SPARSE ELIMINATION THEORY*

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1. Introduction.

Deciding the solvability of systems of polynomial equations is a fundamental problem in computer algebra. Most symbolic algorithms for this problem are based on *multivariate resultants* or *Gröbner bases*, and the study of these methods is an active area of current research. A key issue which has received comparatively little attention so far is that of *sparseness*. The significance of sparseness stems from the fact that polynomial systems arising in practise often have only few monomials appearing with non-zero coefficients.

In this article we present a new elimination theory which is custom-tailored for sparse systems of polynomials equations. Our approach is based on recent advances in combinatorics which were achieved by Gelf'and, Kapranov and Zelevinsky in the context of generalized hypergeometric functions [10],[11]. This line of research deals with *secondary polytopes* [3],[4],[13],[14], *$\mathcal{A}$-discriminants* [12],[13] and *$\mathcal{A}$-resultants* [15],[19]. A key ingredient from algebraic geometry is the theory of *toric varieties* [25],[30].

The objective of this paper is threefold. First, we wish to give a self-contained introduction to the above developments and illustrate their computational significance. Secondly, we derive several new results (mainly on Gröbner bases of toric varieties), and we give a new proof for the asymptotics of the $\mathcal{A}$-resultant [19, Theorem 5.3]. The results about Cayley-Koszul complexes on which the earlier proof was based are now replaced by elementary combinatorial arguments. Thirdly, we show that the $\mathcal{A}$-resultant can be computed in single-exponential time, and we present practical algorithms for doing so.

Let $\mathbf{C}[\mathbf{x}] := \mathbf{C}[x_1,\ldots,x_m]$ denote the polynomial ring in m complex variables. Each monomial $\mathbf{x}^a$ in $\mathbf{C}[\mathbf{x}]$ is identified with a lattice point $a \in \mathbf{Z}_+^m$. In order to model sparseness, we fix a finite set of lattice points $\mathcal{A} = \{a_1, a_2, \ldots, a_n\} \subset \mathbf{Z}_+^m$. We assume that the corresponding monomials are *quasi-homogeneous*, which means there exists a linear functional $h : \mathbf{Z}_+^m \to \mathbf{Q}$ with $h(a_1) = \ldots = h(a_n) = 1$. Let k be the rank of the sublattice of $\mathbf{Z}^m$ which is spanned by $\mathcal{A}$. We consider the following system of polynomial equations:

$$\begin{array}{rclcl} f_1(\mathbf{x}) & = & c_{11}\mathbf{x}^{a_1} + c_{12}\mathbf{x}^{a_2} + \ldots + c_{1n}\mathbf{x}^{a_n} & = & 0 \\ f_2(\mathbf{x}) & = & c_{21}\mathbf{x}^{a_1} + c_{22}\mathbf{x}^{a_2} + \ldots + c_{2n}\mathbf{x}^{a_n} & = & 0 \\ \ldots & & \ldots \quad \ldots \quad \ldots \quad \ldots & & \ldots \\ f_k(\mathbf{x}) & = & c_{k1}\mathbf{x}^{a_1} + c_{k2}\mathbf{x}^{a_2} + \ldots + c_{kn}\mathbf{x}^{a_n} & = & 0 \end{array} \qquad (1.1)$$

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Under which condition on the coefficient matrix (c_{ij}) does (1.1) have a "non-trivial" zero ?

Let $\mathcal{Z}_0(\mathcal{A})$ denote the set of all matrices (c_{ij}) in $\mathbf{C}^{k\times n}$ such that (1.1) has a zero $\mathbf{x} \in (\mathbf{C}^*)^m$ with <u>all</u> coordinates non-zero. The set $\mathcal{Z}_0(\mathcal{A})$ is usually not Zariski closed, and we need to consider its closure $\mathcal{Z}(\mathcal{A})$ in the affine space $\mathbf{C}^{k\times n}$.

The *Newton polytope* of the polynomials in (1.1) is the $(k-1)$-dimensional polytope $Q = conv(\mathcal{A})$. Let $Vol(\cdot)$ denote the unique volume form on the affine hull of Q which is normalized by requiring that the non-zero simplex volumes $Vol(conv(a_{i_1},\ldots,a_{i_k}))$, ~p12 $1 \le i_1 < \ldots < i_k \le n$, are relative prime integers. Our starting point is the following theorem, which is essentially due to Kushnirenko [20], but was first stated in this form in [19]. We will give a proof in Section 3.3. Note that (b) is a consequence of the classical *First Fundamental Theorem of Invariant Theory* (see e.g. [31]).

Theorem 1.1.

(a) *The variety $\mathcal{Z}(\mathcal{A})$ is a hypersurface in $\mathbf{C}^{k\times n}$, which is defined by a unique (up to sign) irreducible polynomial $\mathcal{R}_\mathcal{A}(c_{ij})$ with integer coefficients, called the $\mathcal{A}$-resultant.*

(b) *The $\mathcal{A}$-resultant $\mathcal{R}_\mathcal{A}$ can be expressed as a polynomial $\tilde{\mathcal{R}}_\mathcal{A}$ in the brackets*

$$[i_1 i_2 \ldots i_k] \quad := \quad det \begin{pmatrix} c_{1i_1} & c_{1i_2} & \ldots & c_{1i_k} \\ \vdots & \vdots & \ddots & \vdots \\ c_{ki_1} & c_{ki_2} & \ldots & c_{ki_k} \end{pmatrix} \qquad (1 \le i_1 < i_2 < \ldots < i_k \le n).$$

(c) *The degree of $\tilde{\mathcal{R}}_\mathcal{A}$ as a polynomial function in the brackets equals $Vol(Q)$.*

This article is organized as follows. Section 2 deals exclusively with examples of $\mathcal{A}$-resultants. For specific choices of $\mathcal{A}$ we obtain the *Sylvester resultant*, the *Bezout resultant*, the classical *multivariate resultant*, the *Dixon resultant* and the *hyperdeterminant*. In studying these examples, we will encounter remarkable connections between the bracket terms occurring in $\tilde{\mathcal{R}}_\mathcal{A}$ and *triangulations* of the Newton polytope Q.

In Section 3 we explain our polyhedral discoveries in a systematic fashion, and we give complete proofs for all results stated (including Theorem 1.1). To this end we first prove an asymptotic formula (Corollary 3.5) for the *Chow form* of an arbitrary projective variety in terms of its Gröbner bases. This formula is applied to the toric variety associated with $\mathcal{A}$, whose Chow form is seen to equal the $\mathcal{A}$-resultant. As an important technical tool we introduce the concept of *regular triangulations of integer monoids*.

In Section 4 we investigate the complexity of computing the $\mathcal{A}$-resultant, and we describe a practical algorithm for computing $\tilde{\mathcal{R}}_\mathcal{A}$ in small cases. We also introduce a perturbation technique for evaluating the multivariate resultant for given polynomials with rational coefficients. We close in Section 5 with a few suggestions for future research.

2. Examples of $\mathcal{A}$-resultants

In this section we present four classes of examples of $\mathcal{A}$-resultants for specific sets $\mathcal{A}$, and we relate the extreme terms of $\tilde{\mathcal{R}}_{\mathcal{A}}$ to the regular triangulations of $Q = conv(\mathcal{A})$.

2.1. *The Sylvester resultant.* Our first example is the resultant of two binary forms

$$
\begin{aligned}
f_1(x_1,x_2) &= c_{11}x_1^{n-1} + c_{12}x_1^{n-2}x_2 + c_{13}x_1^{n-3}x_2^2 + \ldots + c_{1n}x_2^{n-1}, \\
f_2(x_1,x_2) &= c_{21}x_1^{n-1} + c_{22}x_1^{n-2}x_2 + c_{23}x_1^{n-3}x_2^2 + \ldots + c_{2n}x_2^{n-1}
\end{aligned}
\tag{2.1}
$$

of degree $n-1$. Their monomials correspond to n equidistant points on the affine line:

$$
\mathcal{A} \;=\; \{(n-1,0),(n-2,1),(n-3,2),\ldots,(0,n-1)\} \;\subset\; \mathbf{Z}^2, \tag{2.2}
$$

and the Newton polytope $Q = conv(\mathcal{A})$ is a line segment of length $Vol(Q) = n-1$. Here the $\mathcal{A}$-resultant equals the determinant of the $(2n-2)\times(2n-2)$-Sylvester matrix

$$
\mathcal{R}_{\mathcal{A}}(c_{ij}) \;=\; det\begin{pmatrix}
c_{11} & c_{12} & c_{13} & \cdots & c_{1n} & 0 & 0 & \cdots & 0 \\
c_{21} & c_{22} & c_{23} & \cdots & c_{2n} & 0 & 0 & \cdots & 0 \\
0 & c_{11} & c_{12} & c_{13} & \cdots & c_{1n} & 0 & \cdots & 0 \\
0 & c_{21} & c_{22} & c_{23} & \cdots & c_{2n} & 0 & \cdots & 0 \\
0 & 0 & c_{11} & c_{12} & c_{13} & \cdots & c_{1n} & \cdots & 0 \\
0 & 0 & c_{21} & c_{22} & c_{23} & \cdots & c_{2n} & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & c_{11} & c_{12} & c_{13} & \cdots & c_{1n} \\
0 & 0 & \cdots & 0 & c_{21} & c_{22} & c_{23} & \cdots & c_{2n}
\end{pmatrix}. \tag{2.3}
$$

This determinant can be expanded as a homogeneous polynomial of degree $2\cdot Vol(Q) = 2n-2$ in the variables c_{ij}. For a complete combinatorial description of the monomials occurring in this expansion we refer to [14].

A more economical expansion of this determinant is the Laplace expansion with respect to adjacent pairs of rows. This gives a formula for $\mathcal{R}_{\mathcal{A}}$ as a homogeneous polynomial of degree $Vol(Q) = n-1$ in the brackets $[\,ij\,] = det\begin{pmatrix} c_{1i} & c_{1j} \\ c_{2i} & c_{2j}\end{pmatrix}$. An explicit such formula is the *Bezout resultant*; see e.g. [27, Lesson IX.84], [17, Corollary 5.1].

For instance, for $n=5$ the Bezout resultant equals

$$
\tilde{\mathcal{R}}_{\mathcal{A}}([\,ij\,]) \;=\; det\begin{pmatrix}
[12] & [13] & [14] & [15] \\
[13] & [14]+[23] & [24]+[15] & [25] \\
[14] & [24]+[15] & [25]+[34] & [35] \\
[15] & [25] & [35] & [45]
\end{pmatrix}. \tag{2.4}
$$

For $n=5$ the complete expansion of $\mathcal{R}_{\mathcal{A}}$ has 219 monomials of degree 8 in the c_{ij} while

n=3 $\tilde{\mathcal{R}}_{\mathcal{A}}([ij]) = \begin{vmatrix} [12] & [13] \\ [13] & [23] \end{vmatrix} = \mathcal{R}_{\mathcal{A}}(c_{ij})$

the expansion of the Bezout resultant $\tilde{\mathcal{R}}_{\mathcal{A}}$ has 36 bracket monomials of degree 4:

$$
\begin{aligned}
&-[12][14][35]^2 + [12][14][25][45] + [12][14][34][45] - [12][15]^2[45] + 2[12][15][25][35] \\
&+ [12][23][25][45] + \underline{[12][23][34][45]} - \underline{[12][23][35]^2} - 2[12][24][15][45] + 2[12][24][25][35] \\
&- \underline{[12][25]^3} - [12][25]^2[34] - \underline{[12][24]^2[45]} + 2[13][14][15][45] + 2[13][14][24][45] \\
&- 2[13][14][25][35] - 2[13][15]^2[35] + 2[13][15][25]^2 - 2[13][15][24][35] + 2[13][15][25][34] \\
&+ \underline{[13]^2[35]^2} - [13]^2[25][45] - \underline{[13]^2[34][45]} + 2[14]^2[15][35] - 3[14][15]^2[25] \\
&- 2[14][15][24][25] - [14][15]^2[34] + 2[14][15][23][35] - [14]^2[23][45] + [14]^2[25]^2 \\
&- \underline{[14]^3[45]} - [15]^2[23][25] - [15]^2[23][34] + 2[15]^3[24] + [15]^2[24]^2 + \underline{[15]^4}
\end{aligned}
$$

The eight underlined terms play the following special role. We define the *weight* of a bracket monomial $\prod[\,ij\,]$ to be the integer vector $\sum(e_i+e_j)$ where e_i denotes the i-th unit vector in $\mathbf{Z}^5$. Thus the weight of $\prod[\,ij\,]$ is its column degree with respect to the 2×5-matrix (c_{ij}). For instance, we have $weight([12][14][35]^2) = (2,1,2,1,2)$, $weight([12][23][34][45]) = (1,2,2,2,1)$, and $weight([14][15]^2[25]) = (3,1,0,1,3)$. Let $\Sigma(\mathcal{A})$ denote the convex hull in $\mathbf{R}^5$ of the set of all weights occurring in the $\mathcal{A}$-resultant $\mathcal{R}_{\mathcal{A}}$. Following [13], we call $\Sigma(\mathcal{A})$ the *secondary polytope*. (The "primary polytope" is the Newton polytope Q). In our example $\Sigma(\mathcal{A})$ is a 3-dimensional polytope combinatorially isomorphic to a cube. The underlined bracket monomials correspond to the vertices.

