UC Berkeley Math 10B, Spring 2015: Final Exam

Prof. Sturmfels, May 14

Name: ______________________________

SID: ______________________________

Neighbors: Please write the names of the students next to you (or "None"):

Left: ______________________________

Right: ______________________________

Section: Circle your discussion section below:

Sec	Time	Room	GSI
101	MWF 8-9	121 Latimer	Jason Ferguson
102	MWF 9-10	104 Barrows	Jason Ferguson
103	MWF 10-11	237 Cory	Minseon Shin
104	MWF 11-12	121 Latimer	Indraneel Kasmalkar
105	MWF 12-1	237 Cory	Indraneel Kasmalkar
106	MWF 1-2	206 Wheeler	Joe Kileel
107	MWF 2-3	283 Dwinelle	Joe Kileel
108	MWF 9-10	110 Wheeler	Minseon Shin
109	MWF 1-2	250 Dwinelle	Elina Robeva
110	MWF 2-3	228 Dwinelle	Elina Robeva

Other/none, explain: ______________________________

Grading

1	/ 8
2	/ 8
3	/ 8
4	/ 8
5	/ 8
6	/ 8
7	/ 8
8	/ 8
9	/ 8
10	/ 8
	/80

Instructions:

- Closed book: No notes, no books, no calculators.
- The exam time is 180 minutes. Do all 10 problems.
- You must justify your answers for full credit.
- Write your answers in the space below each problem.
- Answers in complete sentences are encouraged.
- If you need more space, use reverse side or scratch pages. Indicate clearly where to find your answers.

1. (8 points)

(a) What does the Central Limit Theorem say? State it in your own words.

(b) A coin is tossed 100 times, resulting in 30 heads and 70 tails. Find a 95% confidence interval for the true probability p of getting heads.

(c) Explain in what sense your interval has a 95% chance of containing p.

2. (8 points) Find *all* eigenvalues and *all* eigenvectors of the matrix A, where:

$$A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$

3. (8 points) A quiz question asks:

How many seven-digit bitstrings have at least three 1s?

A student gives the following solution:

> You have to choose which three of the seven digits are 1s, and there are $\binom{7}{3}$ ways to do that. Once you choose those three digits, each of the other four digits can be either 0 or 1, so overall there are $\binom{7}{3} \cdot 2^4 = 35 \cdot 16 = \boxed{560}$ such bitstrings.

(a) Compute the total number of seven-digit bitstrings. Fully simplify your answer.

(b) Using your answer to (a), explain why the student's answer cannot be correct.

(c) What is the correct answer to the quiz question? Fully simplify your answer.

(d) Where is the mistake in the student's solution? Explain carefully.

4. (8 points) Determine all pairs of functions $y_1(t), y_2(t)$ which satisfy:

$$y_1'(t) = -2y_1(t) + 4y_2(t), \qquad y_1(0) = 1,$$

$$y_2'(t) = -2y_1(t) + 2y_2(t), \qquad y_2(0) = 2.$$

Your final answer should not contain the complex number $i = \sqrt{-1}$.

5. (8 points)

(a) Find a solution to the initial value problem

$$y' = t\sqrt{y}, \quad y(0) = 0.$$

(b) Find a solution to the initial value problem

$$y' = t\sqrt{y}, \quad y(0) = 1.$$

(c) Does the following initial value problem have a solution and, if yes, is it unique?

$$y' = 2\sqrt{y}, \quad y(0) = 0.$$

6. (8 points) Let I_2 denote the 2×2-identity matrix. For each part, check that the example you give really is an example.

(a) Give two 2×2-matrices A and B such that $AB \neq BA$.

(b) Give a 2×2-matrix A such that $A \neq I_2$, $A \neq -I_2$, and $AA = I_2$.

(c) Give a 2×2-matrix A such that $A \neq I_2$, $A \neq -I_2$, and $A^T A = I_2$.

(d) Give two 2×3-matrices A and B such that $AB^T = I_2$.

7. (8 points) You want to put 9 indistinguishable apples into 4 distinguishable boxes, where some of the boxes may be empty.

(a) How many ways can you do this? Simplify your answer completely.

(b) A raccoon finds the box with the most apples and eats all of the apples in it. [If there is a tie, the raccoon picks one of the boxes with the most apples.] What is the minimum number of apples that the raccoon is guaranteed to eat? Explain carefully.

8. (8 points)

(a) State the mathematical problem that is solved by the *Least Squares Method.*

(b) Find the least square line $y = \beta_0 + \beta_1 x$ for the data points given by the table

x	0	1	2	3	4
y	6	3	1	0	0

(c) Estimate the value of y when $x = 7/2$.

9. (8 points) We define two random variables X and Y as follows. You pick a number X with equal probability from the set $\{1, 2, 3\}$, and you set

$$Y = \begin{cases} 1 & \text{if } X \text{ is even,} \\ 0 & \text{if } X \text{ is odd.} \end{cases}$$

(a) Show that the covariance $\text{Cov}[X, Y]$ is zero.
Reminder: For any random variables X and Y, $\text{Cov}[X, Y] = E[XY] - E[X]E[Y]$.

(b) Are X and Y independent? Why or why not? Explain carefully.

10. (8 points) You put a cup of coffee initially at 85°C into a refrigerator, which stays at 5°C. After 15 minutes, your coffee has cooled to 65°C. Find a formula for the temperature in Celsius of the coffee t minutes after you put it in the refrigerator.

Reminder: Newton's law of cooling states the rate of change of the temperature of an object is proportional to the difference between the temperature of the object and the temperature of its surroundings.

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