SHEAFINESS CRITERIA FOR HUBER RINGS

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Abstract. In Huber's theory of adic spaces, one associates to certain topological rings an associated topological space and a presheaf of rings. However, the analogy with the theory of schemes breaks down because the resulting presheaf is not always a sheaf unless one adds some additional assumption on the original ring, such as the strong noetherian property. Generalizing work of Scholze and Kedlaya–Liu, which established the sheaf property for perfectoid rings (which are typically not noetherian), Buzzard–Verberkmoes and Mihara established the sheaf property for stably uniform Huber rings. However, the stably uniform property can be difficult to verify in examples, and in particular is not (known to be) preserved under general étale extensions. In this paper, we exhibit several classes of Huber rings contained among the stably uniform rings over $\mathbb{Q}_p$, for which membership (and hence preservation under étale extensions) can be tested more easily; these include *sousperfectoid* rings, which admit module-split embeddings into perfectoids, and *diamantine* rings, which satisfy certain conditions on pro-étale cohomology (or equivalently v-cohomology).

Contents

1. Introduction

For much of its history, nonarchimedean analytic geometry was limited to spaces locally of (topological) finite type over a field; for instance, this is true in the foundational works of

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Tate [46], Raynaud [35], and Berkovich [6, 7]. In retrospect, one reason for this can be seen in Huber's general framework [22, 23] of *adic spectra* associated to *f-adic rings* (hereafter called *Huber rings*): by contrast with what happens for schemes, the structure presheaf on an adic spectrum is in general not a sheaf. When it is a sheaf, we say that the underlying Huber ring is *sheafy*. Even after Huber's original work (including his treatise [24]), sheafiness was only known under strong noetherian hypotheses. (See [28, Lecture 1] for further discussion.)

This state of affairs changed drastically with the introduction of *perfectoid rings* [29, 37], which are sheafy but generally not noetherian. With this development, it became imperative to establish more clearly the distinction between sheafy and nonsheafy Huber rings. Since any perfectoid ring is *uniform* (its power-bounded elements form a bounded subring; equivalently, its topology is defined by a supremum over the adic spectrum), much of the work in this direction has centered on deciding whether sheafiness holds for various subclasses among the uniform Huber rings. In one direction, Buzzard–Verberkmoes [13] and Mihara [33] gave examples of uniform Huber rings which are not sheafy, but these examples have a somewhat pathological flavor. In the other direction, Buzzard–Verberkmoes and Mihara showed that a Huber ring which is *stably uniform* (its rational localizations are all uniform) is sheafy; this includes the case of perfectoid rings.

In this paper, we develop some additional sheafiness criteria for Huber rings, with an eye towards ease of verification in cases of practical interest. A closely related desideratum is stability under some basic operations, such as formation of finite étale extensions and adjunction of a convergent power series variable. Note that the criterion of Buzzard–Verberkmoes–Mihara falls short on these points: if A is a stably uniform Huber ring, it is not known in general whether a finite étale extension B/A is stably uniform, nor whether $A\langle T\rangle$ is stably uniform.

For the rest of this discussion, fix a prime p and consider only Huber rings which are complete $\mathbb{Q}_p$-algebras. We define a Huber ring A to be *sousperfectoid*[1] if there exists a continuous morphism $A \to B$ of Huber rings with B perfectoid which splits in the category of topological A-modules. It is easily seen that the sousperfectoid property both implies uniformity and is preserved under rational localization; consequently, any sousperfectoid ring is stably uniform and hence sheafy by Buzzard–Verberkmoes–Mihara. (For technical reasons, it is useful to also consider a slightly more flexible notion of *weakly sousperfectoid* rings, in which the existence of a splitting as above is replaced with a uniformity condition on splittings of finite étale A-algebras.)

We define a Huber ring A to be *diamantine*[2] if A coincides with the ring of global sections of the structure sheaf on the pro-étale site (or equivalently, on the v-site) of A, and some additional cohomological conditions hold (see Definition 11.1). Somewhat by design, this definition is of étale-local nature (Theorem 11.14) and implies uniformity, so any diamantine ring is stably uniform and sheafy. Moreover, the property of being diamantine is stable under passage to rational localizations and adjunction of a convergent power series variable.

The class of diamantine rings includes both perfectoid rings and some, but not all, affinoid algebras over nonarchimedean fields; for example, any smooth affinoid algebra is diamantine.

[1]We view *subperfectoid*, which is more consistent with conventional English morphology, as an acceptable but less mellifluous synonym.

[2]That is, having characteristic qualities of a *diamond* in the sense of Scholze [41, 39]. The variant form *adamantine* is more common; but besides obscuring the etymological relationship with diamonds, this word admits a distracting association with a fictional metallic alloy.

In fact, we determine exactly which affinoid algebras are diamantine: they are the ones which are reduced and *seminormal* in the sense of Swan (Theorem 10.3); this relies crucially on a geometric form of the Ax–Sen–Tate theorem due to Kedlaya–Liu [30, Theorem 8.2.3].

By analogy with perfectoid rings, we define a *sousperfectoid space* or *diamantine space* to be an adic space covered by the adic spectra of sousperfectoid rings or diamantine rings, respectively (and hence admitting a neighborhood basis of such subspaces). Such spaces are also stable under the formation of finite étale covers, and hence have well-behaved étale sites. In fact, we conjecture (Conjecture 12.3) that the category of diamantine spaces is stable under formation of profinite étale covers, such as the total spaces of étale $\mathbb{Q}_p$-local systems. It would be interesting to directly verify the diamantine property in some key special cases, such as Shimura varieties of infinite level.

2. Huber, Banach, and perfectoid rings

We begin by summarizing basic algebraic facts about Huber (f-adic) rings, Banach rings, and perfectoid rings. We postpone discussion of geometric facts to §3.

Definition 2.1. A *Huber ring* is a (commutative) topological ring A containing an open subring A_0 on which the subspace topology coincides with the I-adic topology for some finitely generated ideal I of A_0. Let A° denote the subring of power-bounded elements of A.

Throughout this paper, we only consider Huber rings which are *complete* and *Tate*; the latter condition asserts that A contains a topologically nilpotent unit. For instance, if K is a *nonarchimedean field* (i.e., a field complete with respect to some nontrivial multiplicative nonarchimedean absolute value), then any Banach algebra over K is a Huber ring. A key special case is when A is an *affinoid algebra* (a quotient of the Tate algebra $K\langle T_1, \dots, T_n\rangle$, the completion of $K[T_1, \dots, T_n]$ for the Gauss norm) or more generally a *Berkovich affinoid algebra* (a quotient of a weighted Tate algebra $K\langle T_1/r_1, \dots, T_n/r_n\rangle$ in which T_i is assigned the norm $r_i > 0$); recall that in these cases, A is noetherian (see [12, Theorem 5.2.6/1] for affinoid algebras and [6, Proposition 2.1.3] for Berkovich affinoid algebras).

Remark 2.2. In [28], the discussion includes Huber rings which are not Tate but only *analytic*, meaning that the topologically nilpotent elements generate the unit ideal. For certain technical reasons (especially Proposition 2.14), we do not allow this extra level of generality here except in a few isolated instances (specified individually). In any case, since the adic spectra of analytic rings admit open coverings by adic spectra of Tate rings, the distinction makes no difference at the level of adic spaces.

Definition 2.3. A Huber ring A is *uniform* if A° is bounded in A. If $A \to B$ is a strict inclusion of Huber rings with B uniform, then A is also uniform. (Recall that a morphism of topological abelian groups is *strict* if the subspace and quotient topologies on the image coincide.)

Any uniform Huber ring is reduced. Conversely, while there exist reduced Huber rings which are not uniform (see [13, Proposition 14] for an example), any reduced affinoid algebra over a nonarchimedean field is uniform [12, Theorem 6.2.4/1].

Remark 2.4. For A a Huber ring, any finitely generated A-module M admits a *natural topology* as a topological A-module (inherited from any surjection from a finite free module), but this topology need not be Hausdorff in general. When it is, M is complete for the natural topology; for example, this always happens if M is projective [29, Lemma 2.2.12].

The classical Banach open mapping theorem admits the following analogue.

Definition 2.5. Let A be a Huber ring (which here must be complete and Tate), let A_0 be a ring of definition of A, and let I be an ideal of definition of A_0. Let M be an A_0-module.

We say M is *torsion* if every element is killed by some power of I. We say M is *uniformly torsion* if M itself is killed by some power of I. Both the torsion modules and the uniformly torsion modules form (thick) Serre subcategories of the category of A_0-modules.

In case M is chosen from a set $\{M_j\}_{j \in J}$ and M is killed by some power of I which can be chosen *independently of* j, we will say that M is j*-uniformly torsion*.

We say M is *derived complete* if $\operatorname{Ext}^i_{A_0}(A_{0f}, M) = 0$ for all $f \in I$ and all nonnegative integers i. See [44, Tag 091P] for other equivalent conditions; in particular, it suffices to check the definition for $i = 0, 1$. Any complete A_0-module is derived complete [44, Tag 091T].

Lemma 2.6. *With notation as in Definition 2.5, the category of derived complete A_0-modules forms a weak Serre subcategory of the category of A_0-modules. In particular, it is an abelian category.*

Proof. See [44, Tag 091U]. □

Theorem 2.7. *With notation as in Definition 2.5, a derived complete A_0-module is torsion if and only if it is uniformly torsion.*

Proof. See [10, Theorem 2.3]. □

Corollary 2.8 (Open mapping theorem)**.** *For A a Huber ring (which here must be complete and Tate), any surjective continuous homomorphism of completely metrizable topological A-modules is strict (and hence a quotient mapping).*

Proof. Set notation as in Definition 2.5, and let $f \colon M \to N$ be such a homomorphism. By hypothesis, f is the base extension of a morphism $f_0 \colon M_0 \to N_0$ of complete A_0-modules M_0, N_0 which are open in M, N. By Lemma 2.6, $\operatorname{coker}(f_0)$ is a derived complete A_0-module. Since f is surjective, $\operatorname{coker}(f_0)$ is also torsion; hence by Theorem 2.7, $\operatorname{coker}(f_0)$ is uniformly torsion. This implies that f is strict. (See also [20], [28, Appendix], or [17, Appendix 0.B.2].) □

Definition 2.9. For A a uniform Huber ring, any finite étale A-algebra B, equipped with its natural topology as an A-module, is again a uniform Huber ring [29, Proposition 2.8.16].

A morphism $A \to B$ of uniform Huber rings is *profinite étale* (resp. *faithfully profinite étale*) if B is the completion of a direct limit of subalgebras which are finite étale (resp. finite étale and faithfully flat, or for short *faithfully finite étale*) over A. We add the adverb *countably* in case the direct limit is taken over a countable index set.

Definition 2.10. A *Banach ring* is a (commutative) ring A equipped with a submultiplicative nonarchimedean norm with respect to which it is complete. As for Huber rings, we only consider Banach rings which are *Tate* in the sense of containing a topologically nilpotent unit. (Again, in [28] this restriction is weakened to the condition that the topologically nilpotent elements generate the unit ideal.)

A Banach ring is *uniform* if its associated spectral seminorm

$$|x|_{\mathrm{sp}} = \lim_{n \to \infty} |x^n|^{1/n}$$

defines the same topology as the original norm.

Remark 2.11. The relationship between Huber and Banach rings is described in detail in [28, §1.5]. We summarize this discussion as follows.

- Every (Tate) Banach ring has underlying topological ring which is a Huber ring.
- Every (Tate) Huber ring can be promoted to a Banach ring in some fashion. (We will often exploit this fact for expository purposes.)
- For A a Banach ring, the forgetful functor from the category of Banach rings over A with bounded morphisms to the category of Huber rings over A with continuous morphisms is an equivalence.
- A Banach ring is uniform if and only if its underlying Huber ring is uniform.

Remark 2.12. One may also consider Banach modules over Banach rings. A bounded morphism of Banach modules over a Banach ring is strict if and only if it admits a bounded set-theoretic section on its image.

Definition 2.13. Fix a prime number p. We use the term *perfectoid ring* in the sense of Fontaine [15], [30, §3.3], to mean a uniform Huber ring A containing a topologically nilpotent unit ϖ such that $p \in \varpi^p A^\circ$ and the Frobenius map $A^\circ/(\varpi) \to A^\circ/(\varpi^p)$ is surjective. (This definition is extended to analytic rings in [28, Lecture 2].)

Proposition 2.14. *For any uniform Huber ring A in which p is topologically nilpotent, there exists a faithfully profinite étale morphism $A \to A'$ with A' perfectoid.*

Proof. For the case where p is invertible is A, see [29, Lemma 3.6.26]. For the general case, see [30, Lemma 3.3.28]. □

Proposition 2.15. *Let A be a perfectoid ring. Let B be a finite étale A-algebra, topologized as a finite A-module as per Definition 2.9.*

(a) *The ring B is again a perfectoid ring.*
(b) *The A°-module B°/A° is* almost finite projective*: for any topologically nilpotent element x of A, there exist a finite free A°-module F and some A°-linear maps $B^\circ/A^\circ \to F \to B^\circ/A^\circ$ whose composition is multiplication by x. (A similar statement then holds with B°/A° replaced by B°.)*

Proof. Part (a) follows from [30, Theorem 3.3.18]. For part (b), [30, Corollary 3.3.24] implies that B° is almost finite projective as an A°-module; using the fact that $B^\circ \otimes_{A^\circ} B^\circ \to (B \otimes_A B)^\circ$ is an almost isomorphism [30, Theorem 3.3.18], we deduce from this that B° is an almost finite étale A°-algebra. (See also [37, Theorem 7.9] for the case where A is an algebra over a perfectoid field, [29, Theorem 3.6.21] for the case where A is an algebra over $\mathbb{Q}_p$, and [28, Theorem 2.5.9] for an extension to analytic rings.) □

Remark 2.16. Proposition 2.15 includes a strong generalization of the *almost purity* theorem of Faltings. This generalization is intimately related to applications of perfectoid rings to commutative algebra, as in the resolution of Hochster's direct summand conjecture [1, 2, 8, 9].

3. Spectra of Huber rings

We next recall some definitions and results concerning the geometric spaces associated to Huber rings.

Definition 3.1. For A a Huber ring, a *ring of integral elements* of A is an open, integrally closed subring of A°. A *Huber pair* is a pair (A, A^+) in which A is a Huber ring and A^+ is a ring of integral elements of A. Given such a pair, Huber defines the *adic spectrum* $\mathrm{Spa}(A, A^+)$ of A as the set of equivalence classes of continuous valuations on A centered in A^+. A *rational subspace* of $\mathrm{Spa}(A, A^+)$ is one of the form

$$U = \{v \in \mathrm{Spa}(A, A^+) \colon v(f_1), \dots, v(f_n) \leq v(g)\}$$

for some $f_1, \dots, f_n, g \in A$ generating the unit ideal. (Here we are relying on our running hypothesis that A contains a topologically nilpotent unit to ensure that the only open ideal of A is the unit ideal.) The rational subspaces form a basis for a topology on $\mathrm{Spa}(A, A^+)$ under which this space is *spectral* in the sense of Hochster [22, Theorem 3.5].

Convention 3.2. For $\mathcal{P}$ a property of Huber rings, unless otherwise specified the phrasing "a Huber pair (A, A^+) has property $\mathcal{P}$" will mean that the ring A has property $\mathcal{P}$.

For Banach rings, one has the following analogue of the construction of the adic spectrum.

Definition 3.3. For A a Banach ring, the *Gel'fand spectrum* of A, denoted $\mathcal{M}(A)$, is the set of bounded multiplicative seminorms on A equipped with the evaluation topology, with respect to which it is compact. For $\alpha \in \mathcal{M}(A)$, let $\mathcal{H}(\alpha)$ denote the *residue field* of α, i.e., the completion of $A/\ker(\alpha)$ for the induced multiplicative norm. (Here $\ker(\alpha)$, the *kernel* of α, is defined as the prime ideal $\alpha^{-1}(0)$.) For A^+ a ring of integral elements of the underlying Huber ring of A, there is a natural but not continuous injection $\mathcal{M}(A) \to \mathrm{Spa}(A, A^+)$.

Remark 3.4. A common refinement of the definitions of the adic spectrum and the Gel'fand spectrum is given by the construction of a *reified adic spectrum* of [27]; many of our results can be transposed to that context with little effort (either by redoing the proofs, or using descent techniques). We leave this process as an exercise for the interested reader.

Proposition 3.5. *For A a Banach ring, the spectral seminorm coincides with the supremum over the Gel'fand spectrum.*

Proof. See [29, Theorem 2.3.10]. □

Definition 3.6. With notation as in Definition 3.1, let $A\langle \frac{f_1}{g}, \dots, \frac{f_n}{g} \rangle$ be the quotient of $A\langle T_1, \dots, T_n \rangle$ by the closure of the ideal $(f_1 - gT_1, \dots, f_n - gT_n)$. Let $A\langle \frac{f_1}{g}, \dots, \frac{f_n}{g} \rangle^+$ be the integral closure of the image of $A^+\langle T_1, \dots, T_n \rangle$ in $A\langle \frac{f_1}{g}, \dots, \frac{f_n}{g} \rangle$. The morphism

$$(A, A^+) \to \left(A \left\langle \frac{f_1}{g}, \dots, \frac{f_n}{g} \right\rangle, A \left\langle \frac{f_1}{g}, \dots, \frac{f_n}{g} \right\rangle^+ \right) \tag{3.6.1}$$

of Huber pairs is then initial among morphisms $(A, A^+) \to (B, B^+)$ for which the image of $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ is contained in U; moreover, the induced map

$$\mathrm{Spa} \left(A \left\langle \frac{f_1}{g}, \dots, \frac{f_n}{g} \right\rangle, A \left\langle \frac{f_1}{g}, \dots, \frac{f_n}{g} \right\rangle^+ \right) \to U$$

is a homeomorphism [23, Lemma 1.5]. We call (3.6.1) the *rational localization* corresponding to U (it being unique up to unique isomorphism). In some cases where the rings of integral elements are not crucial, we refer to the underlying map $A \to A\langle \frac{f_1}{g}, \dots, \frac{f_n}{g} \rangle$ also as a *rational localization*.

By a *rational covering* of (A, A^+), we will mean a family $\{(A, A^+) \to (B_i, B_i^+)\}_i$ of rational localizations such that the spaces $\mathrm{Spa}(B_i, B_i^+)$ form a covering of $\mathrm{Spa}(A, A^+)$. By quasicompactness, any such covering is refined by some finite covering.

Lemma 3.7. *Suppose that A is a uniform Huber ring. Choose $f, g \in A$ generating the unit ideal.*

(a) *The ideals*
$$(f - gT)A\langle T \rangle, \qquad (f - gT)A\langle T^{\pm} \rangle$$
are closed; we thus have
$$A\left\langle \frac{f}{g} \right\rangle \cong A\left\langle T \right\rangle/(f - gT), \quad A\left\langle \frac{f}{g}, \frac{g}{f} \right\rangle \cong A\langle T^{\pm} \rangle/(f - gT).$$

(b) *The sequence*
$$0 \to A \to A\left\langle \frac{f}{g} \right\rangle \oplus A\left\langle \frac{g}{f} \right\rangle \to A\left\langle \frac{f}{g}, \frac{g}{f} \right\rangle \to 0$$
is exact.

(c) *If $A\langle \frac{f}{g}, \frac{g}{f} \rangle$ is uniform, then so are $A\langle \frac{f}{g} \rangle$ and $A\langle \frac{g}{f} \rangle$.*

Proof. Parts (a) and (b) are [28, Lemma 1.7.3] and [28, Lemma 1.7.2, Lemma 1.7.3], respectively; we thus focus on (c). Promote A to a uniform Banach ring equipped with a spectral norm; equip $A\langle T \rangle$, $A\langle U \rangle$, $A\langle T^{\pm} \rangle$ with the Gauss extensions of the norm on A (which are again spectral norms); and equip $A\langle \frac{f}{g} \rangle$, $A\langle \frac{g}{f} \rangle$, $A\langle \frac{f}{g}, \frac{g}{f} \rangle$ with the quotient norms from $A\langle T \rangle$, $A\langle U \rangle$, $A\langle T^{\pm} \rangle$. To show that $A\langle \frac{f}{g} \rangle$ is uniform, for arbitrary $x \in A\langle \frac{f}{g} \rangle$ we must establish a bound of the form $\left|x\right|^2_{A\langle \frac{f}{g} \rangle} \leq c_0 \left|x^2\right|_{A\langle \frac{f}{g} \rangle}$ for some $c_0 > 0$ which does not depend on x.