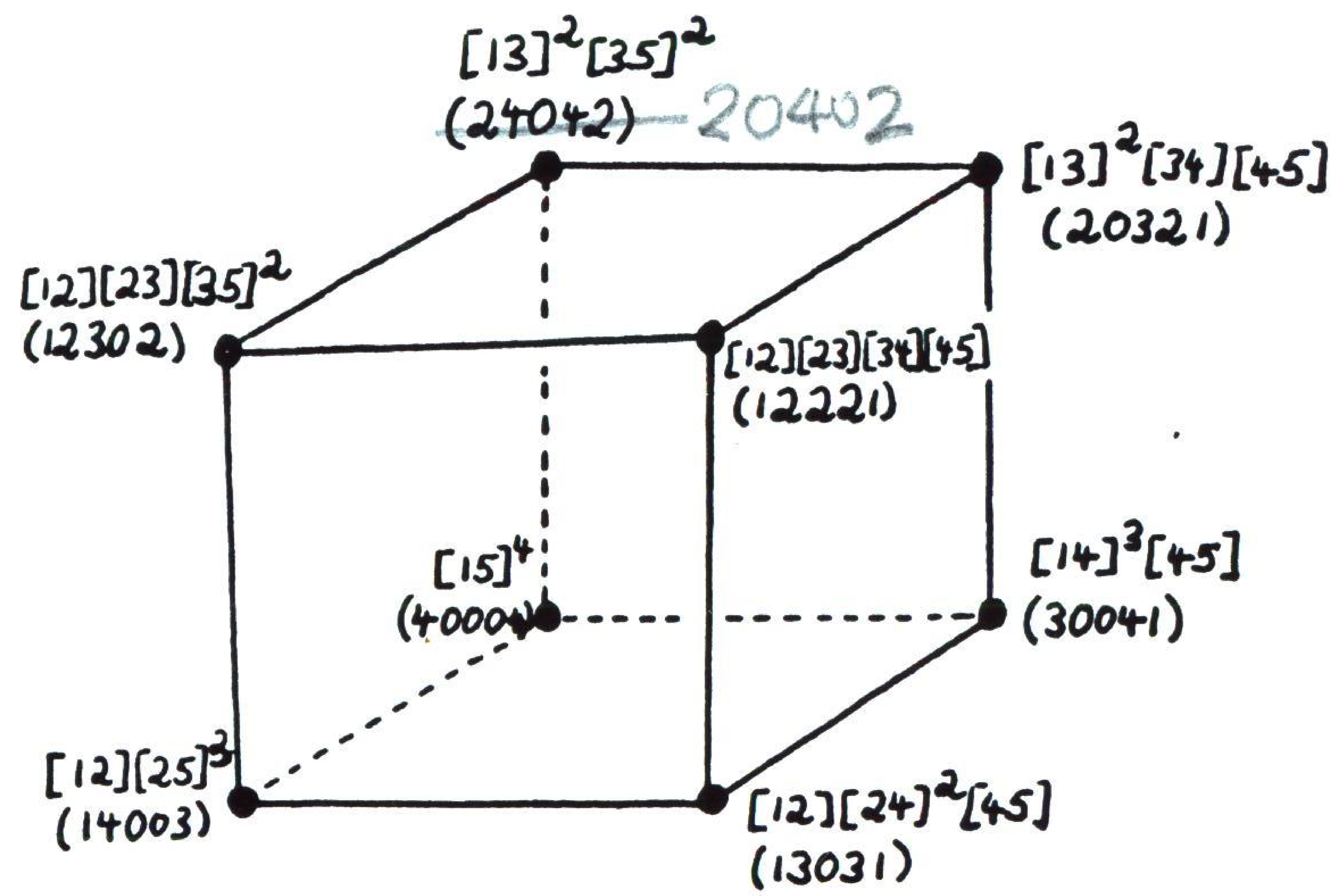


Figure 1. The secondary polytope $\Sigma(\mathcal{A})$ of five equidistant points on a line.

The vertices of the cube $\Sigma(\mathcal{A})$ in Figure 1 are labeled with the eight extreme terms in $\mathcal{R}_{\mathcal{A}}$ and their weights. The weights of all other 28 bracket monomials lie in the convex hull of these eight weights. In this example we can make the following interesting geometric observation (see Figure 2.) The triangulations of the Newton polytope Q with vertices in $\mathcal{A}$ are precisely the extreme bracket monomials in $\mathcal{R}_{\mathcal{A}}$. Here each 1-simplex $conv\{a_i, a_j\}$ together with its volume $Vol(ij)$ appears as a bracket power $[ij]^{Vol(ij)}$.

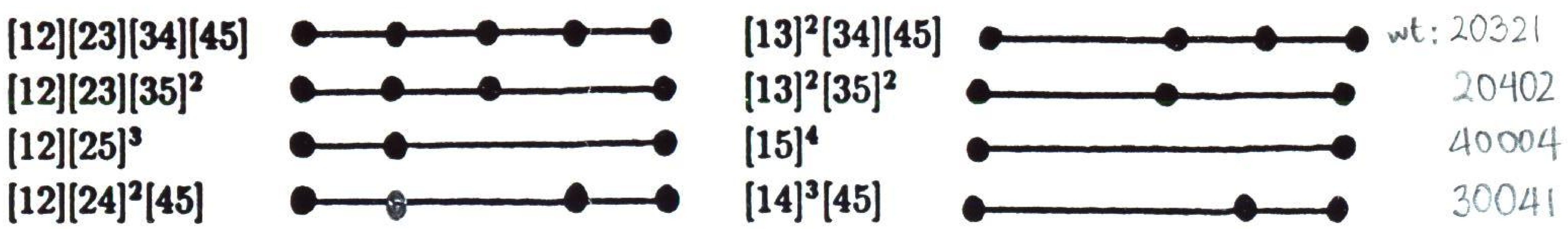


Figure 2. The triangulations of the Newton polytope $Q = conv(\mathcal{A})$.

Let us now consider a system of two sparse polynomials with five monomials:

$$
\begin{aligned}
f_1(x,y) &= c_{11}xy^{e_1} + c_{12}xy^{e_2} + c_{13}xy^{e_3} + c_{14}xy^{e_4} + c_{15}xy^{e_5} \\
f_2(x,y) &= c_{21}xy^{e_1} + c_{22}xy^{e_2} + c_{23}xy^{e_3} + c_{24}xy^{e_4} + c_{25}xy^{e_5},
\end{aligned}
\tag{2.6}
$$

where $e_1 < e_2 < e_3 < e_4 < e_5$. While here the (quasi-)homogenization is different from the one in (2.1), the set $\mathcal{A}$ still consists of five points on the affine line. If these five points are equidistant, then their $\mathcal{A}$-resultant is identical to $\tilde{\mathcal{R}}_{\mathcal{A}}$ as before in (2.4).

For a general configuration of five points we let g denote the greatest common divisor of the differences $e_2-e_1, e_3-e_2, e_4-e_3, e_5-e_4$, and we introduce the "normalized volumes" $d_{ij} := (e_j - e_i)/g$, $1 \leq i < j \leq 5$. The $\mathcal{A}$-resultant is a bracket polynomial of degree d_{15}, which has the same structure of eight extreme bracket terms as before:

$$
\begin{aligned}
\tilde{\mathcal{R}}_{\mathcal{A}} = {} & [12]^{d_{12}}[23]^{d_{23}}[34]^{d_{34}}[45]^{d_{45}} + [13]^{d_{13}}[34]^{d_{34}}[45]^{d_{45}} + [12]^{d_{12}}[25]^{d_{25}} + [13]^{d_{13}}[35]^{d_{35}} \\
& + [12]^{d_{12}}[23]^{d_{23}}[35]^{d_{35}} + [15]^{d_{15}} + [12]^{d_{12}}[24]^{d_{24}} + [14]^{d_{14}}[45]^{d_{45}} + \text{many interior terms} \ldots
\end{aligned}
$$

In particular, the secondary polytope $\Sigma(\mathcal{A})$ is combinatorially isomorphic to the 3-cube.

2.2. *The dense multivariate resultant.* We next discuss the classical *multivariate resultant* for dense homogeneous polynomials of degree d. Here $\mathcal{A}$ is the set of all $n := \binom{m+d-1}{d}$ lattice points $(i_1, i_2, \ldots, i_m) \in \mathbf{Z}^m_+$ with $i_1 + i_2 + \ldots + i_m = d$. The Newton polytope Q is a regular $(m-1)$-simplex. Its normalized volume equals $Vol(Q) = d^{m-1}$.

As an example we consider the case $m = 3, d = 2$. Here the $\mathcal{A}$-resultant $\mathcal{R}_{\mathcal{A}}$ agrees with usual resultant of three ternary quadrics

$$
\begin{aligned}
f_1(x,y,z) &= c_{11}x^2 + c_{12}y^2 + c_{13}z^2 + c_{14}xy + c_{15}xz + c_{16}yz, \\
f_2(x,y,z) &= c_{21}x^2 + c_{22}y^2 + c_{23}z^2 + c_{24}xy + c_{25}xz + c_{26}yz, \\
f_3(x,y,z) &= c_{31}x^2 + c_{32}y^2 + c_{33}z^2 + c_{34}xy + c_{35}xz + c_{36}yz.
\end{aligned}
\tag{2.7}
$$

The exponent set equals $\mathcal{A} = \{(2,0,0),(0,2,0),(0,0,2),(1,1,0),(1,0,1),(0,1,1)\}$, and $Q = conv(\mathcal{A})$ is a triangle with normalized area $Vol(Q) = 4$. We list a complete expansion of the $\mathcal{A}$-resultant $\tilde{R}_{\mathcal{A}}$ as a bracket polynomial of degree 4:

$$
\begin{aligned}
& \underline{[145][246][356][456]} - \underline{[146][156][246][356]} - \underline{[145][245][256][356]} - \underline{[145][246][346][345]} \\
& + [125][126][356][456] - 2[124][156][256][356] + [134][136][246][546] - 2[135][146][346][246] \\
& + [325][324][154][654] - 2[326][354][254][154] - \underline{[126]^2[156][356]} - \underline{[125]^2[256][356]} \\
& - \underline{[134]^2[246][346]} - \underline{[136]^2[146][246]} - \underline{[145][245][235]^2} - \underline{[145][345][234]^2} \\
& + 2[123][124][356][456] - [123][125][346][456] - [123][134][256][456] + 2[123][135][246][456] \\
& - 2[123][145][246][356] - 2[124]^2[356]^2 + 2[124][125][346][356] + 2[124][134][256][356] \\
& + 3[124][135][236][456] - 4[124][135][246][356] - [125]^2[346]^2 + 2[125][135][246][346] \\
& - [134]^2[256]^2 + 2[134][135][246][256] - 2[135]^2[246]^2 - [123][126][136][456] \\
& + 2[123][126][146][356] - 2[124][136]^2[256] - 2[125][126][136][346] \\
& - [213][215][235][465] + 2[213][215][245][365] - 2[214][235]^2[165] - 2[216][215][235][345] \\
& - [321][324][314][654] + 2[321][324][364][154] - 2[326][314]^2[254] - 2[325][324][314][164] \\
& - 3[163][125][523][126] + 3[126][153][623][125] + [163][125]^2[623] + [126]^2[153][523] \\
& - 3[143][163][126][423] + 3[124][143][163][623] + [143]^2[126][623] + [124][163]^2[423] \\
& + 3[124][153][423][523] - 3[143][423][523][125] + [153][423]^2[125] + [124][523]^2[143] \\
& - 3[123][124][153][623] - [123][143][523][126] - [123][153][126][423] - [123][143][623][125] \\
& - [123][163][125][423] - [123][124][523][163] - 2[123]^2[126][136] + 2[123]^2[125][235] \\
& - \underline{[136]^2[126]^2} - \underline{[125]^2[235]^2} - \underline{[134]^2[234]^2} - 2[123]^2[134][234] - \underline{[123]^4}
\end{aligned}
$$

This expansion was computed using the algorithm to be presented in Section 4.2.

The secondary polytope $\Sigma(\mathcal{A})$ is the convex hull in $\mathbf{R}^6$ of all weights occurring in the above expansion, e.g.,

$$
weight([145][246][356][456]) = (1,1,1,3,3,3), \quad weight([123]^2[134][234]) = (3,3,4,2,0,0).
$$

We find that $\Sigma(\mathcal{A})$ is a 3-dimensional polytope with 14 vertices, which are the weights of the underlined bracket monomials. These extreme bracket monomials are precisely the triangulations of the triangle Q with vertices in $\mathcal{A}$. In Figure 3 we depict a Schlegel diagram of the polytope $\Sigma(\mathcal{A})$. Each of the fourteen vertices is labeled with the triangulation of the corresponding extreme bracket monomial. This polytope is isomorphic to the one depicted in [28, Figure 4]; see also [19, Figure 1].

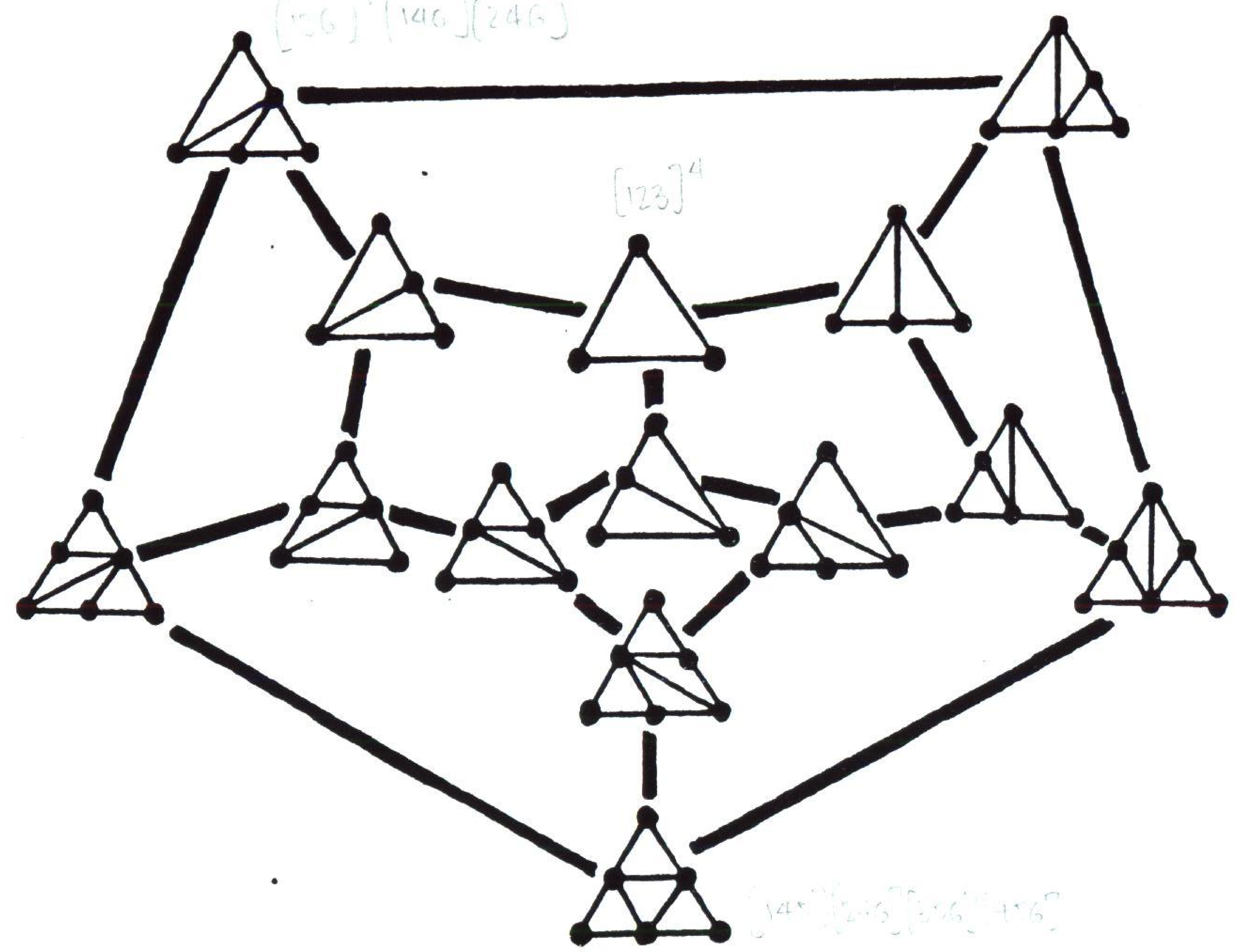


Figure 3. The secondary polytope corresponding to three ternary quadrics.

