Math 128-B: Spring 1999, J. Strain.

Final Exam, 19 May 1999, 1230–1530.

The following problems are worth 30 points each. Please solve enough to get 90 points.

1. Assume a fundamental matrix $Y(t)$ for $y' = P(t)y$ is known on the interval $[a, b]$.
(a) Find a formula for the solution of the initial value problem

$$y' = P(t)y + f(t)$$

satisfying $y(0) = \eta$.
(b) Derive a formula for the Green function $G(t, s)$ of the boundary value problem

$$y' = P(t)y + f(t) \qquad a < t < b$$

$$Ay(a) + By(b) = 0,$$

assuming A and B are $n \times n$ matrices such that the matrix $D = AY(a) + BY(b)$ is invertible.

2. Consider the nonlinear two-point boundary value problem

$$u'' - \exp(u) = 0 \qquad 0 < x < 1$$

$$u(0) = u(1) = 0.$$

Discretize by centered differences with $h = 1/(N+1)$ to get

$$F(v)_i = v_{i+1} - 2v_i + v_{i-1} - h^2 \exp(v_i) = 0$$

$$v_0 = v_{N+1} = 0$$

where $v_i \approx u(ih)$.
(a) Write down Newton's method for solving the nonlinear system $F(v) = 0$.
(b) Let B be the matrix of the linear system in (a). Prove that B is invertible.
(c) Define and evaluate the local truncation error of the centered scheme above.
(d) Define and evaluate the "stability" of this scheme and explain how it relates to the accuracy of the numerical solution in the limit $h \to 0$.

3. Consider the linear ODE

$$y'' = q(x)y \qquad 0 < x < 1$$

with $q(x) > 0$ on $0 \le x \le 1$, and the finite difference scheme

$$(u_{j+1} - 2u_j + u_{j-1})/h^2 = q_j u_j \qquad 0 < j < N+1$$

where $q_j = q(jh)$ and $(N+1)h = 1$.

(a) Let y be a solution of the ODE with $y(0) = y(1) = 0$. Prove or disprove: y must be identically zero.
(b) Let u be a solution of the finite difference scheme with $u_0 = u_{N+1} = 0$. Prove or disprove: u must be identically zero.
(c) Prove or disprove: the matrix

$$S = \begin{bmatrix} -2-h^2q_1 & 1 & 0 & \dots & 0 \\ 1 & -2-h^2q_2 & 1 & \dots & 0 \\ 0 & 1 & -2-h^2q_3 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 1 & -2-h^2q_N \end{bmatrix}$$

is invertible.

4. Let

$$m = (1/n)\sum_{i=1}^{n} x_i.$$

Decide which formula for the variance is likely to give a better relative error bound in floating-point arithmetic and explain the analysis behind your choice: (a)

$$(n-1)S^2 = \sum_{i=1}^{n} x_i^2 - nm^2$$

or (b)

$$(n-1)S^2 = \sum_{i=1}^{n} (x_i - m)^2.$$

5. Suppose A is an n by n real invertible matrix and $r = b - Ay$ where $b = Ax$ is nonzero. Prove that

$$||x-y||/||x|| \le \text{cond}(A)||r||/||b||.$$

6. Prove that floating point arithmetic with machine epsilon ϵ produces an inner product (summed in the natural order) satisfying

$$\text{fl}(x^T y) = x^T(y+e)$$

where

$$|e_i| \le 2n\epsilon|y_i|,$$

as long as $n\epsilon \le 1/2$. Under what conditions on x and y does this bound guarantee small relative error in $x^T y$?

7. Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}.$$

(a) Find a symmetric tridiagonal matrix T with the same eigenvalues as A.
(b) Find an orthogonal matrix Q and an upper triangular matrix R with positive diagonal entries such that $T - T_{33}I = QR$.
(c) Compute $RQ + T_{33}I$ and verify that it is closer to diagonal than T by computing the sum of squares of off-diagonal elements for both.

8. Let A be a number in the range $0.25 \leq A \leq 1$ and consider the Newton iteration

$$x_{k+1} = (x_k + A/x_k)/2$$

for computing $x_\infty = \sqrt{A}$. Assume that the initial guess $x_0 = (1 + 2A)/3$ has error less than 0.05 for any such A.
(a) Prove that the error $e_k = x_k - \sqrt{A}$ satisfies

$$e_{k+1} = e_k^2/2x_k.$$

(b) Determine the number k of steps necessary to guarantee sixteen-digit accuracy in $\sqrt{A}$.

9. (a) Let A be an unknown matrix. Given a known matrix B and known vectors p and $q = Ap$, find a rank-one update $C = B + uv^T$ of B which makes

$$||C - A||_2 \leq ||B - A||_2$$

and

$$Cp = q.$$

Show that your update satisfies these two requirements.
(b) Suppose the QR factorization of a nonsingular matrix $A = QR$ is given, where Q is orthogonal and R is upper triangular with positive diagonal elements. Under what conditions on A, u and v will the rank-one update $B = A + uv^T$ have a QR factorization and what algorithmic sequence of steps can be used to find it?

10. For an arbitrary differentiable function $F : R^n \to R^n$, write Newton's method as a fixed point iteration

$$x_{k+1} = G(x_k).$$

Determine G in terms of F and DF. Now restrict your analysis to the one-dimensional case $n = 1$, assume F has two continuous derivatives, and use Taylor expansion to prove superlinear convergence: as any two distinct points u and v approach a zero of $F(x)$ where $F'(x)$ is nonzero,

$$|G(u) - G(v)|/|u - v| \to 0.$$

11. Suppose a square matrix A has a factorization $A = U\Sigma V^T$ where U and V are orthogonal and Σ is diagonal with diagonal elements $\sigma_1 \geq \sigma_2 \geq \ldots \sigma_r > 0 = 0 = \ldots = 0$. Use this factorization to find a vector x which minimizes $||x||_2^2$ subject to the requirement that $||Ax - b||_2^2$ be minimum.

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