Math 1A: Calculus Worksheets

7^{th} Edition

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Preface

This booklet contains the worksheets for Math 1A, U.C. Berkeley's calculus course.

Christine Heitsch, David Kohel, and Julie Mitchell wrote worksheets used for Math 1AM and 1AW during the Fall 1996 semester. David Jones revised the material for the Fall 1997 semesters of Math 1AM and 1AW. The material was further updated by Zeph Grunschlag and Tom Insel, with help from the comments and corrections provided by David Lippel, Max Oks, and Sarah Reznikoff. Tom Insel coordinated the 1998 edition with much assistance and new material from Cathy Kessel and in consultation with William Stein. Cathy Kessel and Michael Wu have further revised the 1999 and 2000 edition respectively. Michael Hutchings made tiny changes in 2012.

In 1997, the engineering applications were written by Reese Jones, Bob Pratt, and Professors George Johnson and Alan Weinstein, with input from Tom Insel and Dave Jones. In 1998, applications authors were Michael Au, Aaron Hershman, Tom Insel, George Johnson, Cathy Kessel, Jason Lee, William Stein, and Alan Weinstein.

About the worksheets

This booklet contains the worksheets that you will be using in the discussion section of your course. Each worksheet contains Questions, and most also have Problems and Additional Problems. The Questions emphasize qualitative issues and answers for them may vary. The Problems tend to be computationally intensive. The Additional Problems are sometimes more challenging and concern technical details or topics related to the Questions and Problems.

Some worksheets contain more problems than can be done during one discussion section. Do not despair! You are not intended to do every problem of every worksheet.

Why worksheets?

There are several reasons to use worksheets:

- *Communicating to learn.* You learn from the explanations and questions of the students in your class as well as from lectures. Explaining to others enhances your understanding and allows you to correct misunderstandings.

- *Learning to communicate.* Research in fields such as engineering and experimental science is often done in groups. Research results are often described in talks and lectures. Being able to communicate about science is an important skill in many careers.

- *Learning to work in groups.* Industry wants graduates who can communicate *and* work with others.

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1. Graphing a Journey

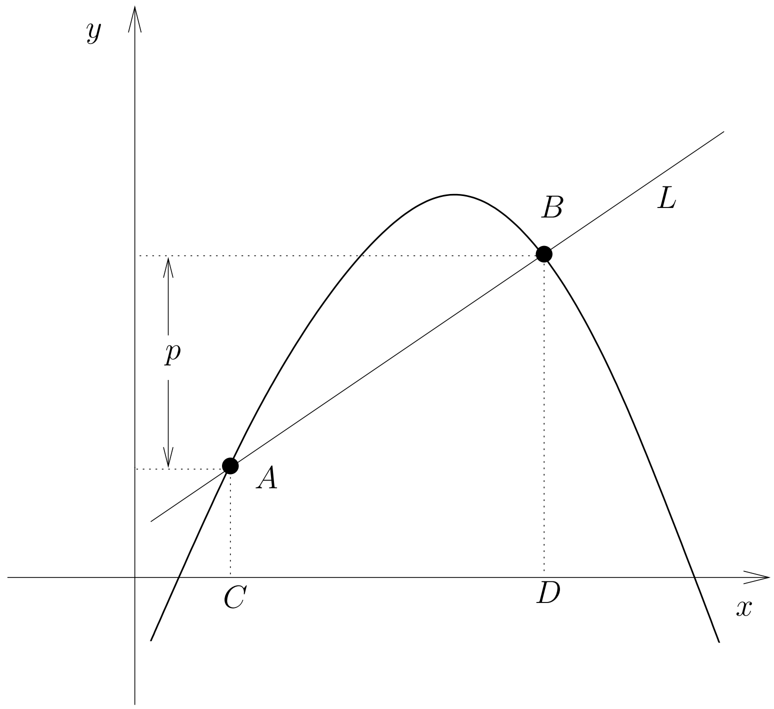
Questions

1. Before you came to UC Berkeley you probably lived somewhere else (another country, state, part of California, or part of Berkeley). Sketch a graph that shows the *speed* of your journey to UC Berkeley as a function of time. (For example, if you came by car this graph would show speedometer reading as a function of time.) Label the axes to show speed.

   Ask someone outside of your group to read your graph. See if that person can tell from your graph what form (or forms) of transportation you used.

2. Using the same labeling on the x-axis, sketch the graph of the *distance* you traveled on your trip to Berkeley as a function of time. (For example, if you traveled by car, this would be the odometer reading as a function of time—if you'd set the odometer to zero at the beginning of your trip.)