To begin with, note that $\left|x^2\right|_{A\langle \frac{f}{g}, \frac{g}{f} \rangle} \leq \left|x^2\right|_{A\langle \frac{f}{g} \rangle}$. Since $A\langle \frac{f}{g}, \frac{g}{f} \rangle$ is uniform, there exists $c_1 > 1$ independent of x such that $\left|x\right|^2_{A\langle \frac{f}{g}, \frac{g}{f} \rangle} \leq c_1 \left|x^2\right|_{A\langle \frac{f}{g}, \frac{g}{f} \rangle}$. By exactness on the right in (b) plus Corollary 2.8, there exists $c_2 > 1$ independent of x such that for some $y \in A$,
$$\left|x - y\right|_{A\langle \frac{f}{g} \rangle}, \left|y\right|_{A\langle \frac{g}{f} \rangle} \leq c_2 \left|x\right|_{A\langle \frac{f}{g}, \frac{g}{f} \rangle}.$$

Put $c_0 := c_1^{1/2} c_2$, so that
$$\left|x - y\right|_{A\langle \frac{f}{g} \rangle}, \left|y\right|_{A\langle \frac{g}{f} \rangle} \leq c_2 \left|x\right|_{A\langle \frac{f}{g}, \frac{g}{f} \rangle} \leq c_0 \left|x^2\right|^{1/2}_{A\langle \frac{f}{g} \rangle}.$$

For $\alpha \in \mathcal{M}(A)$, if $\alpha(f) \geq \alpha(g)$, then
$$\alpha(y) \leq \left|y\right|_{A\langle \frac{g}{f} \rangle} \leq c_2 \left|x\right|_{A\langle \frac{f}{g}, \frac{g}{f} \rangle} \leq c_0 \left|x^2\right|^{1/2}_{A\langle \frac{f}{g} \rangle};$$
if $\alpha(f) < \alpha(g)$, then since $\alpha(x)^2 = \alpha(x^2)$ we have
$$\alpha(y) \leq \max\{\alpha(x^2)^{1/2}, \alpha(x - y)\} \leq \max\{\left|x^2\right|^{1/2}_{A\langle \frac{f}{g} \rangle}, \left|x - y\right|_{A\langle \frac{f}{g} \rangle}\} \leq c_0 \left|x^2\right|^{1/2}_{A\langle \frac{f}{g} \rangle}.$$

It follows that
$$\left|y\right|_{A\langle \frac{f}{g} \rangle} \leq \left|y\right|_A \leq c_0 \left|x^2\right|^{1/2}_{A\langle \frac{f}{g} \rangle},$$

and so

$$|x|_{A\langle \frac{f}{g}\rangle} \leq \max\{|y|_{A\langle \frac{f}{g}\rangle}, |x-y|_{A\langle \frac{f}{g}\rangle}\} \leq c_0 \left|x^2\right|^{1/2}_{A\langle \frac{f}{g}\rangle}.$$

This proves that $A\langle \frac{f}{g}\rangle$ is uniform; a similar argument implies that $A\langle \frac{g}{f}\rangle$ is uniform. (Note that all of these arguments apply to analytic rings in the sense of Remark 2.2.) $\square$

Proposition 3.8. *Let A be a perfectoid ring (for some p). Then for any rational localization $A \to B$, B is again a perfectoid ring.*

Proof. See [37, Theorem 6.3] for the case where A is an algebra over a perfectoid field; [29, Theorem 3.6.14] for the case where A is an algebra over $\mathbb{Q}_p$; [30, Theorem 3.3.18] for the general case considered here; and [28, Theorem 2.5.3] for an extension to analytic rings. $\square$

Definition 3.9. For (A, A^+) a Huber pair, the *structure presheaf* on $\mathrm{Spa}(A, A^+)$ is the presheaf $\mathcal{O}$ assigning to an open subset V to the inverse limit of B as $(A, A^+) \to (B, B^+)$ runs over all rational localizations with $\mathrm{Spa}(B, B^+) \subseteq V$. In particular, if $V = \mathrm{Spa}(B, B^+)$ is itself a rational subspace, then $\mathcal{O}(V) = B$.

We say that (A, A^+) is *sheafy* if $\mathcal{O}$ is a sheaf. Note that for given A, sheafiness is independent of the choice of A^+ [28, Remark 1.6.9]; we may thus also treat sheafiness as a property of Huber rings subject to Convention 3.2. See [13, 33] for various examples of Huber rings which are not sheafy.

Since $\mathcal{O}$ can fail to be a sheaf, we must keep track of the distinction between the sections of a presheaf $\mathcal{F}$ on an open subset U, denoted $\mathcal{F}(U)$, and the locally-defined sections of $\mathcal{F}$ on U (i.e., the sections of the sheafification of $\mathcal{F}$ on U). We write $H^0(U, \mathcal{F})$ for the latter; we also write $H^i(U, \mathcal{F})$ for $i > 0$ to mean the i-th cohomology of the sheafification of $\mathcal{F}$ on U.

Proposition 3.10 (Kedlaya–Liu)**.** *Let (A, A^+) be a sheafy Huber pair. Then $H^i(\mathrm{Spa}(A, A^+), \mathcal{O}) = 0$ for all $i > 0$. (The same then holds on every rational subspace of $\mathrm{Spa}(A, A^+)$.)*

Proof. See [29, Theorem 2.4.23], or [28, Theorem 1.3.4] for an extension to analytic rings. $\square$

As noted earlier, in Huber's original development, sheafiness was only established under certain noetherian hypotheses. Here is an example of one of these hypotheses.

Definition 3.11. A Huber ring A is *strongly noetherian* if the Tate algebras $A\langle T_1, \dots, T_n\rangle$ are noetherian for all $n \geq 0$. For example, any affinoid algebra over a nonarchimedean field is strongly noetherian.

Unlike for polynomial rings in ordinary commutative algebra, a noetherian Huber ring need not be strongly noetherian (that is, there is no general Hilbert basis theorem in this context.) For example, there exists a Huber ring A which is a field (but not a nonarchimedean field) for which $A\langle T\rangle$ is not noetherian [16, §8.3].

Proposition 3.12 (Huber)**.** *Any strongly noetherian Huber ring is sheafy. In particular, any affinoid algebra over a nonarchimedean field is sheafy.*

Proof. See [23, Theorem 2.5], or [28, Theorem 1.2.11] for an extension to analytic rings. $\square$

Subsequently, it was discovered that sheafiness can also be established in some nonnoetherian cases. The basic result in this direction is the following.

Definition 3.13. A Huber ring A is *stably uniform* if for every rational localization $A \to B$, the ring B is again uniform. For examples of Huber rings which are uniform but not stably uniform, see [13, Proposition 17], [33]. If A is stably uniform, then any finite étale extension B of A is again uniform (see Definition 2.9), but it is not known in general whether B must again be stably uniform.

Proposition 3.14 (Buzzard–Verberkmoes, Mihara)**.** *Any stably uniform Huber ring is sheafy.*

Proof. The original references are [13, Theorem 7], [33, Theorem 4.9] (the results were obtained independently). The proof of [13] is reproduced in [29, Theorem 2.8.10]. See also [28, Theorem 1.2.13] for an extension to analytic rings. □

Corollary 3.15. *Any perfectoid ring is sheafy.*

Proof. This follows from Proposition 3.8 and Proposition 3.14. □

Remark 3.16. We do not know whether or not a uniform Huber ring that is sheafy is necessarily stably uniform. Compare [29, Remark 2.8.11] and Remark 6.19 below.

Definition 3.17. Let $\mathcal{P}$ be a property of Huber rings. We say that $\mathcal{P}$ is *of local nature* if the following statements hold.

(i) For any rational localization $A \to B$, if A has property $\mathcal{P}$, then so does B.
(ii) Let A^+ be a ring of integral elements of A. Let $\{(A, A^+) \to (B_i, B_i^+)_i\}$ be a rational covering such that

$$0 \to A \to \prod_i B_i \to \prod_{i,j} B_i \widehat{\otimes}_A B_j \to \cdots$$

is exact. If each B_i has property $\mathcal{P}$, then so does A.

For example, the property of being stably uniform is of local nature.

Remark 3.18. In condition (b) of Definition 3.17, the exactness assumption is automatic if A is sheafy. Since our aim here is to establish sheafiness in cases where it is not previously known, we do not wish to impose sheafiness as a hypothesis; nonetheless, we need some hypothesis to rule out the case where the same B_i can occur for different choices of A (see Remark 3.19).

Remark 3.19. An outstanding open problem is whether the perfectoid property is of local nature. The example of [13, Proposition 13] shows that if one does not require $A \to H^0(\mathrm{Spa}(A, A^+), \mathcal{O})$ to be an isomorphism, it is possible for a nonsheafy Huber ring A to admit a rational covering by perfectoid rings.

It is not known whether there exists a similar example in which A is uniform but the map $A \to H^0(\mathrm{Spa}(A, A^+), \mathcal{O})$ (which must now be injective) fails to be surjective. It may be possible to exhibit such an example by adapting the construction of [13, Proposition 16], which gives a Huber ring A for which $A \to H^0(\mathrm{Spa}(A, A^+), \mathcal{O})$ is injective but not surjective.

If $A \to H^0(\mathrm{Spa}(A, A^+), \mathcal{O})$ is an isomorphism and A admits a rational covering by perfectoid rings with vanishing Čech cohomology, then by Corollary 4.5 below, we know that A is sheafy. However, this is still not enough to deduce that A is perfectoid; see Remark 11.16.

The following is [28, Theorem 1.2.22(a)], but we give an independent proof to address a gap in the proof of that result; see Remark 4.6.

Lemma 3.20. *Let (A, A^+) be a Huber pair such that $X := \mathrm{Spa}(A, A^+)$ admits a covering by rational subspaces $\mathrm{Spa}(B, B^+)$ for each of which B is sheafy. Define*

$$\tilde{A} := H^0(X, \mathcal{O}), \qquad \tilde{A}^+ := H^0(X, \mathcal{O}^+).$$

(a) *The pair $(\tilde{A}, \tilde{A}^+)$ is a Huber pair for the subspace topology on $\tilde{A}$.*
(b) *For $\tilde{X} := \mathrm{Spa}(\tilde{A}, \tilde{A}^+)$, the natural map $\pi\colon \tilde{X} \to X$ is a homeomorphism; more precisely, every rational subspace pulls back to a rational subspace.*
(c) *The map π admits a section $s\colon X \to \tilde{X}$ in the category of locally ringed spaces.*
(d) *For each $x \in X$, the local homomorphism $\pi_x^\sharp : \mathcal{O}_{X,x} \to \mathcal{O}_{\tilde{X},\pi^{-1}(x)}$ is injective with dense image.*

Proof. By Proposition 3.10, $\tilde{A}$ may be computed as the zeroth Čech cohomology for some covering by rational subspaces. This yields (a).

By definition, $\tilde{A}$ is a subring of $\prod_{x\in X} \mathcal{O}_{X,x}$. Consequently, for each $x \in X$ we can extend the associated valuation to $\tilde{A}$ by pulling back along $\tilde{A} \to \mathcal{O}_{X,x}$. This gives a map $s\colon X \to \tilde{X}$ such that $X \xrightarrow{s} \tilde{X} \xrightarrow{\pi} X$ is the identity.

For $\tilde{x} \in \tilde{X}$ with $\pi(\tilde{x}) = x$, we claim that $\tilde{x} = s(x)$. To see this, let $v_x, v_{\tilde{x}}$ be the valuations on $\tilde{A}$ corresponding to $s(x), \tilde{x}$, respectively, and choose any $f \in \tilde{A}$. As an element of $H^0(X, \mathcal{O})$, f restricts on some neighborhood of x to an element of a rational localization of A. In particular, we can find $g_1, g_2 \in A$ with $v_x(g_2) \neq 0$ such that $v_x(f - g_1/g_2)$ and $v_{\tilde{x}}(f - g_1/g_2)$ are less than 1. Since v_x and $v_{\tilde{x}}$ agree on A, we conclude that $v_x(f) \leq 1$ if and only if $v_{\tilde{x}}(f) \leq 1$, which shows that $\tilde{x} = s(x)$.

We conclude that the map $\pi\colon \tilde{X} \to X$ is a continuous bijection between spectral spaces and that specializations lift along π. By [44, Tag 09XU], π is a homeomorphism; this proves (b).

By hypothesis, X admits a basis of rational subspaces U corresponding to rational localizations $(A, A^+) \to (B, B^+)$ with B sheafy. For each such map, we have a restriction map $\tilde{A} = H^0(X, \mathcal{O}) \to H^0(U, \mathcal{O}) = B$, which then induces a section of the map $B \to H^0(\pi^{-1}(U), \mathcal{O})$. These sections being compatible with restriction, we may use them to upgrade s to a map of locally ringed spaces, yielding (c); the same analysis yields (d). □

Remark 3.21. By contrast with Lemma 3.20, if (A, A^+) is an arbitrary Huber pair and $X := \mathrm{Spa}(A, A^+)$, then $H^0(X, \mathcal{O})$ is formed as a colimit over finite rational coverings and therefore is not guaranteed to be complete. One can form its completion for a suitable norm, but it is unclear whether the spectrum of this ring has any meaningful relationship with the original spectrum.

4. Strongly sheafy Huber rings

We next introduce a refined version of the sheafy condition, which turns out to be much easier to control.

Definition 4.1. We say that a Huber ring A is *strongly sheafy* if for every nonnegative integer n, the ring $A\langle T_1, \dots, T_n\rangle$ is sheafy. For example, by Proposition 3.12, any strongly noetherian Huber ring is strongly sheafy.

The main goal of this section is to give the following characterization of strongly sheafy Huber rings. The fact that this characterization does not require an explicit quantification over rational localizations (although it does implicitly refer to them via cohomology) may be a bit surprising; we see it as being analogous to Proposition 3.12.

Theorem 4.2. *Let (A, A^+) be a Huber pair. For each nonnegative integer n, put*

$$X_n := \operatorname{Spa}(A\langle T_1, \dots, T_n\rangle, A^+\langle T_1, \dots, T_n\rangle).$$

Then A is strongly sheafy if and only if for each nonnegative integer n, the following conditions hold.

(i) *The map $A\langle T_1, \dots, T_n\rangle \to H^0(X_n, \mathcal{O})$ is an isomorphism of rings.*
(ii) *For all $i > 0$, $H^i(X_n, \mathcal{O}) = 0$.*

Note that if A is strongly sheafy, then the given conditions hold by Proposition 3.10. The content of the theorem is thus the converse implication.

Lemma 4.3. *Let (A, A^+) be a Huber pair in which A satisfies conditions (i), (ii) of Theorem 4.2. Then for any $f \in A$, the following statements hold.*

(a) *For each nonnegative integer n, the elements $T_{n+1}-f$ and $1-fT_{n+1}$ in $A\langle T_1, \dots, T_{n+1}\rangle$ are not zero-divisors, and each one generates a closed ideal.*
(b) *The rings $A\langle f\rangle$ and $A\langle f^{-1}\rangle$ also satisfy conditions (i), (ii) of Theorem 4.2.*

Proof. We first set some notation so that the two cases can subsequently be treated in parallel.

- In one case, set $w = T_{n+1} - f$, $U_n := \{v \in X_n : v(f) \leq 1\}$, and $B = A\langle f\rangle$.
- In the other case, set $w = 1 - T_{n+1}f$, $U_n = \{v \in X_n : v(f) \geq 1\}$, and $B = A\langle f^{-1}\rangle$.

For each n, let $\pi\colon X_{n+1} \to X_n$ be the projection morphism. Put $V_n := \{v \in X_{n+1}\colon v(w) = 0\}$; then π induces a homeomorphism $V_n \cong U_n$, via which we may define a structure presheaf $\mathcal{O}_{V_n}$. Let $j\colon V_n \to X_{n+1}$ be the canonical inclusion.

We now verify that the sequence of sheaves

$$0 \to \tilde{\mathcal{O}}_{X_{n+1}} \xrightarrow{\times w} \tilde{\mathcal{O}}_{X_{n+1}} \to j_*\tilde{\mathcal{O}}_{V_n} \to 0 \tag{4.3.1}$$

on X_{n+1}, where the tildes denote sheafification, is exact. Exactness at the middle is true by construction; we check exactness at the left and right by examination of the stalk at $v \in X_{n+1}$.

- If $v(w) = 0$, then $\pi(v) \in U_n$ and so taking stalks at v in (4.3.1) factors through restriction from X_{n+1} to $\pi^{-1}(U_n)$. The resulting sequence can be identified with the sequence
$$0 \to \tilde{\mathcal{O}}_{\pi^{-1}(U_n)} \xrightarrow{\times w} \tilde{\mathcal{O}}_{\pi^{-1}(U_n)} \to \pi^{-1}\tilde{\mathcal{O}}_{U_n} \to 0$$
which is surjective on the right, split by the natural map $\pi^{-1}\tilde{\mathcal{O}}_{U_n} \to \tilde{\mathcal{O}}_{\pi^{-1}(U_n)}$, and injective on the left because for any rational localization $(A, A^+) \to (C, C^+)$, the element $T_{n+1} - f$ of $C\langle T_1, \dots, T_{n+1}\rangle$ is not a zero-divisor. (This is analogous to the fact that when f is an element in a ring R, the sequence
$$0 \to R[T] \stackrel{\times(T-f)}{\to} R[T] \to R[T]/(T-f) \to 0$$
is surjective on the right, split by the composition of the identification $R[T]/(T-f) \cong R$ with the structure morphism $R \to R[T]$, and injective on the left because $T - f$ is not a zero-divisor in $R[T]$.)

- If $v(w) > 0$, then w is a unit in $\tilde{\mathcal{O}}_{X_{n+1},v}$ and $(j_*\tilde{\mathcal{O}}_{V_n})_v = 0$; taking stalks at v in (4.3.1) yields the trivially exact sequence
$$0 \to \tilde{\mathcal{O}}_{X_{n+1},v} \overset{\times \text{unit}}{\to} \tilde{\mathcal{O}}_{X_{n+1},v} \to 0 \to 0.$$

For each $i > 0$, one segment of the long exact sequence in cohomology associated to (4.3.1) is
$$H^i(X_{n+1}, \tilde{\mathcal{O}}_{X_{n+1}}) \xrightarrow{\times w} H^i(X_{n+1}, \tilde{\mathcal{O}}_{X_{n+1}}) \to H^i(X_{n+1}, j_*\tilde{\mathcal{O}}_{V_n}) \to H^{i+1}(X_{n+1}, \tilde{\mathcal{O}}_{X_{n+1}}).$$
Since $H^i(X_{n+1}, \mathcal{O}) = 0$ by hypothesis, we deduce that
$$H^i(U_n, \mathcal{O}) = H^i(V_n, \mathcal{O}) = H^i(X_{n+1}, j_*\tilde{\mathcal{O}}_{V_n}) = 0.$$
Another segment of the long exact sequence may be written as
$$0 \to A\langle T_1, \dots, T_{n+1}\rangle \xrightarrow{\times w} A\langle T_1, \dots, T_{n+1}\rangle \to H^0(U_n, \mathcal{O}) \to 0;$$
this implies that the composition of the maps
$$A\langle T_1, \dots, T_{n+1}\rangle/(w) \to B\langle T_1, \dots, T_n\rangle \to H^0(U_n, \mathcal{O})$$
is an isomorphism. Since the first map is surjective, all of the maps must be isomorphisms, which yields both (a) and (b). □

Proof of Theorem 4.2. For any rational localization $(A, A^+) \to (B, B^+)$ we can write B is the form $A\langle f^{-1}\rangle\langle f_1, \dots, f_n\rangle$ for some $f \in A$ and some $f_1, \dots, f_n \in A\langle f^{-1}\rangle$. We may thus deduce the claim by Lemma 4.3. □

Remark 4.4. As per our running assumptions, in Theorem 4.2 the Huber ring A is required to be Tate; however, it should be possible to modify the proof to eliminate this assumption.

Corollary 4.5. *Let (A, A^+) be a Huber pair satisfying the following conditions.*

(i) *The map $A \to H^0(\mathrm{Spa}(A, A^+), \mathcal{O})$ is an isomorphism of rings.*
(ii) *There exists a covering of* $\mathrm{Spa}(A, A^+)$ *by rational subspaces* $\mathrm{Spa}(B, B^+)$ *for each of which B is strongly sheafy.*
(iii) *For some covering as in (ii), the higher Čech cohomology vanishes.*

Then A is strongly sheafy.

We note in passing that the proof applies without change to analytic rings in the sense of Remark 2.2. (The statement is probably true even without analyticity, but we did not check this.)

Proof. Let $\{\mathrm{Spa}(B_i, B_i^+)\}$ be a covering as in (ii) and (iii). Put $X_n := \mathrm{Spa}(A\langle T_1, \dots, T_n\rangle, A^+\langle T_1, \dots, T_n\rangle)$; this space then admits a covering by rational subspaces $\mathrm{Spa}(B_i\langle T_1, \dots, T_n\rangle, B_i^+\langle T_1, \dots, T_n\rangle)$ for which $B_i\langle T_1, \dots, T_n\rangle$ is strongly sheafy. Moreover, the higher Čech cohomology of this covering again vanishes. In particular, we may use this covering to compute that $H^0(X_n, \mathcal{O}) = A\langle T_1, \dots, T_n\rangle$ and $H^i(X_n, \mathcal{O}) = 0$ for $i > 0$; by Theorem 4.2, $A\langle T_1, \dots, T_n\rangle$ is sheafy. □

Remark 4.6. Corollary 4.5 is a weak form of [28, Theorem 1.2.22(b)], which involve sheafy rather than strongly sheafy Huber rings and includes no analogue of condition (iii). However, the given proof of (both parts of) [28, Theorem 1.2.22] is incomplete: it treats completely the case of a simple binary rational covering, but the general assertion does not reduce to that case. While part (a) of the theorem is confirmed by Lemma 3.20, we do not have a complete proof of (b). (Thanks to Ofer Gabber for bringing this to our attention.)

Note that [28, Remark 1.2.24] is unaffected by this issue: for the example of Buzzard–Verberkmoes from Remark 3.19, condition (ii) holds for a simple Laurent covering, and for this covering (iii) is automatic by [28, Lemma 1.8.1].

An important consequence of Theorem 4.2 is the preservation of the strongly sheafy property by finite flat extensions. This will be used to show that this property is of étale-local nature; see Theorem 5.6.

Corollary 4.7. *Let $A \to B$ be a finite flat, locally free, locally monogenic morphism of rings. If A is a strongly sheafy Huber ring, then so is B.*

Proof. Extend A to a Huber pair (A, A^+) and let B^+ be the integral closure of A^+ in B. By Corollary 4.5 we may check the claim locally on $\mathrm{Spa}(A, A^+)$; we may thus reduce to the case where $B = A\langle T\rangle/(P(T))$ for some monic polynomial $P(T)$. In particular, $A \to B$ is now faithfully flat.

