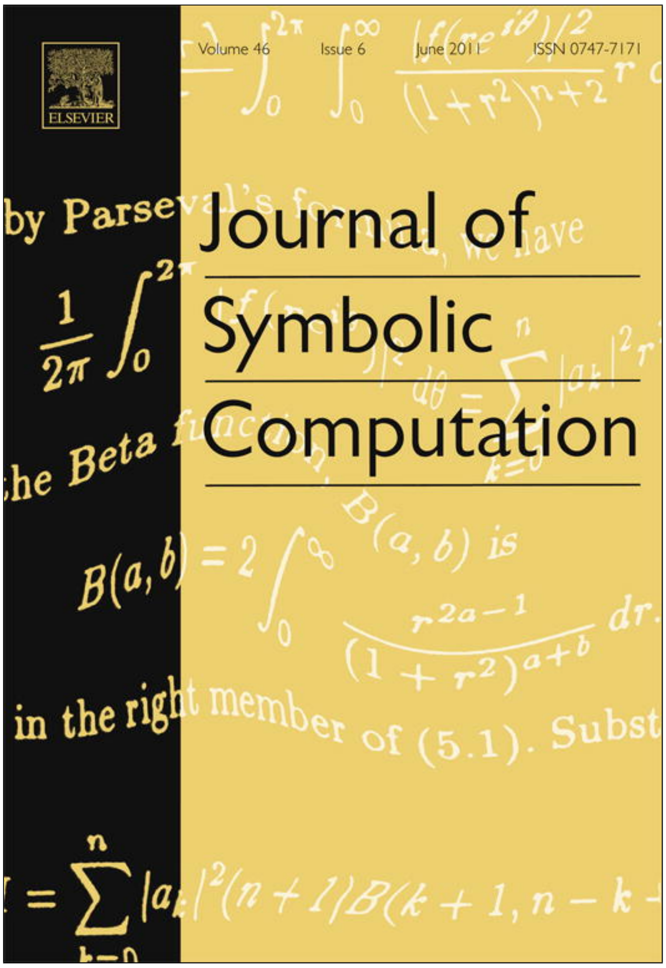
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Quartic curves and their bitangents

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ABSTRACT

A smooth quartic curve in the complex projective plane has 36 inequivalent representations as a symmetric determinant of linear forms and 63 representations as a sum of three squares. These correspond to Cayley octads and Steiner complexes respectively. We present exact algorithms for computing these objects from the 28 bitangents. This expresses Vinnikov quartics as spectrahedra and positive quartics as Gram matrices. We explore the geometry of Gram spectrahedra and we find equations for the variety of Cayley octads. Interwoven is an exposition of much of the 19th century theory of plane quartics.

1. Introduction

We consider smooth curves in the projective plane defined by ternary quartics

$$f(x, y, z) = c_{400}x^4 + c_{310}x^3y + c_{301}x^3z + c_{220}x^2y^2 + c_{211}x^2yz + \cdots + c_{004}z^4, \tag{1.1}$$

whose 15 coefficients c_{ijk} are parameters over the field $\mathbb{Q}$ of rational numbers. Our goal is to devise exact algorithms for computing the two alternate representations

$$f(x, y, z) = \det\big(xA + yB + zC\big), \tag{1.2}$$

where A, B, C are symmetric 4×4-matrices, and

$$f(x, y, z) = q_1(x, y, z)^2 + q_2(x, y, z)^2 + q_3(x, y, z)^2, \tag{1.3}$$

where the $q_i(x, y, z)$ are quadratic forms. The representation (1.2) is of most interest when the real curve $\mathcal{V}_{\mathbb{R}}(f)$ consists of two nested ovals. Following Helton and Vinnikov (2007) and Henrion (2010), one seeks real symmetric matrices A, B, C whose span contains a positive definite matrix. The representation (1.3) is of most interest when the real curve $\mathcal{V}_{\mathbb{R}}(f)$ is empty. Following Hilbert (1888)

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and Powers et al. (2004), one seeks quadrics $q_i(x,y,z)$ with real coefficients. We shall explain how to compute *all* representations (1.2) and (1.3) over $\mathbb{C}$.

The theory of plane quartic curves is a delightful chapter of 19th century mathematics, with contributions by Aronhold, Cayley, Frobenius, Hesse, Klein, Schottky, Steiner, Sturm and many others. Textbook references include Dolgachev (2010), Miller et al. (1916) and Salmon (1879). It started in 1834 with Plücker's result (Plücker, 1834) that the complex curve $\mathcal{V}_{\mathbb{C}}(f)$ has 28 bitangents. The linear form $\ell = \alpha x + \beta y + \gamma z$ of a bitangent satisfies the identity

$$f(x,y,z) = g(x,y,z)^2 + \ell(x,y,z)\cdot h(x,y,z)$$

for some quadric g and some cubic h. This translates into a system of polynomial equations in $(\alpha:\beta:\gamma)$, and our algorithms start out by solving these equations.

Let K denote the corresponding splitting field, that is, the smallest field extension of $\mathbb{Q}$ that contains the coefficients α,β,γ for all 28 bitangents. The Galois group $\mathrm{Gal}(K,\mathbb{Q})$ is very far from being the symmetric group S_{28}. In fact, if the coefficients c_{ijk} are general enough, it is the Weyl group of E_7 modulo its center,

$$\mathrm{Gal}(K,\mathbb{Q}) \cong W(E_7)/\{\pm 1\} \cong \mathrm{Sp}_6(\mathbb{Z}/2\mathbb{Z}). \qquad (1.4)$$

This group has order $8!\cdot 36 = 1451\,520$, and it is not solvable (Harris, 1979, page 18). We will see a combinatorial representation of this Galois group in Section 3 (Remark 3.13). It is based on Miller et al. (1916, Section 19) and Dolgachev and Ortland (1988, Theorem 9). The connection with $\mathrm{Sp}_6(\mathbb{Z}/2\mathbb{Z})$ arises from the theory of theta functions Dolgachev (2010, Section 5). For further information see Harris (1979, Section II.4).

Naturally, the field extensions needed for (1.2) and (1.3) are much smaller for special quartics. As our running example we take the smooth quartic given by

$$E(x,y,z) = 25\cdot(x^4+y^4+z^4) - 34\cdot(x^2y^2+x^2z^2+y^2z^2).$$

We call this the *Edge quartic*. It is one of the curves in the family studied by Edge (1938, Section 14), and it admits a matrix representation (1.2) over $\mathbb{Q}$:

$$E(x,y,z) = \det\begin{pmatrix} 0 & x+2y & 2x+z & y-2z \\ x+2y & 0 & y+2z & -2x+z \\ 2x+z & y+2z & 0 & x-2y \\ y-2z & -2x+z & x-2y & 0 \end{pmatrix}. \qquad (1.5)$$

The sum of three squares representation (1.3) is derived from the expression

$$\begin{pmatrix} x^2 & y^2 & z^2 & xy & xz & yz \end{pmatrix} \begin{pmatrix} 25 & -55/2 & -55/2 & 0 & 0 & 21 \\ -55/2 & 25 & 25 & 0 & 0 & 0 \\ -55/2 & 25 & 25 & 0 & 0 & 0 \\ 0 & 0 & 0 & 21 & -21 & 0 \\ 0 & 0 & 0 & -21 & 21 & 0 \\ 21 & 0 & 0 & 0 & 0 & -84 \end{pmatrix} \begin{pmatrix} x^2 \\ y^2 \\ z^2 \\ xy \\ xz \\ yz \end{pmatrix} \qquad (1.6)$$

by factoring the above rank-3 matrix as $H^T\cdot H$ where H is a complex 3×6-matrix. The real quartic curve $\mathcal{V}_{\mathbb{R}}(E)$ consists of four ovals and is shown in Fig. 1.

Each of the 28 bitangents of the Edge quartic is defined over $\mathbb{Q}$, but the four shown on the right in Fig. 1 are tangent at complex points of the curve. The following theorem and Table 1 summarize the possible shapes of real quartics.

Theorem 1.7. *There are six possible topological types for a smooth quartic curve $\mathcal{V}_{\mathbb{R}}(f)$ in the real projective plane. They are listed in the first column of Table 1. Each of these six types corresponds to only one connected component in the complement of the discriminant Δ in the 14-dimensional projective space of quartics.*

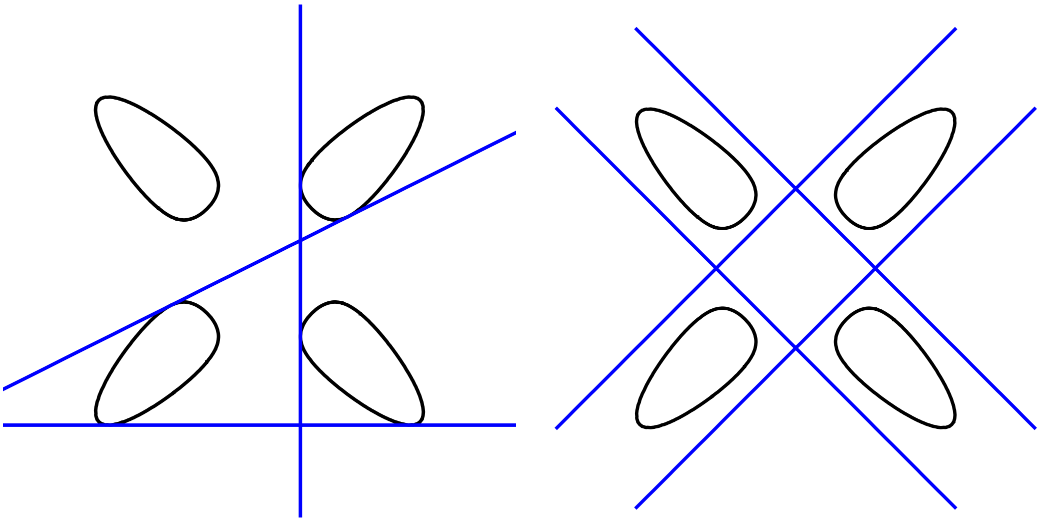


Fig. 1. The Edge quartic and some of its 28 bitangents.

Table 1
The six types of smooth quartics in the real projective plane.

The real curve	Cayley octad	Real bitangents	Real Steiner complexes
4 ovals	8 real points	28	63
3 ovals	6 real points	16	31
2 non-nested ovals	4 real points	8	15
1 oval	2 real points	4	7
2 nested ovals	0 real points	4	15
empty curve	0 real points	4	15

The classification result in Theorem 1.7 is due to Zeuthen (1873). An excellent exposition can be found in Salmon's book (Salmon, 1879, Chapter VI). Klein (1876, Section 5) proved that each type is connected in the complement of the discriminant $\{\Delta = 0\}$. We note that Δ is a homogeneous polynomial of degree 27 in the 15 coefficients c_{ijk} of f. As a preprocessing step in our algorithms, we use the explicit formula for Δ given in Sanyal et al. (2009, Proposition 6.5) to verify that a given quartic curve $\mathcal{V}_{\mathbb{C}}(f)$ is smooth.

The present paper is organized as follows. In Section 2 we present an algorithm, based on Dixon's approach (Dixon, 1902), for computing one determinantal representation (1.2). The resulting 4×4-matrices A, B and C specify three quadratic surfaces in $\mathbb{P}^3$ whose intersection consists of eight points, known as a *Cayley octad*.

In Section 3 we use Cayley octads to compute representatives for all 36 inequivalent classes of determinantal representations (1.2) of the given quartic f. This is accomplished by a combinatorial algorithm developed by Hesse (1855), which realizes the Cremona action (Dolgachev and Ortland, 1988) on the Cayley octads. The output consists of 36 symmetric 8×8-matrices (3.4). These have rank 4 and their 28 entries are linear forms defining the bitangents.

In Section 4 we focus on *Vinnikov quartics*, that is, real quartics consisting of two nested ovals. Helton and Vinnikov (2007) proved the existence of a representation (1.2) over $\mathbb{R}$. We present a symbolic algorithm for computing that representation in practice. Our method uses exact arithmetic and writes the convex inner oval explicitly as a spectrahedron. This settles a question raised by Henrion (2010, Section 1.2).

In Section 5 we identify sums of three squares with *Steiner complexes* of bitangents, and we compute all 63 Gram matrices, i.e. all 6×6-matrices of rank 3 as in (1.6), again using only rational arithmetic over K. This ties in with the results of Powers et al. (2004), where it was proved that a smooth quartic f has precisely 63 inequivalent representations as a sum of three squares (1.3). They strengthened Hilbert's theorem in Hilbert (1888) by showing that precisely eight of these 63 are real when f is positive.

Section 6 is devoted to the boundary and facial structure of the *Gram spectrahedron*. This is the six-dimensional spectrahedron consisting of all sums of squares representations of a fixed positive ternary quartic f. We show that its eight special vertices are connected by 12 edges that form two complete graphs K_4. We also study the structure of the associated semidefinite programming problems.

Section 7 is devoted to the variety of Cayley octads (Dolgachev and Ortland, 1988, Section IX.3). We discuss its defining equations and its boundary strata, we compute the discriminants of (1.2) and (1.3), and we end with a classification of nets of real quadrics in $\mathbb{P}^3$.

We have implemented most of the algorithms presented in this paper in the system SAGE.[1] Our software and supplementary material on quartic curves and Cayley octads can be found at math.berkeley.edu/~cvinzant/quartics.html.

2. Computing a symmetric determinantal representation

We now prove, by way of a constructive algorithm, that every smooth quartic admits a symmetric determinantal representation (1.2). First we compute the 28 bitangents, $\ell = \alpha x + \beta y + \gamma z$. Working on the affine chart $\{\gamma = 1\}$, we equate

$$f\big(x, y, -\alpha x - \beta y\big) = (\kappa_0 x^2 + \kappa_1 xy + \kappa_2 y^2)^2,$$

eliminate $\kappa_0, \kappa_1, \kappa_2$, and solve the resulting system for the unknowns α and β. This constructs the splitting field K for the given f as a finite extension of $\mathbb{Q}$. All further computations in this section are performed via rational arithmetic in K.

