

Math 113 Final Exam, 5/24/03, 8:00 – 11:00 AM *M. Hutchings*

NAME \_\_\_\_\_

To receive full credit you must **justify all answers** except where otherwise stated. The point is to demonstrate that you understand the material. You may cite appropriate theorems that were proved in class. No books, notes, calculators, collaboration, or other aids are permitted. Each part of each question is worth 5 points, except where otherwise noted. There a total of 70 points. Please write your answers on the exam, not in a blue book. You may use the backs of the pages if necessary. Good luck!

Score:

1 \_\_\_\_\_

2 \_\_\_\_\_

3 \_\_\_\_\_

4 \_\_\_\_\_

5 \_\_\_\_\_

6 \_\_\_\_\_

7 \_\_\_\_\_

Total \_\_\_\_\_

1. Let  $\alpha = 1 + \sqrt{2 + \sqrt{3}} \in \mathbb{R}$ .

- (a) Find a degree 4 polynomial  $f \in \mathbb{Q}[x]$  such that  $f(\alpha) = 0$ .
- (b) Is your polynomial  $f$  irreducible over  $\mathbb{Q}$ ?

2. True or false:

- (a) If  $\alpha$  is algebraic over  $\mathbb{Q}$  and  $\beta$  is transcendental over  $\mathbb{Q}$  then  $\alpha + \beta$  is transcendental over  $\mathbb{Q}$ .
- (b) A finite field of order 32 can have a subfield of order 8.

3. Let  $G = \mathbb{Z}_{10} \times \mathbb{Z}_{10}$  and let  $H$  be the subgroup of  $G$  generated by the element  $(2, 4)$ .
- (a) (1 point) List all the elements of  $H$ .
  - (b) (4 points) What is the order of the coset  $(3, 1) + H$  in  $G/H$ ?
  - (c) Which product of cyclic groups of prime-power order is  $G/H$  isomorphic to? (Hint: what is the largest possible order of an element of  $G/H$ ?)

4. True or false:

(a)  $\mathbb{Z}[x]$  is an ideal in  $\mathbb{Q}[x]$ .

(b) Every ideal in  $\mathbb{Z}_n$  is principal. (Hint: division theorem.)

5. (a) Let  $\phi : \mathbb{Z}_{35} \rightarrow \mathbb{Z}_{25}$  be a group homomorphism. What are the possible orders of the kernel and image of  $\phi$ ? (Hint: fundamental homomorphism theorem.)
- (b) How many group homomorphisms  $\phi : \mathbb{Z}_{35} \rightarrow \mathbb{Z}_{25}$  exist? (Hint: what can  $\phi(1)$  be?)

6. Consider the quotient ring  $F = \mathbb{Z}_3[x]/(x^2 + 1)$ .

(a) Show that  $F$  is a field.

(b) Let  $\alpha$  be the equivalence class  $[x + 1] \in F$ . Determine the order of  $\alpha$  in the group of units  $F^*$ .

7. True or false:

(a)  $\sqrt[3]{2} \in \mathbb{Q}(\sqrt[3]{2})$ .

(b) The groups  $S_4$  and  $A_4 \times \mathbb{Z}_2$  are isomorphic. (Hint: think about orders of elements. Or if you want to be fancy, think about centers.)