Put

$$X_n := \mathrm{Spa}(A\langle T_1, \dots, T_n\rangle, A^+\langle T_1, \dots, T_n\rangle), Y_n := \mathrm{Spa}(B\langle T_1, \dots, T_n\rangle, B^+\langle T_1, \dots, T_n\rangle).$$

Let $\pi\colon X_{n+1} \to X_n$ be the projection morphism. Put $V_n = \{v \in X_{n+1} \colon v(P(T_{n+1})) = 0\}$; then $V_n \cong Y_n$ with $\pi\colon V_n \to X_n$ corresponding to the map $A\langle T_1, \dots, T_n\rangle \to B\langle T_1, \dots, T_n\rangle$; via this isomorphism we may define a structure sheaf $\tilde{\mathcal{O}}_{V_n}$, where the tilde denotes sheafification. Let $j\colon V_n \to X_{n+1}$ be the canonical inclusion.

We now verify that the sequence of sheaves

$$0 \to \mathcal{O} \stackrel{\times P(T_{n+1})}{\to} \mathcal{O} \to j_* \tilde{\mathcal{O}}_{V_n} \to 0 \tag{4.7.1}$$

on X_{n+1} is exact by examination of the stalk at $v \in X_{n+1}$. Namely, since $A \to B$ is faithfully flat, we may check exactness after tensoring over A with B, at which point the sequence becomes canonically split exact at the level of stalks (see the proof of Lemma 4.3).

For each $i > 0$, one segment of the long exact sequence in cohomology associated to (4.7.1) is

$$H^i(X_{n+1}, \mathcal{O}) \to H^i(X_{n+1}, \mathcal{O}) \to H^i(X_{n+1}, j_* \tilde{\mathcal{O}}_{V_n}) \to H^{i+1}(X_{n+1}, \mathcal{O}).$$

Since $H^i(X_{n+1}, \mathcal{O}) = 0$ by Theorem 4.2, we deduce that

$$H^i(Y_n, \mathcal{O}) = H^i(V_n, \mathcal{O}) = H^i(X_{n+1}, j_* \tilde{\mathcal{O}}_{V_n}) = 0.$$

Another segment of the long exact sequence may be written as

$$0 \to A\langle T_1, \dots, T_{n+1}\rangle \stackrel{\times P(T_{n+1})}{\to} A\langle T_1, \dots, T_{n+1}\rangle \to H^0(Y_n, \mathcal{O}) \to 0;$$

this implies that the composition of the maps

$$A\langle T_1, \dots, T_{n+1}\rangle/(P(T_{n+1})) \to B\langle T_1, \dots, T_n\rangle \to H^0(Y_n, \mathcal{O})$$

is an isomorphism. Since the first map is an isomorphism, so is the second. We may thus apply Theorem 4.2 to deduce that B is strongly sheafy. $\square$

We mention an instance of Remark 3.4 that will be relevant later.

Remark 4.8. For A a Banach ring, it is natural (although not defined in [27]) to say that A is *really strongly sheafy* if for every nonnegative integer n and every $r_1, \dots, r_n > 0$, the structure presheaf on the reified adic spectrum of $A\langle T_1/r_1, \dots, T_n/r_n\rangle$ is a sheaf. By [27, Theorem 7.14], this implies that the global sections of the structure presheaf equal $A\langle T_1/r_1, \dots, T_n/r_n\rangle$ and the higher cohomology groups vanish; the proof of Theorem 4.2 may be adapted to show that the converse also holds.

5. Étale morphisms

In order to introduce étale morphisms, we first make a temporary definition.

Definition 5.1. A morphism $(A, A^+) \to (B, B^+)$ of Huber pairs is *étale in the naïve sense* if locally on $\operatorname{Spa}(B, B^+)$, the map factors as a composition of rational localizations and finite étale morphisms (in fact, only one finite étale morphism is needed).

Remark 5.2. Let $(A, A^+) \to (B, B^+)$ be a finite morphism of (not necessarily sheafy) Huber pairs. By [29, Theorem 2.6.9], this morphism is étale in the naïve sense if and only if it is étale in the algebraic sense. (See [28, Theorem 1.4.2] for an extension to sheafy analytic rings.)

Definition 5.3. Let (A, A^+) be a Huber pair. By a *naïve étale covering* of (A, A^+), we will mean a family $\{(A, A^+) \to (B_i, B_i^+)\}_i$ of morphisms which are étale in the naïve sense such that the spaces $\operatorname{Spa}(B_i, B_i^+)$ form a set-theoretic covering of $\operatorname{Spa}(A, A^+)$. Using this notion of covering, we obtain a (small) *étale site* $\operatorname{Spa}(A, A^+)_{\mathrm{et}}$ and presheaves $\mathcal{O}, \mathcal{O}^+$.

Definition 5.4. Let $\mathcal{P}$ be a property of Huber rings. By analogy with Definition 3.17, we say that $\mathcal{P}$ is *of étale-local nature* if the following statements hold.

(i) For any rational localization $A \to B$, if A has property $\mathcal{P}$, then so does B.

(ii) For any finite étale morphism $A \to B$, if A has property $\mathcal{P}$, then so does B.

(iii) Let A^+ be a ring of integral elements of A. Let $\{(A, A^+) \to (B_i, B_i^+)_i\}$ be a naïve étale covering such that

$$0 \to A \to \prod_i B_i \to \prod_{i,j} B_i \widehat{\otimes}_A B_j \to \cdots \tag{5.4.1}$$

is exact. If each of the rings B_i has property $\mathcal{P}$, then so does A.

In particular, if $\mathcal{P}$ is of étale-local nature, then it is of local nature.

Proposition 5.5. *Let $\mathcal{P}$ be a property of Huber rings satisfying the following conditions.*

(i) *The property $\mathcal{P}$ is of local nature.*

(ii) *For any finite étale morphism $A \to B$, if A has property $\mathcal{P}$, then so does B.*

(iii) *For any faithfully finite étale morphism $A \to B$, if B has property $\mathcal{P}$, then so does A.*

(iv) *Every Huber ring with property $\mathcal{P}$ is sheafy.*

Then $\mathcal{P}$ is of étale-local nature; moreover, for every naïve étale covering $\{(A, A^+) \to (B_i, B_i^+)_i\}$, if A has property $\mathcal{P}$, then the sequence (5.4.1) *is exact.*

Proof. Let $\mathcal{B}$ be the basis of $\operatorname{Spa}(A, A^+)_{\mathrm{et}}$ consisting of compositions of rational localizations and finite étale morphisms. By (i), (ii), and (iv), if A has property $\mathcal{P}$, then each element of $\mathcal{B}$ is sheafy.

We next check that if A has property $\mathcal{P}$, then for every covering of and by elements of $\mathcal{B}$, (5.4.1) is exact. For this, we check that the conditions of [29, Proposition 8.2.21] are satisfied:

(a) The complex (5.4.1) is exact for a simple Laurent covering. This follows from (iv) and Proposition 3.10.
(b) The complex (5.4.1) is exact for a finite étale covering. This follows from faithfully flat descent for modules.

Now dropping the condition that A has property $\mathcal{P}$, we say that a covering of and by elements of $\mathcal{B}$ has property $\mathcal{P}'$ if condition (iii) of Definition 5.4 holds. We check that $\mathcal{P}'$ satisfies the conditions of [29, Proposition 8.2.20]:

(a) Every covering admitting a refinement having property $\mathcal{P}'$ also has property $\mathcal{P}'$. This follows from (i), (ii), and the previous paragraph (to equate the exactness of (5.4.1) for the original covering and for its refinement, assuming that the terms in the original covering have property $\mathcal{P}$).
(b) Every composition of coverings having property $\mathcal{P}'$ has property $\mathcal{P}'$. This follows from the previous paragraph.
(c) Every rational covering has property $\mathcal{P}'$. This follows from (i).
(d) Any finite étale covering has property $\mathcal{P}'$. This follows from (iii).

We deduce that for every covering of and by elements of $\mathcal{B}$, if (5.4.1) is exact and each of the rings B_i has property $\mathcal{P}$, then so does A.

We now verify that $\mathcal{P}$ is of étale-local nature. Of the conditions in Definition 5.4, (i) is included in our hypothesis (i), while (ii) is precisely our hypothesis (ii). To check (iii), note that given a naïve étale covering $\{(A, A^+) \to (B_i, B_i^+)_i\}$, each pair (B_i, B_i^+) admits a naïve étale covering $\{(B_i, B_i^+) \to (C_{ij}, C_{ij}^+)_j\}$ such that *both* of the maps $(A, A^+) \to (C_{ij}, C_{ij}^+)$ and $(B_i, B_i^+) \to (C_{ij}, C_{ij}^+)$ factor as a composition of rational localizations and finite étale morphisms. If each B_i has property $\mathcal{P}$, then so does each C_{ij}; moreover, the maps $\{(A, A^+) \to (C_{ij}, C_{ij}^+)_{i,j}\}$ form a covering of and by elements of $\mathcal{B}$, and the exactness of (5.4.1) for this covering follows from exactness for the original covering (as in (a) above), so A also has property $\mathcal{P}$. □

We do not know whether the stably uniform or sheafy properties are of étale-local nature. (For the perfectoid property, even local nature is unknown; see Remark 3.19.) By contrast, we have the following result.

Theorem 5.6. *The property of a Huber ring being strongly sheafy is of étale-local nature.*

Proof. We check the criteria of Proposition 5.5: (i) follows from Corollary 4.5, (ii) follows from Corollary 4.7, and (iii) and (iv) are straightforward. □

Definition 5.7. Let (A, A^+) be a strongly sheafy Huber pair. We say that a morphism $(A, A^+) \to (B, B^+)$ is *étale* if it is étale in the naïve sense and moreover B is sheafy. This implies that B^+ is also strongly sheafy: (B, B^+) admits a rational covering by terms which are strongly sheafy, and by Proposition 3.10 this covering is acyclic, so the fact that the strongly sheafy property is of étale-local nature (Theorem 5.6) yields that B is strongly sheafy.

By Corollary 4.7, if $A \to B$ is finite étale as a morphism of rings, then $(A, A^+) \to (B, B^+)$ is étale in this sense. Consequently, the corresponding étale site defines the same topos as with our previous definition.

Proposition 5.8. *Let (A, A^+) be a strongly sheafy Huber pair. Then $H^i(\mathrm{Spa}(A, A^+)_{\mathrm{et}}, \mathcal{O}) = 0$ for all $i > 0$.*

Proof. See [29, Theorem 8.2.22]. $\square$

Remark 5.9. It will follow from Corollary 7.4 below that any perfectoid ring satisfies the conclusion of Proposition 5.8.

Remark 5.10. It should be possible to give a more intrinsic definition of an étale morphism, in which the existence of local factorizations becomes a lemma rather than a definition. See the appendix to [28] for further discussion.

Definition 5.11. By analogy with Definition 5.1, one can also specify what it should mean for a morphism $(A, A^+) \to (B, B^+)$ of Huber pairs to be *smooth in the naïve sense*: locally on $\mathrm{Spa}(B, B^+)$ the morphism should factor as a composition of rational localizations, finite étale coverings, and Tate algebra extensions. If a property of Huber rings is of étale-local nature and is preserved by Tate algebra extensions, then it propagates along smooth morphisms; that is, if A has the property, $(A, A^+) \to (B, B^+)$ is a smooth morphism, and B is sheafy, then B also has the property.

For example, propagation along smooth morphisms holds for the property of being strongly sheafy by Theorem 5.6. It will also hold for the property of being plus-sheafy, or being uniform and plus-sheafy, by Theorem 6.21.

6. Cohomology of the integral structure sheaf

Definition 6.1. For (A, A^+) a Huber pair, the *integral structure presheaf* on $\mathrm{Spa}(A, A^+)$ is the presheaf $\mathcal{O}^+$ assigning to an open subset V the inverse limit of B^+ as $(A, A^+) \to (B, B^+)$ runs over all rational localizations with $\mathrm{Spa}(B, B^+) \subseteq V$. We can and will view $\mathcal{O}^+$ as a subpresheaf of $\mathcal{O}$; it is a sheaf if and only if $\mathcal{O}$ is (using the Tate condition on A in the "only if" direction).

There is no analogue of Proposition 3.10 asserting the acyclicity of $\mathcal{O}^+$ even when A is an affinoid algebra. See Example 6.6 for a mild example and Example 6.11 for a far more serious example; both examples are analyzed using the following algebraic condition.

Definition 6.2. A ring R is *seminormal* if the map

$$R \to \{(y, z) \in R \times R \colon y^3 = z^2\}, \qquad x \mapsto (x^2, x^3)$$

is bijective. Note that injectivity of this map is equivalent to R being reduced, and that any normal (i.e., integrally closed) integral domain is seminormal; in particular, a smooth affinoid algebra over a nonarchimedean field is seminormal. For further discussion and references, see [30, Definition 1.4.1].

Lemma 6.3. *Let $R \to S$ be an étale ring homomorphism. If R is seminormal, then so is S.*

Proof. By writing R as a direct limit of finitely generated $\mathbb{Z}$-algebras, we may reduce to the case where R is excellent, as then is S. In this case, the definition of seminormality given in

Definition 6.2, due to Swan, agrees with the definition of Traverso [47]; we may thus apply [19, Theorem 5.8] to conclude. ☐

Lemma 6.4. *For affinoid algebras, the properties of being reduced and seminormal are both of étale-local nature.*

Proof. With regard to preservation under rational localizations, for the reduced property, see [29, Lemma 2.5.9] and references therein; for the seminormal property, see [30, Proposition 3.7.2]. With regard to preservation under finite étale morphisms, for the reduced property, see [44, Tag 025O]; for the seminormal property, see Lemma 6.3. Condition (iii) of Definition 5.4 is straightforward. ☐

Remark 6.5. We have not checked that seminormality of a Huber ring A implies seminormality of $A\langle T\rangle$. However, for A uniform, this is an easy consequence of Proposition 3.5. For A an affinoid algebra in mixed characteristic, it will be a consequence of Theorem 10.3.

Example 6.6. Let C be a smooth, projective, geometrically irreducible curve of genus at least 2 over $\mathbb{Q}_p$ with good reduction. Let X be the (adic) analytification of C; the space X contains a point x corresponding to the generic point of the special fiber $\overline{C}$ of the smooth model of C over $\mathbb{Z}_p$. Let $\mathrm{Spa}(A, A^+)$ be an affinoid subspace of X whose complement is isomorphic to an open unit disc over $\mathbb{Q}_p$; in particular, this subspace must contain x. Since X is smooth, A is seminormal; it will follow from Theorem 10.3 later that $H^1(\mathrm{Spa}(A, A^+), \mathcal{O}^+)$ is uniformly torsion. However, there exists a surjective map $H^1(\mathrm{Spa}(A, A^+), \mathcal{O}^+) \to H^1(\overline{C}, \mathcal{O})$, and the nonvanishing of the target ensures that $H^1(\mathrm{Spa}(A, A^+), \mathcal{O}^+)$ is nonzero. (See [49, Proposition 2] for a result which can be used to identify the annihilator of $H^1(\mathrm{Spa}(A, A^+), \mathcal{O}^+)$ in this and similar examples.)

The following is an adaptation of an argument suggested by Gabber. On account of Proposition 6.14 below, this generalizes the statement that perfectoid algebras are seminormal [30, Theorem 3.7.4].

Proposition 6.7. *Let (A, A^+) be a sheafy Huber pair with A uniform. If $H^1(\mathrm{Spa}(A, A^+), \mathcal{O}^+)$ is uniformly torsion, then A is seminormal.*

Proof. Note that neither the hypothesis nor the conclusion depends on A^+, so we may assume hereafter that $A^+ = A^\circ$. Choose a pair $(y, z) \in A \times A$ such that $y^3 = z^2$. Define the Banach algebra A' over A by forming the quotient of $A \oplus A$ (equipped with the product topology) by the A-submodule $\{(az, -ay) : a \in A\}$, promoting to an A-algebra by declaring that $(a_0, a_1)(b_0, b_1) = (a_0 b_0 + y a_1 b_1, a_0 b_1 + a_1 b_0)$ and the map $A \to A'$ is $a \mapsto (a, 0)$, then taking the uniform completion as per [29, Definition 2.8.13]. By construction, the element $x = (0, 1) \in A'$ satisfies $x^2 = y, x^3 = z$.

Put $X = \mathrm{Spa}(A, A^\circ)$ and $X' = \mathrm{Spa}(A', A'^\circ)$. Note that any $v \in X$ extends uniquely from A to A' (considering separately the cases $v(y) = 0$, $v(y) \neq 0$); consequently, the map $\pi : X' \to X$ is a homeomorphism inducing isomorphisms on residue fields. By Proposition 3.5, the morphism $A \to A'$ is a strict inclusion (compare Remark 9.2). The diagram

$$\begin{tikzcd} \mathcal{O}_X^+ \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[d] \\ \pi_* \mathcal{O}_{X'}^+ \arrow[r] & \pi_* \mathcal{O}_{X'} \end{tikzcd}$$

is cartesian and cocartesian (the latter because of the isomorphisms of residue fields) and gives rise to an exact sequence

$$0 \to \mathcal{O}_X^+ \to \mathcal{O}_X \oplus \pi_* \mathcal{O}_{X'}^+ \to \pi_* \mathcal{O}_{X'} \to 0.$$

We thus obtain an inclusion

$$\frac{H^0(X', \mathcal{O})}{H^0(X', \mathcal{O}^+) + A} \hookrightarrow H^1(X, \mathcal{O}^+).$$

Note that $A' \to H^0(X', \mathcal{O})$ is not guaranteed to be an isomorphism, but by Proposition 3.5 the preimage of $H^0(X', \mathcal{O}^+)$ in A' equals A'°.

Let $\varpi \in A$ be a topologically nilpotent unit which kills $H^1(X, \mathcal{O}^+)$. For each positive integer n, we apply the previous inclusion to the image of $\varpi^{-n}x$ in $H^0(X', \mathcal{O})$ to find $a_n \in A$ such that $\varpi^{-n+1}x - a_n \in A'^\circ$. The sequence $\{\varpi^{n-1}a_n\}_{n=1}^\infty$ then converges to x in A' and hence in A; that is, $x \in A$ satisfies $x^2 = y, x^3 = z$. It follows that A is seminormal. □

Definition 6.8. Let (A, A^+) be a Huber pair, put $X = \operatorname{Spa}(A, A^+)$, and let $\varpi \in A^+$ be a topologically nilpotent unit of A. We say that (A, A^+) is *plus-sheafy* if

(i) the A^+-modules $\ker(A^+/\varpi^n A^+ \to H^0(X, \mathcal{O}^+/\varpi^n))$, $\operatorname{coker}(A^+/\varpi^n A^+ \to H^0(X, \mathcal{O}^+/\varpi^n))$ are n-uniformly torsion;

(ii) for each positive integer i, the A^+-modules $H^i(X, \mathcal{O}^+/\varpi^n)$ are n-uniformly torsion (but not necessarily i-uniformly torsion).

It is evident that this does not depend on either A^+ or the choice of ϖ.

From the exact sequence

$$0 \to \mathcal{O}^+ \stackrel{\times \varpi^n}{\to} \mathcal{O}^+ \to \mathcal{O}^+/\varpi^n \to 0 \tag{6.8.1}$$

we see that if $A \to H^0(X, \mathcal{O})$ is an isomorphism and $H^i(X, \mathcal{O}^+)$ is uniformly torsion for each positive integer i, then A is plus-sheafy. The converse holds if A is uniform; see Lemma 6.18.

Example 6.9. The ring $A = \mathbb{Z}((T))$ is plus-sheafy: taking $A^+ = \mathbb{Z}[\![T]\!]$ and $\varpi = T$, it is straightforward to check that $A^+/\varpi^n A^+ \cong H^0(X, \mathcal{O}^+)$ and that $H^i(X, \mathcal{O}^+/\varpi^n) = 0$ for $i > 0$. This example will be used in Example 6.28.

Example 6.10. For K a nonarchimedean field and n a positive integer, the ring $A = K\langle T_1, \dots, T_n\rangle/(T_n^2)$ is sheafy and plus-sheafy but not uniform. This follows from the fact that for

$$\begin{aligned} X &= \operatorname{Spa}(K\langle T_1, \dots, T_{n-1}\rangle, K^\circ\langle T_1, \dots, T_{n-1}\rangle), \\ Y &= \operatorname{Spa}(K\langle T_1, \dots, T_n\rangle/(T_n^2), K^\circ\langle T_1, \dots, T_n\rangle/(T_n^2)), \end{aligned}$$

and $f\colon Y \to X$ the natural morphism, we have

$$f_* \mathcal{O}_Y^+ \cong \mathcal{O}_X^+ \oplus \mathcal{O}_X T_n, \qquad R^i f_* \mathcal{O}_Y^+ = 0 \quad (i > 0).$$

While this example is still sheafy, see Example 6.28 for an example where sheafiness fails.

Example 6.11. We recall an example from [5]. Put $A = \mathbb{Q}_p\langle y, z\rangle/(y^3 - z^2)$; then A is integral (and hence uniform; see Definition 2.3) but not seminormal. Put $X = \operatorname{Spa}(A, A^\circ)$. By Proposition 6.7, $H^1(X, \mathcal{O}^+)$ is not killed by any power of p. In particular, although A is strongly sheafy (by virtue of being an affinoid algebra), it will follow from Lemma 6.18(c) that A is not plus-sheafy. See Example 10.5 for additional discussion.

Remark 6.12. Note that in Example 6.11, $H^1(\mathrm{Spa}(A, A^\circ), \mathcal{O}^+)$ is torsion but not uniformly torsion. This does not contradict Theorem 2.7 because $H^1(\mathrm{Spa}(A, A^\circ), \mathcal{O}^+)$ is not a derived complete A°-module: its computation involves a colimit over all finite coverings by rational subspaces, for each of which the Čech cohomology groups are uniformly torsion, but there is no uniformity maintained in the colimit.