2.3. *Multilinear systems and the hyperdeterminant.* Multilinear equations are a very important class of sparse polynomials. Suppose our variables come in p groups $X^{(j)} = \{x_0^{(j)}, x_1^{(j)}, \ldots, x_{k_j}^{(j)}\}$, $j = 1, 2, \ldots, p$. We are interested in the multilinear system

$$\sum_{i_1 i_2 \ldots i_p} c_{l;i_1 i_2 \ldots i_p} \, x_{i_1}^{(1)} x_{i_2}^{(2)} \cdots x_{i_p}^{(p)} \;=\; 0 \qquad (l = 1, 2, \ldots, k) \qquad (2.8)$$

where the sum is over all integer vectors $(i_1, i_2, \ldots, i_p)$ with $0 \leq i_1 \leq k_1, \ldots, 0 \leq i_p \leq k_p$.

In order to describe the set $\mathcal{A}$ of exponent vectors, it is convenient to use the notation Δ_d for the set of unit vectors in $\mathbf{R}^{d+1}$. We view Δ_d geometrically as the vertex set of a regular d-simplex. Then $\mathcal{A} = \Delta_{k_1} \times \Delta_{k_2} \times \ldots \times \Delta_{k_p} \subset \mathbf{Z}^{(k_1+1)+\ldots+(k_p+1)}$. The rank of the lattice spanned by $\mathcal{A}$ equals $k := k_1 + k_2 + \ldots + k_p + 1$, and this is the correct number of equations to get a solvability condition of codimension 1. The Newton polytope $Q = conv(\mathcal{A})$ is a product of simplices. Its normalized volume equals the multinomial coefficient $\binom{k-1}{k_1, k_2, \ldots, k_p} = (k-1)!/(k_1! k_2! \cdots k_p!)$.

The coefficients $(c_{l;i_1 i_2 \ldots i_p})$ can be grouped into a $(p+1)$-dimensional "hypermatrix" of format $k \times (k_1+1) \times \ldots \times (k_p+1)$. Here the $\mathcal{A}$-resultant $\mathcal{R}_\mathcal{A}$ is a polynomial in $c_{l;i_1 i_2 \ldots i_p}$, which is abbreviated $Det(c_{l;i_1 i_2 \ldots i_p})$ and called the *hyperdeterminant*. The hyperdeterminant $Det = \mathcal{R}_\mathcal{A}$ is the natural generalization of the ordinary determinant (the $p = 1$ case). This notion was introduced already by Cayley in 1845.

For a systematic development of the theory of hyperdeterminants we refer to the current work of Gelfand, Kapranov and Zelevinsky [15]. It is known that Det has several essentially different expansions. We are here interested in the expansion of the $k \times k_1 \times \ldots \times k_p$-hyperdeterminant as a polynomial $\tilde{\mathcal{R}}_\mathcal{A}$ in brackets of rank k. Theorem 1.1 implies that $\tilde{\mathcal{R}}_\mathcal{A}$ is a bracket polynomial of degree $\binom{k-1}{k_1, k_2, \ldots, k_p}$ which vanishes if and only if (2.8) has a non-trivial solution. By "non-trivial" we mean that none of the p groups of variables specializes entirely to zero.

$k \times (k_1+1) \times (k_2+1)$

As an example we consider the hyperdeterminant of format $4 \times 3 \times 2$. Here $\mathcal{R}_\mathcal{A}(c_{ijk}) = Det(c_{ijk})$ is the irreducible polynomial which vanishes if and only if the bilinear system

$$\begin{aligned}
c_{111}x_1y_1 + c_{112}x_1y_2 + c_{121}x_2y_1 + c_{122}x_2y_2 + c_{131}x_3y_1 + c_{132}x_3y_2 &= 0,\\
c_{211}x_1y_1 + c_{212}x_1y_2 + c_{221}x_2y_1 + c_{222}x_2y_2 + c_{231}x_3y_1 + c_{232}x_3y_2 &= 0,\\
c_{311}x_1y_1 + c_{312}x_1y_2 + c_{321}x_2y_1 + c_{322}x_2y_2 + c_{331}x_3y_1 + c_{332}x_3y_2 &= 0,\\
c_{411}x_1y_1 + c_{412}x_1y_2 + c_{421}x_2y_1 + c_{422}x_2y_2 + c_{431}x_3y_1 + c_{432}x_3y_2 &= 0.
\end{aligned} \qquad (2.9)$$

has a zero in the product of projective spaces $P^2_{\mathbf{C}} \times P^1_{\mathbf{C}}$. The exponent configuration equals $\mathcal{A} = \{(1,0,0;1,0), (1,0,0;0,1), (0,1,0;1,0), (0,1,0;0,1), (0,0,1;1,0), (0,0,1;0,1)\}$.

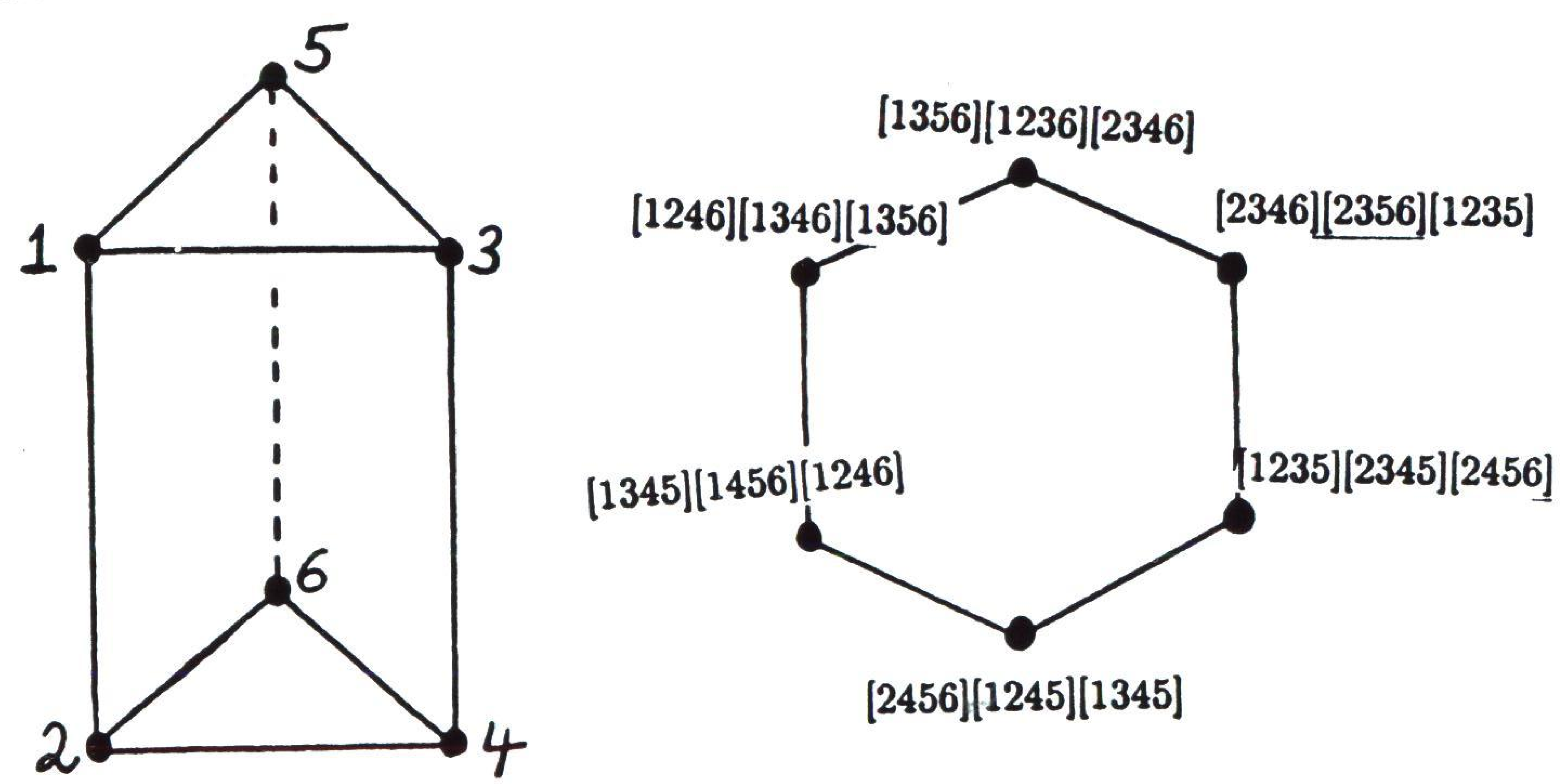


Figure 4. The triangular prism $\Delta_2 \times \Delta_1$ and its secondary polytope $\Sigma(\Delta_2 \times \Delta_1)$.

The Newton polytope $Q = conv(\mathcal{A})$ is the triangular prism $\Delta_2 \times \Delta_1$, that is, the direct product of a triangle Δ_2 with a line segment Δ_1. The normalized volume of Q equals 3; hence the $4 \times 3 \times 2$-hyperdeterminant $Det(c_{ijk})$ is a polynomial of degree 12.

The $4 \times 3 \times 2$-hypermatrix (c_{ijk}) is flattened in (2.9) to an ordinary matrix of size 4×6. The brackets are the maximal minors of the 4×6-coefficient matrix in (2.9). The bracket expansion of $Det = \mathcal{R}_{\mathcal{A}}$ equals

$$\begin{aligned} \tilde{\mathcal{R}}_{\mathcal{A}} \;=\; & -\underline{[1246][1346][1356]} + \underline{[1356][1236][2346]} - \underline{[2346][2356][1235]} \\ & + \underline{[1235][2345][2456]} - \underline{[2456][1245][1345]} + \underline{[1345][1456][1246]} \\ & + [1234][1256][3456] + [1234][1356][2456] + [1235][1246][3456] \\ & - [1235][1346][2456] + [1245][1346][2356] \end{aligned} \tag{2.10}$$

The secondary polytope $\Sigma(\Delta_2 \times \Delta_1)$ is the convex hull in $\mathbf{R}^6$ of the seven occurring weights $(3,1,2,2,1,3)$, $(2,2,3,1,1,3)$, $(1,3,3,1,2,2)$, $(1,3,2,2,3,1)$, $(2,2,1,3,3,1)$, $(3,1,1,3,2,2)$, and $(2,2,2,2,2,2)$. We can see that $\Sigma(\Delta_2 \times \Delta_1)$ is a hexagon, whose vertices are the first six weights. The corresponding underlined bracket monomials are precisely the six triangulations of the triangular prism $\Delta_2 \times \Delta_1$; see Figure 4.

2.4. ***The Dixon resultant.*** Consider the bihomogeneous system

$$\begin{aligned} c_{11}x_1^2y_1 + c_{12}x_1x_2y_1 + c_{13}x_2^2y_1 + c_{14}x_1^2y_2 + c_{15}x_1x_2y_2 + c_{16}x_2^2y_2 &= 0, \\ c_{21}x_1^2y_1 + c_{22}x_1x_2y_1 + c_{23}x_2^2y_1 + c_{24}x_1^2y_2 + c_{25}x_1x_2y_2 + c_{26}x_2^2y_2 &= 0, \\ c_{31}x_1^2y_1 + c_{32}x_1x_2y_1 + c_{33}x_2^2y_1 + c_{34}x_1^2y_2 + c_{35}x_1x_2y_2 + c_{36}x_2^2y_2 &= 0. \end{aligned} \tag{2.11}$$

The exponent vectors are $\mathcal{A} = \{(2,0,1,0), (1,1,1,0), (0,2,1,0), (2,0,0,1), (1,1,0,1), (0,2,0,1)\}$, and so the Newton polytope $Q = conv(\mathcal{A})$ is a lattice rectangle of shape 2×1 and normalized volume $Vol(Q) = 4$:

There is a classical elimination theory due to Dixon [8] for bihomogeneous polynomials. In our example the Dixon resultant can be derived as follows. Let g_1, g_2, g_3 denote the three polynomials in (2.11) and consider the linear map

$$\begin{aligned} \phi \;:\; (\mathbf{C}[x_1,x_2;y_1,y_2]_{3,0})^3 &\rightarrow \mathbf{C}[x_1,x_2;y_1,y_2]_{5,1} \\ (f_1,f_2,f_3) &\mapsto f_1g_1 + f_2g_2 + f_3g_3. \end{aligned} \tag{2.12}$$

Here $\mathbf{C}[x_1,x_2;y_1,y_2]_{5,1}$ is the 12-dimensional vector space of polynomials $h(x_1,x_2;y_1,y_2)$ which are homogeneous of degree 5 in (x_1,x_2) and homogeneous of degree 1 in (y_1,y_2). Similarly, $\mathbf{C}[x_1,x_2;y_1,y_2]_{3,0}$ is the $\mathbf{C}$-linear span of $x_1^3, x_1^2x_2, x_1x_2^2, x_2^3$.

Let $\mathcal{Z}(\mathcal{A})$ denote the set of all coefficient matrices $(c_{ij}) \in \mathbf{C}^{6\times 3}$ such that the system (2.11) has a solution $(x_1,x_2;y_1,y_2)$ in the product of projective lines $P^1_{\mathbf{C}} \times P^1_{\mathbf{C}}$. It can be shown that a coefficient matrix (c_{ij}) lies in $\mathcal{Z}(\mathcal{A})$ if and only if the linear map ϕ is singular. Choosing monomial bases for both 12-dimensional vector spaces in (2.12), we can represent ϕ by the following 12×12-matrix.