Next consider any one of the $\binom{28}{3} = 3276$ triples of bitangents. Multiply their defining linear forms. The resulting polynomial $v_{00} = \ell_1\ell_2\ell_3$ is a *contact cubic* for $\mathcal{V}_{\mathbb{C}}(f)$, which means that the ideal $\langle v_{00}, f\rangle$ in $K[x, y, z]$ defines six points in $\mathbb{P}^2$ each of multiplicity 2. Six points that span three lines in $\mathbb{P}^2$ impose independent conditions on cubics, so the space of cubics in the radical of $\langle v_{00}, f\rangle$ is 4-dimensional over K. We extend $\{v_{00}\}$ to a basis $\{v_{00}, v_{01}, v_{02}, v_{03}\}$ of that space.

Max Noether's Fundamental Theorem (Fulton, 1969, Section 5.5) can be applied to the cubic v_{00} and the quartic f. It implies that a homogeneous polynomial lies in $\langle v_{00}, f\rangle$ if it vanishes to order two at each of the six points of $\mathcal{V}_{\mathbb{C}}\big(\langle v_{00}, f\rangle\big)$. The latter property holds for the sextic forms $v_{0i}v_{0j}$. Hence $v_{0i}v_{0j}$ lies in $\langle v_{00}, f\rangle$ for $1 \le i \le j \le 3$. Using the Extended Buchberger Algorithm, we can compute cubics v_{ij} such that

$$v_{0i}v_{0j} - v_{00}v_{ij} \in \langle f\rangle. \tag{2.1}$$

We now form a symmetric 4×4-matrix V whose entries are cubics in $K[x, y, z]$:

$$V = \begin{pmatrix} v_{00} & v_{01} & v_{02} & v_{03} \\ v_{01} & v_{11} & v_{12} & v_{13} \\ v_{02} & v_{12} & v_{22} & v_{23} \\ v_{03} & v_{13} & v_{23} & v_{33} \end{pmatrix}.$$

The following result is due to Dixon (1902), and it almost solves our problem.

Proposition 2.2. *Each entry of the adjoint V^{adj} is a linear form times f^2, and*

$$\det(f^{-2} \cdot V^{\mathrm{adj}}) = \gamma \cdot f(x, y, z) \quad \textit{for some constant } \gamma \in K.$$

Hence, if $\det(V) \neq 0$ *then* $f^{-2} \cdot V^{\mathrm{adj}}$ *gives a linear matrix representation* (1.2).

Proof. Since $v_{00} \notin \langle f\rangle$, the condition (2.1) implies that, over the quotient ring $K[x, y, z]/\langle f\rangle$, the matrix V has rank 1. Hence, in the polynomial ring $K[x, y, z]$, the cubic f divides all 2×2 minors of V. This implies that f^2 divides all 3×3 minors of V, and f^3 divides $\det(V)$. As the entries of V^{adj} have degree 9, it follows that $V^{\mathrm{adj}} = f^2 \cdot W$, where W is a symmetric matrix whose entries are linear

[1] www.sagemath.org.

forms. Similarly, as $\det(V)$ has degree 12, we have $\det(V) = \delta f^3$ for some $\delta \in K$, and $\delta \neq 0$ unless $\det(V)$ is identically zero. Let I_4 denote the identity matrix. Then

$$\delta f^3 \cdot I_4 = \det(V) \cdot I_4 = V \cdot V^{\mathrm{adj}} = f^2 \cdot V \cdot W.$$

Dividing by f^2 and taking determinants yields

$$\delta^4 f^4 = \det(V) \cdot \det(W) = \delta f^3 \cdot \det(W).$$

This implies the desired identity $\det(W) = \delta^3 f$. □

We now identify the conditions to ensure that $\det(V)$ is not the zero polynomial.

Theorem 2.3. *The determinant of V vanishes if and only if the six points of $\mathcal{V}_{\mathbb{C}}(f, \ell_1\ell_2\ell_3)$, at which the bitangents ℓ_1, ℓ_2, ℓ_3 touch the quartic curve $\mathcal{V}_{\mathbb{C}}(f)$, lie on a conic in $\mathbb{P}^2$. This happens for precisely* 1260 *of the* 3276 *triples of bitangents.*

Proof. Dixon (1902) proves the first assertion. The census of triples appears in the table on page 233 in Salmon's book (Salmon, 1879, Section 262). It is best understood via the Cayley octads in Section 3. For further information see Dolgachev's notes (Dolgachev, 2010, Section 6.1). □

Remark 2.4. Let ℓ_1, ℓ_2, ℓ_3 be any three bitangents of $\mathcal{V}_{\mathbb{C}}(f)$. If the six intersection points with $\mathcal{V}_{\mathbb{C}}(f)$ lie on a conic, the triple $\{\ell_1, \ell_2, \ell_3\}$ is called *syzygetic*, otherwise *azygetic*. A smooth quartic f has 1260 syzygetic and 2016 azygetic triples of bitangents. Similarly, a quadruple $\{\ell_1, \ell_2, \ell_3, \ell_4\}$ of bitangents is called *syzygetic* if its eight contact points lie on a conic and *azygetic* if they do not. Every syzygetic triple ℓ_1, ℓ_2, ℓ_3 determines a fourth bitangent ℓ_4 with which it forms a syzygetic quadruple. Indeed, if the contact points of ℓ_1, ℓ_2, ℓ_3 lie on a conic with defining polynomial q, then q^2 lies in the ideal $\langle f, \ell_1\ell_2\ell_3\rangle$, so that $q^2 = \gamma f + \ell_1\ell_2\ell_3\ell_4$, and the other two points in $\mathcal{V}_{\mathbb{C}}(f, q)$ must be the contact points of the bitangent ℓ_4.

Algorithm 2.5. Given a smooth ternary quartic $f \in \mathbb{Q}[x, y, z]$, we compute the splitting field K over which the 28 bitangents of $\mathcal{V}_{\mathbb{C}}(f)$ are defined. We pick a random triple of bitangents and construct the matrix V via the above method. If $\det(V) \neq 0$, we compute the adjoint of V and divide by f^2, obtaining the desired determinantal representation of f over K. If $\det(V) = 0$, we pick a different triple of bitangents. On each iteration, the probability for $\det(V) \neq 0$ is $\frac{2016}{3276} = \frac{8}{13}$.

Example 2.6. The diagram on the left of Fig. 1 shows an azygetic triple of bitangents to the Edge quartic. Here, the six points of tangency do not lie on a conic. The representation of the Edge quartic in (1.5) is produced by Algorithm 2.5 starting from the cubic $v_{00} = 2(y + 2z)(-2x + z)(x - 2y)$. □

3. Cayley octads and the Cremona action

Algorithm 2.5 outputs a matrix $M = xA + yB + zC$ where A, B, C are symmetric 4×4-matrices with entries in the subfield K of $\mathbb{C}$ over which all 28 bitangents of $\mathcal{V}_{\mathbb{C}}(f)$ are defined. Given one such representation (1.2) of the quartic f, we shall construct a representative from each of the 35 other equivalence classes. Two representations (1.2) are considered *equivalent* if they are in the same orbit under the action of $\mathrm{GL}_4(\mathbb{C})$ by conjugation $M \mapsto U^T M U$. We shall present an algorithm for the following result. It performs rational arithmetic over the splitting field K of the 28 bitangents, and it constructs one representative for each of the 36 orbits.

Theorem 3.1 (*Hesse, 1855*)**.** *Every smooth quartic curve f has exactly* 36 *equivalence classes of linear symmetric determinantal representations* (1.2)*.*

Our algorithm begins by intersecting the three quadric surfaces seen in M:

$$uAu^T = uBu^T = uCu^T = 0 \quad \text{where } u = (u_0 : u_1 : u_2 : u_3) \in \mathbb{P}^3(\mathbb{C}). \tag{3.2}$$

These equations have eight solutions $O_1, \ldots, O_8$. This is the *Cayley octad* of M. In general, a Cayley octad is the complete intersection of three quadrics in $\mathbb{P}^3(\mathbb{C})$.

The next proposition gives a bijection between the 28 bitangents of $\mathcal{V}_{\mathbb{C}}(f)$ and the lines $\overline{O_iO_j}$ for $1 \leq i \leq j \leq 8$. The combinatorial structure of this configuration of 28 lines in $\mathbb{P}^3$ plays an important role for our algorithms.

Proposition 3.3. *Let $O_1, \ldots, O_8$ be the Cayley octad defined above. Then the 28 linear forms $O_i M O_j^T \in \mathbb{C}[x, y, z]$ are the equations of the bitangents of $\mathcal{V}_{\mathbb{C}}(f)$.*

Proof. Fix $i \neq j$. After a change of basis on $\mathbb{C}^4$ given by a matrix $U \in \mathrm{GL}_4(\mathbb{C})$ and replacing M by $U^T M U$, we may assume that $O_i = (1, 0, 0, 0)$ and $O_j = (0, 1, 0, 0)$. The linear form $b_{ij} = O_i M O_j^T$ now appears in the matrix:

$$M = \left(\begin{array}{cc|c} 0 & b_{ij} & \multirow{2}{*}{M'} \\ b_{ij} & 0 & \\ \hline \multicolumn{2}{c|}{(M')^T} & * \end{array} \right).$$

Expanding $\det(M)$ and sorting for terms containing b_{ij} shows that $f = \det(M)$ is congruent to $\det(M')^2$ modulo $\langle b_{ij} \rangle$. This means that b_{ij} is a bitangent. □

Let O be the 8×4-matrix with rows given by the Cayley octad. The symmetric 8×8-matrix OMO^T has rank 4, and we call it the *bitangent matrix* of M. By the definition of O, the bitangent matrix has zeros on the diagonal, and, by Proposition 3.3, its 28 off-diagonal entries are precisely the equations of the bitangents:

$$OMO^T = \begin{pmatrix} 0 & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} & b_{17} & b_{18} \\ b_{12} & 0 & b_{23} & b_{24} & b_{25} & b_{26} & b_{27} & b_{28} \\ b_{13} & b_{23} & 0 & b_{34} & b_{35} & b_{36} & b_{37} & b_{38} \\ b_{14} & b_{24} & b_{34} & 0 & b_{45} & b_{46} & b_{47} & b_{48} \\ b_{15} & b_{25} & b_{35} & b_{45} & 0 & b_{56} & b_{57} & b_{58} \\ b_{16} & b_{26} & b_{36} & b_{46} & b_{56} & 0 & b_{67} & b_{68} \\ b_{17} & b_{27} & b_{37} & b_{47} & b_{57} & b_{67} & 0 & b_{78} \\ b_{18} & b_{28} & b_{38} & b_{48} & b_{58} & b_{68} & b_{78} & 0 \end{pmatrix}. \quad (3.4)$$

Remark 3.5. We can see that the octad $O_1, \ldots, O_8$ consists of K-rational points of $\mathbb{P}^3$: To see this, let K' be the field of definition of the octad over K. Then any element σ of $\mathrm{Gal}(K' : K)$ acts on the octad by permutation, and thus permutes the indices of the bitangents, b_{ij}. On the other hand, as all bitangents are defined over K, σ must fix b_{ij} (up to a constant factor). Thus the permutation induced by σ on the octad must be the identity and $\mathrm{Gal}(K' : K)$ is the trivial group.

Example 3.6. The symmetric matrix M in (1.5) determines the Cayley octad

$$O^T = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 1 & 1 & 3 \\ 0 & 1 & 0 & 0 & 3 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 3 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 & 3 & -1 \end{pmatrix}.$$

All the 28 bitangents of $E(x, y, z)$ are revealed in the bitangent matrix

$$OMO^T = \begin{pmatrix} 0 & x+2y & 2x+z & y-2z & 5x+5y+3z & 5x-3y+5z & 3x+5y-5z & -x+y+z \\ x+2y & 0 & y+2z & -2x+z & x-y+z & 3x+5y+5z & -5x+3y+5z & 5x+5y-3z \\ 2x+z & y+2z & 0 & x-2y & -3x+5z+5y & x-z+y & 5x+3z-5y & 5x+5z+3y \\ y-2z & -2x+z & x-2y & 0 & -3y+5z-5x & -5y-3z+5x & -y-z-x & 5y-5z-3x \\ 5x+5y+3z & x-y+z & -3x+5z+5y & -3y+5z-5x & 0 & 24y+12z & -12x+24z & 24x+12y \\ 5x-3y+5z & 3x+5y+5z & x-z+y & -5y-3z+5x & 24y+12z & 0 & 24x-12y & 12x+24z \\ 3x+5y-5z & -5x+3y+5z & 5x+3z-5y & -y-z-x & -12x+24z & 24x-12y & 0 & 24y-12z \\ -x+y+z & 5x+5y-3z & 5x+5z+3y & 5y-5z-3x & 24x+12y & 12x+24z & 24y-12z & 0 \end{pmatrix}.$$

Each principal 4×4-minors of this matrix is a multiple of $E(x, y, z)$, as in (3.7). □

Each principal 3×3-minor of the bitangent matrix (3.4) is a contact cubic $2b_{ij}b_{ik}b_{jk}$ of $\mathcal{V}_{\mathbb{C}}(f)$ and can serve as the starting point for the procedure in Section 2. Hence, each principal 4×4-minor M_{ijkl} of (3.4) represents the same quartic:

$$\begin{aligned} \det(M_{ijkl}) &= \text{a non-zero scalar multiple of } f(x, y, z) \\ &= b_{ij}^2 b_{kl}^2 + b_{ik}^2 b_{jl}^2 + b_{il}^2 b_{jk}^2 - 2(b_{ij}b_{ik}b_{jl}b_{kl} + b_{ij}b_{il}b_{jk}b_{kl} + b_{ik}b_{il}b_{jk}b_{jl}). \end{aligned} \quad (3.7)$$

However, all these $\binom{8}{4} = 70$ realizations of (1.2) lie in the same equivalence class.