Remark 6.13. For A a Banach ring, the cohomology groups of $\mathcal{O}^+$ are the same whether computed on the adic spectrum or the reified adic spectrum of A (see the proof of [5, Satz 3.1]); by the same token, the global sections of $\mathcal{O}^+/\varpi^n$ or $\mathcal{O}$ are the same in either case. Consequently, the condition that A is plus-sheafy can be formulated equivalently using the reified adic spectrum. (That is, there is no need to separately define the condition of being *really plus-sheafy.*)

Proposition 6.14. *For (A, A^+) a perfectoid Huber pair and every positive integer i, the A^+-module $H^i(\mathrm{Spa}(A, A^+), \mathcal{O}^+)$ is* almost zero*: it is killed by* every *topologically nilpotent unit in A. In particular, A is plus-sheafy.*

Proof. See [30, Theorem 3.5.5]. That result includes a much stronger statement which we will use later; see Theorem 9.3. □

We do not expect the converse of Proposition 6.7 to hold even for affinoid algebras. This leads to the following question.

Question 6.15. Is the plus-sheafy property for a Huber pair (A, A^+) in which A is an affinoid algebra over a nonarchimedean field equivalent to some ring-theoretic property of A, and if so what property?

Remark 6.16. It was shown by Bartenwerfer [5, Theorem, Folgerung 3] (building on [4]) that a smooth affinoid algebra over a nonarchimedean field is plus-sheafy. Following Definition 5.11, we will recover this result by proving a relative version; see Corollary 6.27.

Lemma 6.17. *Let A be a plus-sheafy Huber ring. Then for any $f \in A$, $A\langle f\rangle$ and $A\langle f^{-1}\rangle$ are also plus-sheafy. If moreover A is uniform, then so are $A\langle f\rangle$ and $A\langle f^{-1}\rangle$.*

Proof. Choose an extension of A to a Huber pair (A, A^+). As in the proof of Lemma 4.3, we set notation so that the two cases can be treated in parallel.

- In one case, set $w = T - f$ and $B = A\langle f\rangle$.
- In the other case, set $w = 1 - fT$ and $B = A\langle f^{-1}\rangle$.

Put $X := \mathrm{Spa}(A, A^+)$, $U := \mathrm{Spa}(B, B^+)$, and let $j\colon U \to X$ denote the canonical inclusion. Let $\mathcal{O}\langle T\rangle, \mathcal{O}^+\langle T\rangle$ be the presheaves on X characterized by

$$(\mathcal{O}\langle T\rangle)(\mathrm{Spa}(C, C^+)) = C\langle T\rangle, \qquad (\mathcal{O}^+\langle T\rangle)(\mathrm{Spa}(C, C^+)) = C^+\langle T\rangle.$$

We have an isomorphism of A^+-modules

$$A^+\langle T\rangle/\varpi^n \cong (A^+/\varpi^n A^+)[T] \cong \bigoplus_{i=0}^{\infty} (A^+/\varpi^n A^+)T^i;$$

by the same token, we have an isomorphism of $\mathcal{O}^+$-modules

$$\mathcal{O}^+\langle T\rangle/\varpi^n \cong \bigoplus_{i=0}^{\infty} (\mathcal{O}^+/\varpi^n)T^i. \tag{6.17.1}$$

Consequently, our hypotheses imply that the A^+-modules

$$\operatorname{coker}(A^+\langle T\rangle/\varpi^n A^+\langle T\rangle \to H^0(X, \mathcal{O}^+\langle T\rangle/\varpi^n))$$

are n-uniformly torsion; and for each positive integer i, the A^+-modules $H^i(X, \mathcal{O}^+\langle T\rangle/\varpi^n)$ are n-uniformly torsion.

Choose a positive integer c such that $\varpi^c w \in A^+$; for each positive integer n, we then have a sequence

$$0 \to \tilde{\mathcal{O}}^+\langle T\rangle/\varpi^n \stackrel{\times\varpi^c w}{\to} \tilde{\mathcal{O}}^+\langle T\rangle/\varpi^n \to j_*\tilde{\mathcal{O}}^+/\varpi^n \to 0 \tag{6.17.2}$$

(where the tildes denote sheafification, and more precisely $\tilde{\mathcal{O}}^+\langle T\rangle$ means the sheafification of $\mathcal{O}^+\langle T\rangle$). We claim that in the sequence (6.17.2), the cohomology at the left is a sheaf killed by ϖ; the cohomology at the middle is a sheaf killed by ϖ^{c+1}; and the cohomology at the right is a sheaf killed by ϖ^c. As in the proof of Lemma 4.3, we see this by inspecting the stalks at a point $v \in X$.

- If $v \in U$, then we have an exact sequence
$$0 \to \mathcal{H}(v)\langle T\rangle \stackrel{w}{\to} \mathcal{H}(v)\langle T\rangle \to \mathcal{H}(v) \to 0.$$
We read off the claim using the fact that the Gauss norm on $\mathcal{H}(v)\langle T\rangle$ is multiplicative.
- If $v \notin U$, then w is a unit in $\mathcal{H}(v)\langle T\rangle$ and $j_*\tilde{\mathcal{O}}^+_v = 0$. We again use the multiplicativity of the Gauss norm on $\mathcal{H}(v)\langle T\rangle$ to conclude.

Now let i and n be positive integers. Taking cohomology in (6.17.2), and using the fact that $R^i j_*(\mathcal{O}^+/\varpi^n) = 0$ for $i > 0$, shows that the A^+-modules $H^i(U, \mathcal{O}^+/\varpi^n)$ are n-uniformly torsion. It also gives rise to a sequence

$$H^0(X, \mathcal{O}^+\langle T\rangle/\varpi^n) \to H^0(U, \mathcal{O}^+/\varpi^n) \to H^1(X, \mathcal{O}^+\langle T\rangle/\varpi^n) \tag{6.17.3}$$

which is exact modulo n-uniformly torsion modules; in this sequence, the term on the right is n-uniformly torsion, as then is the cokernel of the left arrow. Combining with the fact that $\operatorname{coker}(A^+\langle T\rangle/\varpi^n A^+\langle T\rangle \to H^0(X, \mathcal{O}^+\langle T\rangle/\varpi^n))$ is n-uniformly torsion, we see that $\operatorname{coker}(B^+/\varpi^n B^+ \to H^0(U, \mathcal{O}^+/\varpi^n))$ is n-uniformly torsion.

Backing up (6.17.3) by one term, we see that the composition of maps

$$A^+\langle T\rangle/(\varpi^c w, \varpi^n) \to B^+/(\varpi^n) \to H^0(U, \mathcal{O}^+/\varpi^n) \tag{6.17.4}$$

is an isomorphism modulo n-uniformly torsion modules; since the first map is surjective modulo n-uniformly torsion modules, both maps must be isomorphisms modulo n-uniformly torsion modules.

Now suppose in addition that A is uniform. Then A^+ is bounded in A and $A^+ \cong \varprojlim_n A^+/(\varpi^n)$. By (6.17.4), the composition

$$A^+\langle T\rangle/(\varpi^c w) \to B^+ \to \varprojlim_n B^+/(\varpi^n)$$

has uniformly torsion kernel and cokernel; inverting ϖ yields that the composition

$$A\langle T\rangle/(w) \to B \to \left(\varprojlim_n B^+/(\varpi^n)\right)[\varpi^{-1}]$$

is an isomorphism. In particular, the map $A\langle T\rangle/(w) \to B$, which is *a priori* only a surjection, is in fact an isomorphism; therefore $B \to \left(\varprojlim_n B^+/(\varpi^n)\right)[\varpi^{-1}]$ is also an isomorphism. This in turn implies that $B^+ \to \varprojlim_n B^+/(\varpi^n)$ is injective, so B^+ is bounded in B; hence B is uniform. $\square$

Lemma 6.18. *Let (A, A^+) be a uniform plus-sheafy Huber pair and put $X = \mathrm{Spa}(A, A^+)$.*

- (a) *For any rational localization $(A, A^+) \to (B, B^+)$, (B, B^+) is again uniform and plus-sheafy. Consequently, A is stably uniform, and hence sheafy by Proposition 3.14.*
- (b) *The map $A^+ \to H^0(X, \mathcal{O}^+)$ is injective with uniformly torsion cokernel.*
- (b) *For $i > 0$, $H^i(X, \mathcal{O}^+)$ is uniformly torsion.*

Proof. As in the proof of Theorem 4.2, we may deduce (a) from Lemma 6.17.

To deduce (b), apply (a) to deduce that $A \to H^0(X, \mathcal{O})$ is an isomorphism; this at once implies that $A^+ \to H^0(X, \mathcal{O}^+)$ is injective. Meanwhile, we can choose a nonnegative integer c so that for any element $\alpha \in H^0(X, \mathcal{O}^+)$, $\varpi^c\alpha$ maps to zero in $\mathrm{coker}(A^+/\varpi^n A^+ \to H^0(X, \mathcal{O}^+/\varpi^n))$ for all n. That is, there exist elements $y_n \in A^+$ such that $y_n - \varpi^c\alpha$ vanishes as an element of $H^0(X, \mathcal{O}^+/\varpi^n)$, and thus equals ϖ^n times an element $z_n \in H^0(X, \mathcal{O}^+)$. Since $A^+ \cap \varpi^n H^0(X, \mathcal{O}^+) = \varpi^n A^+$, we have $y_n - y_{n+1} \in \varpi^n A^+$, and so the elements y_n converge to a limit $y \in A^+$. Since $\varpi^c\alpha - y \in H^0(X, \mathcal{O}) \cong A = A^+[\varpi^{-1}]$, we may choose a nonnegative integer d such that $\varpi^{c+d}\alpha - \varpi^d y \in A^+$; now $\varpi^{c+d}\alpha - \varpi^d y$ belongs to $\bigcap_n (A^+ \cap \varpi^n H^0(X, \mathcal{O}^+)) = \bigcap_n \varpi^n A^+ = 0$. We deduce that $\varpi^c\alpha = y$, and so $\mathrm{coker}(A^+ \to H^0(X, \mathcal{O}^+))$ is killed by ϖ^c.

To prove (c), combine (a) with Proposition 3.10 to deduce that $H^i(X, \mathcal{O}) = 0$. This implies that $H^i(X, \mathcal{O}^+)$ is torsion; in other words,

$$H^i(X, \mathcal{O}^+) = \bigcup_{n \geq 1} H^i(X, \mathcal{O}^+)[\varpi^n]. \tag{6.18.1}$$

From (6.8.1), for $i > 1$ we obtain a surjective map

$$H^{i-1}(X, \mathcal{O}^+/\varpi^n) \to H^i(X, \mathcal{O}^+)[\varpi^n];$$

for $i = 1$, we similarly obtain a map

$$\mathrm{coker}(A^+/\varpi^n A^+ \to H^0(X, \mathcal{O}^+/\varpi^n)) \to H^1(X, \mathcal{O}^+)[\varpi^n]$$

with n-uniformly torsion cokernel. We thus deduce that $H^i(X, \mathcal{O}^+)[\varpi^n]$ is n-uniformly torsion; by (6.18.1), $H^i(X, \mathcal{O}^+)$ is uniformly torsion. $\square$

Remark 6.19. Following up on Remark 3.16, note that Lemma 6.18 implies that a uniform Huber ring which is sheafy but not stably uniform cannot be plus-sheafy.

Lemma 6.20. *Let $A \to B$ be a finite étale morphism of rings. If A is a plus-sheafy Huber ring, then so is B. (As per Definition 2.9, if A is uniform then B is also uniform, and hence stably uniform by Lemma 6.18(a).)*

Proof. Extend A to a Huber pair (A, A^+) and let B^+ be the integral closure of A^+ in B. Fix a topologically nilpotent unit ϖ in A. Put $X := \mathrm{Spa}(A, A^+)$, $Y := \mathrm{Spa}(B, B^+)$, and let $\pi \colon Y \to X$ be the canonical projection; then by computing at stalks we see that $R^i\pi_*\tilde{\mathcal{O}}_Y^+$ (where the tilde denotes sheafification) is killed by ϖ for $i > 0$, and similarly after reducing modulo ϖ^n for every positive integer n. Consequently, for each $i \geq 0$, the kernel and cokernel of $H^i(Y, \tilde{\mathcal{O}}_Y^+/\varpi^n) \to H^i(X, \pi_*\tilde{\mathcal{O}}_Y^+/\varpi^n)$ are n-uniformly torsion. By writing B (in the category of A-modules) as a direct summand of a finite free A-module, we deduce that B is plus-sheafy. $\square$

Theorem 6.21. *The property of a Huber ring being plus-sheafy is of local nature. The property of a Huber ring being uniform and plus-sheafy is of local nature and étale-local nature.*

Proof. We first establish local nature: in Definition 3.17, condition (i) is Lemma 6.18, and condition (ii) is straightforward. We next establish étale-local nature using the criteria of Proposition 5.5: (i) is the previous discussion, (ii) is Lemma 6.20, (iii) is straightforward, and (iv) follows from Lemma 6.18(a). □

Remark 6.22. Using Remark 6.13, the proof of Theorem 6.21 can be extended to show that the property of a Banach ring being plus sheafy, or of being uniform and plus-sheafy, is also of local nature and étale-local nature for the reified adic spectrum.

We next consider passage to Tate algebras. The main result here (the final conclusion of Lemma 6.24) has been generalized by Gabber to the case of a morphism which is "smooth of good reduction" (private communication).

Lemma 6.23. *Let $\pi : Y \to X$ be a morphism of ringed spaces and fix a basis $\mathcal{B}$ of Y closed under intersections. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_Y$-modules with the property that for all $x \in X$, all $V \in \mathcal{B}$, and all $i > 0$,*

$$\varinjlim_{x \in U} \check{H}^i(\pi^{-1}(U) \cap V, \mathcal{F}) = 0$$

where U runs over neighborhoods of x in X and $\check{H}^i$ denotes the colimit of Čech cohomology over all coverings. Then $R^i\pi_*\mathcal{F} = 0$ for all $i > 0$.

Proof. This is a relative version of [44, Tag 01EV], whose proof we follow. We prove the claim for $i = 1, \dots, k$ by induction on k, with the base case $k = 1$ following from [44, Tag 09V1]. Given the claim for some $k \geq 1$, form an exact sequence

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{E} \to 0$$

of $\mathcal{O}_Y$-modules with $\mathcal{I}$ injective. Then $\pi_*\mathcal{I}$ is flasque [44, Tag 09SX] and

$$0 \to \pi_*\mathcal{F} \to \pi_*\mathcal{I} \to \pi_*\mathcal{E} \to R^1\pi_*\mathcal{F}$$

is exact. Moreover, our assumption for $i = 1$ ensures on one hand that $R^1\pi_*\mathcal{F} = 0$, and on the other hand that we may transfer the initial hypothesis from $\mathcal{F}$ to $\mathcal{E}$ (using [44, Tag 01EU]). Applying the induction hypothesis to $\mathcal{E}$, we deduce that $R^i\pi_*\mathcal{F} = 0$ for $i = 2, \dots, k+1$, thus completing the induction. □

Lemma 6.24. *Let A be a (not necessarily sheafy) Banach ring and put $X = \mathrm{Spa}(A, A^\circ)$. Let B be one of $A\langle T/r \rangle$ or $A\langle s/T, T/r \rangle$ for any $0 < s \leq r$, put $Y = \mathrm{Spa}(B, B^\circ)$, and let $\pi : Y \to X$ be the projection. Let $\mathcal{O}_Y^{++}$ denote the presheaf of topologically nilpotent sections of $\mathcal{O}_Y$. Then for all rational subspaces V of Y, all $x \in X$, and all $i, n > 0$,*

$$\varinjlim_{x \in U} \check{H}^i(\pi^{-1}(U) \cap V, \tilde{\mathcal{O}}_Y^{++}) = 0$$

(where the tilde denotes sheafification); consequently, by Lemma 6.23, $R^i\pi_*\tilde{\mathcal{O}}_Y^{++} = 0$ for all $i, n > 0$.

Proof. In the case where X is a point, this is a result of Bartenwerfer [4, Theorem, Folgerung 2]; we follow the presentation in [48, Proposition 3.5] to relativize the argument. Fix an open subset U_0 of X, an element $x \in X$, a rational subset V_0 of $\pi^{-1}(U_0)$, and a covering $\mathfrak{V}$ of V_0. We will show that there exists a refinement $\mathfrak{V}'$ of $\mathfrak{V}$ such that

$$\varinjlim_{U} \check{H}^i(\pi^{-1}(U) \cap V_0, \tilde{\mathcal{O}}_Y^{++}; \mathfrak{V}'|_{\pi^{-1}(U)\cap V_0}) = 0 \qquad (i > 0),$$

where the colimit is taken over all rational subspaces of U_0 containing x.

By Tate's reduction [28, §1.6], we can choose a refinement $\mathfrak{V}'$ of $\mathfrak{V}$ which is a composition of simple Laurent coverings. We may thus reduce the claim to the case where $\mathfrak{V}$ is the simple Laurent covering defined by some $f \in \mathcal{O}(V_0)$; in this case we will prove the claim with $\mathfrak{V}' = \mathfrak{V}$. Let V_1, V_2 be the two terms in the covering and put $V_{12} := V_1 \cap V_2$.

We first recall the proof of this case when X is a point, as described in [48, Proposition 3.5]. Put $K := \mathcal{H}(x)$ and let $V_{j,x}$ be the pullback of V_j to $\mathrm{Spa}(B\widehat{\otimes}_A K, (B\widehat{\otimes}_A K)^\circ)$. The set $V_{j,x}$ is a rational subspace of the analytic projective line $\mathbf{P}^1_{K,\mathrm{an}}$; its complement is the union of finitely many connected components $W_{j,0}, \dots, W_{j,n(j)}$, which we can label so that $W_{j,0}$ is the component containing ∞. For $i = 0, \dots, n(j)$, the set $V_{j,x} \cup W_{j,i}$ is again a rational subspace of $\mathbf{P}^1_{K,\mathrm{an}}$. Moreover,

$$W_{12,j} \in \{W_{1,j_1} : j_1 \in \{0, \dots, n(1)\}\} \cup \{W_{2,j_2} : j_2 \in \{0, \dots, n(2)\}\} \quad (j = 0, \dots, n(12)).$$

Every $g \in \mathcal{O}(V_{j,x})$ has a unique Mittag-Leffler decomposition $g = g_0 + \cdots + g_n$ where $g_i \in \mathcal{O}(V_{j,x} \cup W_i)$ and $g_i(\infty) = 0$ for $i > 0$. Moreover, the supremum norm of g is equal to the maximum of the supremum norms of the g_i. The acyclicity statement in this case is the surjectivity of $\mathcal{O}(V_{1,x}) \oplus \mathcal{O}(V_{2,x}) \to \mathcal{O}(V_{12,x})$; we obtain a splitting using the Mittag-Leffler decomposition and the identification of each $W_{12,j}$ with some W_{1,j_1} or W_{2,j_2}.

Returning to the general case, note that for some neighborhood U of x in U_0, we may lift the decomposition of the complement of $V_{i,x}$ to the complement of $V_i \cap \mathbf{P}^1_{U,\mathrm{an}}$, at which point we again have Mittag-Leffler decompositions and a candidate splitting map for the map $\mathcal{O}(\pi^{-1}(U) \cap V_1) \oplus \mathcal{O}(\pi^{-1}(U) \cap V_2) \to \mathcal{O}(\pi^{-1}(U) \cap V_{12})$. While this map no longer preserves norms, we can still say that for any $g \in \mathcal{O}(V_{12})$ and any $\epsilon > 0$, for U sufficiently small the image of g under the splitting has norm at most $1 + \epsilon$ times the norm of g. This yields the claimed result. $\square$

Lemma 6.25. *If A is a plus-sheafy Banach ring, then so are the weighted Tate algebras $A\langle T/r\rangle$ and $A\langle s/T, T/r\rangle$ for all $0 < s \le r$. In particular, if A is a plus-sheafy Huber ring, then so are $A\langle T\rangle$ and $A\langle T^{\pm}\rangle$.*

Proof. We treat the case of $A\langle T\rangle$ in detail, the other cases being similar (using Remark 4.8 and Remark 6.13 to handle $A\langle T/r\rangle$ and $A\langle s/T, T/r\rangle$). Put

$$X := \mathrm{Spa}(A, A^\circ), \qquad Y := \mathrm{Spa}(A\langle T\rangle, A\langle T\rangle^\circ)$$

and let $\pi : Y \to X$ be the projection. Let $\varpi \in A$ be a topologically nilpotent unit. The map $\mathcal{O}_X^+\langle T\rangle \to \pi_*\tilde{\mathcal{O}}_Y^+$ (where the tilde denotes sheafification) is injective and (because every Tate algebra over a nonarchimedean field is sheafy) its cokernel is killed by every topologically nilpotent element of A; since $A \to H^0(X, \mathcal{O}_X)$ is bijective, we may deduce from this that $A^+\langle T\rangle/\varpi^n \to H^0(X, \tilde{\mathcal{O}}_Y^+/\varpi^n)$ has n-uniformly torsion kernel and cokernel.