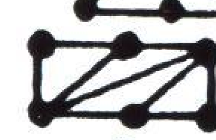
$$\begin{pmatrix} c_{11} & 0 & 0 & 0 & c_{21} & 0 & 0 & 0 & c_{31} & 0 & 0 & 0 \\ c_{14} & 0 & 0 & 0 & c_{24} & 0 & 0 & 0 & c_{34} & 0 & 0 & 0 \\ c_{12} & c_{11} & 0 & 0 & c_{22} & c_{21} & 0 & 0 & c_{32} & c_{31} & 0 & 0 \\ c_{15} & c_{14} & 0 & 0 & c_{25} & c_{24} & 0 & 0 & c_{35} & c_{34} & 0 & 0 \\ c_{13} & c_{12} & c_{11} & 0 & c_{23} & c_{22} & c_{21} & 0 & c_{33} & c_{32} & c_{31} & 0 \\ c_{16} & c_{15} & c_{14} & 0 & c_{26} & c_{25} & c_{24} & 0 & c_{36} & c_{35} & c_{34} & 0 \\ 0 & c_{13} & c_{12} & c_{11} & 0 & c_{23} & c_{22} & c_{21} & 0 & c_{33} & c_{32} & c_{31} \\ 0 & c_{16} & c_{15} & c_{14} & 0 & c_{26} & c_{25} & c_{24} & 0 & c_{36} & c_{35} & c_{34} \\ 0 & 0 & c_{13} & c_{12} & 0 & 0 & c_{23} & c_{22} & 0 & 0 & c_{33} & c_{32} \\ 0 & 0 & c_{16} & c_{15} & 0 & 0 & c_{26} & c_{25} & 0 & 0 & c_{36} & c_{35} \\ 0 & 0 & 0 & c_{13} & 0 & 0 & 0 & c_{23} & 0 & 0 & 0 & c_{33} \\ 0 & 0 & 0 & c_{16} & 0 & 0 & 0 & c_{26} & 0 & 0 & 0 & c_{36} \end{pmatrix} \tag{2.13}$$

The determinant $\mathcal{R}_{\mathcal{A}}(c_{ij})$ of this matrix is called the ***Dixon resultant***. It vanishes if and only if $(c_{ij}) \in \mathcal{Z}(\mathcal{A})$. By complete expansion of the determinant in (2.13) we find that $\mathcal{R}_{\mathcal{A}}(c_{ij})$ is a degree 12 polynomial with 20,791 monomials.

A more economical representation of the Dixon resultant is its expansion $\tilde{\mathcal{R}}_{\mathcal{A}}$ in terms of brackets. Here $[i\,j\,k]$ denotes the 3×3-minor with column indices i,j,k of the 6×3-coefficient matrix (c_{ij}). The following expression equals $\tilde{\mathcal{R}}_{\mathcal{A}}$:

$-\,[124][235][245][356]$ 

$+\,[124][236][245][256]$ 

$+\,[124][234][345][356]$ 

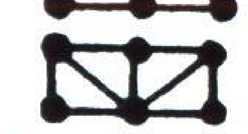
$-\,[136]^2[145][156]$ 

$-\,[124][234][346]^2$ 

$-\,[124][236][246]^2$ 

$+\,[134]^2[346]^2$ 

$-\,[125][145][236][256]$ 

$+\,[125][145][235][356]$ 

$+\,[126][145][156][236]$ 

$-\,[134]^2[345][356]$ 

$-\,[126][146]^2[236]$ 

$-\,[135]^2[145][356]$ 

$+\,[136]^2[146]^2$ 

$$\begin{aligned} & +2[123][145][345][356] + [134][135][245][356] \\ & +2[124][126][236][456] - [125][126][246][346] + [125][136][246][246] \\ & -2\,[134][136][146][346] \\ & +2[123][145][156][356] - [134][135][156][256] + [135][135][146][256] \\ & +2[124][234][236][456] + [124][235][246][346] \end{aligned}$$

$$
\begin{aligned}
&- 3[123][123][456][456] + 8[123][124][356][456] - 2[123][125][346][456] \\
&- 2[123][134][256][456] + 3[123][135][246][456] - [123][145][246][356] \\
&- 2[124][124][356][356] + 2[124][125][346][356] + 2[124][134][256][356] \\
&- [124][135][236][456] - 2[124][135][246][356] + [125][125][346][346] \\
&- [125][135][246][346] + [134][134][256][256] - [134][135][246][256] \\
+ [123][134][346][456] &- 2[123][145][346][346] - 2[134][134][236][456] - [134][135][246][346] \\
+ [123][146][246][346] &- 2[124][126][346][346] + 2[124][136][246][346] + [134][136][246][246] \\
- 2[134][134][156][356] &+ [134][135][136][456] + 2[134][135][146][356] + [135][135][146][346] \\
+ [123][136][146][456] &- 2[123][146][146][356] - 2[124][136][136][456] - [135][136][146][246]
\end{aligned}
$$

The first fourteen bracket monomials in this expansion are the triangulations of $\mathcal{A}$. The remaining ten weight components are presented as linear combinations of standard bracket monomials (cf. Section 4.2). Their weights are the interior lattice points of the secondary polytope $\Sigma(\mathcal{A})$. It can be seen that $\Sigma(\mathcal{A})$ is a 3-dimensional polytope with 14 vertices which is combinatorially isomorphic to the secondary polytope in Figure 3.

3. $\mathcal{A}$-resultants, Chow forms, and Gröbner bases of toric varieties

This section provides a self-contained treatment of the theory of $\mathcal{A}$-resultants. In the first subsection we recall the definitions of regular triangulations and secondary polytopes, and we state the main result (Theorem 3.1) about the asymptotics of the $\mathcal{A}$-resultant. The second subsection deals with Chow forms and Gröbner bases of general projective varieties. In the third subsection we focus our attention on the toric variety $X_{\mathcal{A}}$, and we characterize its Gröbner bases. We complete the proofs for Theorems 1.1 and 3.1 by constructing the $\mathcal{A}$-resultant $\mathcal{R}_{\mathcal{A}}$ as the Chow form of $X_{\mathcal{A}}$.

3.1. Regular triangulations and the secondary polytope. In the following we assume familiarity with some basic notions in the theory of convex polytopes (cf. [16],[21]). A ***polyhedral complex*** is a finite collection of polytopes in $\mathbf{R}^m$ having the property that the intersection of any two is a face of each and is itself in the collection. Let $\mathcal{A} = \{a_1, a_2, \ldots, a_n\} \subset \mathbf{Z}_+^m$ and $Q = conv(\mathcal{A})$ as in the introduction. A ***subdivision*** of $\mathcal{A}$ is a collection Δ of subsets of $\mathcal{A}$, called ***cells***, whose convex hulls form a polyhedral complex whose union equals Q. If each cell in Δ is linearly independent in $\mathbf{R}^m$ then Δ is a ***triangulation*** of $\mathcal{A}$.

Every vector $\omega = (\omega_1, \ldots, \omega_n) \in \mathbf{R}^n$ induces a subdivision Δ_ω of $\mathcal{A}$ as follows. For any point $q \in Q$ we choose a vector $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbf{R}_+^n$ satisfying $\sum_{i=1}^n \lambda_i a_i = q$ and such that the inner product $\sum_{i=1}^n \lambda_i \omega_i$ is minimal with this property. These requirements determine the support $\Delta_\omega(q) := \{a_i \in \mathcal{A} : \lambda_i \neq 0\}$ uniquely as a function of q and ω.

$\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$

We define $\Delta_\omega := \{\Delta(q) \subseteq \mathcal{A} : q \in Q\}$.

The collection Δ_ω is a subdivision of $\mathcal{A}$, and, if the weight vector ω is chosen generic, then Δ_ω is a triangulation of $\mathcal{A}$. A triangulation of $\mathcal{A}$ is said to be ***regular*** if it equals Δ_ω for some $\omega \in \mathbf{R}^n$. For any regular triangulation Δ of $\mathcal{A}$, the set $\mathcal{C}(\Delta) := \{\omega \in \mathbf{R}^n \mid \Delta_\omega = \Delta\}$ is an open convex polyhedral cone in $\mathbf{R}^n$ (cf. [3],[13, Section 3A]).

Let us now assume the validity of Theorem 1.1. We define the ***weight*** of any monomial $m = \prod c_{ij}^{\nu_{ij}}$ in the $\mathcal{A}$-resultant $\mathcal{R}_{\mathcal{A}}(c_{ij})$ to be its column degree $\textit{weight}(m) := (\sum_{i=1}^k \nu_{i1}, \ldots, \sum_{i=1}^k \nu_{in}) \in \mathbf{Z}_+^n$. The ***weight*** of a bracket monomial $\prod_\sigma [\sigma]$ is the weight of any of the monomials occurring in its expansion into c_{ij}'s, namely, $\textit{weight}(\prod_\sigma [\sigma]) = \sum_\sigma (e_{\sigma_1} + e_{\sigma_2} + \ldots + e_{\sigma_k})$. We define the ***secondary polytope*** $\Sigma(\mathcal{A}) \subset \mathbf{R}^n$ to be the convex hull of all the weights occurring in $\mathcal{R}_{\mathcal{A}}$.

For any $\mathbf{v} \in \Sigma(\mathcal{A}) \cap \mathbf{Z}_+^n$, let $\mathcal{R}_{\mathcal{A},\mathbf{v}}$ denote the sum of all terms of weight $\mathbf{v}$ in $\mathcal{R}_{\mathcal{A}}$. The following result due to Kapranov, Sturmfels and Zelevinsky [19] gives a complete description for all extreme terms of the $\mathcal{A}$-resultant.

Theorem 3.1. [19] *Let $\mathbf{v}$ be any vertex of the secondary polytope $\Sigma(\mathcal{A})$. Then its inner normal cone equals $\mathcal{C}(\Delta)$ for some regular triangulation Δ of $\mathcal{A}$. The corresponding extreme term of the $\mathcal{A}$-resultant equals the bracket monomial $\mathcal{R}_{\mathcal{A},\mathbf{v}} = \pm \prod_{\sigma \in \Delta} [\sigma]^{Vol(\sigma)}$.*

Here the product is over all maximal cells σ in Δ, and $Vol(\sigma)$ is the normalized volume defined in the introduction. The ***inner normal cone*** of $\Sigma(\mathcal{A})$ at $\mathbf{v}$ is defined as

not only vtx, but interior (cone) ✓ ⇔ v vtx

$$
\{\omega \in \mathbf{R}^n \ : \ \omega \cdot \mathbf{v} > \omega \cdot \mathbf{v}' \ \text{ for all } \ \mathbf{v}' \in \Sigma(\mathcal{A}) \setminus \{\mathbf{v}\}\}. \qquad ⇨ [3] \tag{3.1}
$$

It is clear from our construction that the closed polyhedral cones $\overline{\mathcal{C}(\Delta)}$ cover $\mathbf{R}^n$, as Δ runs over all triangulations of $\mathcal{A}$, and that the intersection of the $\overline{\mathcal{C}(\Delta)}$ is a linear subspace of dimension k in $\mathbf{R}^n$. These geometric considerations and Theorem 3.1 imply the following.

Corollary 3.2. *The secondary polytope $\Sigma(\mathcal{A})$ has dimension $n - k$, and its vertices are in one-to-one correspondence with the regular triangulations of $\mathcal{A}$.*

We briefly illustrate Theorem 3.1 and the subtle geometric concept of *regular* triangulations for the classical resultant of three ternary quartics:

$$
\begin{aligned}
&c_{i,1}x^4 + c_{i,2}x^3y + c_{i,3}x^3z + c_{i,4}x^2y^2 + c_{i,5}x^2yz + c_{i,6}x^2z^2 + c_{i,7}xy^3 + c_{i,8}xy^2z \\
&\quad + c_{i,9}xyz^2 + c_{i,10}xz^3 + c_{i,11}y^4 + c_{i,12}y^3z + c_{i,13}y^2z^2 + c_{i,14}yz^3 + c_{i,15}z^4. \qquad (i = 1, 2, 3)
\end{aligned}
$$

Here $\mathcal{A} = \{(i, j, k) \in \mathbf{Z}_+^3 : i + j + k = 4\}$, $n = 15$, and the Newton polytope $Q = conv(\mathcal{A})$ is a triangle with $Vol(Q) = 16$. Consider the following three triangulations of $\mathcal{A}$:

⇨ convex super-polyt. of Risler

eg. $\forall q$ in cell (123), $\lambda = (\lambda_1 \lambda_2 \lambda_3 0 \dots 0)$

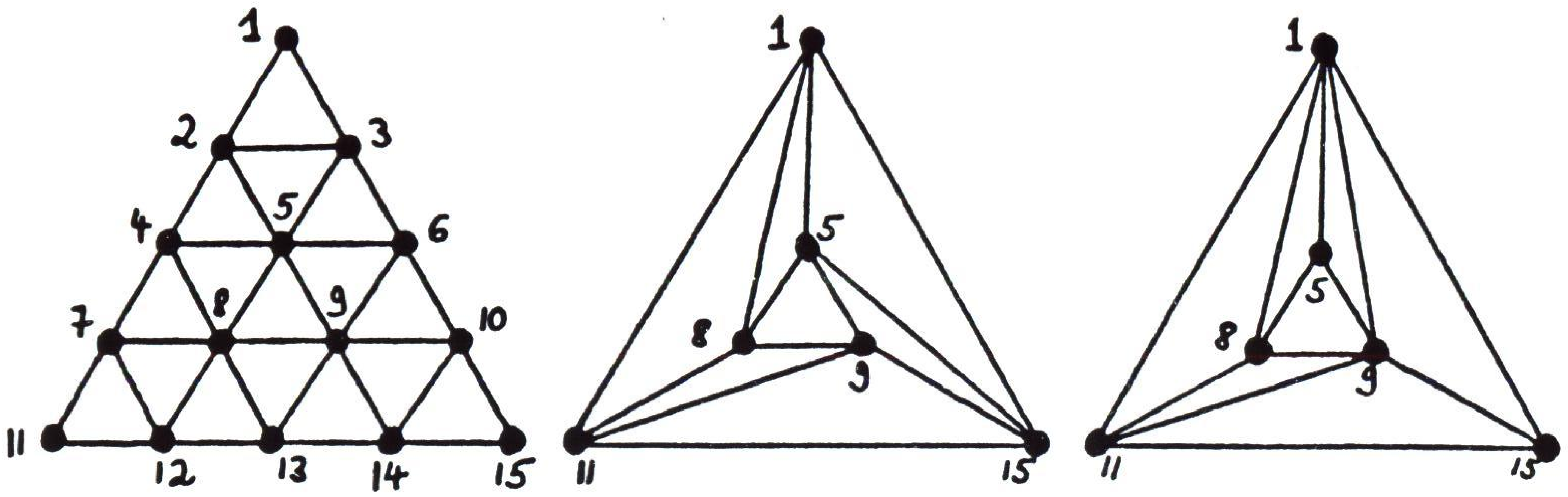


Figure 5. Three triangulations of the set of monomials in a ternary quartic

Each of these triangulations is encoded by a bracket monomial of degree 16:

$$\Delta^{(1)} := [1\,2\,3][2\,3\,5][2\,4\,5][3\,5\,6][4\,5\,8][4\,7\,8][5\,6\,9][5\,8\,9][6\,9\,10]\cdot$$ wt = (1,3,3,3,6,

$$\cdot [7\,8\,12][7\,11\,12][8\,9\,13][8\,12\,13][9\,10\,14][9\,13\,14][10\,14\,15]$$ 3,3,6,6,3,1,3,3,3)

$$\Delta^{(2)} := [1\,5\,8][1\,5\,15]^4[1\,8\,11]^4[5\,8\,9][5\,9\,15][8\,9\,11][9\,11\,15]^4$$

$$\Delta^{(3)} := [1\,5\,8][1\,5\,9][1\,8\,11]^4[5\,8\,9][1\,9\,15]^4[8\,9\,11][9\,11\,15]^4$$

By Theorem 1.1, the resultant $\mathcal{R}_{\mathcal{A}}(c_{ij})$ has degree $3 \cdot 16 = 48$. The triangulation Δ_1 is regular; for instance, $\Delta^{(1)} = \Delta_\omega$ for $\omega = (9,4,4,1,0,1,4,0,0,4,9,4,1,4,9)$. Theorem 3.1 implies that $\mathbf{v} = weight(\Delta^{(1)}) \in \mathbb{Z}_+^{16}$ is a vertex of the 13-dimensional polytope $\Sigma(\mathcal{A})$, and we have $\mathcal{R}_{\mathcal{A},\mathbf{v}} = \pm\Delta^{(1)}$.