In what follows, we present a simple recipe due to Hesse (1855) for finding 35 alternate bitangent matrices, each of which lies in a different $\mathrm{GL}_4(\mathbb{C})$-orbit. This furnishes all 36 inequivalent determinantal representations promised in Theorem 3.1. We begin with a remark that explains the number 1260 in Theorem 2.3.

Remark 3.8. We can use the combinatorics of the Cayley octad to classify syzygetic collections of bitangents. There are 56 triples $\triangle$ of the form $\{b_{ij}, b_{ik}, b_{jk}\}$. Any such triple is azygetic, by the if-direction in Theorem 2.3, because the cubic $b_{ij}b_{ik}b_{jk}$ appears on the diagonal of the adjoint of the invertible matrix M_{ijkl}. Every product of an azygetic triple of bitangents appears as a 3×3 minor of exactly one of the 36 inequivalent bitangent matrices, giving $36 \cdot 56 = 2016$ azygetic triples of bitangents and $\binom{28}{3} - 2016 = 1260$ syzygetic triples.

A quadruple of bitangents of type $\square$ is of the form $\{b_{ij}, b_{jk}, b_{kl}, b_{il}\}$. Any such quadruple is syzygetic. Indeed, Eq. (3.7) implies $f + 4(b_{ij}b_{jk}b_{kl}b_{il}) = (b_{ij}b_{kl} - b_{ik}b_{jl} + b_{il}b_{jk})^2$, and this reveals a conic containing the eight points of contact.

Consider the following matrix which is obtained by permuting the entries of M_{ijkl}:

$$M'_{ijkl} = \begin{pmatrix} 0 & b_{kl} & b_{jl} & b_{jk} \\ b_{kl} & 0 & b_{il} & b_{ik} \\ b_{jl} & b_{il} & 0 & b_{ij} \\ b_{jk} & b_{ik} & b_{ij} & 0 \end{pmatrix}.$$

This procedure does not change the determinant: $\det(M'_{ijkl}) = \det(M_{ijkl}) = f$. This gives us 70 linear determinantal representations (1.2) of the quartic f, one for each quadruple $I = \{i, j, k, l\} \subset \{1, \ldots, 8\}$. These are equivalent in pairs:

Theorem 3.9. *If $I \neq J$ are quadruples in $\{1, \ldots, 8\}$, then the symmetric matrices M'_I and M'_J are in the same $\mathrm{GL}_4(\mathbb{C})$-orbit if and only if I and J are disjoint. None of these orbits contains the original matrix $M = xA + yB + zC$.*

Proof. Fix $I = \{1, 2, 3, 4\}$ and note the following identity in $K[x, y, z, u_0, u_1, u_2, u_3]$:

$$\begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}^T \begin{pmatrix} 0 & b_{12} & b_{13} & b_{14} \\ b_{12} & 0 & b_{23} & b_{24} \\ b_{13} & b_{23} & 0 & b_{34} \\ b_{14} & b_{24} & b_{34} & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$= u_0u_1u_2u_3 \begin{pmatrix} u_0^{-1} \\ u_1^{-1} \\ u_2^{-1} \\ u_3^{-1} \end{pmatrix}^T \begin{pmatrix} 0 & b_{34} & b_{24} & b_{23} \\ b_{34} & 0 & b_{14} & b_{13} \\ b_{24} & b_{14} & 0 & b_{12} \\ b_{23} & b_{13} & b_{12} & 0 \end{pmatrix} \begin{pmatrix} u_0^{-1} \\ u_1^{-1} \\ u_2^{-1} \\ u_3^{-1} \end{pmatrix}.$$

This shows that the Cayley octad of M'_{1234} is obtained from the Cayley octad of M_{1234} by applying the *Cremona transformation* at O_1, O_2, O_3, O_4. Equivalently, observe that the standard basis vectors of $\mathbb{Q}^4$ are the first four points in the Cayley octads of both M_{1234} and M'_{1234}, and if $O_i = (\alpha_i : \beta_i : \gamma_i : \delta_i)$ for $i = 5, 6, 7, 8$ belong to the Cayley octad of M_{1234}, then $O'_i = (\alpha_i{}^{-1} : \beta_i^{-1} : \gamma_i{}^{-1} : \delta_i^{-1})$ for $i = 5, 6, 7, 8$ belong to the Cayley octad O' of M'_{1234}.

Thus the transformation from M_{ijkl} to M'_{ijkl} corresponds to the Cremona action $\mathrm{cr}_{3,8}$ on Cayley octads, as described on page 107 in the book of Dolgachev and Ortland (1988). Each Cremona transformation changes the projective equivalence class of the Cayley octad, and altogether we recover the 36 distinct classes. That M'_I is equivalent to M'_J when I and J are disjoint can be explained by the following result due to Coble (1929). See Dolgachev and Ortland (1988, Section III.3) for a derivation in modern terms. □

Theorem 3.10. *Let O be an unlabeled configuration of eight points in linearly general position in $\mathbb{P}^3$. Then O is a Cayley octad (i.e. the intersection of three quadrics) if and only if O is self-associated (i.e. fixed under Gale duality; cf. Eisenbud and Popescu (2000)).*

The Cremona action on Cayley octads was known classically as the *bifid substitution*, a term coined by Arthur Cayley himself. We can regard this as a combinatorial rule that permutes and scales the 28 entries of the 8×8 bitangent matrix:

Corollary 3.11. *The entries of the two bitangent matrices $OM_{1234}O^T = (b_{ij})$ and $O'M'_{1234}O'^T = (b'_{ij})$ are related by non-zero scalars in the field K as follows:*

$$\text{The linear form } b'_{ij} \text{ is a scalar multiple of } \begin{cases} b_{kl} & \text{if } \{i,j,k,l\} = \{1,2,3,4\}, \\ b_{ij} & \text{if } |\{i,j\} \cap \{1,2,3,4\}| = 1, \\ b_{kl} & \text{if } \{i,j,k,l\} = \{5,6,7,8\}. \end{cases}$$

Proof. The first case is the definition of M'_{1234}. For the second case we note that

$$\begin{aligned} b_{15} &= O_1 M_{1234} O_5^T = \beta_5 b_{12} + \gamma_5 b_{13} + \delta_5 b_{14} \\ \text{and } b'_{15} &= O'_1 M'_{1234} O_5'^T = \beta_5^{-1} b_{34} + \gamma_5^{-1} b_{24} + \delta_5^{-1} b_{23}, \end{aligned} \tag{3.12}$$

by Proposition 3.3. The identity $O_5 M_{1234} O_5^T = 0$, when combined with (3.12), translates into $\alpha_5 b_{15} + \beta_5\gamma_5\delta_5 b'_{15} = 0$, and hence $b'_{15} = -\alpha_5\beta_5^{-1}\gamma_5^{-1}\delta_5^{-1} b_{15}$. For the last case we consider any pair $\{i,j\} \subset \{5,6,7,8\}$. We know that $b'_{ij} = \nu b_{kl}$, for some $\nu \in K^*$ and $\{k,l\} \subset \{5,6,7,8\}$, by the previous two cases. We must exclude the possibility $\{k,l\} \cap \{i,j\} \neq \emptyset$. After relabeling this would mean $b'_{56} = \nu b_{56}$ or $b'_{56} = \nu b_{57}$. If $b'_{56} = \nu b_{56}$ then the lines $\{b'_{12}, b'_{25}, b'_{56}, b'_{16}\}$ and $\{b_{34}, b_{25}, b_{56}, b_{16}\}$ coincide. This is impossible because the left quadruple is syzygetic while the right quadruple is not, by Remark 3.8. Likewise, $b'_{56} = \nu b_{57}$ would imply that the azygetic triple $\{b'_{15}, b'_{56}, b'_{16}\}$ corresponds to the syzygetic triple $\{b_{15}, b_{57}, b_{16}\}$. □

Remark 3.13. The 35 bifid substitutions of the Cayley octad are indexed by partitions of $[8] = \{1,2,\ldots,8\}$ into pairs of 4-sets. They are discussed in modern language in Dolgachev and Ortland (1988, Proposition 4, page 172). Each bifid substitution determines a permutation of the set $\binom{[8]}{2} = \big\{\{i,j\} : 1 \leq i < j \leq 8\big\}$. For instance, the bifid partition 1234|5678 determines the permutation in Corollary 3.11. Hesse (1855, page 318) wrote these 35 permutations of $\binom{[8]}{2}$ explicitly in a table of format 35×28. Hesse's remarkable table is a combinatorial realization of the Galois group (1.4). Namely, $W(E_7)/\{\pm 1\}$ is the subgroup of column permutations that fixes the rows.

We conclude this section with a remark on the real case. Suppose that f is given by a real symmetric determinantal representation (1.2), i.e. $f = \det(M)$ where $M = xA + yB + zC$ and A, B, C are real symmetric 4×4-matrices. By Vinnikov (1993, Section 0), such a representation exists for every smooth real quartic f. Then the quadrics $uAu^T, uBu^T, uCu^T \in K[u_0, u_1, u_2, u_3]_2$ defining the Cayley octad are real, so that the points $O_1, \ldots, O_8$ are either real or come in conjugate pairs.

Corollary 3.14. *Let $M = xA + yB + zC$ be a real symmetric matrix representation of f with Cayley octad $O_1, \ldots, O_8$. Then the bitangent $O_i^T M O_j$ is defined over $\mathbb{R}$ if and only if O_i and O_j are either real or form a conjugate pair, $O_i = \overline{O_j}$.*

From the possible numbers of real octad points we can infer the numbers of real bitangents stated in Table 1. If $2k$ of the eight points are real, then there are $4 - k$ complex conjugate pairs, giving $\binom{2k}{2} + 4 - k = 2k^2 - 2k + 4$ real bitangents.

4. Spectrahedral representations of Vinnikov quartics

The symmetric determinantal representations $f = \det(M)$ of a ternary quartic $f \in \mathbb{Q}[x,y,z]$ are grouped into 36 orbits under the action of $\mathrm{GL}_4(\mathbb{C})$ given by $M \mapsto T^T M T$. The algorithms in Sections 2 and 3 construct representatives for all 36 orbits. If we represent each orbit by its 8×8-bitangent matrix (3.4), then this serves as a classifier for the 36 orbits. Suppose we are given any other symmetric linear matrix representation $M = xA + yB + zC$ of the same quartic f, and our task is to identify in which of the 36 orbits it lies. We do this by computing the Cayley octad O of M and the resulting bitangent matrix OMO^T. That 8×8-matrix can be located in our list of 36 bitangent matrices by comparing

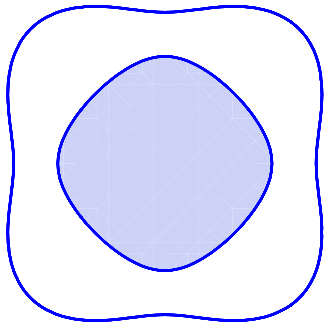


Fig. 2. The Vinnikov quartic in Example 4.1.

principal minors of size 3×3. These minors are products of azygetic triples of bitangents, and they uniquely identify the orbit since there are $2016 = 36 \cdot 56$ azygetic triples.

We now address the problem of finding matrices A, B and C whose entries are real numbers. Theorem 1.7 shows that this is not a trivial matter because none of the 36 bitangent matrices in (3.4) has only real entries, unless the curve $\mathcal{V}_{\mathbb{R}}(f)$ consists of four ovals (as in Fig. 1). We discuss the case when the curve is a *Vinnikov quartic*, which means that $\mathcal{V}_{\mathbb{R}}(f)$ consists of two nested ovals.

As shown in Helton and Vinnikov (2007), the region bounded by the inner oval corresponds exactly to

$$\left\{(x, y, z) \in \mathbb{R}^3 : xA + yB + zC \text{ is positive definite}\right\},$$

a convex cone. This means that the inner oval is a *spectrahedron*. The study of such *spectrahedral representations* is of considerable interest in convex optimization. Recent work by Henrion (2010) underscores the difficulty of this problem for curves of genus $g \geq 2$, and in the last two paragraphs of Henrion (2010, Section 1.2), he asks for the development of a practical implementation. This section constitutes a definitive computer algebra solution to Henrion's problem for smooth quartic curves.

Example 4.1. The following smooth quartic is a Vinnikov curve:

$$f(x, y, z) = 2x^4 + y^4 + z^4 - 3x^2y^2 - 3x^2z^2 + y^2z^2.$$

Running the algorithm in Section 2, we find that the coefficients of the 28 bitangents are expressed in radicals over $\mathbb{Q}$. However, only four of the bitangents are real. Using Theorem 4.3 below, we conclude that there exists a real matrix representation (1.2) with entries expressed in radicals over $\mathbb{Q}$. One such representation is

$$f(x, y, z) = \det \begin{pmatrix} ux + y & 0 & az & bz \\ 0 & ux - y & cz & dz \\ az & cz & x + y & 0 \\ bz & dz & 0 & x - y \end{pmatrix} \quad \text{with} \tag{4.2}$$

$$\begin{aligned} a &= -0.5746420320929616054803275247826307148584936344936 7\ldots, \\ b &= 1.0349259519639555405811894425822590453912925799696 9\ldots, \\ c &= 0.6997059709130126292355709389225602795109611461192 5\ldots, \\ d &= 0.4800486503802432010856027835498806214572648351951\ldots, \\ u &= \sqrt{2} = 1.4142135623730950488016887242096980785696718\ldots. \end{aligned}$$

The expression in radicals is given by the following maximal ideal in $\mathbb{Q}[a, b, c, d, u]$:

$$\begin{aligned} \langle u^2 - 2,\ & 256d^8 - 384d^6u + 256d^6 - 384d^4u + 672d^4 - 336d^2u + 448d^2 - 84u + 121, \\ & 23c + 7584d^7u + 10688d^7 - 5872d^5u - 8384d^5 + 1806d^3u + 2452d^3 - 181du - 307d, \\ & 23b + 5760d^7u + 8192d^7 - 4688d^5u - 6512d^5 + 1452d^3u + 2200d^3 - 212du - 232d, \\ & 23a - 1440d^7u - 2048d^7 + 1632d^5u + 2272d^5 - 570d^3u - 872d^3 + 99du + 81d\rangle. \end{aligned}$$

A picture of the curve $\mathcal{V}_{\mathbb{R}}(f)$ in the affine plane $\{x = 1\}$ is shown in Fig. 2. □

The objective of this section is to establish the following algorithmic result:

Theorem 4.3. *Let $f \in \mathbb{Q}[x, y, z]$ be a quartic whose curve $\mathcal{V}_{\mathbb{C}}(f)$ is smooth. Suppose $f(x, 0, 0) = x^4$ and $f(x, y, 0)$ is squarefree, and let K be the splitting field for its 28 bitangents. Then we can compute a determinantal representation*

$$f(x, y, z) = \det(xI + yD + zR) \tag{4.4}$$

where I is the identity matrix, D is a diagonal matrix, R is a symmetric matrix, and the entries of D and R are expressed in radicals over K. Moreover, there exist such matrices D and R with real entries if and only if $\mathcal{V}_{\mathbb{R}}(f)$ is a Vinnikov curve containing the point $(1 : 0 : 0)$ inside the inner oval.