By Lemma 6.24, $R^i\pi_*(\tilde{\mathcal{O}}_Y^{++}/\varpi^n) = 0$ for all $i, n > 0$, so the map $H^i(X, \pi_*(\tilde{\mathcal{O}}_Y^{++}/\varpi^n)) \to H^i(Y, \tilde{\mathcal{O}}_Y^{++}/\varpi^n)$ is a bijection for all $i, n > 0$. Since $\tilde{\mathcal{O}}_Y^{+}/\tilde{\mathcal{O}}_Y^{++}$ is killed by every topologically nilpotent element of A, the kernel and cokernel of the map $H^i(X, \pi_*(\tilde{\mathcal{O}}_Y^{+}/\varpi^n)) \to H^i(Y, \tilde{\mathcal{O}}_Y^{+}/\varpi^n)$ are n-uniformly torsion. As in the previous paragraph, $\pi_*(\tilde{\mathcal{O}}_Y^{+}/\varpi^n)$ differs from $\mathcal{O}_X^+\langle T\rangle/\varpi^n$ by an n-uniformly torsion A^+-module, and $\mathcal{O}_X^+\langle T\rangle/\varpi^n$ may be rewritten as $(\mathcal{O}_X^+/\varpi^n)[T]$ as in (6.17.1); hence $H^i(Y, \tilde{\mathcal{O}}_Y^{+}/\varpi^n)$ differs from $H^i(X, \mathcal{O}_X^+/\varpi^n)[T]$ by an n-uniformly torsion A^+-module. Combining this with the fact that A is plus-sheafy yields that $A\langle T\rangle$ is plus-sheafy. ☐

Corollary 6.26. *Let A be a uniform plus-sheafy Huber ring.*

(a) *The ring A is strongly sheafy. (Example 6.11 shows that the converse fails.)*
(b) *If A is a Banach ring, then for every nonnegative integer n and every $\rho_1, \dots, \rho_n > 0$, $A\langle T_1/\rho_1, \dots, T_n/\rho_n\rangle$ is sheafy.*

Proof. This is immediate from Lemma 6.25. ☐

Corollary 6.27. *Let $(A, A^+) \to (B, B^+)$ be a morphism of uniform Huber rings which is smooth in the naïve sense. If A is plus-sheafy, then so is B. In particular, any smooth affinoid algebra over a nonarchimedean field is plus-sheafy.*

Proof. As per Definition 5.11, this follows by combining Theorem 6.21 with Lemma 6.25. ☐

The following example was suggested by Yutaro Mikami.

Example 6.28. Let A be the completion of the nonsheafy Huber ring given at the end of [23, §1] (which Huber attributes to Rost). Explicitly, A is the completion of $A^\triangleright = \mathbb{Z}[X_1, X_2, X_3]_{X_1X_2}$ for the topology under which $\{X_2^n B \colon n = 0, 1, \dots\}$ form a fundamental system of neighborhoods of 0, where B is the subring

$$\mathbb{Z}[X_2, X_1X_2, X_1^{-1}X_2, X_1^nX_2^nX_3, X_1^{-n}X_2^{-n}X_3 \colon n = 0, 1, \dots] \subset A.$$

Note that X_2 is a topologically nilpotent unit in A, so A is Tate.

Huber shows that A is nonsheafy as follows. Promote A to a Huber pair (A, A^+), put $X = \mathrm{Spa}(A, A^+)$, and let $(A, A^+) \to (B_1, B_1^+), (A, A^+) \to (B_2, B_2^+)$ be the rational localizations defined by the conditions $v(X_1) \leq 1$ and $v(X_1) \geq 1$. Then $X_3 \in A$ is nonzero but maps to zero in B_1 and B_2, so $A \to H^0(X, \mathcal{O})$ fails to be injective. Note that this also shows that A is not uniform.

Now note that since X_3 defines the zero section of $\mathcal{O}$, neither X nor the cohomology of $\mathcal{O}$ changes if we replace A by $A/(X_3) \cong \mathbb{Z}((X_2))\langle X_1X_2, X_2/X_1\rangle$. By Example 6.9 and Lemma 6.25, we deduce that A is plus-sheafy but not sheafy.

7. Sousperfectoid rings

We now describe a criterion for strong sheafiness which is easily checked in some key examples. This construction is related to the use of *preperfectoid* and *relatively perfectoid* rings in [40, Definition 2.3.4] and [29, §3.7]. Hereafter, fix a prime number p.

Definition 7.1. For A a Huber ring in which p is topologically nilpotent, a *perfectoid frame* for A consists of a morphism $A \to A'$ of Huber rings with A' perfectoid, together with a splitting $\sigma \colon A' \to A$ in the category of topological A-modules. We summarize this relationship using the notation $A \overset{\sigma}{\leftrightarrows} A'$.

We say that A is *sousperfectoid* if there exists a perfectoid frame for A. In this case, $A \to A'$ is a strict inclusion, so A is uniform. Moreover, if $A \to B$ is a rational localization, then by Proposition 3.8 we obtain another perfectoid frame $B \overset{\sigma'}{\leftrightarrows} A'\widehat{\otimes}_A B$ by base extension, so B is again sousperfectoid. Consequently, any sousperfectoid ring is stably uniform, and hence sheafy by Proposition 3.14. However, we do not know whether the sousperfectoid property is of local nature; see Remark 8.16 for a related discussion.

We extend the preceding definitions to a Banach ring by applying them to the underlying Huber ring.

Remark 7.2. The existence of a splitting of $A \to A'$ in the category of A-modules is stable under taking completed tensor products over A with a Huber ring B over A. However, such tensor products generally do not preserve uniformity, and passing to the uniform completion may destroy the splitting; see Example 10.5, and also Remark 8.3 for a related phenomenon.

Any perfectoid ring is sousperfectoid. We may generate additional examples using stability under rational localization and the following lemmas.

Lemma 7.3. *If A is a sousperfectoid Banach ring, then so are the weighted Tate algebras $A\langle T/r\rangle$ and $A\langle s/T, T/r\rangle$ for all $0 < s \le r$. In particular, if A is a sousperfectoid Huber ring, then so are $A\langle T\rangle$ and $A\langle T^{\pm}\rangle$.*

Proof. Let $A \overset{\sigma}{\leftrightarrows} A'$ be a perfectoid frame. We may then construct perfectoid frames

$$A\left\langle \frac{T}{r}\right\rangle \overset{\sigma'}{\leftrightarrows} A'\left\langle \frac{T^{p^{-\infty}}}{r^{p^{-\infty}}}\right\rangle, \qquad A\left\langle \frac{s}{T}, \frac{T}{r}\right\rangle \overset{\sigma'}{\leftrightarrows} A'\left\langle \frac{s^{p^{-\infty}}}{T^{p^{-\infty}}}, \frac{T^{p^{-\infty}}}{r^{p^{-\infty}}}\right\rangle$$

by the formula $\sigma'(\sum_n x_n T^n) = \sum_{n\in\mathbb{Z}} \sigma(x_n) T^n$, i.e., by declaring that σ' acts via σ on coefficients and kills all non-integral powers of T. $\square$

Corollary 7.4. *Any sousperfectoid ring, and in particular any perfectoid ring, is strongly sheafy.*

Proof. As noted in Definition 7.1 above, any sousperfectoid ring is sheafy. By Lemma 7.3, if A is sousperfectoid, then for any n the ring $A\langle T_1, \dots, T_n\rangle$ is again sousperfectoid, and hence sheafy. $\square$

Lemma 7.5. *Let $f : A \to B$ be a finite étale morphism of Huber rings.*

(a) *If A is sousperfectoid, then so is B.*
(b) *Suppose that f is also faithfully flat. If B is sousperfectoid, then so is A.*

Proof. Suppose that A is sousperfectoid. Let $A \overset{\sigma}{\leftrightarrows} A'$ be a perfectoid frame. By Proposition 2.15, by base extension we obtain a perfectoid frame $B \overset{\sigma'}{\leftrightarrows} B \otimes_A A'$. This implies (a).

Suppose that f is faithfully flat and B is sousperfectoid. Let $B \overset{\sigma}{\leftrightarrows} B'$ be a perfectoid frame. By composing σ with any A-linear splitting of B, we obtain a perfectoid frame $A \overset{\sigma'}{\leftrightarrows} B'$. This implies (b). $\square$

Lemma 7.6. *Let A be a sousperfectoid Huber ring. Then every* countably *faithfully profinite étale morphism $A \to B$ splits in the category of topological A-modules. In particular, if there exists such a morphism with B perfectoid, then this morphism extends to a perfectoid frame.*

Proof. Let $A \overset{\sigma}{\leftrightarrows} A'$ be a perfectoid frame; then $A' \to B' := B\widehat{\otimes}_A A'$ is again countably profinite étale, and so B' is again perfectoid by Proposition 2.15. Moreover, by applying Proposition 2.15 countably many times (as in [30, Lemma 3.4.4]), the morphism $A' \to B'$ splits in the category of topological A-modules. We thus obtain a splitting of $A \to B'$ which we may restrict to B. $\square$

Remark 7.7. We do not know if Lemma 7.6 remains true if the word "countably" is omitted, due to difficulties with taking inverse limits over uncountable inverse sets, as in the errata to [38]. One approach to remedying this difficulty is to work with *weakly sousperfectoid* Huber rings, as in §8.

As an example of the difficulty caused by Remark 7.7, we consider the case of a nonarchimedean field.

Remark 7.8. Let K be a nonarchimedean field. Then every separable Banach space over K admits a Schauder basis, i.e., it is isomorphic to a finite or countable product of copies of K topologized with the supremum norm [12, Proposition 2.7.2/3]. Consequently, any countably faithfully profinite étale morphism $K \to L$ splits in the category of topological K-vector spaces. In particular, if there exists such a morphism with L perfectoid (e.g., if K is topologically countably generated over $\mathbb{Q}_p$), then K is sousperfectoid.

We now turn briefly to the case of positive characteristic.

Definition 7.9. A ring (without topology) of characteristic p is *Frobenius-split* (or *F-split*) if the absolute Frobenius endomorphism of A admits a splitting. This concept has been extensively studied for both rings and schemes, and has a diverse range of applications. See [43] for a detailed survey.

Lemma 7.10. *Let A be a uniform Huber ring of characteristic p.*

(a) *If A is sousperfectoid, then A is F-split.*
(b) *If A is F-finite (that is, the Frobenius endomorphism is finite) and F-split, then A is sousperfectoid.*

Proof. Suppose that A admits a perfectoid frame $A \overset{\sigma}{\leftrightarrows} A'$. Then restricting σ to $A^{1/p} \subseteq A'$ yields a Frobenius splitting. This proves (a).

If A is F-finite, then any splitting of Frobenius is automatically continuous. By iterating the splitting, we get a continuous splitting of the map from A to its perfect closure, which then extends to the completion and gives rise to a perfectoid frame. This yields (b). $\square$

Example 7.11. Let K be a perfect nonarchimedean field of characteristic p. Let X be a smooth projective variety over K with ample canonical bundle (e.g., a curve of genus at least 2). Then the Frobenius map on the structure sheaf $\mathcal{O}_X$ does not admit a global splitting [43, Corollary 2.18], so the homogeneous coordinate ring of X for the canonical embedding is not Frobenius-split [43, Theorem 2.12].

Let Y be the affine cone over X for the canonical embedding. If we take a connected affinoid neighborhood of the origin in the analytification of Y, we obtain a normal affinoid algebra A over K. We expect that a suitable variation of the proof of [43, Theorem 2.12] can be used to show that A is neither Frobenius-split nor sousperfectoid; however, we did not attempt to write out the details.

8. Weakly sousperfectoid rings

To circumvent several technical issues, including those associated to taking uncountable inverse limits (Remark 7.7), we offer a weaker version of the sousperfectoid property. It is motivated by the following observation.

Remark 8.1. Let A be a Banach ring. Let $\{A_i\}_i$ be a filtered system of Banach A-algebras with submetric morphisms, and let A' be the completed direct limit of the A_i [29, Definition 2.6.1]. Suppose that there exists $c > 0$ such that for each i, the map $A \to A_i$ admits a splitting in the category of Banach A-modules of operator norm at most c. Then the map $A \to A'$ is strict.

Definition 8.2. Let A be a uniform Banach ring equipped with the spectral norm. For B a faithfully finite étale A-algebra, view B as a Banach module over A (and thus as a uniform Banach ring; see Definition 2.9), and equip B with its spectral norm. Let $c_{B/A}$ be the infimum of the operator norms of all splittings of the map $A \to B$ in the category of A-modules.

Remark 8.3. In connection with Remark 7.2, beware that the operator norm of a splitting is not stable under base change. That is, with notation as in Definition 8.2, for $A \to A'$ a morphism of uniform Banach rings, there is no immediate relationship between $c_{B/A}$ and $c_{(B \otimes_A A')/A'}$.

For example, let $\mathbb{C}_p$ be a completed algebraic closure of $\mathbb{Q}_p$ and take $A = \mathbb{C}_p\langle T \rangle$, $B = \mathbb{C}_p\langle T^{1/p^n} \rangle$, $A' = B$. Let $\sigma : B \to A$ be the reduced trace; this is an A-module splitting of $A \to B$ of operator norm 1. However, $B \otimes_A A'$ splits as a direct sum of copies of A' with the induced map $\sigma' \colon B \otimes_A A' \to A'$ being the averaging map, which has operator norm p^n. With some more effort, one can show that there exists *no* splitting of $A \to B$ whose base extension to A' has operator norm less than p^n.

Lemma 8.4. *Let $A \to A'$ be a morphism of Banach rings with A uniform. Assume either that*

- (i) *A' is the completed direct limit of a filtered system $\{A'_j\}_{j \in J}$ of faithfully finite étale A-algebras (for the spectral norm on each A'_j) for which $\{c_{A'_j/A}\}_{j \in J}$ is bounded; or*
- (ii) *A' is uniform and $A \to A'$ splits in the category of topological A-modules.*

Then the following statements hold.

- (a) *If $A' \to H^0(\mathrm{Spa}(A', A'^\circ), \mathcal{O})$ is an isomorphism of rings, then so is $A \to H^0(\mathrm{Spa}(A, A^\circ), \mathcal{O})$.*
- (b) *If A' is sheafy (resp. strongly sheafy), then A is sheafy (resp. strongly sheafy).*
- (c) *If A' is sheafy and $H^1(\mathrm{Spa}(A', A'^\circ), \mathcal{O}^+)$ is uniformly torsion (so in particular if A' is plus-sheafy, by Lemma 6.18(c)), then $H^1(\mathrm{Spa}(A, A^\circ), \mathcal{O}^+)$ is uniformly torsion.*

Proof. We first treat (a). In case (ii), taking the composition

$$H^0(\mathrm{Spa}(A, A^\circ), \mathcal{O}) \to H^0(\mathrm{Spa}(A', A'^\circ), \mathcal{O}) \cong A' \to A$$

yields a splitting of the map $A \to H^0(\mathrm{Spa}(A, A^\circ), \mathcal{O})$; since the construction is compatible with localization, the splitting map is injective. In case (i), we make an analogous construction as follows. Choose splittings $\pi_j \colon A'_j \to A$ with j-uniformly bounded operator norm. Given $f \in H^0(\mathrm{Spa}(A, A^\circ), \mathcal{O})$, we again map to $H^0(\mathrm{Spa}(A', A'^\circ), \mathcal{O}) \cong A'$; we then write the result as the limit of a convergent sequence $\{g_n\}$ with $g_n \in A'_{j_n}$ for some $j_n \in J$. The sequence $\pi_{j_n}(g_n)$ then converges to a limit which does not depend on the choices of the g_n.

To prove (b), note that conditions (i) and (ii) of the present theorem are both preserved by passing from A to a rational localization or to $A\langle T\rangle$; we may thus deduce (b) from (a).

To prove (c), note that condition (ii) implies that the exact sequence

$$0 \to A^+ \to A'^+ \to A^+/A'^+ \to 0$$

defines a torsion class in $\mathrm{Ext}(A^+/A'^+, A^+)$; we thus obtain an exact sequence

$$* \to H^1(\mathrm{Spa}(A, A^\circ), \mathcal{O}^+) \to H^1(\mathrm{Spa}(A', A'^\circ), \mathcal{O}^+)$$

in which the unspecified first term is uniformly torsion. In case (i), we similarly obtain an exact sequence

$$* \to H^1(\mathrm{Spa}(A, A^\circ), \mathcal{O}^+) \to H^1(\mathrm{Spa}(A'_j, A'^\circ_j), \mathcal{O}^+)$$

in which the unspecified first term is j-uniformly torsion. ☐

Definition 8.5. Let A be a uniform Banach ring in which p is topologically nilpotent. We say that A is *weakly sousperfectoid* if the quantities $c_{B/A}$ are uniformly bounded as B varies over all faithfully finite étale A-algebras. If we need to specify a particular bound $c \geq 1$, we will say that A is *c-weakly sousperfectoid.*

This definition admits a topological reformulation: A is weakly sousperfectoid if and only if there exists a topologically nilpotent element x of A such that for every faithfully finite étale A-algebra B, the inclusion $A \to B$ admits a splitting in the category of A-modules carrying xB° into A°. We may thus say that a uniform Huber ring is *weakly sousperfectoid* if some (hence any) promotion of A to a uniform Banach ring is weakly sousperfectoid.

Lemma 8.6. *Any weakly sousperfectoid ring is strongly sheafy.*

Proof. Apply Proposition 6.14 and Lemma 8.4. ☐

Remark 8.7. Note that one could formulate the definition of a weakly sousperfectoid Huber or Banach ring without requiring p to be topologically nilpotent. However, at this level of generality, we do not know whether this definition demonstrates any reasonable behavior; in particular, we do not know that it implies sheafiness.

In contrast with Remark 7.8, every nonarchimedean field is weakly sousperfectoid.

Lemma 8.8. *Suppose that A is a nonarchimedean field of residue characteristic p.*

(a) *Let V be a finite-dimensional A-vector space. Then for every Banach norm on V and every $c > 1$, there exists a weighted supremum norm on V with respect to some basis which differs from the given norm (in either direction) by a multiplicative factor of at most c.*
(b) *We have $c_{B/A} = 1$. Consequently, A is 1-weakly sousperfectoid.*

Proof. Part (a) is well-known; see any of [12, Proposition 2.6.2/3], [25, Lemma 1.3.7], [32, §3, Lemme 2]. Part (b) is an immediate consequence of part (a). ☐

We next set about reconciling the definitions of sousperfectoid and weakly sousperfectoid rings, starting with the fact that perfectoid rings are weakly sousperfectoid.

Lemma 8.9. *Let A be a perfectoid ring. For every faithfully finite étale A-algebra B, we have $c_{B/A} = 1$. Consequently, A is 1-weakly sousperfectoid.*

Proof. This is an immediate consequence of Proposition 2.15. □

Corollary 8.10. *Any sousperfectoid Huber ring (in particular, any perfectoid ring) is weakly sousperfectoid. More precisely, if A is a Banach ring admitting a perfectoid frame $A \overset{\sigma}{\leftrightarrows} A'$ in which σ has operator norm c, then A is weakly c-sousperfectoid.*

Proof. It suffices to prove the second assertion. Let $A \to B$ be a faithfully finite étale morphism and put $B' := A' \otimes_A B$; then $A' \to B'$ is again faithfully finite étale. By Lemma 8.9, we have $c_{B'/A'} = 1$; consequently, for any $c' > c$ we can find a splitting of $A \to A' \to B' = A \to B \to B'$ of operator norm at most c'. □

Remark 8.11. If A is a 1-sousperfectoid Banach ring, then $H^1(\mathrm{Spa}(A, A^+), \mathcal{O}^+)$ is annihilated by every topologically nilpotent element of A. This observation can be used to exhibit examples of Banach rings which are weakly sousperfectoid but not weakly 1-sousperfectoid, e.g., using Example 6.6.

We now give a crucial reformulation of the definition of a weakly sousperfectoid ring, which will allow us to emulate some basic argument about sousperfectoid rings. It essentially asserts that to establish that A is weakly sousperfectoid, we only need to check the uniform boundedness of the constants $c_{B/A}$ for B running over a convenient family of faithfully finite étale A-algebras.

Lemma 8.12. *The following statements hold.*

- (a) *Let A be a uniform Banach ring. Let A' be a ring which is the completed direct limit of a filtered system $\{A'_i\}_{i \in I}$ of faithfully finite étale A-algebras (for the spectral norm on each A'_i). Suppose that A' is weakly sousperfectoid and $\{c_{A'_i/A}\}_{i \in I}$ is bounded. Then A is weakly sousperfectoid.*
- (b) *Let $A \to A'$ be a morphism of uniform Huber rings (resp. uniform Banach rings) which splits in the category of topological A-modules (resp. Banach A-modules). If A' is weakly sousperfectoid (so in particular by (a), if A' is perfectoid), then A is weakly sousperfectoid.*

Proof. To prove (a), let $A \to B$ be a faithfully finite étale A-algebra; then any splitting of $A' \to (B \otimes_A A')$ can be approximated to within any desired accuracy by a splitting taking B into A'_i for some i. In particular, this approximation can be made without increasing the operator norm. It follows that

$$c_{B/A} \leq c_{(B \otimes_A A')/A'} \sup_i \{c_{A'_i/A}\},$$

so A is weakly sousperfectoid.