   Ask someone else outside of your group to read your graph. See if that person can tell from your graph what form (or forms) of transportation you used.



3. (a) In the graph above, A has coordinates $(2, 3)$ and B has coordinates $(4, 8)$. Calculate the slope of the line L through A and B and the value of p.

   (b) The point D (connected to B) moves toward C. What happens to the slope of L and to the the value of p?

4. Graph

   (a) $y = \frac{1}{x}$.

   (b) $y = \sin x$. What are the x-intercepts of this graph? (I.e., where does the graph cross the x-axis? A related question is: What are the zeros of the function $y = \sin x$?)

   (c) $y = \sin \frac{1}{x}$. What are the x-intercepts of this graph? What is the domain of $\sin \frac{1}{x}$?

5. True or false? Between every two distinct rational numbers there is a rational number. Explain your answer.

6. True or false? Between every two distinct rational numbers there is an irrational number. Explain your answer.

7. True or false? For all real numbers a and b, $|a + b| \leq |a| + |b|$.

8. True or false? For all functions f and g, $|f(x) + g(x)| \leq |f(x)| + |g(x)|$ for every x in the domains of f and g.

2. Graphical Problems

Questions

1. Is there a function all of whose values are equal to each other? If so, graph your answer. If not, explain why.

Problems

1. (a) Find all x such that $f(x) \leq 2$ where

$$f(x) = -x^2 + 1 \qquad f(x) = (x-1)^2 \qquad f(x) = x^3$$

Write your answers in interval notation and draw them on the graphs of the functions.

(b) Using the functions in part a, find all x such that $|f(x)| \leq 2$. Write your answers in interval notation and draw them on the graphs of the functions.

(c) Can you find *upper bounds* for the functions in part a? That is, for each function f is there a number M such that for all x, $f(x) \leq M$?

(d) What about *lower bounds* for the functions in part a? That is, for each f can you find a number m such that for all x, $f(x) \geq m$?

(e) What about finding upper and lower bounds for these functions restricted to the interval $[-1, 1]$? That is, for each f can you find numbers M and m such that for all x in $[-1, 1]$, $m \leq f(x) \leq M$?

(f) True or false? If M is an upper bound for the function f and M' is an upper bound for the function g, then for all x which are in the domains of both f and g,

$$|f(x) + g(x)| \leq M + M'.$$

2. (a) Graph the functions below. Find their maximum and minimum values, if they exist. You don't need calculus to do this!

$$y = -x^2 + 1 \qquad y = x^2 - 1 \qquad y = (x-1)^2$$
$$y = \sin x - 1 \qquad y = \sin(x-1)$$

(b) Suppose $f(x) = x^2$ and $g(x) = \sin x$.

i. Write the functions in part a in terms of f and g. (For example, if $h(x) = 2x^2$ you can write h in terms of f as $h(x) = 2f(x)$.) If you find more than one way of writing these functions in terms of f and g, show that they are equivalent.

ii. How can you change the graph of f to obtain the graphs of the first three functions? Use your work from part a to help you.

iii. How can you change the graph of g to obtain the graphs of the last two functions?

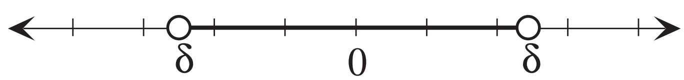
3. Tangent Lines and ε–δ Preliminaries

Questions

1. Can the graph of a function have more than one tangent at a given point? If so, graph your answer. If not, explain why.
2. Is there a function whose graph doesn't have a tangent at some point? If so, graph your answer. If not, explain why.

Problems

1. Suppose δ is a positive real number (δ is the lowercase Greek letter delta). How do you describe all real numbers x that are within δ of 0 as pictured on the line below?



   (a) Using inequalities ($<, >$).

   (b) Using absolute value notation.

   (c) Using interval notation.

2. This problem is like problem 1 except that we are using a instead of 0.

   Suppose a is a real number and δ is a positive real number. How do you describe all real numbers x that are within δ of a?

   (a) Graphically, as on the line above.
   (b) Using inequalities.
   There are at least two ways to do this. Find both.
   (c) Using absolute value notation.

   (d) Using interval notation.

3. This problem is like problem 2 except that we are asking that x not be equal to a.

   Suppose a is a real number and δ is a positive real number. How do you describe all real numbers x that are within δ of a, but not equal to a:

   (a) Graphically.
   (b) Using inequalities.
   (c) Using absolute value notation.
   (d) Using interval notation.