The triangulation $\Delta^{(2)}$ is not regular (cf. [3, Fig. 1]) and hence does not occur as an extreme term in the resultant $\mathcal{R}_{\mathcal{A}}$. On the other hand, the triangulation $\Delta^{(3)}$ is regular.

(i) *3.2. The Chow variety and the Hilbert scheme.* The Chow form is a classical tool for encoding an irreducible projective variety $X \hookrightarrow P^{n-1}$ of dimension $k-1$ by a single polynomial [5],[9],[26, Sect. I.6]. Consider k generic linear forms in $\mathbb{C}[\mathbf{y}] = \mathbb{C}[y_1,\dots,y_n]$:

$$\ell_i(\mathbf{y}) := c_{i1}y_1 + c_{i2}y_2 + \dots + c_{in}y_n \qquad (i = 1,2,\dots,k),$$

defining an $(n-k-1)$-flat ξ in P^{n-1}. Let $\mathcal{Z}(X)$ denote the set of matrices $(c_{ij}) \in \mathbb{C}^{k\times n}$ with $\xi \cap X \neq \emptyset$. Using a dimension count and classical elimination theory, we see that $\mathcal{Z}(X)$ is a hypersurface which is defined by a homogeneous polynomial $\mathcal{R}_X(c_{ij})$ with coefficients in the field of definition of X. We call $\mathcal{R}_X$ the *Chow form* of X.

Note that the Chow form depends only on the $(n-k-1)$-flat ξ and not on the specific matrix (c_{ij}). Hence $\mathcal{R}_X(c_{ij})$ can be expressed as a homogeneous polynomial $\tilde{\mathcal{R}}_X$ in the brackets $[i_1 i_2 \dots i_k] := det(c_{\mu i_\nu})_{1\le\mu,\nu\le k}$. These are the Plücker coordinates of the $(n-k-1)$-flat ξ, and we may also write $\tilde{\mathcal{R}}_X(\xi)$ for the bracket expansion of the Chow form. The degree d of the bracket polynomial $\tilde{\mathcal{R}}_X(\xi)$ equals the *degree* of the variety X. The latter is usually defined as the cardinality of $X \cap \eta$ for any generic $(n-k)$-flat η, but, as we can choose $\eta \supset \xi$, it follows that $\#(X \cap \eta) = d$.

It is instructive to see how the variety X is recovered from its Chow form $\mathcal{R}_X$ (see [9] for details). A point $y = (y_1 : \dots : y_n) \in P^{n-1}$ lies in X if and only if $span(\Psi, y)$ meets X for all $(n-k-2)$-flats Ψ. If Ψ is chosen to be generic in P^{n-1} and $\xi := span(\Psi, y)$, then $\mathcal{R}_X(\xi)$ can be expanded as a polynomial in the Plücker coordinates of Ψ with coefficients in $\mathbb{C}[y_1,\dots,y_n]$. The variety defined by the set of coefficient polynomials is precisely X. This construction also shows that $\mathcal{R}_X$ is irreducible whenever X is irreducible.

Let V denote the vector space $S^d \wedge^k \mathbb{C}^n$ of all degree d bracket monomials modulo the ideal of *Grassmann-Plücker relations* in degree d, and let $P(V)$ denote the projectivization of V. A basis for the vector space V is given, for instance, by the standard Young tableaux of rectangular shape $d \times k$ (see e.g. [31]). Let $C^{k,d}_{irr}(P^{n-1})$ denote the set of all irreducible subvarieties of dimension $k-1$ and degree d in P^{n-1}. By the above argument, the map $X \mapsto \mathcal{R}_X$ defines an embedding of this set into the projective space $P(V)$. The closure $C^{k,d}(P^{n-1})$ of $C^{k,d}_{irr}(P^{n-1})$ in $P(V)$ is called the *Chow variety*. Its elements are *algebraic cycles* of degree d and dimension $k-1$. Each cycle is coded by a product of Chow forms $\prod \mathcal{R}_{X_i}^{e_i}$, where the X_i are irreducible $(k-1)$-dimensional varieties in P^{n-1} with $\sum e_i\, deg(X_i) = d$.

A meta-theorem of symbolic computation states that geometric computations with projective varieties can be done in single-exponential time while algebraic problems may have double-exponential complexity in the worst case (see e.g. [7]). This discrepancy is explained by the appearance of embedded and lower-dimensional components in subschemes of P^{n-1}. From this perspective algebraic cycles and the Chow variety are appealing since they ignore such "nasty" components. We refer to the recent work of Caniglia [5] on *"how to compute the Chow form of an unmixed polynomial ideal in subexponential time"*.

Let us take a closer look at the relation between Gröbner bases and Chow forms. (ii) We assume that the reader is familiar with the connection between Gröbner bases and Hilbert schemes described in [1]. We say that two homogeneous ideals $I = \oplus_{r=0}^{\infty} I_r$ and $J = \oplus_{r=0}^{\infty} J_r$ in $\mathbb{C}[\mathbf{y}]$ are *equivalent* if $I_r = J_r$ for all but finitely many degrees $r \in \mathbb{Z}_+$. By a *subscheme* of P^{n-1} we mean an equivalence class of homogeneous ideals.

For each numerical polynomial $h \in \mathbb{Q}[r]$, let $\mathcal{HILB}_h(P^{n-1})$ denote the set of subschemes of P^{n-1} having Hilbert polynomial h. This set is called the *Hilbert scheme*. It is naturally equipped with the structure of a projective variety as follows. We fix a sufficiently large integer $r \gg 0$ which depends only on h. Consider the vector space $W := \wedge^{h(r)} S^r \mathbb{C}^n$, and let $P(W)$ denote its projectivization. To each subscheme I we associate its *r-th Hilbert point* $\wedge^{h(r)} I_r \in P(W)$, which is the Plücker coordinate vector of the $h(r)$-dimensional linear subspace I_r of $S^r\mathbf{C}^n$. Then the map $I \mapsto \wedge^{h(r)} I_r$ is a closed embedding of $\mathcal{HILB}_h(P^{n-1})$ into $P(W)$.

Recall that the degree d and the dimension $k-1$ of a subscheme I can be read off from the Hilbert polynomial via $h(r) = \frac{d}{(k-1)!} r^{k-1} + O(r^{k-2})$. The general linear group $GL_n(\mathbf{C})$ acts naturally ("by linear change of coordinates") on the vector spaces V and W and their projectivization. The following result is well known (see e.g. [24, Section 5.4]).

Theorem 3.3. *The exists a unique $GL_n(\mathbf{C})$-equivariant morphism of algebraic varieties*

$$\begin{array}{rcl} \phi : \mathcal{HILB}_h(P^{n-1}) & \to & \mathcal{C}^{k,d}(P^{n-1}) \\ I & \mapsto & \prod \mathcal{R}_X^{mult(X,I)} \end{array}$$

from the Hilbert scheme to the Chow variety. Here the product is over all irreducible subvarieties X in P^{n-1} and $mult(X,I)$ denotes the geometric multiplicity of X in I.

The main theme of Gröbner basis theory is the reduction of problems about ideals to the case of monomial ideals. It is thus useful to have an explicit description of the map ϕ for *monomial schemes*, i.e., equivalence classes of ideals which are generated by monomials.

With each k-set $\sigma = \{\sigma_1 < \ldots < \sigma_k\} \in \binom{[n]}{k}$ we associate the coordinate $(k-1)$-flat $E_\sigma = span(e_{\sigma_1}, \ldots, e_{\sigma_k})$. The Chow form $\mathcal{R}_{E_\sigma}$ of this linear variety equals the single bracket $[\sigma] := [\sigma_1\sigma_2\ldots\sigma_k]$.

Let M be any monomial ideal with Hilbert polynomial $h(r) = \frac{d}{(k-1)!} r^{k-1} + O(r^{k-2})$. Each $(k-1)$-dimensional irreducible component of M is a coordinate flat E_σ. The *support* of a monomial m in $\mathbf{C}[\mathbf{y}]$ is the set $supp(m) := \{ i \in [n] : y_i \text{ divides } m \}$. A monomial m is called *standard* if $m \notin M$. We define a *standard pair* to be a pair (m, σ) consisting of a monomial m and an index set $\sigma \in \binom{[n]}{k}$ such that $supp(m) \cap \sigma = \emptyset$ and such that $m \cdot m'$ is standard for all monomials m' with $supp(m') \subseteq \sigma$.

Proposition 3.4. *The Chow form of a $(k-1)$-dimensional monomial ideal M equals*

$$\phi(M) \;=\; \prod_{\sigma \in \binom{[n]}{k}} [\sigma]^{C_{\sigma,M}}, \text{ where } C_{\sigma,M} \text{ counts the number of standard pairs } (\cdot, \sigma).$$

Proof: We need to show that $C_{\sigma,M}$ equals the geometric multiplicity of E_σ in M. This number is the $\mathbf{C}$-dimension of the algebra $\mathbf{C}[\mathbf{y}]_P/M_P$ where $P = < y_i : i \notin \sigma >$ is the prime ideal of E_σ. We compute this algebra as the quotient of $\mathbf{C}[y_i : i \notin \sigma]$ modulo the monomial ideal M' which is the image of M under the specialization $y_{\sigma_1} \mapsto 1, \ldots, y_{\sigma_k} \mapsto 1$. It can be seen that a monomial $m \in \mathbf{C}[y_i : i \notin \sigma]$ is standard modulo M' if and only if (m, σ) is a standard pair modulo M. This proves Proposition 3.4. ◁

We next consider Gröbner bases for an arbitrary subscheme I of dimension $k-1$ in P^{n-1}. Each term order on $\mathbf{C}[\mathbf{y}]$ can be represented by a weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in \mathbf{Z}^n$, or, equivalently, with a diagonal *one-parameter subgroup*

$$\omega : \mathbf{C}^* \to GL(\mathbf{C}^n), \quad t \mapsto diag(t^{\omega_1}, t^{\omega_2}, \ldots, t^{\omega_n}). \tag{3.2}$$

Let $\mathcal{G}$ be a Gröbner basis for I with respect to ω and $init_\omega(I) := < init_\omega(g) : g \in \mathcal{G} >$ the initial ideal of I. A monomial m is called *standard* if $m \notin init_\omega(I)$. Recall that the set of standard monomials forms a $\mathbf{C}$-basis for the residue ring $\mathbf{C}[\mathbf{y}]/I$. By a *standard pair* for I with respect to ω we mean a standard pair (m, σ) for the initial ideal $init_\omega(I)$. Let $C_{\sigma,\omega}(I)$ denote the number of standard pairs of the form $(\cdot, \sigma)$.

By Theorem 3.3, the morphism ϕ is equivariant with respect to the action of one-parameter subgroup ω on the Hilbert scheme and the Chow variety. Using Proposition 3.4, we can now read off the asymptotics of the Chow form of I from the Gröbner basis $\mathcal{G}$:

Corollary 3.5. *The Chow form of a homogeneous ideal I satisfies the asymptotic formula:*

$$\phi(\omega(\mathbf{t}) \cdot I) \;=\; \Big(\prod_{\sigma \in \binom{[n]}{k}} [\sigma]^{C_{\sigma,\omega}(I)} \Big) \cdot \mathbf{t}^{\sum_\sigma C_{\sigma,\omega}(I)(\sigma_1 + \ldots + \sigma_k)} \quad + \quad \textit{higher terms in } \mathbf{t}.$$

3.3. Gröbner bases for toric varieties. Let $\mathcal{A} = \{a_1, \ldots, a_n\} \subset \mathbf{Z}_+^m$ and $h : \mathbf{Z}_+^m \to \mathbf{Q}$ as before. The set $\mathcal{A}$ spans a rank k monoid $\mathcal{M}(\mathcal{A}) := \{\sum \lambda_i a_i : \lambda_i \in \mathbf{Z}_+\}$ in $\mathbf{Z}_+^m$, and we write $\mathbf{C}[\mathcal{A}] := \mathbf{C}[\mathbf{x}^{a_1}, \ldots, \mathbf{x}^{a_n}] \hookrightarrow \mathbf{C}[\mathbf{x}]$ for the monoid algebra of $\mathcal{M}(\mathcal{A})$. Both $\mathcal{M}(\mathcal{A})$ and $\mathbf{C}[\mathcal{A}]$ are graded by the linear functional h.

We define the *toric variety* $X_\mathcal{A}$ to be the projective spectrum of the graded algebra $\mathbf{C}[\mathcal{A}]$. Equivalently, $X_\mathcal{A}$ is the closure of the set $\{ (\mathbf{x}^{a_1} : \ldots : \mathbf{x}^{a_n}) : \mathbf{x} \in (\mathbf{C}^*)^m \}$ in P^{n-1}. Thus $X_\mathcal{A}$ is a $(k-1)$-dimensional irreducible subvariety in P^{n-1}. It is our goal to study the Gröbner bases of its prime ideal $\mathcal{I}_\mathcal{A} := kernel\,(\mathbf{C}[\mathbf{y}] \to \mathbf{C}[\mathbf{x}],\, y_i \mapsto \mathbf{x}^{a_i})$.

Our main new result (Theorem 3.7) is a natural bijection between the distinct initial ideals of $\mathcal{I}_\mathcal{A}$ and the regular triangulations of the monoid $\mathcal{M}(\mathcal{A})$. In Proposition 3.11 we give an explicit formula for the multiplicity $C_{\sigma,\omega}(\mathcal{I}_\mathcal{A})$ of any coordinate flat E_σ in any initial scheme $init_\omega(\mathcal{I}_\mathcal{A})$. This sharpens [30, Theorem 3.1] where $C_{\sigma,\omega}(\mathcal{I}_\mathcal{A})$ was shown to be non-zero if and only if σ is a maximal simplex in the regular triangulation Δ_ω.