To prove (b), we need only consider the Banach case. For B a faithfully finite étale A-algebra, we may split $A \to B$ by composing the inclusion $B \to B' := B \otimes_A A'$, a splitting of $A' \to B'$, and the splitting $A' \to A$. The quantity $c_{B/A}$ is thus bounded by a fixed constant (the operator norm of $A' \to A$) times the supremum of the quantities $c_{B'/A'}$. □

Corollary 8.13. *If A is a weakly sousperfectoid Banach ring, then so are the weighted Tate algebras $A\langle T/r\rangle$ and $A\langle s/T, T/r\rangle$ for all $0 < s \leq r$. In particular, if A is a weakly sousperfectoid Huber ring, then so are $A\langle T\rangle$ and $A\langle T^{\pm}\rangle$.*

Proof. For simplicity, we treat only the case of $A\langle T\rangle$, the other cases being similar. By Lemma 8.12(b), it suffices to check that $A\langle T^{p^{-\infty}}\rangle$ is weakly sousperfectoid. To check this, apply Proposition 2.14 to construct a faithfully profinite étale morphism $A \to A'$ with A' perfectoid, then apply Lemma 8.12(a) to the family $\{B\langle T^{p^{-\infty}}\rangle\}$ where B runs over faithfully finite étale A-subalgebras of A'. □

Theorem 8.14. *Let $A \to B$ be a rational localization of Huber rings. If A is weakly sousperfectoid, then so is B. Consequently, any weakly sousperfectoid Huber ring is stably uniform.*

Proof. Promote A and B to Banach rings. Apply Proposition 2.14 to construct a faithfully profinite étale morphism $A \to A'$ with A' perfectoid; by Proposition 3.8, the morphism $B \to B' := B\widehat{\otimes}_A A'$ has the same properties. For A_i running over the faithfully finite étale subalgebras of A', the maps $B \to B \otimes_A A_i$ are uniformly split for the tensor product norm on $B \otimes_A A_i$; consequently, $B \to B'$ is strict (because the spectral norm is equivalent to the tensor product norm). It follows that B is uniform. By Lemma 8.12 applied to the $B \otimes_A A_i$, B is weakly sousperfectoid. □

Lemma 8.15. *Let $f : A \to B$ be a finite étale morphism of Huber rings.*

(a) *If A is weakly sousperfectoid, then so is B.*
(b) *Suppose that f is also faithfully flat. If B is weakly sousperfectoid, then so is A.*

Proof. Promote A and B to Banach rings. Suppose that A is weakly sousperfectoid and equip it with its spectral norm. Apply Proposition 2.14 to construct a faithfully profinite étale morphism $A \to A'$ with A' perfectoid and put $B' = A' \otimes_A B$; by Proposition 2.15, the tensor product norm is equivalent to its associated spectral norm. Consequently, $c_{(A_i \otimes_A B)/B}$ is uniformly bounded as A_i runs over faithfully finite étale A-subalgebras of A'. By Lemma 8.12, we deduce that B is weakly sousperfectoid.

Suppose that f is faithfully flat and B is weakly sousperfectoid. Fix an A-linear splitting of $A \to B$; using this splitting, we see that $c_{B_i/A}$ is uniformly bounded as B_i runs over all faithfully finite étale B-algebras. By Lemma 8.12, we deduce that A is weakly sousperfectoid. □

Remark 8.16. While the weakly sousperfectoid property does avoid some complications associated with the sousperfectoid property, neither one is known to be of local nature (Definition 3.17). For a statement about local nature limited to affinoid algebras, see Theorem 10.3.

9. Fine topologies on (pre)adic spaces

In order to formulate some subsequent statements, we must consider some finer topologies than the étale topology. Note that at several points below, we write $\mathrm{Spa}(A, A^+)$ where it would be more appropriate to write $\widetilde{\mathrm{Spa}}(A, A^+)$ as in [40, Definition 2.1.5] or [29, Definition 8.2.3] (to denote the sheafification of the representable functor); this should not cause any confusion.

Definition 9.1. For (A, A^+) a Huber pair, the *v-topology* on $X = \mathrm{Spa}(A, A^+)$ is the Grothendieck topology[3] whose objects are preadic spaces over $\mathrm{Spa}(A, A^+)$ (i.e., the adic spaces of [40, Definition 2.1.5]), and where a family of morphisms $\{f_i : U_i \to X\}_{i \in I}$ is a covering if every quasicompact open subset of X is contained in the image of some quasicompact open subset of $\sqcup U_i$. Let $\mathrm{Spa}(A, A^+)_{\mathrm{v}}$ denote the resulting site; it carries natural presheaves $\mathcal{O}$ and $\mathcal{O}^+$. Note that the category of preadic spaces admits fiber products, so this definition makes sense.

When A is a Tate ring over $\mathbb{Z}_p$, the associated diamond $\mathrm{Spd}(A, A^+)$ has a v-site as defined in [39], and there is a natural map of sites $\mathrm{Spd}(A, A^+)_{\mathrm{v}} \to \mathrm{Spa}(A, A^+)_{\mathrm{v}}$. This map is not an equivalence, but it does induce an equivalence of topoi. Likewise, if A is perfectoid, there is a natural map from $\mathrm{Spa}(A, A^+)_{\mathrm{v}}$ to the v-site defined in [30]; again, this is not an equivalence of sites, but it induces an equivalence of topoi.

Remark 9.2. Let $A \to B$ be a morphism of uniform Banach rings such that $\mathcal{M}(B) \to \mathcal{M}(A)$ is surjective. By Proposition 3.5, $A \to B$ is an isometry for the spectral norms, and hence a strict inclusion.

Now let $(A, A^+) \to (B, B^+)$ be a morphism of uniform Huber rings which is a covering in the v-topology. By the previous paragraph, $A \to B$ is again a strict inclusion.

For our purposes, the most important example of a v-cover is given by Proposition 2.14, on account of the following statement.

Theorem 9.3. *Let (A, A^+) be a perfectoid Huber pair.*

(a) *The presheaves $\mathcal{O}, \mathcal{O}^+$ on $\mathrm{Spa}(A, A^+)_{\mathrm{v}}$ are sheaves.*
(b) *For all $i > 0$, $H^i(\mathrm{Spa}(A, A^+)_{\mathrm{v}}, \mathcal{O}) = 0$.*
(c) *For all $i > 0$, the A^+-module $H^i(\mathrm{Spa}(A, A^+)_{\mathrm{v}}, \mathcal{O}^+)$ is* almost zero*: it is killed by* every *topologically nilpotent unit of A.*

Proof. See [30, Theorem 3.5.5]. □

Remark 9.4. Theorem 9.3 implies a corresponding assertion for any topology between the analytic topology and the v-topology. For example, this includes the *pro-étale topology* in the original sense of Scholze [38, §3] (see also [29, §9.1]), as well as the more general sense used in Scholze's theory of diamonds [41, 39]. It also implies that for any Huber pair (A, A^+), the pro-étale cohomology and the v-cohomology of the structure presheaf on $\mathrm{Spa}(A, A^+)$ coincide; this will be used when making some citations into the literature.

Definition 9.5. For $A \to A'$ a morphism of Huber rings, define the complex

$$C_{A'/A} \colon 0 \to C^0_{A'/A} \xrightarrow{d^0} C^1_{A'/A} \to \cdots$$

by taking $C^i_{A'/A}$ to be the uniform completion of the $(i+1)$-fold tensor product of A' over A (so that $C^0_{A'/A}$ is the uniform completion of A') and d^i is the alternating sum of the base extension maps. Also, let

$$C^+_{A'/A} \colon 0 \to C^{0,+}_{A'/A} \xrightarrow{d^0} C^{1,+}_{A'/A} \to \cdots$$

[3]As usual, large topologies require some set-theoretic care; for a rigorous treatment, one should first express everything within a fixed Grothendieck universe, then check that no answers depend on this choice. Compare [44, Tag 0F4R].

denote the corresponding complex of plus subrings. Since $C^+_{A'/A}$ is a complex of complete A^+-modules, Lemma 2.6 and Theorem 2.7 together imply that $h^i(C^+_{A'/A})$ is a derived complete A^+-module.

In the case where $A \to A'$ is a v-covering with A' perfectoid, each term in the complex $C_{A'/A}$ is perfectoid by [29, Corollary 3.6.18]. Combining this observation with Theorem 9.3 yields canonical isomorphisms

$$h^i(C_{A'/A}) \cong H^i(\mathrm{Spa}(A, A^+)_{\mathrm{v}}, \mathcal{O})$$

via which we may topologize the right-hand side using the subquotient topology on the left-hand side. This topology does not depend on the choice of A'. By the same token, in the category of A^+-modules modulo the subcategory of modules annihilated by all topologically nilpotent units of A, we may identify $h^i(C^+_{A'/A})$ with $H^i(\mathrm{Spa}(A, A^+)_{\mathrm{v}}, \mathcal{O}^+)$.

Definition 9.6. For (A, A^+) a Huber pair in which p is topologically nilpotent, define the *v-completion* $(\breve{A}, \breve{A}^+)$ to be the Huber pair given by $\breve{A} = H^0(\mathrm{Spa}(A, A^+)_{\mathrm{v}}, \mathcal{O})$ and $\breve{A}^+ = H^0(\mathrm{Spa}(A, A^+)_{\mathrm{v}}, \mathcal{O}^+)$. We say a Huber pair is *v-complete* if the natural map $(A, A^+) \to (\breve{A}, \breve{A}^+)$ is an isomorphism.

Note that $\breve{A}$ depends only on A, and that v-completeness is independent of A^+. Concretely, we can compute $\breve{A}$ and $\breve{A}^+$ by the following recipe.

Lemma 9.7. *Given a Huber ring A in which p is topologically nilpotent, let A_i be a directed system of finite étale Galois A-algebras with Galois groups G_i, such that the completion $\widehat{A}_\infty$ of the direct limit $A_\infty = \varinjlim A_i$ is perfectoid. Set $G = \varprojlim G_i$, so G operates naturally on A_∞ and $\widehat{A}_\infty$. Then $\breve{A} \cong \widehat{A}_\infty^G$.*

Likewise, if $A^+ \subset A$ is a ring of integral elements, and $\widehat{A}^+_\infty$ denotes the completion of the integral closure of A^+ in A_∞, then $\breve{A}^+ \cong \widehat{A}^{+G}_\infty$.

Proof. Apply Definition 9.5 with $A' = \widehat{A}_\infty$. It is easy to check in this case that $C_{A'/A}$ is the usual complex of continuous cochains

$$0 \to \widehat{A}_\infty \to C(G, \widehat{A}_\infty) \to C(G \times G, \widehat{A}_\infty) \to \dots,$$

so the formula for $\breve{A}$ follows immediately. The analogous result for rings of integral elements follows by a similar argument. □

Lemma 9.8. *Let A be a Huber ring in which p is topologically nilpotent.*

(a) *Let $A \to B$ be a finite étale ring map. Then the natural map $\breve{A} \otimes_A B \to \breve{B}$ is an isomorphism.*
(b) *The natural map $\breve{A} \to \breve{\breve{A}}$ is an isomorphism.*
(c) *For any ring of integral elements A^+, the natural map $(A, A^+) \to (\breve{A}, \breve{A}^+)$ induces an isomorphism* $\mathrm{Spd}(\breve{A}, \breve{A}^+) \cong \mathrm{Spd}(A, A^+)$. *In particular, the natural map* $\mathrm{Spa}(\breve{A}, \breve{A}^+) \to \mathrm{Spa}(A, A^+)$ *is a homeomorphism inducing isomorphisms of completed residue fields, and* $\mathrm{Hom}(A, B) = \mathrm{Hom}(\breve{A}, B)$ *for any perfectoid Huber ring B.*

Proof. Choose a directed system A_i and all attendant notation as in Lemma 9.7. Also set $H_i := \ker(G \to G_i)$, so $A_i = A_\infty^{H_i}$ and $\breve{A}_i \cong \widehat{A_\infty}^{H_i}$.

For (a), set $B_i = A_i \otimes_A B$ and $B_\infty = \varinjlim B_i$. Since $A \to B$ is finite étale, $\widehat{A}_\infty \otimes_A B$ is a perfectoid Tate ring, and in particular is already complete for the natural topology, so the natural map $\widehat{A}_\infty \otimes_A B \to \widehat{B}_\infty$ is an isomorphism. Thus

$$\breve{B} \cong \widehat{B}_\infty^G \cong (\widehat{A}_\infty \otimes_A B)^G \cong \widehat{A}_\infty^G \otimes_A B \cong \breve{A} \otimes_A B$$

as desired.

For (b), note that $\breve{A}_i := \widehat{A}_\infty^{H_i}$ is a directed system of finite étale Galois $\breve{A}$-algebras (apply (a) to the maps $A \to A_i$ and $A \to A_i \otimes_A A_i \cong C(G_i, A_i)$). Moreover, the composition of the natural maps

$$A_\infty = \varinjlim A_\infty^{H_i} \to \varinjlim \widehat{A}_\infty^{H_i} \to \widehat{A}_\infty$$

has dense image, so the natural map

$$A_\infty \to \varinjlim \widehat{A}_\infty^{H_i} = \varinjlim \breve{A}_i$$

has dense image and becomes an isomorphism after completion. In particular, the completion $\widehat{\varinjlim \breve{A}_i}$ is perfectoid, so $\breve{\breve{A}} \cong \widehat{\varinjlim \breve{A}_i}^G$ by Lemma 9.7. On the other hand, we've already observed that $\widehat{\varinjlim \breve{A}_i}^G \cong \widehat{A}_\infty^G \cong \breve{A}$, so the result follows.

For (c), combine the proof of (b) with [39, Proposition 15.4], noting in particular that $\mathrm{Spd}(\widehat{A}_\infty, \widehat{A}_\infty^+)$ is simultaneously a $\underline{G}$-torsor over $\mathrm{Spd}(A, A^+)$ and over $\mathrm{Spd}(\breve{A}, \breve{A}^+)$. The preservation of residue fields uses the Ax–Sen theorem; see Proposition 9.15 below. □

Remark 9.9. Beware that Lemma 9.8 does not imply that v-completion commutes with rational localization. The proof of Lemma 11.7 will show that this would follow from Conjecture 9.12.

Lemma 9.10. *Let (A, A^+) be a Huber pair in which p is topologically nilpotent, and suppose that A is v-complete.*

(a) *The ring A is uniform and seminormal.*
(b) *The map $A \to H^0(\mathrm{Spa}(A, A^+), \mathcal{O})$ is an isomorphism of rings.*

Proof. By the open mapping theorem (Corollary 2.8), the map $A \to \breve{A}$ is an isomorphism of topological rings, not just underlying rings. Since the target is evidently uniform, so then is A. To see that A is seminormal, apply [30, Theorem 3.7.4] (or Proposition 6.7) for the case where A is perfectoid, and [30, Corollary 3.7.5] for the general case. This proves (a).

Let $A \to A'$ be a v-covering with A' perfectoid. Since A is uniform, by Proposition 3.5 the map $A \to H^0(\mathrm{Spa}(A, A^+), \mathcal{O})$ is injective. To check that this map is surjective, take an arbitrary element of $H^0(\mathrm{Spa}(A, A^+), \mathcal{O})$; map it to $H^0(\mathrm{Spa}(A', A'^+), \mathcal{O}) = A'$ (using Theorem 9.3(a)); check pointwise (using Proposition 3.5 again) that further applying d^0 gives the zero element of $C^1_{A'/A}$; and use the posited equality $A = \ker(d^0)$. This proves (b). □

Definition 9.11. Let (A, A^+) be a Huber pair in which p is topologically nilpotent. By Corollary 2.8, for any $i > 0$, the following conditions on A are equivalent.

(a) The A-module $H^i(\mathrm{Spa}(A, A^+)_{\mathrm{v}}, \mathcal{O})$ is complete for the subquotient topology.
(b) The A-module $H^i(\mathrm{Spa}(A, A^+)_{\mathrm{v}}, \mathcal{O})$ is Hausdorff for the subquotient topology.
(c) The torsion submodule of $H^i(\mathrm{Spa}(A, A^+)_{\mathrm{v}}, \mathcal{O}^+)$ is uniformly torsion. (We do not require the whole module to be torsion; see Remark 9.13.)

(d) The image of d^i is closed.
(e) The map d^i is strict.

When these conditions hold, we say that A is *i-strict*.

While the following conjecture seems overly optimistic, we have no counterexample against it. For example, it is true for any affinoid algebra in mixed characteristic (see Lemma 10.1).

Conjecture 9.12. *Let (A, A^+) be a Huber pair in which p is topologically nilpotent. Then A is i-strict for every $i > 0$.*

Remark 9.13. One trivial way for A to be i-strict is to have $H^i(\mathrm{Spa}(A, A^+)_{\mathrm{v}}, \mathcal{O}) = 0$. However, this generally does not happen except when A is perfectoid (Theorem 9.3).

We record some interactions between Theorem 9.3 and the Galois cohomology of nonarchimedean fields.

Remark 9.14. In case K is a nonarchimedean field, the extension in Proposition 2.14 may be taken to be a completed algebraic closure L of K. By Theorem 9.3, it follows that $H^i(\mathrm{Spa}(K, K^\circ)_{\mathrm{v}}, \mathcal{O})$ may be identified with $H^i_{\mathrm{cont}}(G_K, L)$, the i-th cohomology group of the complex of continuous inhomogeneous L-valued cochains on G_K.

Proposition 9.15 (Ax–Sen)**.** *Let K be a nonarchimedean field in which $|p| = p^{-1}$. Let L be a completed algebraic closure of K.*
(a) *We have $L^{G_K} = K$.*
(b) *The torsion submodule of the K°-module $H^1_{\mathrm{cont}}(G_K, L^\circ)$ is killed by every element of K of norm at most $p^{-p/(p-1)^2}$.*

In particular, by Remark 9.14, $K = H^0(\mathrm{Spa}(K, K^\circ)_{\mathrm{v}}, \mathcal{O})$ and K is 1-strict.

Proof. This follows from a result of Ax [3, §2, Proposition 1]. In the case where K is the completion of an algebraic extension of a complete discretely valued field with perfect residue field, Sen [42] showed that the norm bound in (b) can be improved to $p^{-1/(p-1)}$; we will not need this improvement here. □

Corollary 9.16. *Let (A, A^+) be a Huber pair in which p is a topologically nilpotent unit. Put $X := \mathrm{Spa}(A, A^+)$ and let $\nu\colon X_{\mathrm{v}} \to X$ be the canonical projection. Then $\ker(R^1\nu_*\tilde{\mathcal{O}}^+_{X_{\mathrm{v}}} \to R^1\nu_*\tilde{\mathcal{O}}_{X_{\mathrm{v}}})$ (where the tildes denote sheafification) is killed by every $x \in A$ for which $x^{(p-1)^2}p^{-p}$ is topologically nilpotent.*

Proof. Promote A to a Banach ring over $\mathbb{Q}_p$. We compute the stalk at $v \in X$ projecting to $\alpha \in \mathcal{M}(A)$. Per Proposition 2.14, let $A \to A'$ be a faithfully profinite étale morphism with A' perfectoid. An element of the stalk of the kernel at v can be represented, for some rational localization $(A, A^+) \to (B, B^+)$ with $v \in \mathrm{Spa}(B, B^+)$, by an element y of $C^{1,+}_{B'/B}$ (where $B' = A'\widehat{\otimes}_A B$) whose image in $C^1_{B'/B}$ equals $d^0(z)$ for some $z \in C^0_{B'/B}$; to say that this element of the stalk is killed by x is to assert that xy is itself in the image of d^0. By Proposition 9.15(b), this holds after base extension from A to $\mathcal{H}(\alpha)$; that is, the image of xy in $C^{1,+}_{(A'\widehat{\otimes}_A\mathcal{H}(\alpha))/\mathcal{H}(\alpha)}$ equals $d^0(z')$ for some $z' \in C^{0,+}_{(A'\widehat{\otimes}_A\mathcal{H}(\alpha))/\mathcal{H}(\alpha)} = (A'\widehat{\otimes}_A\mathcal{H}(\alpha))^+$. Now $d^0(xz - z') = 0$, so by Proposition 9.15(a) we have $xz - z' \in \mathcal{H}(\alpha)$; in other words, we can adjust the choice of z' so that $xz = z' \in (A'\widehat{\otimes}_A\mathcal{H}(\alpha))^+$. At the expense of replacing (B, B^+) with another localization, we can ensure that $xz \in C^{0,+}_{B'/B}$, proving the claim. □

Remark 9.17. We do not know whether a similar uniform bound on the torsion of $H^i_{\mathrm{cont}}(G_K, L^\circ)$ exists for any $i > 1$. For K a local field, such a bound has been recently announced by Barthel–Schlank–Stapleton–Weinstein.

Remark 9.18. Proposition 9.15(a) fails when K is of characteristic p, as $H^0(\mathrm{Spa}(K, K^\circ)_{\mathrm{v}}, \mathcal{O})$ equals not K but its completed perfect closure (see [3]).

Adapting the proof of Proposition 6.7 to the v-topology yields the following.

Lemma 9.19. *For (A, A^+) a uniform Huber pair in which p is a topologically nilpotent unit, the following conditions are equivalent.*

(i) *The torsion submodule of $H^1(\mathrm{Spa}(A, A^+), \mathcal{O}^+)$ is uniformly torsion.*

(ii) *The ring $H^0(\mathrm{Spa}(A, A^+)_{\mathrm{v}}, \mathcal{O})$ is 1-strict and equals $H^0(\mathrm{Spa}(A, A^+), \mathcal{O})$.*

Proof. To lighten notation, we write $\mathcal{O}^+, \mathcal{O}$ throughout where we really mean the sheafifications thereof. Put $X := \mathrm{Spa}(A, A^+)$ and let $\nu\colon X_{\mathrm{v}} \to X$ be the canonical projection. The commutative diagram

$$\begin{tikzcd} \mathcal{O}_X^+ \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[d] \\ \nu_*\mathcal{O}^+_{X_{\mathrm{v}}} \arrow[r] & \nu_*\mathcal{O}_{X_{\mathrm{v}}} \end{tikzcd}$$

is cartesian and cocartesian (the latter by Proposition 9.15(a)), and hence gives rise to a short exact sequence

$$0 \to \mathcal{O}_X^+ \to \mathcal{O}_X \oplus \nu_*\mathcal{O}^+_{X_{\mathrm{v}}} \to \nu_*\mathcal{O}_{X_{\mathrm{v}}} \to 0. \tag{9.19.1}$$

Note that the kernel of $H^1(X, \mathcal{O}^+) \to H^1(X, \mathcal{O})$ equals the torsion submodule $H^1(X, \mathcal{O}^+)_{\mathrm{tors}}$, and similarly with X replaced by X_{v}. Consequently, taking cohomology in (9.19.1) yields an exact sequence

$$0 \to \frac{H^0(X_{\mathrm{v}}, \mathcal{O})}{H^0(X_{\mathrm{v}}, \mathcal{O}^+) + H^0(X, \mathcal{O})} \to H^1(X, \mathcal{O}^+)_{\mathrm{tors}} \to H^1(X, \nu_*\mathcal{O}^+_{X_{\mathrm{v}}})_{\mathrm{tors}} \to 0. \tag{9.19.2}$$

(Here we are using that A is uniform to identify $H^0(X, \mathcal{O})$ with a subgroup of $H^0(X_{\mathrm{v}}, \mathcal{O})$.) In particular, any element of $H^0(X_{\mathrm{v}}, \mathcal{O})$ not contained in $H^0(X, \mathcal{O})$ contributes a family of torsion elements to $H^1(X, \mathcal{O}^+)$ which are not killed by any single power of p.