4. Let $f(x)$ be x^2.

   (a) Find all the positive numbers x such that $f(x)$ is within 1 of 9. ("Within" means the same thing it did in Problems 1 and 2, but here it refers to numbers on the y-axis.) Give your answer:
      i. On a graph of $y = x^2$.
      ii. Using inequalities.
      iii. Using interval notation.

   (b) It's difficult to express *all* numbers x such that $f(x)$ is within 1 of 9 using absolute value notation. (Why?) Instead:
      i. Find a real number δ such that whenever x is within δ of 3, $f(x)$ is within 1 of 9.
         Write this number using the min notation ("min" is for "minimum"). If a and b are two numbers, then $\min\{a, b\}$ is the smaller of a and b. For example, $\min\{5, 4\} = 4$. If a and b are equal, then $\min\{a, b\}$ is just a (or b). For example, $\min\{\sqrt{4}, 2\} = 2$.
      ii. Using absolute value notation and the value of δ that you have found, write an expression for x such that x is within δ of 3.

   (c) i. Find a real number δ such that whenever x is within δ of 3, $f(x)$ is within 1/2 of 9. Write this number using the min notation.
      ii. Using absolute value notation and the value of δ that you have found, write an expression for x such that x is within δ of 3.

   (d) Is it true that for any positive number ε, there is a positive δ so that $f(x)$ is within ε of 9 whenever x is within δ of 3? (ε is the lowercase Greek letter epsilon and stands for "error.")

      If your answer is yes, show how you can write an expression in terms of ε for a δ that works. Explain why your δ works.

      If your answer is no, show that there is a positive number ε for which the statement above is not true.

Additional Problems

1. Graph $y = 2x - x^2$. For which points a is the tangent line to the curve at the point $(a, 2a - a^2)$ a horizontal line? How is the value of $2x - x^2$ at these points related to the values of $2x - x^2$ at other points?

2. Look back at Problem 1 above. The point $(2, 0)$ is on the curve $y = 2x - x^2$.

   (a) Draw the tangent line to the curve at the point $(2, 0)$.

   (b) Try to determine the equation of this tangent line. What information do you need in order to determine the equation of a line? What information do you know in this situation? What information do you need?

4. Limits of Functions

Questions

1. Suppose f is a function and L and a are real numbers.
   (a) Describe what is meant by $\lim_{x\to a} f(x) = L$ as best you can.
   (b) Graph a function $y = f(x)$ such that $\lim_{x\to a} f(x) = L$, and $f(a) = L$.
   (c) Graph a function $y = f(x)$ such that $\lim_{x\to a} f(x) = L$, but $f(a) \neq L$.
   (d) Graph a function $y = f(x)$ such that $\lim_{x\to a} f(x) = L$, and $f(a)$ does not exist.

2. Is ∞ a number? What does $\lim_{x\to a} f(x) = \infty$ mean? Does $\lim_{x\to a} f(x) = \infty$ mean that the limit exists? Graph a function $y = f(x)$ with a point $x = a$ where $\lim_{x\to a} f(x) = \infty$. Can you write down a formula for such a function?

3. What does the Squeeze Theorem say? Draw a picture illustrating this theorem. What conditions do you need to check before applying the Squeeze Theorem?

4. What do we mean by $\lim_{x\to a^-} f(x) = L$? What is meant by $\lim_{x\to a^+} f(x) = L$? Graph a function $y = f(x)$ and some point $x = a$ where $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$. Find an algebraic expression for such a function.

Problems

1. (a) Is the limit of a sum *always* the sum of the limits? Give an example where $\lim_{x\to a}[f(x) + g(x)]$ exists even though neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exist.
   (b) Is the limit of a product *always* the product of the limits? Give an example where $\lim_{x\to a}[f(x)g(x)]$ exists even though neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exist.

2. (a) Find $\lim_{x\to 0^+}(\frac{1}{x} - \frac{1}{|x|})$, $\lim_{x\to 0^-}(\frac{1}{x} - \frac{1}{|x|})$, and $\lim_{x\to 0}(\frac{1}{x} - \frac{1}{|x|})$, if they exist. Graph $f(x) = (\frac{1}{x} - \frac{1}{|x|})$ to check your answers. (**Hint:** What does $|x|$ equal when $x < 0$?)
   (b) Show that $\lim_{x\to 0} |x| = 0$. Graph $g(x) = |x|$ to check your answer.
   (c) Show that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist. Graph $h(x) = \frac{|x|}{x}$ to check your answer.