We say that a weight vector $\omega \in \mathbf{Z}^n$ is *admissible* if $init_\omega(\mathcal{I}_\mathcal{A})$ is monomial ideal, i.e., if no ties occur during a Gröbner basis computation for $\mathcal{I}_\mathcal{A}$ with respect to ω. Two admissible vectors ω, ω' are *equivalent* if they give rise to the same initial ideal $init_\omega(\mathcal{I}_\mathcal{A}) = init_{\omega'}(\mathcal{I}_\mathcal{A})$. Recall from [1],[22],[30] that the equivalence classes are the lattice points in the open cells of the *Gröbner fan*, which is the normal fan of the *state polytope* of $\mathcal{I}_\mathcal{A}$.

Given $\omega \in \mathbb{Z}^n$ admissible, we define $\tilde{\omega} \in \mathbb{Z}^{m+n}$ to be an *elimination order* on $\mathbb{C}[\mathbf{x},\mathbf{y}]$ which extends ω. This means that $\mathbf{x}^\alpha \mathbf{y}^\beta \succ_{\tilde{\omega}} \mathbf{y}^\gamma$ if $\alpha \neq 0$, or if $\alpha = 0$ and $\omega \cdot \beta > \omega \cdot \gamma$. In order to compute the reduced Gröbner basis $\mathcal{G}$ for $\mathcal{I}_\mathcal{A}$ with respect to ω, we introduce the ideal $\mathcal{J}_\mathcal{A}$ generated by $\mathbf{x}^{\mathbf{a}_1} - y_1, \ldots, \mathbf{x}^{\mathbf{a}_n} - y_n$ in $\mathbb{C}[\mathbf{x},\mathbf{y}]$, and we compute the reduced Gröbner basis $\tilde{\mathcal{G}}$ for $\mathcal{J}_\mathcal{A}$ with respect to $\tilde{\omega}$. Note that a monomial $\mathbf{y}^\lambda$ is *standard* with respect to $\mathcal{G}$ (i.e., $\mathbf{y}^\lambda \in \mathbb{C}[\mathbf{y}] \setminus init_\omega(\mathcal{I}_\mathcal{A})$) if and only if $\mathbf{y}^\lambda$ is standard with respect to $\tilde{\mathcal{G}}$.

The ring map $\mathbb{C}[\mathbf{y}] \to \mathbb{C}[\mathbf{x}]$, $y_i \mapsto \mathbf{x}^{\mathbf{a}_i}$ corresponds to an epimorphism of monoids

$$\pi \ : \ \mathbb{Z}_+^n \twoheadrightarrow \mathcal{M}(\mathcal{A}), \ \lambda = (\lambda_1, \ldots, \lambda_n) \ \mapsto \ \sum_{i=1}^n \lambda_i a_i. \tag{3.3}$$

If $\alpha \in \mathcal{M}(\mathcal{A})$ and $\lambda \in \pi^{-1}(\alpha)$, then we say that λ is a *representation* of α.

Lemma 3.6. *A weight vector $\omega \in \mathbb{Z}^n$ is admissible if and only each $\alpha \in \mathcal{M}(\mathcal{A})$ has a unique representation λ of minimum weight, i.e., $\omega \cdot \lambda < \omega \cdot \mu$, for all $\mu \in \pi^{-1}(\alpha) \setminus \{\lambda\}$.*

Proof: Suppose ω is admissible, and let $\tilde{\mathcal{G}}$ and $\mathcal{G}$ be as above. Each element of $\tilde{\mathcal{G}}$ is a difference of monomials, since this property of the input set $\{\mathbf{x}^{\mathbf{a}_i} - y_i\}$ is preserved by the Buchberger algorithm. In particular, each element of $\mathcal{G}$ has the form $\mathbf{y}^\beta - \mathbf{y}^\gamma$. This implies that the unique normal form of any monomial $\mathbf{y}^\mu$ must be some standard monomial $\mathbf{y}^\lambda$ with $\pi(\lambda) = \pi(\mu)$ and $\omega \cdot \lambda < \omega \cdot \mu$. For, let λ' be any other representation of $\pi(\lambda)$. Then $\mathbf{y}^{\lambda'} - \mathbf{y}^\lambda$ lies in $\mathcal{I}_\mathcal{A}$, and therefore $\mathbf{y}^\lambda$ must be the unique normal form of $\mathbf{y}^{\lambda'}$. This implies $\omega \cdot \lambda < \omega \cdot \lambda'$.

Suppose ω is not admissible. During the execution of the Buchberger algorithm we run into a polynomial $\mathbf{y}^\lambda - \mathbf{y}^{\lambda'} \in \mathcal{I}_\mathcal{A}$ with $\omega \cdot \lambda = \omega \cdot \lambda'$ such that neither $\mathbf{y}^\lambda$ nor $\mathbf{y}^{\lambda'}$ is a multiple of an ω-initial term of some other element of $\mathcal{I}_\mathcal{A}$. This means that $\alpha = \pi(\lambda) = \pi(\lambda')$ has no representation of weight less that $\omega \cdot \lambda = \omega \cdot \lambda'$. ◁

We define a *section* to be a map $S : \mathcal{M}(A) \to \mathbb{Z}_+^n$ such that $\pi \circ S$ is the identity on $\mathcal{M}(\mathcal{A})$. By Lemma 3.6, each admissible $\omega \in \mathbb{Z}^n$ defines a section S_ω which maps each $\alpha \in \mathcal{M}(\mathcal{A})$ to its representation λ of minimum weight. A section S is said to be a *regular triangulation* of the monoid $\mathcal{M}(\mathcal{A})$ provided $S = S_\omega$ for some admissible $\omega \in \mathbb{Z}^n$. The next statement is a direct consequence of the proof of Lemma 3.6.

Theorem 3.7. *The regular triangulations S_ω of the monoid $\mathcal{M}(\mathcal{A})$ are in one-to-one correspondence with the initial ideals $init_\omega(\mathcal{I}_\mathcal{A})$. For any admissible $\omega \in \mathbb{Z}^n$, the set of standard monomials in $\mathbb{C}[\mathbf{y}]$ modulo $\mathcal{I}_\mathcal{A}$ equals $S_\omega(\mathcal{M}(\mathcal{A})) := \{\mathbf{y}^{S_\omega(\alpha)} : \alpha \in \mathcal{M}(\mathcal{A})\}$.*

Let us justify the geometric term "triangulation" by showing that the regular triangulation Δ_ω of the polytope $Q = conv(\mathcal{A})$ can be recovered uniquely from the regular triangulation S_ω of the monoid $\mathcal{M}(\mathcal{A})$. Here we will apply some known facts from Stanley's decomposition theory for integer monoids [29, Section 4.6].

Let $\sigma = \{\sigma_1, \ldots, \sigma_t\} \subset [n]$ such that $\mathcal{A}_\sigma = \{a_{\sigma_1}, \ldots, a_{\sigma_t}\}$ is linearly independent in $\mathbb{R}^m$. Its convex hull $Q_\sigma := conv(\mathcal{A}_\sigma)$ is a $(t-1)$-simplex, its positive hull $\widehat{Q}_\sigma := pos(\mathcal{A}_\sigma)$ is a t-dimensional simplicial cone, and its monoid $\mathcal{M}(\mathcal{A}_\sigma)$ is a *simplicial monoid* of rank t. In the case of maximal dimension $t = k$ we write c_σ for the index of the (lattice generated by) $\mathcal{M}(\mathcal{A}_\sigma)$ in the (lattice generated by) the ambient monoid $\mathcal{M}(\mathcal{A})$.

Lemma 3.8. *The index c_σ of a simplicial submonoid $\mathcal{M}(\mathcal{A}_\sigma)$ in $\mathcal{M}(\mathcal{A})$ equals the normalized volume $Vol(Q_\sigma)$ of the corresponding $(k-1)$-simplex Q_σ.*

Proof: See [29, pages 227 and 239]. ◁

For each independent set σ we abbreviate $\mathcal{M}(\mathcal{A})_\sigma := rel\,int(\widehat{Q}_\sigma) \cap \mathcal{M}(\mathcal{A})$, and we define

$$\mathcal{D}_\sigma \ := \ \{\gamma \in \mathcal{M}(\mathcal{A}) \ : \ \gamma = \lambda_1 a_{\sigma_1} + \ldots + \lambda_t a_{\sigma_t} \text{ where } 0 < \lambda_1, \ldots, \lambda_t \leq 1\}. \tag{3.4}$$

Here "*rel int*" stands for the relative interior of a polyhedron in its affine span. The next lemma states that $\mathcal{D}_\sigma$ is a system of representatives for the residue classes of $\mathcal{M}(\mathcal{A})_\sigma$ modulo its simplicial submonoid $\mathcal{M}(\mathcal{A}_\sigma)$. In particular, if $t = k$, then Lemma 3.9 implies that $|\mathcal{D}_\sigma| = c_\sigma = Vol(Q_\sigma)$; see also [29, page 227].

Lemma 3.9. *We have the direct sum decomposition $\mathcal{M}(\mathcal{A})_\sigma = \mathcal{M}(\mathcal{A}_\sigma) \oplus \mathcal{D}_\sigma$, that is, every $\alpha \in \mathcal{M}(\mathcal{A})_\sigma$ can be written uniquely as $\alpha = \beta + \gamma$, where $\beta \in \mathcal{M}(\mathcal{A}_\sigma)$ and $\gamma \in \mathcal{D}_\sigma$.*

Proof: See [29, Lemma 4.6.7]. ◁

Fix an admissible $\omega \in \mathbb{Z}^n$, and let $\alpha \in \mathcal{M}(\mathcal{A})$. There exists a unique independent set $\sigma = \sigma(\omega, \alpha)$ such that $\alpha = \lambda_1 a_{\sigma_1} + \ldots + \lambda_t a_{\sigma_t}$ with positive rational coefficients $\lambda_1, \ldots, \lambda_t > 0$ of minimum weight $\lambda_1 \omega_{\sigma_1} + \ldots + \lambda_t \omega_{\sigma_t}$. For, if σ were not uniquely minimal, then some integer multiple of α would have two integral representations of minimum weight, in contradiction to Lemma 3.6.

The collection $\{\sigma(\omega, \alpha) : \alpha \in \mathcal{M}(\mathcal{A})\}$ is a polyhedral subdivision of the polytope Q. By construction, it agrees with the polyhedral subdivision Δ_ω defined above, and the argument in the previous paragraph shows that Δ_ω is in fact a triangulation.

Proposition 3.10. *The assignment $S_\omega \mapsto \Delta_\omega$ is a well-defined surjective map from the set of regular triangulations of $\mathcal{M}(\mathcal{A})$ onto the set of regular triangulations of Q.*

Proof: It remains to be seen that the map $S_\omega \mapsto \Delta_\omega$ is surjective. For any regular triangulation Δ of Q the set $C(\Delta)$ of weight vectors ω with $\Delta_\omega = \Delta$ is an open polyhedral cone in $\mathbb{R}^n$. The subset of admissible rational weight vectors is dense in this cone, and, after scaling, we can find an admissible $\omega \in \mathbb{Z}^n \cap C(\Delta)$. ◁

Fix any admissible $\omega \in \mathbb{Z}^n$ and let $S_\omega \mapsto \Delta_\omega$ as above. We have shown that the monoid $\mathcal{M}(\mathcal{A})$ is the disjoint union of the submonoids $\mathcal{M}(\mathcal{A})_\sigma$, where σ ranges over all simplices in Δ_ω. Fix a simplex σ in Δ_ω, and let $\alpha \in \mathcal{M}(\mathcal{A})_\sigma$ of sufficiently large degree

$h(\alpha)$. It follows from Theorem 3.7 and Lemma 3.9 that the unique normal form of $\mathbf{x}^\alpha$ modulo $\mathcal{J}_\mathcal{A}$ with respect to $\tilde{\omega}$ equals $\mathbf{y}^\beta \mathbf{y}^\gamma$ where $supp(\mathbf{y}^\beta) = \sigma$ and $\gamma \in S_\omega(\mathcal{D}_\sigma)$. In this case $(\mathbf{y}^\gamma, \sigma)$ is a standard pair for $\mathcal{J}_\mathcal{A}$ with respect to $\tilde{\omega}$, and hence also for $\mathcal{J}_\mathcal{A}$ with respect to ω.

Proposition 3.11. *For any $\tau \in \binom{[n]}{k}$, the multiplicity of the $(k-1)$-flat E_τ in the initial ideal $init_\omega(\mathcal{I}_\mathcal{A})$ equals*

$$C_{\tau,\omega}(\mathcal{I}_\mathcal{A}) \quad = \quad \begin{cases} Vol(Q_\tau) & \text{if } \tau \in \Delta_\omega \\ 0 & \text{otherwise} \end{cases} .$$

Proof: Let $\tau \in \binom{[n]}{k}$. It follows from the decomposition in Lemma 3.9 that a pair (m, τ) is a standard pair if and only if $\tau = \sigma$ is one of the maximal cells in the regular triangulation Δ_ω and m is a monomial whose exponent vector lies in $S_\omega(\mathcal{D}_\sigma)$. $\triangleleft$

We are now prepared to prove our main results about the $\mathcal{A}$-resultant.

Proof of Theorems 1.1 and 3.1. The toric variety $X_\mathcal{A}$ is an irreducible variety of dimension $k-1$ in P^{n-1}. The variety $\mathcal{Z}(\mathcal{A})$ in $\mathbf{C}^{k\times n}$ defined in the introduction coincides with the irreducible hypersurface $\mathcal{Z}(X_\mathcal{A})$ defined in Section 3.2. This hypersurface is defined by the Chow form $\mathcal{R}_{X_\mathcal{A}}$, which therefore agrees with the $\mathcal{A}$-resultant $\mathcal{R}_\mathcal{A}$. This proves part (a) and (b) of Theorem 1.1.

Let $\mathbf{v}$ be any vertex of the secondary polytope $\Sigma(\mathcal{A})$, and choose any admissible vector $\omega \in \mathbf{Z}^n$ in the inner normal cone of $\Sigma(\mathcal{A})$ at $\mathbf{v}$. By Theorem 3.3 and Corollary 3.5, the weight component $\mathcal{R}_{\mathcal{A},\mathbf{v}}$ equals the algebraic cycle underlying the monomial scheme $init_\omega(\mathcal{I}_\mathcal{A})$. By Proposition 3.11, the Chow form encoding this algebraic cycle equals

$$\phi(init_\omega(\mathcal{I}_\mathcal{A})) \quad = \quad \pm \prod_{\sigma\in\Delta_\omega} [\sigma]^{Vol(\sigma)} \tag{3.5}$$

This completes the proof of Theorem 3.1, and it also proves part (c) of Theorem 1.1 since $Vol(Q) = \sum_{\sigma\in\Delta_\omega} Vol(\sigma)$. $\triangleleft$

Remark 3.12. The above results imply a nice combinatorial description of the state polytope of a toric variety $X_\mathcal{A}$. Theorem 3.7 implies that the *Gröbner fan* (cf. [22],[30]) of the toric ideal $\mathcal{I}_\mathcal{A}$ is the common refinement of the normal fans of the lattice polytopes $conv(\pi^{-1}(\alpha)) \subset \mathbf{R}^n$ as α runs over the monoid $\mathcal{M}(\mathcal{A})$. In this common refinement it suffices to consider finitely many α of the form $\alpha = \pi(\lambda) = \pi(\mu)$ where $\mathbf{y}^\lambda - \mathbf{y}^\mu$ runs over the universal Gröbner basis given in [30, Corollary 2.6]. In the language of fiber polytopes [4] this means that the state polytope of $X_\mathcal{A}$ equals the integral (with respect to a suitable measure on $\mathcal{M}(\mathcal{A})$) of the corresponding projection bundle of lattice polytopes:

$$State(X_\mathcal{A}) \quad = \quad \int_{\mathcal{M}(\mathcal{A})} conv(\pi^{-1}(\alpha))\, d\alpha \tag{3.6}$$

4. Computing the $\mathcal{A}$-resultant

In this section we discuss algorithms for computing the $\mathcal{A}$-resultant and their complexity.