The hypotheses in Theorem 4.3 impose no loss of generality. Any smooth quartic will satisfy them after a linear change of coordinates $(x : y : z)$ in $\mathbb{P}^2$.

Proof. Using the method in Section 2, we find a first representation $f(x, y, z) = \det(xA + yB + zC)$ over the field K. However, the resulting matrices A, B, C might have non-real entries. The matrix A is invertible because we have assumed $\det(xA) = f(x, 0, 0) = x^4$, which implies $\det(A) = 1$.

The binary form $f(x, y, 0) = \det(xA+yB)$ is squarefree. That assumption guarantees that the 4×4-matrix $A^{-1}B$ has four distinct complex eigenvalues. Since its entries are in K, its four eigenvalues lie in a radical extension field L over K. By choosing a suitable basis of eigenvectors, we find a matrix $U \in \mathrm{GL}_4(L)$ such that $U^{-1}A^{-1}BU$ is a diagonal matrix $D_1 = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ over the field L.

We claim that $D_2 = U^T AU$ and $D_3 = U^T BU$ are diagonal matrices. For each column u_i of U we have $A^{-1}Bu_i = \lambda_i u_i$, so $Bu_i = \lambda_i Au_i$. For $1 \le i < j \le 4$ this implies $u_j^T Bu_i = \lambda_i u_j^T Au_i$ and, by switching indices, we get $u_i^T Bu_j = \lambda_j u_i^T Au_j$. Since B is symmetric, the difference of the last two expressions is zero, and we conclude $(\lambda_i - \lambda_j) \cdot u_i^T Au_j = 0$. By assumption, we have $\lambda_i \neq \lambda_j$ and therefore $u_i^T Au_j = 0$ and $u_i^T Bu_j = 0$. This means that D_2 and D_3 are diagonal.

Let D_4 be the diagonal matrix whose entries are the reciprocals of the square roots of the entries of D_2. These entries are also expressed in radicals over K. Then $D_4D_2D_4 = I$ is the identity matrix, $D_4D_3D_4 = D$ is also diagonal, and

$$D_4U^T MUD_4 = xI + yD + zR$$

is the real symmetric matrix representation required in (4.4).

In order for the entries of D and R to be real numbers, it is necessary (by Helton and Vinnikov (2007)) that $\mathcal{V}_{\mathbb{R}}(f)$ be a Vinnikov curve. We now assume that this is the case. The existence of a real representation (4.4) is due to Vinnikov (1993, Section 0). A transcendental formula for the matrix entries of D and R in terms of theta functions is presented in (4.2) and (4.3) of Helton and Vinnikov (2007, Section 4). We need to show how our algebraic construction above can be used to compute Vinnikov's matrices D and R.

Given a quartic $f \in \mathbb{Q}[x, y, z]$ with leading term x^4, the identity (4.4) translates into a system of 14 polynomial equations in 14 unknowns, namely the four entries of D and the ten entries of R. For an illustration of how to solve them see Example 4.6. We claim that these equations have at most $24 \cdot 8 \cdot 36 = 6912$ complex solutions and all solutions are expressed in radicals over K. Indeed, there are 36 conjugation orbits, and per orbit we have the freedom to transform (4.4) by a matrix T such that $T^T T = I$ and $T^T DT$ is diagonal. Since the entries of D are distinct, these constraints imply that T is a permutation matrix times a diagonal matrix with entries ± 1. There are $24 \cdot 16$ possible choices for T, but T and $-T$ yield the same triple (I, D, R), so the number of solutions per orbit is $24 \cdot 8$.

We conclude that, for each of the 36 orbits, either all representations (4.4) are real or none of them is. Hence, by applying this method to all 36 inequivalent symmetric linear determinantal representations constructed in Section 3, we are guaranteed to find Vinnikov's real matrices D and R. See also Plaumann et al. (2010, Section 2) for additional examples and a more detailed discussion. □

The above argument for the simultaneous diagonalizability of A and B is taken from Greub's linear algebra text book (Greub, 1975). We could also handle the exceptional case when $A^{-1}B$ does not have four distinct eigenvalues. Even in that case there exists a matrix U in radicals over K such that $U^T AU$ and $U^T BU$ are diagonal, but the construction of U is more difficult. The details are found in Greub (1975, Section IX.3).

Corollary 4.5. *Every smooth Vinnikov curve has a real determinantal representation* (1.2) *in radicals over the splitting field K of its* 28 *bitangents.*

We close with the remark that the representation (4.4) generally does not exist over the field K itself but the passage to a radical extension field is necessary.

Example 4.6. All 6912 matrix representations $xI + yD + zR$ of the Edge quartic $E(x, y, z) = 25 \cdot (x^4 + y^4 + z^4) - 34 \cdot (x^2y^2 + x^2z^2 + y^2z^2)$ are non-real and have degree 4 over $\mathbb{Q}$. The entries of D are the four complex zeros of the irreducible polynomial $x^4 - \frac{34}{25}x^2 + 1$. After fixing D, we have 192 choices for R, namely, selecting one of the 36 orbits fixes R up to conjugation by $\mathrm{diag}(\pm 1, \pm 1, \pm 1, \pm 1)$. For the orbit of the matrix $xA + yB + zC$ in (1.5), our algorithm gives the representation

$$D = \begin{pmatrix} -\sqrt{21}/5 - 2i/5 & 0 & 0 & 0 \\ 0 & \sqrt{21}/5 + 2i/5 & 0 & 0 \\ 0 & 0 & -\sqrt{21}/5 + 2i/5 & 0 \\ 0 & 0 & 0 & \sqrt{21}/5 - 2i/5 \end{pmatrix}$$

$$R = \begin{pmatrix} 0 & -\frac{2}{5}(\sqrt{3/7} + i) & -\sqrt{27/35} & 0 \\ -\frac{2}{5}(\sqrt{3/7} + i) & 0 & 0 & \sqrt{27/35} \\ -\sqrt{27/35} & 0 & 0 & -\frac{2}{5}(\sqrt{3/7} - i) \\ 0 & \sqrt{27/35} & -\frac{2}{5}(\sqrt{3/7} - i) & 0 \end{pmatrix}. \quad \square$$

5. Sums of three squares and Steiner complexes

Our next goal is to write the given quartic f as the sum of three squares of quadrics. Such representations (1.3) are classified by Gram matrices of rank 3. A *Gram matrix* for f is a symmetric 6×6 matrix G with entries in $\mathbb{C}$ such that

$$f = v^T \cdot G \cdot v \quad \text{where } v = (x^2, y^2, z^2, xy, xz, yz)^T.$$

We can write $G = H^T \cdot H$, where H is an $r \times 6$-matrix and $r = \mathrm{rank}(G)$. Then the factorization $f = (Hv)^T \cdot (Hv)$ expresses f as the sum of r squares.

It can be shown that no Gram matrix with $r \leq 2$ exists when f is smooth, and there are infinitely many for $r \geq 4$. For $r = 3$ their number is 63 by Theorem 5.1.

Gram matrices classify the representations (1.3): two distinct representations

$$f = q_1^2 + q_2^2 + q_3^2 = p_1^2 + p_2^2 + p_3^2$$

correspond to the same Gram matrix G of rank 3 if and only if there exists an orthogonal matrix $T \in \mathrm{O}_3(\mathbb{C})$ such that $T \cdot (p_1, p_2, p_3)^T = (q_1, q_2, q_3)^T$. The objective of this section is to present an algorithmic proof for the following result.

Theorem 5.1. *Let $f \in \mathbb{Q}[x, y, z]$ be a smooth quartic and K the splitting field for its* 28 *bitangents. Then f has precisely* 63 *Gram matrices of rank* 3*, all of which we compute using rational arithmetic over the field K.*

The fact that f has 63 Gram matrices of rank 3 is a known result due to Coble (1929, Chapter 1, Section 14); see also Powers et al. (2004, Proposition 2.1). Our contribution is a new proof that yields a K-rational algorithm for computing all rank-3 Gram matrices. Instead of appealing to the Jacobian threefold of f, as in Powers et al. (2004), we shall identify the 63 Gram matrices with the 63 Steiner complexes of bitangents (see Salmon (1879, Section VI) and Dolgachev (2010, Section 6)).

We begin by constructing a representation $f = q_1^2 + q_2^2 + q_3^2$ from any pair of bitangents. Let ℓ, ℓ' be distinct bitangents of f, and let $p \in \mathbb{C}[x, y, z]_2$ be a non-singular quadric passing through the four contact points of $\ell\ell'$ with f. By Max Noether's Fundamental Theorem (Fulton, 1969, Section 5.5), the ideal $\langle \ell\ell', f \rangle$ contains p^2, thus

$$f = \ell\ell' u - p^2, \tag{5.2}$$

for some quadric $u \in \mathbb{C}[x, y, z]_2$, after rescaling p by a constant. Over $\mathbb{C}$, the identity (5.2) translates directly into one of the form:

$$f = \left(\frac{1}{2}\ell\ell' + \frac{1}{2}u\right)^2 + \left(\frac{1}{2i}\ell\ell' - \frac{1}{2i}u\right)^2 + (ip)^2. \tag{5.3}$$

Remark 5.4. Just as systems of contact cubics to $\mathcal{V}_{\mathbb{C}}(f)$ were behind the formula (1.2), systems of contact conics to $\mathcal{V}_{\mathbb{C}}(f)$ are responsible for the representations (1.3). The simplest choice of a contact conic is a product of two bitangents.

In (5.3) we wrote f as a sum of three squares over $\mathbb{C}$. There are $\binom{28}{2} = 378$ pairs $\{\ell, \ell'\}$ of bitangents. We will see Theorem 5.10 that each pair forms a syzygetic quadruple with 5 other pairs. This yields $378/6 = 63$ equivalence classes. More importantly, there is a combinatorial rule for determining these 63 classes from a Cayley octad. This allows us to compute the 63 Gram matrices over K.

Eq. (5.2) can also be read as a quadratic determinantal representation

$$f = \det\begin{pmatrix} q_0 & q_1 \\ q_1 & q_2 \end{pmatrix} \tag{5.5}$$

with $q_0 = \ell\ell'$, $q_1 = p$, and $q_2 = u$. This expression gives rise to the quadratic system of contact conics $\{\lambda_0^2 q_0 + 2\lambda_0\lambda_1 q_1 + \lambda_1^2 q_2^2 \;:\; \lambda \in \mathbb{P}^1(\mathbb{C})\}$. The implicitization of this quadratic system is a quadratic form on span$\{q_0, q_1, q_2\}$. With respect to the basis (q_0, q_1, q_2), it is represented by a symmetric 3×3 matrix C. Namely,

$$C = \begin{pmatrix} 0 & 0 & 2 \\ 0 & -1 & 0 \\ 2 & 0 & 0 \end{pmatrix} \quad \text{and its inverse is} \quad C^{-1} = \begin{pmatrix} 0 & 0 & 1/2 \\ 0 & -1 & 0 \\ 1/2 & 0 & 0 \end{pmatrix}.$$

The formula (5.5) shows that $f = q_0q_2 - q_1^2 = (q_0, q_1, q_2) \cdot C^{-1} \cdot (q_0, q_1, q_2)^T$. We now extend q_0, q_1, q_2 to a basis $q = (q_0, q_1, q_2, q_3, q_4, q_5)$ of $\mathbb{C}[x, y, z]_2$. Let T denote the matrix that takes the monomial basis $v = (x^2, y^2, z^2, xy, xz, yz)$ to q. If $\widetilde{G}$ is the 6×6 matrix with C^{-1} in the top left block and zeros elsewhere, then

$$f = (q_0, q_1, q_2) \cdot C^{-1} \cdot (q_0, q_1, q_2)^T = v^T \cdot T^T \cdot \widetilde{G} \cdot T \cdot v. \tag{5.6}$$

Thus, $G = T^T\widetilde{G}T$ is a rank-3 Gram matrix of f. This construction is completely reversible, showing that every rank-3 Gram matrix of f is obtained in this way.