By Corollary 9.16, the sheaf $\ker(R^1\nu_*\mathcal{O}_X^+ \to R^1\nu_*\mathcal{O}_X)$ is killed by some power of p. Consequently, in the previous sequence, the last nonzero term becomes isomorphic to $H^1(X_{\mathrm{v}}, \mathcal{O}^+)_{\mathrm{tors}}$ in the quotient category modulo groups killed by powers of p. This yields the desired equivalence. □

Remark 9.20. Set notation as in the proof of Lemma 9.19, but assume in addition that A is sheafy. By Proposition 3.10 we have $H^i(X, \mathcal{O}) = 0$ for $i > 0$. Now taking cohomology in (9.19.1) shows that for $i > 0$, we have an exact sequence

$$0 \to \mathrm{coker}(H^i(X, \nu_*\mathcal{O}^+_{X_{\mathrm{v}}}) \to H^i(X, \nu_*\mathcal{O}_{X_{\mathrm{v}}})) \to H^{i+1}(X, \mathcal{O}^+)_{\mathrm{tors}} \to H^{i+1}(X, \nu_*\mathcal{O}^+_{X_{\mathrm{v}}})_{\mathrm{tors}} \to 0. \tag{9.20.1}$$

It follows that for $i > 1$, $H^i(X, \mathcal{O}^+)$ is uniformly torsion if and only if $H^{i-1}(X, \nu_*\mathcal{O}_{X_{\mathrm{v}}}) = 0$ and $H^i(X, \nu_*\mathcal{O}^+_{X_{\mathrm{v}}})$ is uniformly torsion.

Keep in mind however that we do not have a direct comparison of $H^i(X, \nu_* \mathcal{O}^+_{X_v})_{\mathrm{tors}}$ with $H^i(X_v, \mathcal{O}^+)_{\mathrm{tors}}$ due to the possible contribution of $R^j \nu_* \mathcal{O}^+_{X_v}$ for $j > 1$. We do not control the latter even when X is a point (Remark 9.17).

10. Affinoid algebras revisited

We next determine which affinoid algebras over mixed-characteristic nonarchimedean fields have the weakly sousperfectoid property. We do not attempt to test for the sousperfectoid property due to complications already occurring for nonarchimedean fields; see Remark 10.8.

In the process, we give a partial answer to Problem 6.15 in mixed characteristics; note that while this statement makes no reference to anything other than affinoid algebras, our proof requires perfectoid algebras, and in particular some results of [30] extending the Ax–Sen theorem (Proposition 9.15). Somewhat vexingly, this approach limits us to handling the mixed-characteristic case of Problem 6.15; it may be possible to handle the case of positive characteristic using similar techniques, but Theorem 10.3 does not directly generalize (see Example 7.11).

Lemma 10.1. *Let A be an affinoid algebra over a nonarchimedean field K.*

(a) *If K is perfectoid, then A is i-strict for all $i > 0$.*
(b) *If K is of mixed characteristics, then A is 1-strict.*

Proof. In case (a), we have by [30, Theorem 8.6.2] (and the comparison of topologies given by Remark 9.4) that for all $i > 0$, $H^i(\mathrm{Spa}(A, A^+)_v, \mathcal{O})$ is a finite A-module and hence a complete A-module because A is noetherian [29, Remark 2.2.11].

In case (b), let K' be a completed algebraic closure of K and put $A' = A \widehat{\otimes}_K K'$. By Proposition 9.15, K is 1-strict. By this plus [29, Lemma 2.2.9],

$$H^1_{\mathrm{cont}}(G_K, K') \widehat{\otimes}_K A \cong H^1_{\mathrm{cont}}(G_K, A'),$$

from which it follows that $H^1_{\mathrm{cont}}(G_K, A')$ is separated and so $H^1_{\mathrm{cont}}(G_K, A'^{\circ})$ is uniformly torsion. By the previous paragraph, $H^1(\mathrm{Spa}(A', A'^{\circ})_v, \mathcal{O}^+)$ is uniformly torsion. By the Hochschild–Serre spectral sequence, we have an exact sequence

$$0 \to H^1_{\mathrm{cont}}(G_K, A'^{\circ}) \to H^1(\mathrm{Spa}(A, A^{\circ})_v, \mathcal{O}^+) \to H^1(\mathrm{Spa}(A', A'^{\circ})_v, \mathcal{O}^+)^{G_K};$$

this yields the desired result. □

Lemma 10.2. *Let A be an affinoid algebra over a nonarchimedean field (of any characteristic). Let M be a finitely generated A°-module. Then any A°-submodule N of M is* almost finitely generated*: for every topologically nilpotent unit x of A, there exists a finitely generated submodule P of N containing xN.*

Proof. See [31, Satz 5.1]. □

Theorem 10.3. *Let A be a reduced (hence uniform) affinoid algebra over a nonarchimedean field of mixed characteristics. Then the following conditions are equivalent.*

(a) *The ring A is weakly sousperfectoid. (By Corollary 8.10, this holds if A is sousperfectoid.)*
(b) *The ring A is seminormal.*
(c) *The ring A is v-complete.*
(d) *The A°-module $H^1(\mathrm{Spa}(A, A^{\circ}), \mathcal{O}^+)$ is uniformly torsion.*

In particular, by (b) plus Lemma 6.4, all of these properties (on affinoid algebras) are of étale-local nature.

Proof. By Lemma 8.6, (a) implies (d). By Proposition 6.7, (d) implies (b). By the geometric Ax–Sen–Tate theorem [30, Theorem 8.2.3], (b) implies (c).

It thus remains to check that (c) implies (a). Promote A to a Banach ring. Let $A \to B$ be a faithfully finite étale morphism; our goal is to define a splitting $B \to A$ of bounded operator norm. Apply Proposition 2.14 to construct a faithfully profinite étale morphism $A \to A'$ with A' perfectoid. Put

$$B' := B \otimes_A A', \qquad A'' := C^1_{A'/A}, \qquad B'' := C^1_{B'/B}.$$

By Lemma 8.9, for any $c_1 > 1$ we may choose a splitting $\pi \colon B' \to A'$ of operator norm at most c_1. By base extension, we obtain two splittings $\pi_0, \pi_1 \colon B'' \to A''$. The difference $\pi_0 - \pi_1$ gives an element of $\mathrm{Hom}_{A''}(B''/A'', A'') = A'' \otimes_A \mathrm{Hom}_A(B/A, A)$; the class of this element in $\mathrm{coker}(d^0) \otimes_A \mathrm{Hom}_A(B/A, A)$ does not depend on the choice of π. Since one possible choice would be the base extension of a splitting $B \to A$, it follows that $\pi_0 - \pi_1 \in \mathrm{image}(d^0) \otimes_A \mathrm{Hom}_A(B/A, A) = \mathrm{Hom}_A(B/A, \mathrm{image}(d^0))$.

Let M be the image of B° in B/A. By Lemma 10.2, $\mathrm{Hom}(M, A^\circ)$ is an almost finitely generated A°-module; for any $c_2 > 1$, we can thus write $\pi_0 - \pi_1$ as a finite sum $\sum_i a_i'' \psi_i$ in which $\psi_i \in \mathrm{Hom}(M, A^\circ)$ and $a_i'' \in \mathrm{image}(d^0)$ has norm at most $c_1 c_2$.

By Lemma 10.1, there exists $c_3 > 1$ (independent of B) such that each a_i'' can be lifted to some $a_i' \in A'$ of norm at most $c_1 c_2 c_3$. Put $\psi := \sum_i a_i' \psi_i$; this is an A'-linear map from B'/A' to A' of operator norm at most $c_1 c_2 c_3$. Viewing ψ as a map $B' \to A'$, we may form the difference $\pi - \psi$; this map carries B/A into $\ker(d^0) = A$ thanks to (c). We thus deduce that $c_{B/A} \leq c_1 c_2 c_3$, and so (a) holds as desired. $\square$

Corollary 10.4. *For A as in Theorem 10.3, the v-completion of A is equal to the seminormalization of A.*

Example 10.5. As in Example 6.11, put $A = \mathbb{Q}_p\langle y, z\rangle/(y^3 - z^2)$. By Theorem 10.3, A is not weakly sousperfectoid. Note that $H^0(\mathrm{Spa}(A, A^\circ)_{\mathrm{v}}, \mathcal{O}) = \mathbb{Q}_p\langle x\rangle$ where $y = x^2, z = x^3$.

This example makes Remark 7.2 explicit: whereas $\mathbb{Q}_p\langle y, z\rangle$ is sousperfectoid, it cannot admit a perfectoid frame $\mathbb{Q}_p\langle y, z\rangle \overset{\sigma}{\leftrightarrows} B$ that admits a base extension to A. That is because the uniform completion of $B \otimes_{\mathbb{Q}_p\langle y,z\rangle} A$ receives a morphism with dense image from the perfectoid ring B, and therefore is itself perfectoid [30, Theorem 3.3.18(ii)]; the existence of a frame as described would then have the untenable consequence that A is sousperfectoid.

Remark 10.6. In Example 10.5, we can write $A = A_1 \widehat{\otimes}_{A_0} A_2$ for

$$A_0 = \mathbb{Q}_p\langle x, y, z\rangle, \quad A_1 = \mathbb{Q}_p\langle x, y, z\rangle/(x), \qquad A_2 = \mathbb{Q}_p\langle x, y, z\rangle/(y^3 - z^2 - x).$$

Since A_0, A_1, A_2 are all smooth affinoid algebras over $\mathbb{Q}_p$, they are all weakly sousperfectoid by Theorem 10.3. This shows the category of weakly sousperfectoid Huber rings is not closed under the formation of completed tensor products even in cases where the completed tensor product is uniform.

However, if A is a *perfectoid* Huber ring and B, C are sousperfectoid (resp. weakly sousperfectoid) Huber rings over A, then $B \widehat{\otimes}_A C$ is again sousperfectoid (resp. weakly sousperfectoid). This follows from the fact that the completed tensor product of two perfectoid rings over a third one is again perfectoid (e.g., see [30, Theorem 3.3.13]); we leave further details to the reader.

We mention the following variant of Example 10.5 in the context of perfectoid spaces.

Example 10.7. Suppose $p > 2$. Let K be a perfectoid field of characteristic 0 and put

$$A_0 := K\langle T^{p^{-\infty}} \rangle, \qquad A := A_0[T^{1/2}], \qquad A' := K\langle (T^{1/2})^{p^{-\infty}} \rangle.$$

The ring A, topologized as a finite free A_0-module, is a closed subring of A' and hence uniform. By Corollary 4.7 and Corollary 7.4, A is also sheafy (see [28, Exercise 2.5.8] for an alternate argument in the equal-characteristic case). However, A is evidently not seminormal. The latter conclusion may also be seen directly from the definition: the map $A \to A'$ induces an isomorphism $\mathcal{M}(A') \to \mathcal{M}(A)$, from which it follows that $H^0(\mathrm{Spa}(A, A^+)_{\mathrm{v}}, \mathcal{O})$ equals A' rather than A.

Remark 10.8. Note that Theorem 10.3 does not give a criterion for testing whether an affinoid algebra A over a nonarchimedean field K of mixed characteristics is sousperfectoid. This is due to difficulties involving uncountable inverse limits (see Remark 7.7 and Remark 7.8), which mean in particular that we do not even know whether K itself is sousperfectoid. One may hope to prove that A is sousperfectoid if and only if K is sousperfectoid and A is seminormal, but we did not attempt to establish this.

11. Diamantine rings

For general Huber rings, we do not know whether either the sousperfectoid or weakly sousperfectoid properties is of local nature. To remedy this, we formulate one more sheafiness condition.

Definition 11.1. Let A be a Huber ring over $\mathbb{Q}_p$ (that is, a Huber ring in which p is a topologically nilpotent unit). We say that A is *diamantine* if A is plus-sheafy and v-complete. Any such ring is uniform and seminormal (by Lemma 9.10), strongly sheafy (by Corollary 6.26), and 1-strict (by Lemma 9.19). In fact, if A is plus-sheafy and uniform, then it is v-complete by Lemma 6.18(c) and Lemma 9.19, and hence diamantine.

Remark 11.2. Because a diamantine Huber ring is required to be v-complete, the v-topology on diamantine spaces is subcanonical. This implies that the functor from diamantine spaces over $\mathbb{Q}_p$ to Scholze's category of *diamonds* (i.e., pro-étale sheaves on the category of perfectoid spaces of characteristic p which are locally the quotients of perfectoid spaces by pro-étale equivalence relations) is fully faithful, and explains the choice of terminology here.

Proposition 11.3. *Let A be a Huber ring over $\mathbb{Q}_p$.*

(a) *If A is perfectoid, then A is diamantine.*
(b) *If A is a nonarchimedean field, then A is diamantine.*

Proof. In case (a), the ring A is plus-sheafy by Proposition 6.14 and v-complete by Theorem 9.3. In case (b), the ring A is plus-sheafy trivially and v-complete by Proposition 9.15. $\square$

Lemma 11.4. *Let A be a Banach ring in which p is topologically nilpotent.*

(a) *If A is weakly sousperfectoid (which holds by Corollary 8.10 if A is sousperfectoid), then A is v-complete.*

(b) *For all* $0 < s \leq r$*, the v-completion of* $A\langle T/r\rangle$ *(resp.* $A\langle s/T, T/r\rangle$*) is equal to* $\check{A}\langle T/r\rangle$ *(resp.* $\check{A}\langle s/T, T/r\rangle$*).*

Proof. Assertion (a) follows from Lemma 8.4(a). For (b), we treat the case of $A\langle T\rangle$ in detail, the other cases being similar. Let $A \to A'$ be a v-covering with A' perfectoid. By Lemma 7.3, $A'\langle T\rangle$ is sousperfectoid, and so by (a) is v-complete. We may thus identify the v-completion of $A\langle T\rangle$ with the equalizer of the two maps $A'\langle T\rangle \to C^1_{A'/A}\langle T\rangle$, which is equal to $\check{A}\langle T\rangle$. □

Corollary 11.5. *If* A *is a diamantine Banach ring, then so are the weighted Tate algebras* $A\langle T/r\rangle$ *and* $A\langle s/T, T/r\rangle$ *for all* $0 < s \leq r$*. In particular, if* A *is a diamantine Huber ring, then so are* $A\langle T\rangle$ *and* $A\langle T^{\pm}\rangle$*.*

Proof. Let B be one of the rings in question. By Lemma 11.4, B is v-complete; by Lemma 6.25, it is also plus-sheafy. □

Definition 11.6. For M a Banach module over a Huber ring A, let $M\langle T\rangle$ be the set of formal sums $\sum_{n=0}^{\infty} m_n T^n$ in which $\{m_n\}$ is a null sequence in M. This module may be identified with $M\widehat{\otimes}_A A\langle T\rangle$.

Lemma 11.7. *Let* A *be a diamantine Huber ring. Then for any* $f \in A$*,* $B := A\langle f^{-1}\rangle$ *is also diamantine.*

Proof. Extend $A \to B$ to a rational localization $(A, A^+) \to (B, B^+)$ of Huber pairs. By Lemma 6.18, B is again plus-sheafy; it thus remains to check that $B = H^0(\mathrm{Spa}(B, B^+)_{\mathrm{v}}, \mathcal{O})$.

Apply Proposition 2.14 to construct a faithfully profinite étale morphism $A \to A'$ with A' perfectoid. Let M_0, M_1 be the kernel and cokernel of $d^0 \colon C^0_{A'/A}\langle T\rangle \to C^1_{A'/A}\langle T\rangle$. Since A is 1-strict by Lemma 9.19, $M_0 = A\langle T\rangle$ and M_1 is a Banach module over A.

In the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C^0_{A'/A}\langle T\rangle \arrow[r, "\times(1-fT)"] \arrow[d, "d^0"] & C^0_{A'/A}\langle T\rangle \arrow[r] \arrow[d, "d^0"] & C^0_{A'/A}\langle f^{-1}\rangle \arrow[r] \arrow[d, "d^0"] & 0 \\
0 \arrow[r] & C^1_{A'/A}\langle T\rangle \arrow[r, "\times(1-fT)"] & C^1_{A'/A}\langle T\rangle \arrow[r] & C^1_{A'/A}\langle f^{-1}\rangle \arrow[r] & 0
\end{tikzcd} \tag{11.7.1}$$

the rows are strict exact by Lemma 3.7(a). By [30, Remark 1.2.8], the map $M_1\langle T\rangle \overset{\times(1-fT)}{\to} M_1\langle T\rangle$ is injective. We thus have an exact sequence

$$0 \to A\langle T\rangle \overset{\times(1-fT)}{\to} A\langle T\rangle \to H^0(\mathrm{Spa}(B, B^+)_{\mathrm{v}}, \mathcal{O}) \to 0$$

which, by Lemma 3.7(a) applied to A, implies that $A\langle f^{-1}\rangle \to H^0(\mathrm{Spa}(B, B^+)_{\mathrm{v}}, \mathcal{O})$ is an isomorphism. □

Corollary 11.8. *Let* (A, A^+) *be a diamantine Huber pair. Then* $\mathrm{Spa}(A, A^+)$ *admits a neighborhood basis consisting of rational subspaces* $\mathrm{Spa}(B, B^+)$ *for which* B *is diamantine.*

Proof. This follows from Lemma 11.7 via [30, Lemma 3.7.8]. □

Lemma 11.9. *Any diamantine Huber ring is stably uniform.*

Proof. By Lemma 9.10, any diamantine ring is uniform. For any rational localization $(A, A^+) \to (B, B^+)$ we have $B = A\langle f^{-1}\rangle\langle f_1, \dots, f_n\rangle$ for some $f \in A$ and some $f_1, \dots, f_n \in$

$A\langle f^{-1}\rangle$. In light of Lemma 11.7, we know that $A\langle f^{-1}\rangle$ is diamantine; hence to prove that B is uniform, we may reduce to the case where $B = A\langle f_1, \dots, f_n\rangle$ for some $f_1, \dots, f_n \in A$.

We will prove that for any disjoint subsets S, T of $\{1, \dots, n\}$, the ring

$$A_{S,T} := A\langle f_i : i \in S\rangle\langle f_j^{\pm} : j \in T\rangle$$

is uniform; the case $S = \{1, \dots, n\}, T = \emptyset$ will prove the desired result. We proceed by induction on $\#S$, the case $S = \emptyset$ (for T arbitrary) being a consequence of Lemma 11.7. For $S \neq \emptyset$, write S as the disjoint union of S' and $\{i\}$ for some i. By the induction hypothesis, $A_{S',T\cup\{i\}}$ is uniform; by Lemma 3.7(c), this implies that $A_{S,T}$ is uniform. □

Theorem 11.10. *Let (A, A^+) be a diamantine Huber pair. Then for any rational localization $(A, A^+) \to (B, B^+)$, B is also diamantine.*

Proof. By Lemma 6.18, B is plus-sheafy; it thus remains to check that the map $B \to H^0(\mathrm{Spa}(B, B^+)_{\mathrm{v}}, \mathcal{O})$ is an isomorphism. By Lemma 11.9, B is uniform and so the map $B \to H^0(\mathrm{Spa}(B, B^+)_{\mathrm{v}}, \mathcal{O})$ is injective. By Corollary 11.8, $\mathrm{Spa}(A, A^+)$ admits a neighborhood basis consisting of rational subspaces satisfying the conclusion of the theorem; consequently, any global section of $\mathcal{O}$ on $\mathrm{Spa}(B, B^+)_{\mathrm{v}}$ arises from a global section on $\mathrm{Spa}(B, B^+)$. By Lemma 11.9 again, B is sheafy; it follows that $B \to H^0(\mathrm{Spa}(B, B^+)_{\mathrm{v}}, \mathcal{O})$ is surjective. □

Corollary 11.11. *The diamantine property of Huber rings is of local nature.*

Proof. Of the conditions of Definition 3.17, (i) is Theorem 11.10. To check (ii), note that the plus-sheafy condition is of local nature (Theorem 6.21); the rest is straightforward. □

Remark 11.12. In [29, Definition 9.2.12], a Huber ring A was defined to be *pro-sheafy* if A is uniform and sheafy and $\mathcal{O}_X \to \nu_* \mathcal{O}_{X_{\mathrm{v}}}$ is an isomorphism. Any diamantine ring has this property, by the discussion of Definition 11.1 (for the uniform and sheafy conditions) and Theorem 11.10 (for the last condition).

Lemma 11.13. *Let $f : A \to B$ be a finite étale morphism of Huber rings.*

(a) *If A is diamantine, then so is B.*
(b) *If f is faithfully flat and B is diamantine, then so is A.*

Proof. Part (a) follows directly from Theorem 6.21 and Lemma 9.8.