4.1. Computational Complexity. We first give an estimate for the asymptotic complexity of computing the $\mathcal{A}$-resultant. Our input is a set of lattice points $\mathcal{A} = \{a_1, a_2, \ldots, a_n\} \subset \mathbf{Z}^m_+$ of rank k. We write V for the maximum of the normalized volume $Vol(Q)$ of the Newton polytope and the size of the coordinates $log|a_{ij}|$, and we assume $k, n \leq V$.

Theorem 4.1. *The $\mathcal{A}$-resultant $\mathcal{R}_\mathcal{A}(c_{ij})$ can be computed in $V^{O(kn)}$ bit operations.*

In order to prove this bound, we proceed as in [5] and we estimate the degree in a representation of $\mathcal{R}_\mathcal{A}(c_{ij})$ as a rational linear combination of the input polynomials

$$f_i \quad = \quad c_{i1}\mathbf{x}^{a_1} + c_{i2}\mathbf{x}^{a_2} + \ldots + c_{in}\mathbf{x}^{a_n} \qquad (i = 1, 2, \ldots, k). \tag{4.1}$$

Consider the ring $S = \mathbf{Q}[\mathcal{A}][c_{ij}]$ of polynomials in $\mathbf{x}^{a_i}$ with coefficients in the polynomial ring $\mathbf{Q}[c_{ij}]$ over the rational numbers. The $\mathbf{Q}$-algebra S has a natural $\mathbf{Z}^{n+1}_+$-grading via h and the weight of monomials in the c_{ij}. Let S_d denote the subspace of all homogeneous polynomials of $\mathbf{Z}^{n+1}_+$-degree $(d, d, \ldots, d)$ in S. The dimension of the $\mathbf{Q}$-vector space S_d is of order $d^{O(kn)}$, and we can choose a basis for this space consisting of monomials $\mathbf{x}^\alpha \prod c_{ij}^{\nu_{ij}}$, $\alpha \in \mathcal{M}(\mathcal{A})$. We will view $f_1, f_2, \ldots, f_k$ as vectors in S_1.

Lemma 4.2. *For each $j = 1, 2, \ldots, n$ and $d \geq V$ there exists a homogeneous identity*

$$\mathbf{x}^{d\cdot a_i}\mathcal{R}_\mathcal{A}(c_{ij}) \quad = \quad g_1 f_1 + g_2 f_2 + \ldots + g_k f_k \qquad \text{in} \quad S_d.$$

Proof. Fix j and for each $i = 1, \ldots, k$ solve (4.1) for c_{ij}. Substitute the resulting expressions $f_i - \frac{1}{\mathbf{x}^{a_i}} \sum_{l\neq i} c_{il}\mathbf{x}^{a_l}$ into the $\mathcal{A}$-resultant $\mathcal{R}_\mathcal{A}$. Expand the result as a polynomial in c_{il} and f_i to get an expression of the form $\frac{1}{\mathbf{x}^{V\cdot a_i}}(g_1 f_1 + g_2 f_2 + \ldots + g_k f_k)$, $g_i \in S_{V-1}$. $\triangleleft$

Proof of Theorem 4.1. For $d = V$ consider the $\mathbf{Q}$-linear map

$$\phi : (S_{d-1})^k \to S_d\,, \quad (g_1, g_2, \ldots, g_k) \mapsto g_1 f_1 + g_2 f_2 + \ldots + g_k f_k.$$

In time $V^{O(kn)}$ we can write down a matrix for ϕ with respect to the monomial bases. Each entry in this matrix is either 0 or 1. The same holds for the coordinate projection $\theta_i : S_d \mapsto [S / < \mathbf{x}^{d\cdot a_i} >]_d$. In analogy to [5, Corollary 3.6], it now suffices to compute a vector $(g_1, g_2, \ldots, g_k) \in ker(\theta_i \circ \phi) \setminus ker(\phi)$. Using linear algebra over the rationals, this can be done in $V^{O(kn)}$ arithmetic operations, and, since all matrices have only zero-one entries, it can be done in $V^{O(kn)}$ bit operations. $\triangleleft$

4.2. Practical computation. In this subsection we will explain the techniques which were used to compute the bracket expansions of the $\mathcal{A}$-resultants in Section 2. These techniques are most useful for small numbers of monomials, say, $n \leq 8$.

To simplify the computation we first dehomogenize the given equations (1.1). This is done by choosing a $(k-1)$-dimensional coordinate flat onto which the affine span of $\mathcal{A}$ projects one-to-one. After relabeling of indices we may suppose that the span of $e_1, \ldots, e_{k-1}$ has this property. For each exponent vector a_i we substitute its projection b_i onto the first $k-1$ coordinates.

Next we replace (1.1) by the row-reduced system

$$\begin{array}{rcl} h_1(\mathbf{x}) & = & \mathbf{x}^{b_1} \qquad\qquad\qquad + \; d_{1,k+1}\mathbf{x}^{b_{k+1}} + \ldots + d_{1n}\mathbf{x}^{b_n} \\ h_2(\mathbf{x}) & = & \qquad \mathbf{x}^{b_2} \qquad\quad + \; d_{2,k+1}\mathbf{x}^{b_{k+1}} + \ldots + d_{2,n}\mathbf{x}^{b_n} \\ \ldots & & \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ h_k(\mathbf{x}) & = & \qquad\qquad \mathbf{x}^{b_k} \; + \; d_{k,k+1}\mathbf{x}^{b_{k+1}} + \ldots + d_{k,n}\mathbf{x}^{b_n}. \end{array} \tag{4.2}$$

The new coefficients d_{ij} are indeterminates. They are thought of as local coordinates on the Grassmannian with respect to the chart $\{[1\,2\,\ldots\,k] \neq 0\}$.

We are now left in (4.2) with a system of k polynomial equations in the affine variables $x_1, \ldots, x_{k-1}$. Using Gröbner bases (or successive applications of the Sylvester resultant), we eliminate $x_1, \ldots, x_{k-1}$ from (4.2), and we obtain an irreducible polynomial $P_{\mathcal{A}}(d_{ij})$. This polynomial is the $\mathcal{A}$-resultant expressed in local coordinates on the Grassmannian.

In order to recover the $\mathcal{A}$-resultant, it is useful to decompose $P_{\mathcal{A}}$ into weight components $P_{\mathcal{A},\mathbf{v}}$, where $\mathbf{v} \in \mathbf{Z}^n$. The *weight* of each variable d_{ij} is the difference of standard coordinate vectors $e_j - e_i$. The convex hull of weights occurring $P_{\mathcal{A}}(d_{ij})$ is the translated secondary polytope $\Sigma(\mathcal{A}) - d(e_1 + \ldots + e_k)$, where usually, $d = Vol(Q)$.

For each vertex $\mathbf{w}$ of $\Sigma(\mathcal{A})$ we now determine the corresponding regular triangulation $\Delta_{\mathbf{w}}$ of $\mathcal{A}$. If we set $\mathbf{v} = \mathbf{w} - d(e_1 + \ldots + e_k)$, then we know that $\prod_{\sigma \in \Delta_{\mathbf{w}}} [\sigma]^{Vol(\sigma)}$ specializes to either $P_{\mathcal{A},\mathbf{v}}$ or to $-P_{\mathcal{A},\mathbf{v}}$. In the latter case we adjust the sign.

For each interior point $\mathbf{w} = \mathbf{v} + d(e_1 + \ldots + e_k)$, we consider the weight component $P_{\mathcal{A},\mathbf{v}}(d_{ij})$. We substitute $d_{ij} = \frac{[1,2,\ldots i-1,j,i+1,\ldots,k]}{[1,2,3,\ldots,k]}$ and clear denominators to get a bracket polynomial. We now apply the straightening algorithm as in [31, Example 3.3], and we obtain $[1,2,3,\ldots,k]^e \cdot \tilde{\mathcal{R}}_{\mathcal{A},\mathbf{w}}$ where e is some non-negative integer, and $\tilde{\mathcal{R}}_{\mathcal{A},\mathbf{w}}$ is the unique expansion of the desired weight component in terms of standard bracket monomials.

4.3. Example (Computing a pentagonal $\mathcal{A}$-resultant)
Let $n = 5$, $m = k = 3$ and $\mathcal{A} = \{(1,0,0),(1,1,0),(1,4,2),(1,3,2),(1,1,1)\}$. Here the Newton polytope $Q = conv(\mathcal{A})$ is a pentagon with normalized area $Vol(Q) = 5$. In order to compute the $\mathcal{A}$-resultant, we consider the dehomogenized system in local coordinates:

$$d_{31} + d_{32}\,y + y^4 z^2 \;=\; d_{41} + d_{42}\,y + y^3 z^2 \;=\; d_{51} + d_{52}\,y + yz \;=\; 0. \tag{4.3}$$

Eliminating y and z from (4.3), we obtain a polynomial $P_{\mathcal{A}}(d_{31}, d_{32}, d_{41}, d_{42}, d_{51}, d_{52})$ which is the sum of weight components. Here $weight(d_{ij}) = e_j - e_i$. The weights are $\mathbf{v}_1 =$

$(3,4,-3,0,-4)$, $\mathbf{v}_2 = (5,2,-1,-2,-4)$, $\mathbf{v}_3 = (4,2,0,-4,-2)$, $\mathbf{v}_4 = (1,4,-1,-4,0)$, $\mathbf{v}_5 = (1,5,-3,-1,-2)$, $\mathbf{v}_6 = (4,3,-2,-1,-4)$, $\mathbf{v}_7 = (2,4,-2,-2,-2)$, $\mathbf{v}_8 = (3,3,-1,-3,-2)$, and the corresponding weight components are

$$\begin{aligned} P_{\mathcal{A},\mathbf{v}_1} &= d_{52}^4 d_{31}^3 - 2d_{31}^2 d_{52}^3 d_{32} d_{51} + d_{32}^2 d_{51}^2 d_{31} d_{52}^2, \\ P_{\mathcal{A},\mathbf{v}_2} &= d_{41}^2 d_{51}^2 d_{31} d_{52}^2 - 2d_{41} d_{42} d_{51}^3 d_{31} d_{52} + d_{31} d_{42}^2 d_{51}^4, \\ P_{\mathcal{A},\mathbf{v}_3} &= d_{41}^4 d_{52}^2 - 2d_{41}^3 d_{42} d_{52} d_{51} + d_{41}^2 d_{42}^2 d_{51}^2, \\ P_{\mathcal{A},\mathbf{v}_4} &= d_{31} d_{42}^4 - d_{32} d_{42}^3 d_{41}, \\ P_{\mathcal{A},\mathbf{v}_5} &= d_{32}^2 d_{42} d_{31} d_{52}^2 - d_{32}^3 d_{52}^2 d_{41}, \\ P_{\mathcal{A},\mathbf{v}_6} &= 2d_{31}^2 d_{52}^3 d_{41} d_{51} - 2d_{32} d_{51}^2 d_{31} d_{52}^2 d_{41} - 2d_{42} d_{31}^2 d_{52}^2 d_{51}^2 + 2d_{32} d_{42} d_{51}^3 d_{31} d_{52}, \\ P_{\mathcal{A},\mathbf{v}_7} &= 3d_{32}^2 d_{52}^2 d_{41}^2 - 2d_{32}^2 d_{42} d_{52} d_{51} d_{41} - 5d_{32} d_{42} d_{31} d_{52}^2 d_{41} + 2d_{32} d_{42}^2 d_{31} d_{52} d_{51} \\ &\quad + 2d_{42}^2 d_{52}^2 d_{31}^2, \quad \text{and} \quad P_{\mathcal{A},\mathbf{v}_8} = 4d_{42} d_{31} d_{52}^2 d_{41}^2 - 3d_{32} d_{41}^3 d_{52}^2 - 6d_{42}^2 d_{31} d_{52} d_{51} d_{41} \\ &\qquad + 4d_{32} d_{41}^2 d_{42} d_{52} d_{51} + 2d_{31} d_{42}^3 d_{51}^2 - d_{32} d_{42}^2 d_{51}^2 d_{41} \end{aligned}$$

Set $\mathbf{w}_i := \mathbf{v}_i + (0,0,5,5,5)$. Then $\mathbf{w}_1, \ldots, \mathbf{w}_5$ are precisely the weights of the five triangulations of $\mathcal{A}$, while $\mathbf{w}_6, \mathbf{w}_7, \mathbf{w}_8$ are interior vertices of the secondary polytope $\Sigma(\mathcal{A})$.