The key player in the formula (5.6) is the quadratic form given by C. From this, one easily gets the Gram matrix G. We shall explain how to find G geometrically from the pair of bitangents ℓ, ℓ'. The following result is taken from Salmon (1879):

Proposition 5.7. *Let $f = \det(Q)$ where Q is a symmetric 2×2-matrix with entries in $\mathbb{C}[x, y, z]_2$ as in (5.5). Then Q defines a quadratic system of contact conics $\lambda^T Q\lambda$, $\lambda \in \mathbb{P}^1(\mathbb{C})$, that contains exactly six products of two bitangents.*

Sketch of Proof. To see that $\lambda^T Q\lambda$ is a contact conic, note that for any $\lambda, \mu \in \mathbb{C}^2$,

$$(\lambda^T Q\lambda)(\mu^T Q\mu) - (\lambda^T Q\mu)^2 = \sum_{i,j,k,l} \lambda_i\lambda_j\mu_k\mu_l(Q_{ij}Q_{kl} - Q_{ik}Q_{jl}). \tag{5.8}$$

The expression $Q_{ij}Q_{kl} - Q_{ik}Q_{jl}$ is a multiple of $\det(Q) = f$, and hence so is the left hand side of (5.8). This shows that $\lambda^T Q\lambda$ is a contact conic of $\mathcal{V}_{\mathbb{C}}(f)$. The set of singular conics is a cubic hypersurface in $\mathbb{C}[x, y, z]_2$. As $\lambda^T Q\lambda$ is quadratic in λ, we see that there are six points $\lambda \in \mathbb{P}^1(\mathbb{C})$ for which $\lambda^T Q\lambda$ is the product of two linear forms. These are bitangents of f and therefore K-rational. □

Remark 5.9. If the Gram matrix G is real, then it is positive (or negative) semidefinite if and only if the quadratic system $\mathcal{Q} = \{\lambda^T Q\lambda \mid \lambda \in \mathbb{P}^1(\mathbb{C})\}$ does not contain any real conics. For if G is real, we may take a real basis (q_0', q_1', q_2') of span$\{q_0, q_1, q_2\} = \ker(G)^\perp$ in $\mathbb{C}[x, y, z]_2$. If $\mathcal{Q}$ does not contain any real conics, then the matrix C' representing $\mathcal{Q}$ with respect to the basis (q_0', q_1', q_2') is definite. Using C' instead of C in the above construction, we conclude that C'^{-1} is definite and hence G is semidefinite. The converse follows by reversing the argument.

We now come to Steiner complexes, the second topic in the section title.

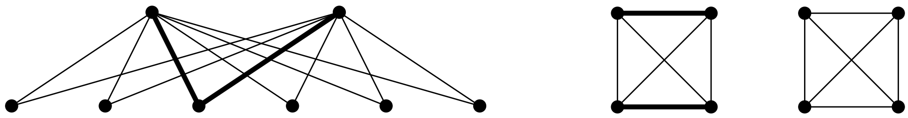
Theorem 5.10. *Let* $\mathcal{S} = \{\{\ell_1, \ell_1'\}, \ldots, \{\ell_6, \ell_6'\}\}$ *be six pairs of bitangents of a smooth quartic* $f \in \mathbb{Q}[x, y, z]$. *Then the following three conditions are equivalent:*

1. *The reducible quadrics* $\ell_1\ell_1', \ldots, \ell_6\ell_6'$ *lie in a system of contact conics* $\lambda^T Q\lambda$, $\lambda \in \mathbb{P}^1(\mathbb{C})$, *for* Q *a quadratic determinantal representation* (5.5) *of* f.
2. *For each* $i \neq j$, *the eight contact points* $\mathcal{V}_{\mathbb{C}}(\ell_i\ell_i'\ell_j\ell_j') \cap \mathcal{V}_{\mathbb{C}}(f)$ *lie on a conic.*
3. *With indices as in the bitangent matrix* (3.4) *for a Cayley octad, either*

$$\mathcal{S} = \{\{b_{ik}, b_{jk}\} \mid \{i, j\} = I \text{ and } k \in I^c\} \quad \text{for a 2-set } I \subset \{1, \ldots, 8\},$$
$$\text{or } \mathcal{S} = \{\{b_{ij}, b_{kl}\} \mid \{i, j, k, l\} = I \text{ or } \{i, j, k, l\} = I^c\} \quad \text{for a 4-set } I \subset \{1, \ldots, 8\}.$$

Proof. This is a classical result due to Hesse (1855). The proof can also be found in the books of Salmon (1879) and Miller et al. (1916, Section 185–186). □

A *Steiner complex* (Steiner, 1855) is a sextuple $\mathcal{S}$ of pairs of bitangents satisfying the conditions of Theorem 5.10. A pair of bitangents in $\mathcal{S}$ is either of the form $\{b_{ik}, b_{jk}\}$ (referred to as type $\bigvee$) or of the form $\{b_{ij}, b_{kl}\}$ (type $||$). The first type of Steiner complex in Theorem 5.10(3) contains pairs of bitangents of type $\bigvee$ and the second type contains pairs of type $||$. There are $\binom{8}{2} = 28$ Steiner complexes of type $\bigvee$ and $\binom{8}{4}/2 = 35$ Steiner complexes of bitangents of type $||$. The two types of Steiner complexes are easy to remember by the following combinatorial pictures:



Type $\bigvee$ Type $||$

This combinatorial encoding of Steiner complexes enables us to derive the last column in Table 1 in the introduction. We represent the quartic as (1.3) with A, B, C real, as in Vinnikov (1993). The corresponding Cayley octad $\{O_1, \ldots, O_8\}$ is invariant under complex conjugation. Let π be the permutation in S_8 that represents complex conjugation, meaning $\overline{O_i} = O_{\pi(i)}$. Then complex conjugation on the 63 Steiner complexes is given by the action of π on their labels. For instance, when all O_i are real, as in the first row of Table 1, then π is the identity. For the other rows we can relabel so that $\pi = (12)$, $\pi = (12)(34)$, $\pi = (12)(34)(56)$ and $\pi = (12)(34)(56)(78)$. We say that a Steiner complex $\mathcal{S}$ is *real* if its labels are fixed under π. For example, if $\mathcal{S}$ is the Steiner complex $\{\{b_{13}, b_{23}\}, \ldots, \{b_{18}, b_{28}\}\}$ of type $\bigvee$ as above, then $\mathcal{S}$ is real if and only if π fixes $\{1, 2\}$. Similarly, if $\mathcal{S}$ is the Steiner complex $\{\{b_{12}, b_{34}\}, \{b_{13}, b_{24}\}, \{b_{14}, b_{23}\}, \{b_{56}, b_{78}\}, \{b_{57}, b_{68}\}, \{b_{58}, b_{67}\}\}$ of type $||$, then $\mathcal{S}$ is real if and only if π fixes the partition $\{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}\}$. For instance, for the empty curve, in the last row Table 1, one can check that exactly 15 Steiner complexes are fixed by $\pi = (12)(34)(56)(78)$, as listed in Section 6.

We now sum up what we have achieved in this section, namely, a recipe for constructing the 63 Gram matrices from the 28 + 35 Steiner complexes $\bigvee$ and $||$.

Proof and Algorithm for Theorem 5.1. We take as input a smooth ternary quartic $f \in \mathbb{Q}[x, y, z]$ and any of the 63 Steiner complexes $\{\{\ell_1, \ell_1'\}, \ldots, \{\ell_6, \ell_6'\}\}$ of bitangents of $\mathcal{V}_{\mathbb{C}}(f)$. From this we can compute a rank-3 Gram matrix for f as follows. The six contact conics $\ell_i\ell_i'$ span a 3-dimensional subspace of $K[x, y, z]_2$, by Theorem 5.10(1), of which $\{\ell_1\ell_1', \ell_2\ell_2', \ell_3\ell_3'\}$ is a basis. The six vectors $\ell_i\ell_i'$ lie on a conic in that subspace, and we compute the symmetric 3×3-matrix $\widetilde{C}$ representing this conic in the chosen basis. We then extend its inverse $\widetilde{C}^{-1}$ by zeroes to a 6×6 matrix $\widetilde{G}$ and fix an arbitrary basis $\{q_4, q_5, q_6\}$ of $\text{span}\{\ell_1\ell_1', \ell_2\ell_2', \ell_3\ell_3'\}^{\perp}$ in $K[x, y, z]_2$. Let $T \in K^{6\times 6}$ be the matrix taking $v = (x^2, y^2, z^2, xy, xz, yz)^T$ to $(\ell_1\ell_1', \ell_2\ell_2', \ell_3\ell_3', q_4, q_5, q_6)^T$. Then $G = T^T\widetilde{G}T$ is the desired rank-3 Gram matrix for f, and all rank-3 Gram matrices arise in this way. Note that G does not depend on the choice of q_4, q_5, q_5. □

Remark 5.11. Given f, finding a Steiner complex as input for the above algorithm is not a trivial task. But when a linear determinantal representation of f is known, and thus a Cayley octad, one can use the criterion in Theorem 5.10(3).

Example 5.12. We consider the quartic $f = \det(M)$ defined by the matrix

$$M = \begin{pmatrix} 52x+12y-60z & -26x-6y+30z & 48z & 48y \\ -26x-6y+30z & 26x+6y-30z & -6x+6y-30z & -45x-27y-21z \\ 48z & -6x+6y-30z & -96x & 48x \\ 48y & -45x-27y-21z & 48x & -48x \end{pmatrix}.$$

The complex curve $\mathcal{V}_{\mathbb{C}}(f)$ is smooth and its set of real points $\mathcal{V}_{\mathbb{R}}(f)$ is empty. The corresponding Cayley octad consists of four pairs of complex conjugates:

$$O^T = \begin{pmatrix} i & -i & 0 & 0 & -6+4i & -6-4i & 3+2i & 3-2i \\ 1+i & 1-i & 0 & 0 & -4+4i & -4-4i & 7-i & 7+i \\ 0 & 0 & i & -i & -3+2i & -3-2i & -\frac{86}{39}-\frac{4}{13}i & -\frac{86}{39}+\frac{4}{13}i \\ 0 & 0 & 1+i & 1-i & 1-i & 1+i & \frac{4}{39}-\frac{20}{39}i & \frac{4}{39}+\frac{20}{39}i \end{pmatrix}.$$

Here the 8×8 bitangent matrix $OMO^T = (b_{ij})$ is defined over the field $K = \mathbb{Q}(i)$ of Gaussian rationals, and hence so are all 63 Gram matrices. According to the lower right entry in Table 1, precisely 15 of the Gram matrices are real, and hence these 15 Gram matrices have their entries in $\mathbb{Q}$. For instance, the representation

$$f = 288 \begin{pmatrix} x^2 \\ y^2 \\ z^2 \\ xy \\ xz \\ yz \end{pmatrix}^T \begin{pmatrix} 45\,500 & 3102 & -9861 & 5718 & -9246 & 4956 \\ 3102 & 288 & -747 & 882 & -18 & -144 \\ -9861 & -747 & 3528 & -864 & -1170 & -504 \\ 5718 & 882 & -864 & 4440 & 1104 & -2412 \\ -9246 & -18 & -1170 & 1104 & 11\,814 & -5058 \\ 4956 & -144 & -504 & -2412 & -5058 & 3582 \end{pmatrix} \begin{pmatrix} x^2 \\ y^2 \\ z^2 \\ xy \\ xz \\ yz \end{pmatrix}$$

is obtained by applying our algorithm for Theorem 5.1 to the Steiner complex

$$\mathcal{S} = \big\{\{b_{13}, b_{58}\},\ \{b_{15}, b_{38}\},\ \{b_{18}, b_{35}\},\ \{b_{24}, b_{67}\},\ \{b_{26}, b_{47}\},\ \{b_{27}, b_{46}\}\big\}.$$

The above Gram matrix has rank 3 and is positive semidefinite, so it translates into a representation (1.3) for f as the sum of three squares of quadrics over $\mathbb{R}$. □

6. The Gram spectrahedron

The *Gram spectrahedron* Gram(f) of a real ternary quartic f is the set of its positive semidefinite Gram matrices. This spectrahedron is the intersection of the cone of positive semidefinite 6×6-matrices with a 6-dimensional affine subspace. By Hilbert's result in Hilbert (1888), Gram(f) is non-empty if and only if f is non-negative. In terms of coordinates on the 6-dimensional subspace given by a fixed quartic

$$f(x, y, z) = c_{400}x^4 + c_{310}x^3y + c_{301}x^3z + c_{220}x^2y^2 + c_{211}x^2yz + \cdots + c_{004}z^4,$$

the Gram spectrahedron Gram(f) is the set of all positive semidefinite matrices

$$\begin{pmatrix} c_{400} & \lambda_1 & \lambda_2 & \frac{1}{2}c_{310} & \frac{1}{2}c_{301} & \lambda_4 \\ \lambda_1 & c_{040} & \lambda_3 & \frac{1}{2}c_{130} & \lambda_5 & \frac{1}{2}c_{031} \\ \lambda_2 & \lambda_3 & c_{004} & \lambda_6 & \frac{1}{2}c_{103} & \frac{1}{2}c_{013} \\ \frac{1}{2}c_{310} & \frac{1}{2}c_{130} & \lambda_6 & c_{220}-2\lambda_1 & \frac{1}{2}c_{211}-\lambda_4 & \frac{1}{2}c_{121}-\lambda_5 \\ \frac{1}{2}c_{301} & \lambda_5 & \frac{1}{2}c_{103} & \frac{1}{2}c_{211}-\lambda_4 & c_{202}-2\lambda_2 & \frac{1}{2}c_{112}-\lambda_6 \\ \lambda_4 & \frac{1}{2}c_{031} & \frac{1}{2}c_{013} & \frac{1}{2}c_{121}-\lambda_5 & \frac{1}{2}c_{112}-\lambda_6 & c_{022}-2\lambda_3 \end{pmatrix}, \quad \text{where } \lambda \in \mathbb{R}^6. \qquad (6.1)$$

The main result of Powers et al. (2004) is that a smooth positive quartic f has exactly eight inequivalent representations as a sum of three real squares, which had been conjectured in Powers and Reznick (2000). These eight representations correspond to rank-3 positive semidefinite Gram matrices. We call these the *vertices of rank* 3 of Gram(f). In Section 5 we compute them using arithmetic over K.

We define the *Steiner graph* of the Gram spectrahedron to be the graph on the eight vertices of rank 3 whose edges represent edges of the convex body Gram(f).