To prove (b), it similarly suffices to check that $A = H^0(\mathrm{Spa}(A, A^\circ)_{\mathrm{v}}, \mathcal{O})$. This is easily seen by computing within B. □

Theorem 11.14. *The diamantine property of Huber rings is of étale-local nature.*

Proof. We establish étale-local nature using the criteria of Proposition 5.5: (i) is Corollary 11.11, and (ii) and (iii) are Lemma 11.13, and (iv) is Lemma 6.18. □

Lemma 11.15. *If A is plus-sheafy, then so is $\check{A}$. Consequently, by Lemma 9.8, the functor $A \mapsto \check{A}$ defines an adjunction between plus-sheafy Huber rings in which p is topologically nilpotent and diamantine rings.*

Proof. Put $X = \mathrm{Spa}(A, A^\circ)$ and $\check{X} = \mathrm{Spa}(\check{A}, \check{A}^\circ)$. Fix a topologically nilpotent unit ϖ of A. By Lemma 9.8, the natural map $\check{X} \to X$ is a homeomorphism. If we use this map to identify the two spaces, then the natural morphisms of sheaves

$$\nu_* \mathcal{O}^+_{X_{\mathrm{v}}} \to \nu_* \mathcal{O}^+_{\check{X}_{\mathrm{v}}}, \qquad \nu_* \mathcal{O}_{X_{\mathrm{v}}} \to \nu_* \mathcal{O}_{\check{X}_{\mathrm{v}}}$$

are isomorphisms. Meanwhile, by Proposition 9.15(a), the maps

$$\mathcal{O}^+_X \to \nu_* \mathcal{O}^+_{X_v}, \qquad \mathcal{O}^+_{\check{X}} \to \nu_* \mathcal{O}^+_{\check{X}_v}$$

are injective and their cokernels are killed by ϖ, and likewise after quotienting everything modulo ϖ^n for any positive integer n. This allows us to transfer conditions (ii) and (iii) of Definition 6.8 from A to $\check{A}$. Since evidently $H^0(\check{X}, \mathcal{O}) = \check{A}$, we deduce that $\check{A}$ is plus-sheafy. □

Remark 11.16. Continuing with Remark 3.19, let A be a sheafy Huber ring which admits a rational covering by perfectoid rings. By Corollary 11.11, A is diamantine; in particular, $H^1(\mathrm{Spa}(A, A^+), \mathcal{O}^+)$ is uniformly torsion. However, this is not enough to deduce that A is perfectoid. It would suffice to show that there exists a topologically nilpotent unit $\varpi \in A^+$ which both divides p and annihilates $H^1(\mathrm{Spa}(A, A^+), \mathcal{O}^+)$, as this would imply the existence of approximate p-th roots in $A^+/(\varpi)$.

Lemma 11.17. *Let A be a diamantine Huber ring. Let G be a finite group acting on A. Then A^G is also diamantine.*

Proof. Choose a topologically nilpotent unit ϖ in A^G (e.g., by taking the norm of a topologically nilpotent unit in A). By Lemma 12.5, A^G is v-complete. To prove that it is plus-sheafy, we must prove that for each positive integer i, there exists a positive integer c such that for each positive integer n, $H^i(\mathrm{Spa}(A^G, A^{G\circ}), \mathcal{O}^+/\varpi^n)$ is killed by ϖ^c. In fact, by the usual spectral sequence [44, Tag 01ES] it is equivalent to prove the corresponding statement with sheaf cohomology replaced by Čech cohomology. To see this, start with a cocycle on $\mathrm{Spa}(A^G, A^{G\circ})$, trivialize it on $\mathrm{Spa}(A, A^\circ)$, then use the trace to return to A^G. □

Theorem 11.18. *Let $(A, A^+) \to (B, B^+)$ be a morphism of Huber rings which is smooth in the naïve sense. If A is diamantine, then so is B. In particular, any smooth affinoid algebra over a nonarchimedean field is diamantine.*

Proof. As per Definition 5.11, the first assertion follows by combining Corollary 11.5 with Theorem 11.14. The second assertion then follows from Proposition 11.3(b). □

12. Conjectures

In this section, we present a series of optimistic conjectures. We begin with the most interesting (and perhaps most plausible) of them.

Conjecture 12.1. *Let A be a Huber ring in which p is topologically nilpotent. Suppose that for every point $x \in X = \mathrm{Spa}(A, A^\circ)$, the completed residue field $K(x)$ at x is a perfectoid field. Then $\check{A}$ is a perfectoid ring.*

The condition on residue fields is not enough to guarantee that A itself is perfectoid, as demonstrated by Example 10.7.

Proposition 12.2. *Conjecture 12.1 implies that perfectoidness is a local property.*

Proof. Suppose that A is sheafy and that $X = \mathrm{Spa}(A, A^+)$ has an open covering by affinoid perfectoid subsets. Then A is diamantine, and in particular v-complete, so $A \cong \check{A}$. But the completed residue fields of X are evidently perfectoid, so Conjecture 12.1 implies that $\check{A}$ is a perfectoid ring. Thus $A \cong \check{A}$ is perfectoid. □

Conjecture 12.1 would also give a new proof of Bhatt-Scholze's theorem that if A is a perfectoid ring and $A \to B$ is an integral ring map, then B has a canonical "perfectoidization", cf. [11, Theorem 1.16(1)]. To see this, observe that integrality implies that any completed residue field L of $\operatorname{Spa}(B, B^\circ)$ is the completion of an algebraic extension of some completed residue field K of $\operatorname{Spa}(A, A^\circ)$; since K is perfectoid by assumption, L is also perfectoid. Conjecture 12.1 then applies.

One of the most intriguing hopes for diamantine rings is that they might give a handle on some of the thorny issues around inverse limits in p-adic geometry.

Conjecture 12.3. *Let A be a diamantine Huber ring. Then for any profinite étale morphism $A \to B$, B is also diamantine.*

Conjecture 12.4. *Let A be a perfectoid Huber ring over $\mathbb{Q}_p$. Let G be a p-adic Lie group acting continuously on A. Suppose in addition that the action map*

$$G \times \operatorname{Spa}(A, A^\circ) \to \operatorname{Spa}(A, A^\circ) \times_{\operatorname{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)} \operatorname{Spa}(A, A^\circ)$$

is a monomorphism. Then A^G is diamantine.

Lemma 12.5. *Let A be a diamantine ring. Let G be a profinite group acting continuously on A. Then $A^G \cong H^0(\operatorname{Spa}(A^G, A^{G\circ})_{\mathrm{v}}, \mathcal{O})$.*

Proof. Straightforward. □

Remark 12.6. Let A be a diamantine ring. Let G be a profinite group acting continuously on A such that for each open subgroup H of G, the map $A^G \to A^H$ is finite étale. We then have a canonical isomorphism

$$H^i_{\mathrm{cont}}(G, A) \cong H^i(\operatorname{Spa}(A^G, A^{G\circ})_{\mathrm{v}}, \mathcal{O})$$

so by Lemma 12.5, A^G is diamantine if and only if $H^i_{\mathrm{cont}}(G, A)$ is complete for each positive integer i.

By contrast, if G is a profinite group acting continuously on A without further conditions, one cannot use the $H^i_{\mathrm{cont}}(G, A)$'s as a reliable indicator of whether A^G is diamantine, and in fact the group $H^i_{\mathrm{cont}}(G, A)$ can be rather unpleasant, as the following example shows.

Example 12.7. Let $\mathbb{C}_p$ be a completed algebraic closure of $\mathbb{Q}_p$. Let $\overline{p}$ be the element of $\mathbb{C}_p^\flat$ corresponding to some sequence $(p^{p^{-n}})_n$. Put $A := \mathbb{C}_p\langle T_i^{p^{-\infty}} : i = 0, 1, \dots \rangle$ and let $\overline{T}_i$ be the element of $A^\flat$ corresponding to the sequence $(T_i^{p^{-n}})_n$. Let G be the group $\mathbb{Z}_p$ acting $\mathbb{C}_p$-linearly on A via the substitution

$$T_{2i} \mapsto T_{2i}, \qquad T_{2i+1} \mapsto \sharp(\overline{T}_{2i+1} + \overline{p}^i \overline{T}_{2i}) \qquad (i = 0, 1, \dots).$$

Then the principal crossed homomorphism corresponding to T_{2i+1} has supremum norm p^{-c_i} for

$$c_i = \min\{j + p^{-j} i : j = 0, 1, \dots\}.$$

Since $c_i \to \infty$ as $i \to \infty$, the map from A to crossed homomorphisms is not strict, so $H^1_{\mathrm{cont}}(G, A)$ is not complete. However, $A^G = \mathbb{C}_p\langle T_{2i}^{p^{-\infty}} \rangle$ is perfectoid and hence diamantine.

We would also like to state a special case of both Conjecture 12.1 and Conjecture 12.3, which we believe is within reach. (For a closely related problem, see [36, Conjecture 4.1.27].)

Conjecture 12.8. *Let $X_i = \mathrm{Spa}(A_i, A_i^\circ)$ be a tower of smooth affinoid adic spaces over a mixed-characteristic perfectoid field K with finite étale transition maps, and suppose the tower is* deeply ramified *in the sense of* [18]. *Then the ring $\widehat{\varinjlim_i A_i}$ is diamantine.*

13. Summary of containments

We conclude by summarizing our results with a diagram of containments among various classes of Huber rings (or equivalently, implications among various properties of Huber rings); see Figure 1. In each case, we have given an internal or external reference for the containment; this includes cases which are true by definition or vacuous, in which case we enclose the reference in parentheses. (Note that in some cases, the implication must be interpreted within the class of Huber rings over $\mathbb{Q}_p$ because one of the properties involved is only defined in that context.) The absence of an indicated relation is not meant to imply non-containment in either direction, but in some cases this are known; see below.

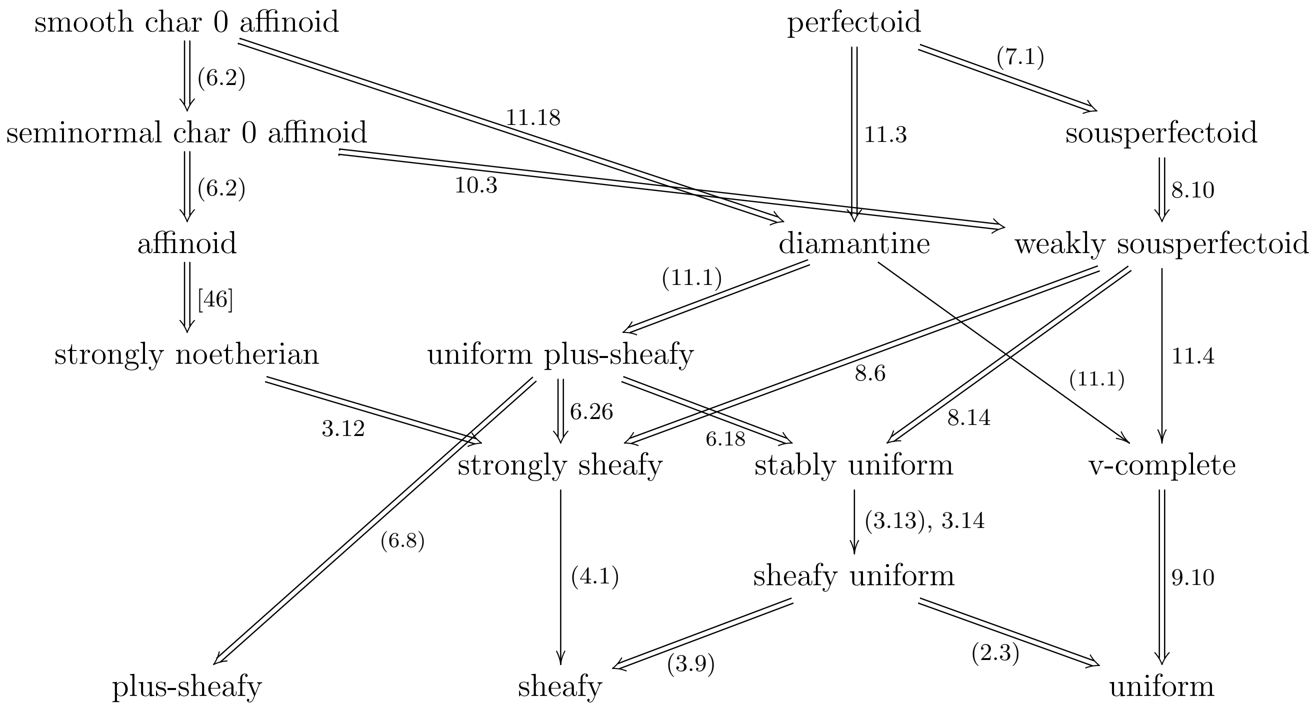

Figure 1. Containments among various classes of Huber rings.

We use a double arrow to indicate containments which are known to be strict, for the following reasons. In the process we also establish some non-containments that are not indicated in the diagram.

- The affinoid algebra $\mathbb{Q}_p\langle T_1, T_2, T_3, T_4\rangle/(T_1T_2 - T_3T_4)$ is seminormal but not smooth.
- For K a perfectoid field, the ring $K\langle T^{p^{-\infty}}\rangle$ is perfectoid but not strongly noetherian.
- The field $\mathbb{Q}_p$ is sousperfectoid, diamantine, and a smooth characteristic-0 affinoid, but not perfectoid.

- Any nonarchimedean field is weakly sousperfectoid by Lemma 8.8, but a nonarchimedean field can fail to be sousperfectoid for reasons of cardinality (see Remark 7.8).
- Example 6.11 is affinoid and stably uniform, but not v-complete (Theorem 10.3).
- The rings considered in [26] are strongly noetherian but not affinoid. They are probably also sousperfectoid and diamantine, but we did not check this. They cannot be perfectoid because a noetherian perfectoid ring is a finite product of perfectoid fields [28, Corollary 2.9.3].
- Example 6.10 is sheafy and plus-sheafy but not uniform; Example 6.28 is plus-sheafy but not sheafy; and the examples of [13, Proposition 18] and [33, Theorem 4.6] are uniform but not sheafy.

This leaves the following containments for which strictness remains an open question.

- Weakly sousperfectoid to v-complete.
- Diamantine to v-complete.
- Strongly sheafy to sheafy.
- Stably uniform to sheafy uniform.

References

[1] Y. André, La "lemme d'Abhyankar" perfectoïde, *Publ. Math. IHÉS* **127** (2018), 1–70.

[2] Y. André, La conjecture du facteur direct, *Publ. Math. IHÉS* **127** (2018), 71–93.

[3] J. Ax, Zeros of polynomials over local fields — the Galois action, *J. Algebra* **15** (1970), 417–428.

[4] W. Bartenwerfer, Die erste "metrische" Kohomologiegruppe glatter affinoider Räume, *Nederl. Akad. Wetensch. Proc. Ser. A* **40** (1978), no. 1, 1–14.

[5] W. Bartenwerfer, Die höheren metrischen Kohomologiegruppen affinoider Räume, *Math. Ann.* **241** (1979), 11–34.

[6] V. Berkovich, *Spectral Theory and Analytic Geometry over Non-Archimedean Fields*, Surveys and Monographs 33, Amer. Math. Soc., Providence, 1990.

[7] V.G. Berkovich, Étale cohomology for non-Archimedean analytic spaces, *Publ. Math. IHÉS* **78** (1993), 5–161.

[8] B. Bhatt, Almost direct summands, *Nagoya Math. J.* **214** (2014), 195–204.

[9] B. Bhatt, On the direct summand conjecture and its derived variant, *Invent. Math.* **212** (2018), 297–317.

[10] B. Bhatt, Torsion completions are bounded, *J. Pure Applied Alg.* **223** (2019), 1940–1945.

[11] B. Bhatt and P. Scholze, Prisms and prismatic cohomology, *Ann. Math.* **196** (2022), 1135–1275.

[12] S. Bosch, U. Güntzer, and R. Remmert, *Non-Archimedean Analysis*, Grundlehren der Math. Wiss. 261, Springer-Verlag, Berlin, 1984.

[13] K. Buzzard and A. Verberkmoes, Stably uniform affinoids are sheafy, *J. reine angew. Math.* **740** (2018), 25–39.

[14] X. Caruso, T. Vaccon, and T. Verron, Gröbner bases over Tate algebras, *ISSAC'19—Proceedings of the 2019 ACM International Symposium on Symbolic and Algebraic Computation*, ACM, New York, 2019, 74–81.

[15] J.-M. Fontaine, Perfectoïdes, presque pureté et monodromie-poids (d'après Peter Scholze), Séminaire Bourbaki, volume 2011/2012, *Astérisque* **352** (2013).

[16] K. Fujiwara, O. Gabber, and F. Kato, On Hausdorff completions of commutative rings in rigid geometry, *J. Algebra* **332** (2011), 293–321.

[17] K. Fujiwara and F. Kato, *Foundations of Rigid Geometry I*, EMS Monographs in Mathematics, European Math. Soc., Zürich, 2018.

[18] O. Gabber and L. Ramero, *Almost Ring Theory*, Lecture Notes in Math. 1800, Springer, New York, 2003.

[19] S. Greco and C. Traverso, On seminormal schemes, *Compos. Math.* **40** (1980), 325–365.

[20] T. Henkel, An Open Mapping Theorem for rings with a zero sequence of units, arXiv:1407.5647v2 (2014).
[21] M. Hochster, Prime ideal structure in commutative rings, *Trans. Amer. Math. Soc.* **142** (1969), 43–60.
[22] R. Huber, Continuous valuations, *Math. Z.* **212** (1993), 455–477.
[23] R. Huber, A generalization of formal schemes and rigid analytic varieties, *Math. Z.* **217** (1994), 513–551.
[24] R. Huber, *Étale Cohomology of Rigid Analytic Varieties and Adic Spaces*, Aspects of Mathematics, E30, Friedr. Vieweg & Sohn, Braunschweig, 1996.
[25] K.S. Kedlaya, *p-adic Differential Equations*, Cambridge Univ. Press, Cambridge, 2010.
[26] K.S. Kedlaya, Noetherian properties of Fargues-Fontaine curves, *Int. Math. Res. Notices* (2015), article ID rnv227.
[27] K.S. Kedlaya, Reified valuations and adic spaces, *Res. Num. Theory* **1** (2015), 1–42.
[28] K.S. Kedlaya, Sheaves, shtukas, and stacks, in *Perfectoid Spaces: Lectures from the 2017 Arizona Winter School*, Mathematical Surveys and Monographs 242, American Mathematical Society, 2019, 58–205.
[29] K.S. Kedlaya and R. Liu, Relative p-adic Hodge theory: Foundations, *Astérisque* **371** (2015); errata, [30, Appendix].
[30] K.S. Kedlaya and R. Liu, Relative p-adic Hodge theory, II: Imperfect period rings, arXiv:1602.06899v3 (2019).
[31] R. Kiehl, Der Endlichkeitssatz für eigentliche Abbildungen in der nichtarchimedische Funktionentheorie, *Invent. Math.* **2** (1967), 191–214.
[32] M. Matignon and M. Reversat, Sous-corps fermés d'un corps valué, *J. Algebra* **90** (1984), 491–515.
[33] T. Mihara, On Tate acyclicity and uniformity of Berkovich spectra and adic spectra, *Israel J. Math.* **216** (2016), 61–105.
[34] D. Mumford, *Abelian Varieties*, corrected reprint of the second (1974) edition, Hindustan Book Agency, New Delhi, 2008.
[35] M. Raynaud, Géométrie analytique rigide d'apres Tate, Kiehl, ..., *Bull. Soc. Math. France* **39/40** (1974), 319–327.
[36] J.E. Rodríguez Camargo, Geometric Sen theory over rigid analytic spaces, arXiv:2205.02016v3 (2023).
[37] P. Scholze, Perfectoid spaces, *Publ. Math. IHÉS* **116** (2012), 245–313.
[38] P. Scholze, p-adic Hodge theory for rigid analytic varieties, *Forum of Math. Pi* **1** (2013), doi:10.1017/fmp.2013.1; corrigendum, *ibid.* **4** (2016), doi:10.1017/fmp.2016.4.
[39] P. Scholze, Étale cohomology of diamonds, preprint (2017) available at `http://www.math.uni-bonn.de/people/scholze/`.
[40] P. Scholze and J. Weinstein, Moduli of p-divisible groups, *Cambridge J. Math.* **1** (2013), 145–237.
[41] P. Scholze and J. Weinstein, *Berkeley Lectures on p-adic Geometry*, Annals of Math Studies 207, Princeton University Press, 2020.
[42] S. Sen, On automorphisms of local fields, *Annals of Math.* **90** (1969), 33–46.
[43] K. Smith and W. Zhang, Frobenius splitting in commutative algebra, in *Commutative Algebra and Noncommutative Algebraic Geometry, Vol. I*, Math. Sci. Res. Inst. Publ. 67, Cambridge Univ. Press, New York, 2015, 291–345.
[44] The Stacks Project Authors, *Stacks Project*, `http://stacks.math.columbia.edu` (retrieved Feb 2020).
[45] R.G. Swan, On seminormality, *J. Algebra* **67** (1980), 210–229.
[46] J. Tate, Rigid analytic spaces, *Invent. Math.* **12** (1971), 257–289.
[47] C. Traverso, Seminormality and Picard group, *Ann. Scuola Norm. Sup. Pisa* **24** (1970), 585–595.
[48] M. van der Put, Cohomology on affinoid spaces, *Compos. Math.* **45** (1982), 165–198.
[49] J.-P. Wintenberger, Une généralisation du théorème du Tate–Sen–Ax, *C. R. Acad. Sci. Paris* **307** (1988), 63–65.