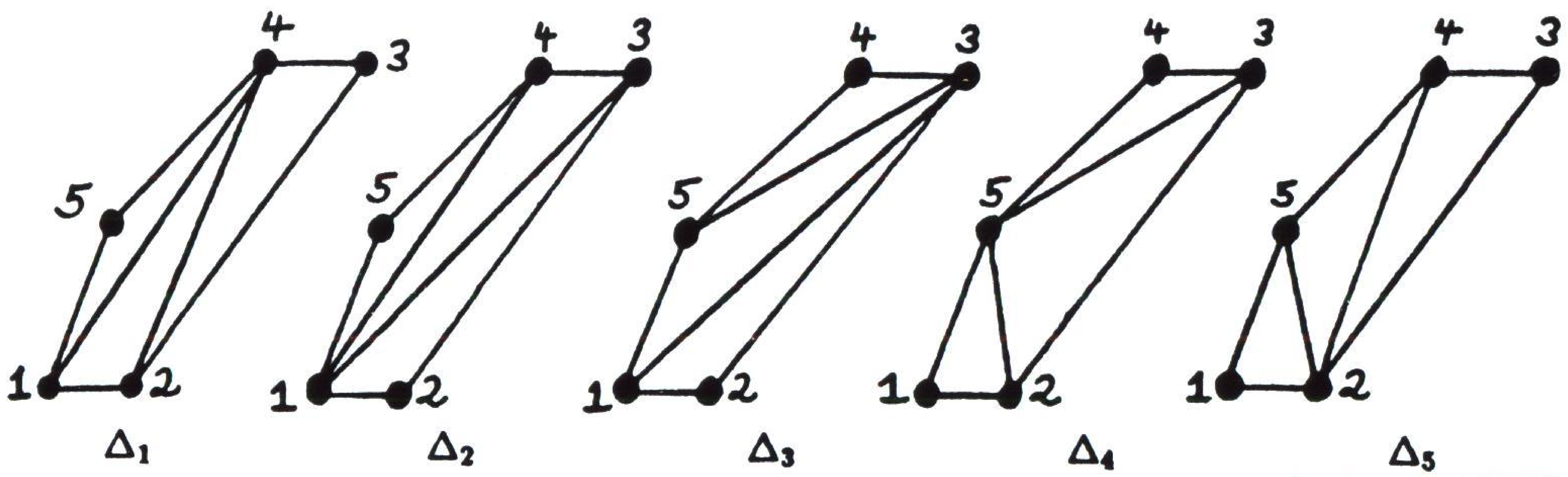


Figure 6. The five triangulations of the pentagonal set $\mathcal{A}$

By construction, the brackets $[i\,j\,k]$ are the maximal minors of the matrix

$$\begin{pmatrix} d_{31} & d_{32} & 1 & 0 & 0 \\ d_{41} & d_{42} & 0 & 1 & 0 \\ d_{51} & d_{52} & 0 & 0 & 1 \end{pmatrix}.$$

From Figure 6 we now get the extreme terms of the $\mathcal{A}$-resultant with their correct signs:

$$\tilde{\mathcal{R}}_{\mathcal{A},\mathbf{w}_1} = [124]^2[145][234]^2, \;\; \tilde{\mathcal{R}}_{\mathcal{A},\mathbf{w}_2} = [123]^2[134]^2[145], \;\; \tilde{\mathcal{R}}_{\mathcal{A},\mathbf{w}_3} = [123]^2[135]^2[345],$$
$$\tilde{\mathcal{R}}_{\mathcal{A},\mathbf{w}_4} = -[125][235]^3[345], \qquad \text{and} \quad \tilde{\mathcal{R}}_{\mathcal{A},\mathbf{w}_5} = [125][245]^2[234]^2.$$

In order to determine the three interior weight components of $\tilde{\mathcal{R}}_{\mathcal{A}}$, we substitute $d_{3j} = \frac{[j\,4\,5]}{[3\,4\,5]}$, $d_{4j} = \frac{[3\,j\,5]}{[3\,4\,5]}$, $d_{5j} = \frac{[3\,4\,j]}{[3\,4\,5]}$, into $P_{\mathcal{A},\mathbf{v}_i}$ for $i = 6,7,8$ and $j = 1,2$. We clear denominators and apply the straightening algorithm as in [31, Example 3.3]. As the result we obtain the following linear combinations of standard bracket monomials.

$$\tilde{\mathcal{R}}_{\mathcal{A},\mathbf{w}_6} = 2[123][423][124][143][145],$$

$$\tilde{\mathcal{R}}_{\mathcal{A},\mathbf{w}_7} = 3[123][123][245][245][345] - 5[123][124][235][245][345] + [123][134][235][245][245] + 2[124][124][235][235][345] - [124][134][235][235][245]$$

$$\tilde{\mathcal{R}}_{\mathcal{A},\mathbf{w}_8} = 2[123][123][125][345][345] + [123][123][135][245][345] - 2[123][124][135][235][345]$$

We now collect terms to get the final answer $\tilde{\mathcal{R}}_{\mathcal{A}} = \sum_{i=1}^{8} \tilde{\mathcal{R}}_{\mathcal{A},\mathbf{w}_i}$ $\triangleleft$

4.3. *Evaluating the $\mathcal{A}$-resultant.* Computing a complete expansion of $\tilde{\mathcal{R}}_{\mathcal{A}}$ as in Example 4.3 makes only sense if the set $\mathcal{A}$ is sufficiently small. Otherwise it will be of interest to represent the $\mathcal{A}$-resultant $\mathcal{R}_{\mathcal{A}}$ as a *black box* which returns a rational number $\mathcal{R}_{\mathcal{A}}(c_{ij})$ whenever we input a rational $k \times n$-matrix (c_{ij}) [18]. Here we discuss a possible implementation for such a black box.

Suppose the system (1.1) has rational coefficients. In order to decide whether $\mathcal{R}_{\mathcal{A}}(c_{ij})$ is zero we may proceed as follows. Using the techniques of Section 3.3, we precompute a Gröbner basis $\mathcal{G}$ for the toric ideal $\mathcal{I}_{\mathcal{A}}$ with respect to any admissible $\omega \in \mathbf{Z}^n$. Now replace (1.1) by the system of linear equations

$$\ell_i(\mathbf{y}) \quad := \quad c_{i1}y_1 + c_{i2}y_2 + \ldots + c_{in}y_n \qquad (i = 1,2,\ldots,k). \tag{4.4}$$

Remark 4.4. *The set $\mathcal{G} \cup \{\ell_1,\ldots,\ell_k\}$ has a zero in P^{n-1} if and only if $\mathcal{R}_{\mathcal{A}}(c_{ij}) = 0$.*

This means that we can decide the vanishing of the $\mathcal{A}$-resultant by computing a Gröbner basis for $\mathcal{G} \cup \{\ell_1,\ldots,\ell_k\}$ with respect to ω. However, from this Gröbner basis computation we cannot get the specific non-zero value of the $\mathcal{A}$-resultant.

For computing $\mathcal{R}_{\mathcal{A}}(c_{ij})$ we suggest a multiplicative perturbation technique. It is based on the validity of the following intriguing conjecture.

Conjecture 4.5. *Let $(c_{ij}) \in \mathbf{Q}^{k\times n}$ such that $\mathcal{R}_{\mathcal{A}}(c_{ij}) \neq 0$. Then there exists a regular triangulation Δ of $\mathcal{A}$ such that $[\sigma] = det(c_{\sigma_i,j})_{1\le i,j\le k}$ is non-zero for all $\sigma \in \Delta$.*

Given (c_{ij}) and Δ as above, then the rational number $\mathcal{R}_{\mathcal{A}}(c_{ij})$ can be computed as follows. Choose an admissible vector $\omega = (\omega_1,\ldots,\omega_n) \in -\mathcal{C}(\Delta) \cap \mathbf{Z}^n_+$ whose negative induces the regular triangulation Δ. Introduce a new variable z and extend ω to an elimination order ω' on $\mathbf{Q}[y_1,\ldots,y_n,z]$ with $y_i \gg z$. Replace (4.4) by the perturbed system

$$\ell'_i(\mathbf{y},z) \quad := \quad c_{i1}\,z^{\omega_1}\,y_1 + c_{i2}\,z^{\omega_2}\,y_2 + \ldots + c_{in}\,z^{\omega_n}\,y_n, \qquad (i = 1,2,\ldots,k) \tag{4.5}$$

and eliminate $y_1,\ldots,y_n$ from $\mathcal{G} \cup \{\ell'_1,\ldots,\ell'_k\}$; for instance by computing a Gröbner basis with respect to ω'. As the result we obtain a univariate monic polynomial $P(z) = z^d + p_{d-1}z^{d-1} + \ldots + p_0$ of degree $d = \sum_{\sigma\in\Delta} Vol(\sigma)(\omega_{\sigma_1} + \ldots + \omega_{\sigma_k})$. Indeed, by construction, $P(z)$ is a scalar multiple of the evaluation of the $\mathcal{A}$-resultant at the perturbed matrix $(c'_{ij}) = (c_{ij}\,z^{\omega_j})$. Now the following is a direct consequence of Theorem 3.1.

Proposition 4.6. *The specialized $\mathcal{A}$-resultant equals $\mathcal{R}_{\mathcal{A}}(c_{ij}) = \pm P(1)\cdot\prod_{\sigma\in\Delta}[\sigma]^{Vol(\sigma)}$.*

Example 4.7. (Evaluating the $\mathcal{A}$-resultant of an octahedral system)
Consider the polynomial equations

$$c_{i1}\,x_1x_2 + c_{i2}\,x_1x_3 + c_{i3}\,x_1x_4 + c_{i4}\,x_2x_3 + c_{i5}\,x_2x_4 + c_{i6}\,x_3x_4 \quad = \quad 0 \quad (i = 1,2,3,4)$$

$$\text{where} \qquad (c_{ij}) \quad := \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ -1 & -1 & 1 & 1 & -1 & -1 \\ 11 & 0 & 13 & 17 & 1 & 23 \\ 1 & 3 & 0 & 0 & 0 & 9 \end{pmatrix}. \tag{4.6}$$

Here $\mathcal{A}$ is the vertex set of a regular octahedron $Q = conv(\mathcal{A})$ with $Vol(Q) = 4$. We choose the regular triangulation $\Delta = \{1236, 1246, 1356, 1456\}$. The value at the corresponding bracket monomial for our matrix in (4.6) equals

$$[1236][1246][1356][1456](c_{ij}) \quad = \quad 197506339956. \tag{4.7}$$

We choose the weight vector $\omega = (2,1,0,0,0,0)$ which lies in $-\mathcal{C}(\Delta)$. Its ω-weight equals $d = \omega \cdot weight([1236][1246][1356][1456]) = 10$, while the two other triangulations of $\mathcal{A}$ have ω-weights 6 and 8.

We now replace the coefficient matrix (4.6) by $(c_{ij}\,z^{\omega_j}) = (c_{ij})\cdot diag(z^2, z^1, 1, 1, 1, 1)$, and we eliminate the variables x_1, x_2, x_3, x_4. This results in the polynomial

$$P(z) \quad = \quad 1219174938\,z^{10} - 362046865\,z^9 + 5758488\,z^8 - 10474542\,z^7 + 35154^6,$$

which has upper degree 10 and lower degree 6, as predicted. Using Proposition 4.6, we conclude

$$\pm\mathcal{R}_{\mathcal{A}}(c_{ij}) \quad = \quad \frac{197506339956}{1219174938}\cdot P(1) \quad = \quad 138096442026.$$

5. Conclusion and future directions

This article has demonstrated how methods from polyhedral geometry can be applied to problems in computational algebra, such as solving sparse systems of polynomial equations. It is clear from the exciting work of Gelfand, Kapranov and Zelevinsky – and our discussion above – that this theory is still in its infancy and that there are many connections yet to be explored and many open problems yet to be investigated. Among the numerous potential applications which could be mentioned at this point, the author was particularly inspired by the work of Warren [32] on toric variety techniques in geometric modeling. Let us briefly sketch four possible directions for future research.

i The polynomial system (1.1) is rather special in the sense that all k polynomials have the same set $\mathcal{A}$ of monomials. For many applications it is more natural to allow each input polynomial $f_i(\mathbf{x})$ to have its own set $\mathcal{A}_i$. In this case the $\mathcal{A}$-resultant would be replaced by the *mixed resultant* $\mathcal{D}_{\mathcal{A}_1,\ldots,\mathcal{A}_k}$ which was introduced in [13, Proposition 1.3.1]. The *volume* of the Newton polytope Q now gets replaced by the *mixed volume* of the individual Newton polytopes $Q_1,\ldots,Q_k$. We refer to the work of Kushnirenko [20] and Bernstein [2] who were the first to express the (mixed) degree of toric varieties as (mixed) volumes. How can the techniques of Section 3 and the algorithms of Section 4 be best generalized to this setting ? Is there a natural notion of a *mixed triangulation* or a *mixed Gröbner basis* ?

ii Modeling sparse systems as (1.1) means geometrically that we embed all varieties in question into a toric variety $X_{\mathcal{A}}$. It would be interesting to have an intrinsic Buchberger algorithm on the toric variety $X_{\mathcal{A}}$, i.e., using only the one-parameter subgroups of the dense torus in $X_{\mathcal{A}}$. Each term order ω on $\mathbf{C}[\mathbf{x}]$ induces a term order on the monoid algebra $\mathbf{C}[\mathcal{A}]$. For each ideal $\mathcal{I}$ in $\mathbf{C}[\mathcal{A}]$, we get an initial ideal $init_\omega(\mathcal{I})$, which, by noetherianity, is generated by finitely many monomials $\mathbf{x}^\alpha$, $\alpha \in \mathcal{M}(\mathcal{A})$. It is possible to generalize the Buchberger algorithm to this setting, but the details are subtle and not well understood. For instance, a satisfactory *intrinsic Gröbner basis theory on toric varieties* should provide an answer to the following question. The *degree* of the algebraic cycle defined by $\mathcal{I}$ on $X_{\mathcal{A}}$ is a certain element (cohomology class) in the Chow ring of $X_{\mathcal{A}}$. How can this element be read off from the generators of the initial ideal $init_\omega(\mathcal{I})$?

iii The work of Gelfand, Kapranov and Zelevinsky on discriminants and resultants originated in their theory of $\mathcal{A}$-hypergeometric functions [10],[11]. More precisely, the $\mathcal{A}$-discriminant describes the singularities of the system of $\mathcal{A}$-hypergeometric differential equations. It may be speculated that this deep connection to analysis will lead to an entirely new approach to elimination theory. Here is a first concrete question in this direction: Is it possible to express the interior weight components $\mathcal{R}_{\mathcal{A},\mathbf{v}}$ of the $\mathcal{A}$-resultant in terms of $\mathcal{A}$-hypergeometric integrals of Euler type ?

iv It is generally agreed that symbolic algorithms for elimination theory can be successful in "real world" applications only in conjunction with numerical computations. A very interesting numerical algorithm is the *homotopy method for multi-homogeneous systems* due to Morgan and Sommese [23]. The basic idea underlying their work is to embed the zero-dimensional variety in question into a product of projective spaces $P^{k_1} \times P^{k_2} \times \ldots \times P^{k_p}$, which frequently results in smaller Bezout numbers (cf. Section 2.3). It would be interesting to extend this approach from $\mathcal{A} = \Delta^{k_1} \times \Delta^{k_2} \times \ldots \times \Delta^{k_p}$ to an arbitrary set $\mathcal{A}$, and to study numerical homotopy continuation methods on any toric variety $X_{\mathcal{A}}$. In order to start the desired homotopies, however, one needs a supply of "nice" explicit systems whose degree attains the Bezout number; in the toric case this is the volume $Vol(Q)$ or a mixed volume as in [2]. Here is a specific problem: For any $\mathcal{A}$, find an explicit system (1.1) with integer coefficients such that $\mathcal{R}_{\mathcal{A}}(c_{ij}) \neq 0$ but each subsystem of $k-1$ equations has $Vol(Q)$ rational roots in $(\mathbf{C}^*)^m$.

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