Theorem 6.2. *The Steiner graph of the Gram spectrahedron* Gram(f) *of a generic positive ternary quartic f is the disjoint union $K_4 \sqcup K_4$ of two complete graphs, and the relative interiors of these edges consist of rank-*5 *matrices.*

This theorem means that the eight rank-3 Gram matrices are divided into two groups of four, and, for G and G' in the same group, we have $\mathrm{rank}(G + G') \leq 5$. The second sentence asserts that $\mathrm{rank}(G + G') = 5$ holds for generic f. For the proof it suffices to verify this for one specific f. This we have done, using exact arithmetic, for the quartic in Example 5.12. For instance, the rank-3 vertices

$$\left(\frac{1}{288}\right) G = \begin{pmatrix} 45\,500 & 3102 & -9861 & 5718 & -9246 & 4956 \\ 3102 & 288 & -747 & 882 & -18 & -144 \\ -9861 & -747 & 3528 & -864 & -1170 & -504 \\ 5718 & 882 & -864 & 4440 & 1104 & -2412 \\ -9246 & -18 & -1170 & 1104 & 11\,814 & -5058 \\ 4956 & -144 & -504 & -2412 & -5058 & 3582 \end{pmatrix}$$

$$\left(\frac{1}{288}\right) G' = \begin{pmatrix} 45\,500 & -2802 & -6666 & 5718 & -9246 & 132 \\ -2802 & 288 & -72 & 882 & 1206 & -144 \\ -6666 & -72 & 3528 & -4878 & -1170 & -504 \\ 5718 & 882 & -4878 & 16\,248 & 5928 & -3636 \\ -9246 & 1206 & -1170 & 5928 & 5424 & -1044 \\ 132 & -144 & -504 & -3636 & -1044 & 2232 \end{pmatrix}$$

both contain the vector $(11\,355, -4241, 47\,584, 8325, 28\,530, 36\,706)^T$ in their kernel, so that $\mathrm{rank}(G + G') \leq 5$. But this vector spans the intersection of the kernels, hence $\mathrm{rank}(G + G') = 5$, and every matrix on the edge has rank 5.

We also know that there exist instances of smooth positive quartics where the rank along an edge drops to 4. One such example is the Fermat quartic, $x^4 + y^4 + z^4$, which has two psd rank-3 Gram matrices whose sum has rank 4. We do not know whether the Gram spectrahedron Gram(f) has proper faces of dimension ≥ 1 other than the twelve edges in the Steiner graph $K_4 \sqcup K_4$. In particular, we do not know whether the Steiner graph coincides with the graph of all edges of Gram(f).

Proof of Theorem 6.2. Fix a real symmetric linear determinantal representation $M = xA + yB + zC$ of f. The existence of such M when f is positive was proved by Vinnikov (1993, Section 0). The Cayley octad $\{O_1, \ldots, O_8\}$ determined by M consists of four pairs of complex conjugate points. Recall from Section 5 that a Steiner complex corresponds to either a subset $I \subset \{1, \ldots, 8\}$ with $|I| = 2$ (type $\bigvee$) or a partition $I|I^c$ of $\{1, \ldots, 8\}$ into two subsets of size 4 (type $||$). We write $\mathcal{S}_I$ for the Steiner complex given by I or $I|I^c$ and G_I for the corresponding Gram matrix. Theorem 6.2 follows from the more precise result in Theorem 6.3 which we shall prove further below. □

Theorem 6.3. *Let f be positive with $\mathcal{V}_{\mathbb{C}}(f)$ smooth and conjugation acting on the Cayley octad by $\overline{O}_i = O_{\pi(i)}$ for $\pi = (12)(34)(56)(78)$. The eight Steiner complexes corresponding to the vertices of rank* 3 *of the Gram spectrahedron* Gram(f) *are*

1357|2468	1368|2457	1458|2367	1467|2358
1358|2467	1367|2458	1457|2368	1468|2357.

The Steiner graph $K_4 \sqcup K_4$ is given by pairs of Steiner complexes in the same row.

Our proof of Theorem 6.3 consists of two parts: (1) showing that the above Steiner complexes give the positive semidefinite Gram matrices and (2) showing how they form two copies of K_4. We will begin by assuming (1) and proving (2):

By Theorem 5.10, for any two pairs of bitangents $\{\ell_1, \ell_1'\}$ and $\{\ell_2, \ell_2'\}$ in a fixed Steiner complex $\mathcal{S}$, there is a conic u in $\mathbb{P}^2$ that passes through the eight contact points of these four bitangents with $\mathcal{V}_{\mathbb{C}}(f)$. In this manner, one associates with every Steiner complex $\mathcal{S}$ a set of $\binom{6}{2} = 15$ conics, denoted conics($\mathcal{S}$).

Lemma 6.4. *Let $\mathcal{S}$ and $\mathcal{T}$ be Steiner complexes with Gram matrices $G_{\mathcal{S}}$ and $G_{\mathcal{T}}$. If* $\text{conics}(\mathcal{S}) \cap \text{conics}(\mathcal{T}) \neq \emptyset$ *then* $\text{rank}(G_{\mathcal{S}} + G_{\mathcal{T}}) \leq 5$.

Proof. Suppose $\mathcal{S} = \{\{\ell_1, \ell_1'\}, \ldots, \{\ell_6, \ell_6'\}\}$. Let Q be a quadratic matrix representation (5.5) such that the six points $\ell_1\ell_1', \ldots, \ell_6\ell_6' \in \mathbb{P}(\mathbb{C}[x, y, z]_2)$ lie on the conic $\{\lambda^T Q \lambda : \lambda \in \mathbb{P}^1(\mathbb{C})\}$. By the construction in the proof of Theorem 5.1, we know that the projective plane in $\mathbb{P}(\mathbb{C}[x, y, z]_2)$ spanned by this conic is $\ker(G_{\mathcal{S}})^{\perp}$.

Consider two pairs $\{\ell_1, \ell_1'\}, \{\ell_2, \ell_2'\}$ from $\mathcal{S}$ and let $u \in \text{conics}(\mathcal{S})$ be the unique conic passing through the eight contact points of these bitangents with the curve $\mathcal{V}_{\mathbb{C}}(f)$. By our choice of Q, we can find $\lambda, \mu \in \mathbb{P}^1$ such that $\lambda^T Q \lambda = \ell_1\ell_1'$ and $\mu^T Q \mu = \ell_2\ell_2'$. Eq. (5.8) then shows that $u = \lambda^T Q \mu$. From this we see that $u \in \text{span}\{Q_{11}, Q_{12}, Q_{22}\} = \ker(G_{\mathcal{S}})^{\perp}$. Therefore, $\text{conics}(\mathcal{S}) \subseteq \ker(G_{\mathcal{S}})^{\perp}$.

If $\text{conics}(\mathcal{S}) \cap \text{conics}(\mathcal{T}) \neq \emptyset$, then the two 3-planes $\ker(G_{\mathcal{S}})^{\perp}$ and $\ker(G_{\mathcal{T}})^{\perp}$ meet nontrivially. Since $\mathbb{C}[x, y, z]_2$ has dimension 6, this implies that $\ker(G_{\mathcal{S}})$ and $\ker(G_{\mathcal{T}})$ meet nontrivially. Hence $\text{rank}(G_{\mathcal{S}} + G_{\mathcal{T}}) \leq 5$. □

For example, conics($\mathcal{S}_{1358}$) and conics($\mathcal{S}_{1457}$) share the conic going through the contact points of b_{15}, b_{26}, b_{38}, and b_{47}. Lemma 6.4 then implies $\text{rank}(G_{1358} + G_{1457}) \leq 5$, as shown above for Example 5.12 with $G = G_{1358}$ and $G' = G_{1457}$.

Using this approach, we only have to check that $\text{conics}(\mathcal{S}_I) \cap \text{conics}(\mathcal{S}_J) \neq \emptyset$ when I and J are in the same row of the table in Theorem 6.3. More precisely:

Lemma 6.5. *Let I and J be subsets of $\{1, \ldots, 8\}$ of size four with $I \neq J$ and $I \neq J^c$. Then* $\text{conics}(\mathcal{S}_I) \cap \text{conics}(\mathcal{S}_J) \neq \emptyset$ *if and only if* $|I \cap J| = 2$.

Proof. Every syzygetic set of four bitangents $\ell_1, \ell_2, \ell_3, \ell_4$ determines a unique conic u passing through their eight contact points with $\mathcal{V}_{\mathbb{C}}(f)$. There are three ways to collect the four bitangents into two pairs, so u appears in conics($\mathcal{S}$) for exactly three Steiner complexes. For two Steiner complexes $\mathcal{S}_I$ and $\mathcal{S}_J$, we have $\text{conics}(\mathcal{S}_I) \cap \text{conics}(\mathcal{S}_J) \neq \emptyset$ if and only if there are bitangents $\ell_1, \ell_2, \ell_3, \ell_4$ such that $\{\ell_1,\ell_2\}, \{\ell_3,\ell_4\} \in \mathcal{S}_I$ and $\{\ell_1,\ell_3\}, \{\ell_2,\ell_4\} \in \mathcal{S}_J$. This translates into $|I \cap J| = 2$. □

To complete the proof of Theorem 6.3, it remains to show that the eight listed Steiner complexes give positive semidefinite Gram matrices. Recall that a Steiner complex $\mathcal{S}_I$ is real if and only if I is fixed by the permutation π coming from conjugation. As stated in Section 3, there are 15 real Steiner complexes, namely,

1. The eight complexes of type || listed in Theorem 6.3.
2. Three more complexes of type ||, namely 1234|5678, 1256|3478, 1278|3456.
3. Four complexes of type $\bigvee$, namely 12, 34, 56, 78.

Since we know from Powers et al. (2004) that exactly eight of these give positive semidefinite Gram matrices, it suffices to rule out the seven Steiner complexes in (2) and (3). Every Steiner complex $\mathcal{S}_I$ gives rise to a system of contact conics $\mathcal{Q}_I = \{\lambda^T Q_I \lambda, \lambda \in \mathbb{P}^1(\mathbb{C})\}$, where Q_I is a symmetric 2×2-matrix as in (5.5), and a rank-3 Gram matrix G_I for f. The following proposition is a direct consequence of Remark 5.9.

Proposition 6.6. *Let $\mathcal{S}_I$ be a real Steiner complex. The Gram matrix G_I is positive semidefinite if and only if the system $\mathcal{Q}_I$ does not contain any real conics.*

It follows that if $\mathcal{S}_I$ is one of the three Steiner complexes in (2), then the Gram matrix G_I is not positive semidefinite, since the system $\mathcal{Q}_I$ contains a product of two of the real bitangents $b_{12}, b_{34}, b_{56}, b_{78}$. Thus it remains to show that if $I = ij$ with $ij \in \{12, 34, 56, 78\}$ as in (3), then the system $\mathcal{Q}_{ij}$ contains a real conic.

The symmetric linear determinantal representation M gives rise to the system $\{\lambda^T M^{\text{adj}} \lambda \mid \lambda \in \mathbb{P}^3(\mathbb{C})\}$ of (azygetic) contact cubics (see Dolgachev (2010, Section 6.3)). The main idea of the following

Table 2
Statistics for semidefinite programming over Gram spectrahedra.

Rank of optimal matrix	3	4	5	Any
Algebraic degree	63	38	1	102
Probability	2.01%	95.44%	2.55%	100%

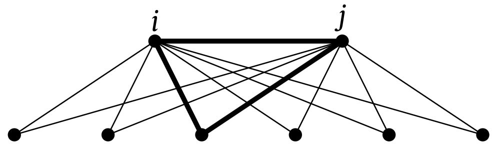
is that multiplying a bitangent with a contact conic of f gives a contact cubic, and if both the bitangent and the cubic are real, then the conic must be real. The next lemma identifies products of bitangents and contact conics inside the system of contact cubics given by M.

Lemma 6.7. *For $i \neq j$ we have $b_{ij} \cdot \mathcal{Q}_{ij} = \{\lambda^T M^{\mathrm{adj}} \lambda \mid \lambda \in \mathrm{span}\{O_i, O_j\}^\perp\}$.*

Proof. After a change of coordinates, we can assume that O_i, O_j, O_k, O_l are the four unit vectors e_1, e_2, e_3, e_4. This means that $M = xA + yB + zC$ takes the form

$$M = \begin{pmatrix} 0 & b_{ij} & b_{ik} & b_{il} \\ b_{ij} & 0 & b_{jk} & b_{jl} \\ b_{ik} & b_{jk} & 0 & b_{kl} \\ b_{il} & b_{jl} & b_{kl} & 0 \end{pmatrix}.$$

Consider the three 3×3-minors complementary to the lower 2×2-block of M. They are $e_3^T M^{\mathrm{adj}} e_3$, $e_3^T M^{\mathrm{adj}} e_4$, $e_4^T M^{\mathrm{adj}} e_4$. We check that all three are divisible by b_{ij}. Therefore $b_{ij}^{-1} \cdot \lambda^T M^{\mathrm{adj}} \lambda$ with $\lambda \in \mathrm{span}\{e_3, e_4\}$ is a system of contact conics. Note that $b_{ik} b_{jk} = b_{ij}^{-1} e_4^T M^{\mathrm{adj}} e_4$. Similarly, we can find the other six products of pairs of bitangents from the Steiner complex $\mathcal{S}_{ij}$, as illustrated by the following picture:



Hence the system of contact conics $\mathcal{Q}_{ij}$ arises from division by b_{ij} as asserted. □

Proof of Theorem 6.3 (and hence of Theorem 6.2). With all the various lemmas in place, only one tiny step is left to be done. Fix any of the four Steiner complexes ij of type $\bigvee$ in (3). Then the bitangent b_{ij} is real. Since M is real and $\overline{O_i} = O_j$, we can pick a real point $\lambda \in \mathrm{span}\{O_i, O_j\}^\perp$. Lemma 6.7 implies that that $\mathcal{Q}_{ij}$ contains the real conic $b_{ij}^{-1} \cdot \lambda^T M^{\mathrm{adj}} \lambda$. Proposition 6.6 now completes the proof. □

Semidefinite programming over the Gram spectrahedron Gram(f) means finding the best sum of squares representation of a positive quartic f, where "best" refers to some criterion that can be expressed as a linear functional on Gram matrices. This optimization problem is of particular interest from the perspective of Tables 1 and 2 in Nie et al. (2010), because $m = n = 6$ is the smallest instance where the Pataki range of optimal ranks has size three. For the definition of *Pataki range* see also (5.16) in Rostalski and Sturmfels (in press, Section 5). The matrix rank of the exposed vertices of a *generic* 6-dimensional spectrahedron of 6×6-matrices can be either 3, 4 or 5.

