:2209.00593v1 (2022).
[6] K.S. Kedlaya, Vector bundles on products of Fargues-Fontaine curves, preprint.