The Gram spectrahedra Gram(f) are not generic but they exhibit the generic behavior as far as the Pataki range is concerned. Namely, if we optimize a linear function over Gram(f) then the rank of the optimal matrix can be either 3, 4 or 5. We obtained the following numerical result for the distribution of these ranks by optimizing a random linear function over Gram(f) for randomly chosen f:

The sampling in Table 2 was done in `matlab`,[2] using the random matrix generator. This distribution for the three possible ranks appears to be close to that of the generic case, as given in Nie et al. (2010, Table 1). The algebraic degree of the optimal solution, however, is much lower than in the generic situation of Nie et al. (2010, Table 2), where the three degrees are 112, 1400 and 32. For example,

2 www.mathworks.com.

while the rank-3 locus on the generic spectrahedron has 112 points over $\mathbb{C}$, our Gram spectrahedron Gram(f) has only 63, one for each Steiner complex.

The greatest surprise in Table 2 is the number 1 for the algebraic degree of the rank-5 solutions. This means that the optimal solution of a rational linear function over the Gram spectrahedron Gram(f) is $\mathbb{Q}$-rational whenever it has rank 5. For a concrete example, consider the problem of maximizing the function

$$159\lambda_1 - 9\lambda_2 + 34\lambda_3 + 73\lambda_4 + 105\lambda_5 + 86\lambda_6$$

over the Gram spectrahedron Gram(f) of the *Fermat quartic* $f = x^4 + y^4 + z^4$. The optimal solution for this instance is the rank-5 Gram matrix (6.1) with coordinates

$$\lambda = \left(\frac{-867799528369}{6890409751681}, \frac{-7785115393679}{13780819503362}, \frac{-2624916076477}{6890409751681}, \right.$$
$$\left. \times \frac{1018287438360}{6890409751681}, \frac{2368982554265}{6890409751681}, \frac{562671279961}{6890409751681} \right).$$

The drop from 1400 to 38 for the algebraic degree of optimal Gram matrices of rank 4 is dramatic. It would be nice to understand the geometry behind this. We finally note that the algebraic degrees 63, 38, 1 in Table 2 were computed using `Macaulay2`[3] by elimination from the KKT equations, as described in Rostalski and Sturmfels (in press, Section 5).

7. The variety of Cayley octads

The Cayley octads form a subvariety of codimension three in the space of eight labeled points in $\mathbb{P}^3$. A geometric study of this variety was undertaken by Dolgachev and Ortland (1988, Section IX.3), building on classical work of Coble (1929). This section complements their presentation with several explicit formulas we found useful for constructing examples and for performing symbolic computations. Besides convex algebraic geometry (Helton and Vinnikov, 2007; Henrion, 2010; Rostalski and Sturmfels, in press), our results have potential applications in *number theory* (e.g. arithmetic of del Pezzo varieties (Dolgachev and Ortland, 1988, Section V)) and *integrable systems* (e.g. 3-phase solutions to the Kadomtsev–Petviashvili equation (Dubrovin et al., 1997)). In Theorem 7.5 we compute the discriminant of the quartics (1.2) and (1.3), and in Proposition 7.8 we discuss an application to nets of real quadrics in $\mathbb{P}^3$.

We begin with the fact that a Cayley octad is determined by any seven of its points. Here is a rational formula for the eighth point in terms of the first seven.

Proposition 7.1. *Consider a general configuration $\mathcal{C}$ of seven points in $\mathbb{P}^3$, with coordinates* (1:0:0:0), (0:1:0:0), (0:0:1:0), (0:0:0:1), (1:1:1:1), $(\alpha_6{:}\beta_6{:}\gamma_6{:}\delta_6)$ *and* $(\alpha_7{:}\beta_7{:}\gamma_7{:}\delta_7)$. *The unique point* $(\alpha_8{:}\beta_8{:}\gamma_8{:}\delta_8)$ *in $\mathbb{P}^3$ which completes $\mathcal{C}$ to a Cayley octad is given by the following rational functions in the eight free parameters:*

$$\alpha_8 = \frac{\beta_6\gamma_7 - \beta_6\delta_7 - \gamma_6\beta_7 + \gamma_6\delta_7 + \delta_6\beta_7 - \delta_6\gamma_7}{\beta_6\gamma_6\beta_7\delta_7 - \beta_6\gamma_6\gamma_7\delta_7 - \beta_6\delta_6\beta_7\gamma_7 + \beta_6\delta_6\gamma_7\delta_7 + \gamma_6\delta_6\beta_7\gamma_7 - \gamma_6\delta_6\beta_7\delta_7},$$
$$\beta_8 = \frac{\alpha_6\gamma_7 - \alpha_6\delta_7 - \gamma_6\alpha_7 + \gamma_6\delta_7 + \delta_6\alpha_7 - \delta_6\gamma_7}{\alpha_6\gamma_6\alpha_7\delta_7 - \alpha_6\gamma_6\gamma_7\delta_7 - \alpha_6\delta_6\alpha_7\gamma_7 + \alpha_6\delta_6\gamma_7\delta_7 + \gamma_6\delta_6\alpha_7\gamma_7 - \gamma_6\delta_6\alpha_7\delta_7},$$
$$\gamma_8 = \frac{\alpha_6\beta_7 - \alpha_6\delta_7 - \beta_6\alpha_7 + \beta_6\delta_7 + \delta_6\alpha_7 - \delta_6\beta_7}{\alpha_6\beta_6\alpha_7\delta_7 - \alpha_6\beta_6\beta_7\delta_7 - \alpha_6\delta_6\alpha_7\beta_7 + \alpha_6\delta_6\beta_7\delta_7 + \beta_6\delta_6\alpha_7\beta_7 - \beta_6\delta_6\alpha_7\delta_7},$$
$$\delta_8 = \frac{\alpha_6\beta_7 - \alpha_6\gamma_7 - \beta_6\alpha_7 + \beta_6\gamma_7 + \gamma_6\alpha_7 - \gamma_6\beta_7}{\alpha_6\beta_6\alpha_7\gamma_7 - \alpha_6\beta_6\beta_7\gamma_7 - \alpha_6\gamma_6\alpha_7\beta_7 + \alpha_6\gamma_6\beta_7\gamma_7 + \beta_6\gamma_6\alpha_7\beta_7 - \beta_6\gamma_6\alpha_7\gamma_7}.$$

[3] www.math.uiuc.edu/Macaulay2.

Proof. This can be verified using linear algebra over the rational function field $K = \mathbb{Q}(\alpha_6, \beta_6, \gamma_6, \delta_6, \alpha_7, \beta_7, \gamma_7, \delta_7)$. We compute three linearly independent quadrics that vanish at the seven points and check that they also vanish at $(\alpha_8{:}\beta_8{:}\gamma_8{:}\delta_8)$. □

The formula in Proposition 7.1 parametrizes the semialgebraic set of real quartics that consist of four ovals. Indeed, if the parameters $\alpha_6, \beta_6, \gamma_6, \delta_6, \alpha_7, \beta_7, \gamma_7, \delta_7$ are real numbers then the corresponding quartic curve (1.2) has 28 real bitangents, so it falls into the first row of Table 1, and all quartics with four ovals arise. Note that this row is the one relevant for applications to periodic water waves (Dubrovin et al., 1997). In practice we usually choose rational numbers for the parameters. This represents all curves whose 28 bitangents are rational, such as the Edge quartic (1.5), and it ensures that the ground field is $K = \mathbb{Q}$ for all computations in Sections 3–6.

Yet the above formula has two disadvantages. First of all, it breaks the symmetry among the eight points in the Cayley octad, and, secondly, it does not offer an arithmetically useful parametrization for the last two rows of Table 1. Indeed, Vinnikov quartics and positive quartics are the lead actors in this paper, and we found ourselves unable to manipulate them properly using Proposition 7.1. For example, for a long time we failed to find a quartic with eight rank-3 Gram matrices over $\mathbb{Q}$. Then we derived Proposition 7.2, and this led us to Example 5.12.

Let O be a configuration of eight points in general position in $\mathbb{P}^3$, represented by a 4×8-matrix. If O^* is another such matrix whose row space equals the kernel of O then the configuration represented by O^* is said to be *Gale dual* or *associated* to O. We refer to Dolgachev and Ortland (1988) and Eisenbud and Popescu (2000) for the basics on Gale duality in the context of algebraic geometry. Both configurations O and O^* are understood as equivalence classes modulo projective transformations of $\mathbb{P}^3$ and relabeling of the eight points. We say that the configuration O is *Gale self-dual* if O and O^* are equivalent in this sense. By a classical result due to Coble (1929), O is Gale self-dual if and only if O is a Cayley octad; see Theorem 3.10.

The variety of Cayley octads is defined by the equation $O = O^*$. We now translate this equation into an algebraic form that is useful for computations. Let p_{ijkl} denote the 4×4-minor of the 4×8-matrix O that represents our configuration of eight points in $\mathbb{P}^3$. Consider the condition that O is mapped to a configuration projectively equivalent to its Gale dual O^* if we relabel the points by the permutation $(18)(27)(36)(45)$. We express this condition using the *Plücker coordinates* p_{ijkl}.

Proposition 7.2. *Eight points in* $\mathbb{P}^3$ *form a Cayley octad if and only if*

$$\begin{aligned}
p_{1234}p_{1256}p_{3578}p_{4678} &= p_{5678}p_{3478}p_{1246}p_{1235}, & p_{1234}p_{1257}p_{3568}p_{4678} &= p_{5678}p_{3468}p_{1247}p_{1235},\\
p_{1234}p_{1267}p_{3568}p_{4578} &= p_{5678}p_{3458}p_{1247}p_{1236}, & p_{1234}p_{1356}p_{2578}p_{4678} &= p_{5678}p_{2478}p_{1346}p_{1235},\\
p_{1234}p_{1457}p_{2568}p_{3678} &= p_{5678}p_{2368}p_{1347}p_{1245}, & p_{1234}p_{1467}p_{2568}p_{3578} &= p_{5678}p_{2358}p_{1347}p_{1246},\\
p_{1235}p_{1267}p_{3468}p_{4578} &= p_{4678}p_{3458}p_{1257}p_{1236}, & p_{1235}p_{1347}p_{2468}p_{5678} &= p_{4678}p_{2568}p_{1357}p_{1234},\\
p_{1235}p_{1367}p_{2468}p_{4578} &= p_{4678}p_{2458}p_{1357}p_{1236}, & p_{1235}p_{1467}p_{2468}p_{3578} &= p_{4678}p_{2358}p_{1357}p_{1246},\\
p_{1236}p_{1347}p_{2458}p_{5678} &= p_{4578}p_{2568}p_{1367}p_{1234}, & p_{1236}p_{1456}p_{2478}p_{3578} &= p_{4578}p_{2378}p_{1356}p_{1246},\\
p_{1245}p_{1267}p_{3468}p_{3578} &= p_{3678}p_{3458}p_{1257}p_{1246}, & p_{1245}p_{1346}p_{2378}p_{5678} &= p_{3678}p_{2578}p_{1456}p_{1234},\\
p_{1245}p_{1356}p_{2378}p_{4678} &= p_{3678}p_{2478}p_{1456}p_{1235}, & p_{1245}p_{1357}p_{2368}p_{4678} &= p_{3678}p_{2468}p_{1457}p_{1235},\\
p_{1245}p_{1367}p_{2368}p_{4578} &= p_{3678}p_{2458}p_{1457}p_{1236}, & p_{1246}p_{1357}p_{2368}p_{4578} &= p_{3578}p_{2468}p_{1457}p_{1236},\\
p_{1246}p_{1357}p_{2458}p_{3678} &= p_{3578}p_{2468}p_{1367}p_{1245}, & p_{1247}p_{1357}p_{2368}p_{4568} &= p_{3568}p_{2468}p_{1457}p_{1237},\\
&\textit{and} & p_{1346}p_{1357}p_{2458}p_{2678} &= p_{2578}p_{2468}p_{1367}p_{1345}.
\end{aligned}$$

Before discussing the proof of this theorem, we first explain why the shape of the above equations is plausible. Consider the condition for six points $(x_i : y_i : z_i)$ in $\mathbb{P}^2$ to be self-dual, in the sense above. This condition means that the six points lie on a conic, and we write this algebraically in terms of Plücker coordinates as

$$\det\begin{pmatrix} x_1^2 & y_1^2 & z_1^2 & x_1y_1 & x_1z_1 & y_1z_1 \\ x_2^2 & y_2^2 & z_2^2 & x_2y_2 & x_2z_2 & y_2z_2 \\ x_3^2 & y_3^2 & z_3^2 & x_3y_3 & x_3z_3 & y_3z_3 \\ x_4^2 & y_4^2 & z_4^2 & x_4y_4 & x_4z_4 & y_4z_4 \\ x_5^2 & y_5^2 & z_5^2 & x_5y_5 & x_5z_5 & y_5z_5 \\ x_6^2 & y_6^2 & z_6^2 & x_6y_6 & x_6z_6 & y_6z_6 \end{pmatrix} = p_{123}p_{145}p_{246}p_{356} - p_{124}p_{135}p_{236}p_{456}. \tag{7.3}$$

This formula appears in Dolgachev and Ortland (1988, Ex. 4, p. 37) and we adapt the derivation given there.

Sketch of proof for Proposition 7.2. The cross ratio $(p_{1234}p_{1256})/(p_{1235}p_{1246})$ is invariant under projective transformations. The permutation (18)(27)(36)(45) of the points transforms that cross ratio into $(p_{8765}p_{8743})/(p_{8764}p_{8753})$. The condition $O = O^*$ implies that these two cross ratios are equal. By clearing denominators, the equality of cross ratios translates into the first of the 21 equations listed above:

$$p_{1234}p_{1256}p_{3578}p_{4678} = p_{5678}p_{3478}p_{1246}p_{1235}.$$

The other 20 equations are found by the same argument for cross ratios. By incorporating the quadratic Plücker relations among the p_{ijkl}, we check that our list of 21 cross ratio identities is complete, in the sense that it ensures $O = O^*$. □

Dolgachev and Ortland (1988, page 176) present the conditions under which a regular Cayley octad O can degenerate. Their analysis exhibits $64 = 28 + 35 + 1$ boundary divisors in the compactified space of Cayley octads. These are as follows:

(1) Two points of O can come together. This gives $28 = \binom{8}{2}$ boundary divisors, e.g., points 1 and 2 come together if and only if $p_{12ij} = 0$ for $3 \le i < j \le 8$.
(2) Four points of O can become coplanar. The equations in Proposition 7.2 then ensure that the other four points become coplanar as well. So, in total there are $35 = \frac{1}{2}\binom{8}{4}$ boundary divisors such as $\{p_{1234} = p_{5678} = 0\}$.
(3) The eight points of O can lie on a twisted cubic curve, which is the intersection of the three quadrics. The condition for seven points in $\mathbb{P}^3$ to lie on a twisted cubic curve has codimension 2. White (1915, Eq. (2)) writes this condition by adding an index to (7.3). This gives 7·15·3 equations like

$$p_{1237}p_{1457}p_{2467}p_{3567} - p_{1247}p_{1357}p_{2367}p_{4567} = 0. \tag{7.4}$$

Applying the symmetric group S_8 to the indices, we obtain equations for the codimension 4 locus of octads that lie on a twisted cubic curve. This locus is a divisor in the compacted space of Cayley octads, as in Dolgachev and Ortland (1988, Section IX.3). Equivalently, the (7.4) imply those in Proposition 7.2.

We now shift gears and examine the three types of boundary divisors from the perspective of the desirable representations (1.2) and (1.3) of a ternary quartic f. In other words, we wish to identify the conditions, expressed algebraically in terms of these two representations, for the quartic curve $\mathcal{V}_{\mathbb{C}}(f)$ to become singular.

Recall that the discriminant Δ of f is a homogeneous polynomial of degree 27, featured explicitly in (Sanyal et al., 2009, Proposition 6.5), in the 15 coefficients c_{ijk} of (1.1). If we take f in the representation (1.2) then each coefficient c_{ijk} is replaced by a polynomial of degree 4 in the $30 = 10 + 10 + 10$ entries of the symmetric matrices A, B and C. The result of performing this substitution in the discriminant $\Delta(c_{ijk})$ is denoted $\Delta(A, B, C)$. This is a homogeneous polynomial of degree 108 in 30 unknowns. We call $\Delta(A, B, C)$ the *Vinnikov discriminant* of a ternary quartic.

Similarly, if we take f in the representation (1.3) then each coefficient c_{ijk} is replaced by a polynomial of degree 2 in the $18 = 6 + 6 + 6$ coefficients of the quadrics q_1, q_2 and q_3. The result of performing this substitution in the discriminant $\Delta(c_{ijk})$ is denoted $\Delta(q_1, q_2, q_3)$. This is a homogeneous polynomial of degree 54 in 18 unknowns. We call $\Delta(q_1, q_2, q_3)$ the *Hilbert discriminant* of a ternary quartic.

Theorem 7.5. *The irreducible factorization of the Vinnikov discriminant equals*

$$\Delta(A, B, C) = \mathbf{M}(A, B, C) \cdot \mathbf{P}(A, B, C)^2, \tag{7.6}$$

where **P** *has degree* 30 *and corresponds to the boundary divisor* (2)*, while* **M** *has degree* 48*, and this is the* mixed discriminant *corresponding to both* (1) *and* (3)*.*

The irreducible factorization of the Hilbert discriminant equals

$$\Delta(q_1, q_2, q_3) = \mathbf{Q}(q_1, q_2, q_3) \cdot \mathbf{R}(q_1, q_2, q_3)^2, \tag{7.7}$$

where **Q** *has degree* 30 *and the degree* 12 *factor* **R** *is the resultant of* q_1, q_2 *and* q_3*.*

This theorem is proved by a computation, the details of which we omit here. It has been pointed out to us by Igor Dolgachev and Giorgio Ottaviani that the factorization (7.6) was already known to Salmon (1879), who refers to $\mathbf{M}(A, B, C)$ as the *tact invariant*. See also Gizatullin (2007, Section 10) for a modern treatment.

We discuss the geometric meaning of the factors in (7.6) and (7.7). The polynomials **M**, **P**, **Q**, and **R** are absolutely irreducible: they do not factor over $\mathbb{C}$. The polynomial **P** represents the condition that the span of A, B and C in the space of 4×4-symmetric matrices contains a rank-2 matrix. Note that the variety of such rank-2 matrices has codimension 3 and degree 10. The *Chow form* of that variety is precisely our polynomial **P**, which explains why **P** has degree $3 \cdot 10 = 30$.

Non-vanishing of the mixed discriminant **M** is the condition for the intersection of three quadrics in $\mathbb{P}^3$ to be zero-dimensional and smooth. A general formula for the degree of such discriminants appears in Nie (2010, Theorem 3.1). It implies that **M** is tri-homogeneous of degree (16, 16, 16) in the entries of (A, B, C), so the total degree of M is 48. Note that vanishing of **M** represents not just condition (1) but it also subsumes condition (3) that the quadrics intersect in a twisted cubic curve.

The resultant **R** of three ternary quadrics (q_1, q_2, q_3) is tri-homogeneous of degree (4, 4, 4) since two quadrics meet in 4 points in $\mathbb{P}^2$. Thus **R** has total degree 12. The extraneous factor **Q** of degree 30 expresses the condition that, at some point in $\mathbb{P}^2$, the vector (q_1, q_2, q_3) is non-zero and lies in the kernel of its Jacobian.

We close this paper by reinterpreting Table 1 as a tool to study linear spaces of symmetric 4×4 matrices. Two matrices A and B determine a *pencil of quadrics* in $\mathbb{P}^3$, and three matrices A, B, C determine a *net of quadrics* in $\mathbb{P}^3$. We now consider these pencils and nets over the field $\mathbb{R}$ of real numbers. A classical fact, proved by Calabi (1964), states that a pencil of quadrics either has a common point or contains a positive definite quadric. This fact is the foundation for an optimization technique known in engineering as the *S-procedure*. The same dichotomy is false for nets of quadrics (Calabi, 1964, Section 4), and for quadrics in $\mathbb{P}^3$ it fails in two interesting ways.

Theorem 7.8. *Let* $\mathcal{N}$ *be a real net of homogeneous quadrics in four unknowns with* $\Delta(\mathcal{N}) \neq 0$*. Then precisely one of the following four cases holds:*

(a) *The quadrics in* $\mathcal{N}$ *have a common point in* $\mathbb{P}^3(\mathbb{R})$*.*
(b) *The net* $\mathcal{N}$ *is definite, i.e. it contains a positive definite quadric.*
(c) *There is a definite net* $\mathcal{N}'$ *with* $\det(\mathcal{N}') = \det(\mathcal{N})$*, but* $\mathcal{N}$ *is nondefinite.*
(d) *The net* $\mathcal{N}$ *contains no singular quadric.*

Proof. For a real net of quadrics, $\mathcal{N} = \mathbb{R}\{A, B, C\}$, the Vinnikov discriminant $\Delta(A, B, C)$ in (7.6) is independent (up to scaling) of the basis $\{A, B, C\}$, and thus can be denoted $\Delta(\mathcal{N})$. If $\Delta(\mathcal{N})$ is non-zero, the polynomial $\det(\mathcal{N}) = \det(xA + yB + zC)$ defines a smooth curve, which depends on the choice of basis $\{A, B, C\}$ only up to projective change of coordinates in $[x : y : z]$. This real quartic falls into precisely one of the six classes in Table 1. The first four classes correspond to our case (a). The fifth class corresponds to our cases (b) and (c) by the Helton–Vinnikov Theorem (Helton and Vinnikov, 2007). As a Vinnikov quartic has definite and non-definite real determinantal representations, both (b) and (c) do occur (Vinnikov, 1993). For an example, see Plaumann et al. (2010, Ex. 5.2). The last class corresponds to our case (d). □

Given a net of quadrics $\mathcal{N} = \mathbb{R}\{A, B, C\}$, one may wish to know whether there is a common intersection point in real projective 3-space $\mathbb{P}^3(\mathbb{R})$, and, if not, one seeks the certificates promised in parts (b)–(d) of Theorem 7.8. Our algorithms in Sections 3–5 furnish a practical method for identifying cases (b) and (d). The difference between (b) and (c) is more subtle and is discussed in detail in Plaumann et al. (2010).

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References

Calabi, E., 1964. Linear systems of real quadratic forms. Proc. Amer. Math. Soc. 15, 844–846.
Coble, A., 1929. Algebraic Geometry and Theta Functions. In: Colloquium Publications, vol. 10. American Math. Society.
Dixon, A.C., 1902. Note on the reduction of a ternary quantic to a symmetrical determinant. Cambridge Proc. 11, 350–351.
Dolgachev, I., 2010. Topics in Classical Algebraic Geometry, Part I, Lecture Notes, Available online at www.math.lsa.umich.edu/~idolga/topics.pdf.
Dolgachev, I., Ortland, D., 1988. Point sets in projective spaces and theta functions. Astérisque (165).
Dubrovin, B., Flickinger, R., Segur, H., 1997. Three-phase solutions of the Kadomtsev–Petviashvili equation. Stud. Appl. Math. 99, 137–203.
Edge, W.L., 1938. Determinantal representations of $x^4 + y^4 + z^4$. Math. Proc. Cambridge Phil. Soc. 34, 6–21.
Eisenbud, D., Popescu, S., 2000. The projective geometry of the Gale transform. J. Algebra 230, 127–173.
Fulton, W., 1969. Algebraic curves. In: Advanced Book Classics. Addison-Wesley, reprinted in 1989.
Gizatullin, M., 2007. On covariants of plane quartic associated to its even theta characteristic. In: Proceedings of the Korea–Japan Conference in honor of Igor Dolgachev's 60th birthday. In: Contemp. Math., vol. 422. Amer. Math. Soc., Providence, RI, pp. 37–74.
Greub, W., 1975. Linear Algebra, fourth ed. In: Graduate Texts in Mathematics, vol. 23. Springer-Verlag, New York.
Harris, J., 1979. Galois groups of enumerative problems. Duke Math. J. 46, 685–724.
Helton, J.W., Vinnikov, V., 2007. Linear matrix inequality representation of sets. Comm. Pure Appl. Math. 60, 654–674.
Henrion, D., 2010. Detecting rigid convexity of bivariate polynomials. Linear Algebra Appl. 432, 1218–1233.
Hesse, O., 1855. Über die Doppeltangenten der Curven vierter Ordnung. J. Reine Angew. Math. 49, 279–332.
Hilbert, D., 1888. Ueber die Darstellung definiter Formen als Summe von Formenquadraten. Math. Ann. 32 (3), 342–350.
Klein, F., 1876. Über den Verlauf der Abelschen Integrale bei den Kurven vierten Grades. Math. Ann. 10, 365–397.
Miller, G.A., Blichfeldt, H.F., Dickson, L.E., 1916. Theory and Applications of Finite Groups. J. Wiley & Sons, New York, reprinted by Dover Publications Inc., New York, 1961.
Nie, J., 2010. Discriminants and nonnegative polynomials, arXiv:1002.2230. Preprint.
Nie, J., Ranestad, K., Sturmfels, B., 2010. The algebraic degree of semidefinite programming. Math. Program. Ser. A 122, 379–405.
Plaumann, D., Sturmfels, B., Vinzant, C., 2010. Computing linear matrix representations of Helton–Vinnikov curves. arXiv:1011.6057. Preprint.
Plücker, J., 1834. Solution d'une question fondamentale concernant la théorie générale des courbes. J. Reine Angew. Math. 12, 105–108.
Powers, V., Reznick, B., 2000. Notes towards a constructive proof of Hilbert's theorem on ternary quartics. In: Quadratic Forms and Their Applications (Dublin, 1999). In: Contemp. Math., vol. 272. Amer. Math. Soc., Providence, RI, pp. 209–227.
Powers, V., Reznick, B., Scheiderer, C., Sottile, F., 2004. A new approach to Hilbert's theorem on ternary quartics. C. R. Math. Acad. Sci. Paris 339 (9), 617–620.
Rostalski, P., Sturmfels, B., Dualities in convex algebraic geometry, Rendiconti di Mathematica e delle sue Applicazioni (2011), arXiv:1006.4894 (in press).
Salmon, G., A Treatise on the Higher Plane Curves: Intended as a Sequel to "A Treatise on Conic Sections", 3rd ed., Dublin, 1879; reprinted by Chelsea Publ. Co., New York, 1960.
Sanyal, R., Sottile, F., Sturmfels, B., 2009. Orbitopes, arXiv:0911.5436. Preprint.
Steiner, J., 1855. Eigenschaften der Curven vierten Grads rücksichtlich ihrer Doppeltangenten. J. Reine Angew. Math. 49, 265–272.
White, H.S., 1915. Seven points on a twisted cubic curve. Proc. Natl. Acad. Sci. USA 8, 464–466.
Vinnikov, V., 1993. Self-adjoint determinantal representations of real plane curves. Math. Ann. 296 (3), 453–479.
Zeuthen, H.G., 1873. Sur les différentes formes des courbes planes du quatrième ordre. Math. Ann. 7, 408–432.