3. For what kinds of functions can one evaluate the limit $\lim_{x\to a} f(x)$ by just plugging in $x = a$? Give an example of a function $f(x)$ for which the limit $\lim_{x\to a} f(x)$ exists and can be evaluated, yet for which this substitution doesn't work.

Additional Problems

1. Below are listed three limits and three methods for evaluating limits. Match the best method to each limit and evaluate each limit. (Find the best method if you did not have a graphing calculator or computer.)

• $\displaystyle\lim_{x\to 0} \frac{x^{1000} - 703x^{506} + \pi x^{17} + 480}{-372x^{75} + 39x^{14} + \sqrt{7}x^5 - 24}$	• graphing
• $\displaystyle\lim_{x\to 4} \frac{x^2 + 2x - 24}{x - 4}$	• algebra
• $\displaystyle\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$	• direct substitution

Explain why the method you chose is the best of the three possibilities.

2. Using a calculator, try to guess $\lim_{n\to\infty} n \sin\left(\frac{\pi}{n}\right)$. (Remember that n is in radians, not degrees!)

3. Draw a circle of radius 1. Inscribe a regular n-gon inside it. Draw line segments from the vertices to the center.
   (a) What is the measure of the angles formed around the center of the circle?
   (b) What is the area of one triangle? What is the area of the polygon?
   (c) Calculate $\lim_{n\to\infty}$ (Area of n-gon). Does this answer make sense intuitively?

5. The Precise Definition of a Limit

Questions

1. An important aspect of mathematical statements is the *order* of the words in the statement.

   (a) "If 4 divides a number, then that number is even." ("a divides b" means that b is divisible by a.)

   (b) "If a number is even, then 4 divides that number."

   Determine if each statement is true or false. If true, explain why. If false, give a counterexample. Why is the order of the two parts in each statement important?

2. Another important concept in mathematical statements is a *quantifier*: a phrase such as "for every" or "there exists." (These are sometimes written $\forall$ and $\exists$.)

   (a) "Every number is even."

   (b) "There exists a number which is even."

   Determine whether statements a and b are true or false. If true, explain why. If false, give a counterexample.

   (c) Now look back at the definition of $\lim_{x\to a} f(x) = L$. What are the quantifiers?

3. Suppose a, L, ε, and δ are real numbers, and that ε and δ are positive.

   (a) What interval is determined by $|x-a| < \delta$? What is the left endpoint? The right endpoint? The midpoint? Sketch the interval on the x-axis.

   (b) Sketch the set given by $0 < |x-a| < \delta$. How does this set of real numbers differ from the interval given in part a?

   (c) On the y-axis, sketch the interval for $|y-L| < \varepsilon$. What are the endpoints and the midpoint?

   (d) Now sketch the *region* in the xy-plane determined by $|x-a| < \delta$ and $|y-L| < \varepsilon$.

   (e) Graph a function f so that $|f(x)-L| < \varepsilon$ whenever $0 < |x-a| < \delta$.

   (f) On the same axes and using the same L and a, graph a function g which does *not* satisfy the statement $|g(x)-L| < \varepsilon$ whenever $0 < |x-a| < \delta$.

Problems

1. (a) In Problem 4d of Worksheet 3 you found the ingredients for a graphically motivated ε–δ proof that $\lim_{x\to 3} x^2 = 9$, i.e. for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|x^2 - 9| < \varepsilon$ whenever $0 < |x - 3| < \delta$. Write out that proof.

   (b) Example 4 of your textbook gives an algebraically motivated ε–δ proof that $\lim_{x\to 3} x^2 = 9$. Here are some questions about the first part of that example: finding the right δ.

   Example 4 starts by stating the result to be proved:
   For every $\varepsilon > 0$ there is a $\delta > 0$ so that $|x^2 - 9| < \varepsilon$ whenever $0 < |x - 3| < \delta$
   and working backward, looking at the statement and finding some of its properties.

   i. How can you factor $|x^2 - 9|$?
   ii. Assuming (as in the statement) that $|x - 3| < \delta$ means that the function $y = |x - 3|$ is bounded on the interval $[3 - \delta, 3 + \delta]$. What are its upper and lower bounds on that interval?
   iii. This also means that the function $y = |x + 3|$ is bounded on the interval $[3 - \delta, 3 + \delta]$. What are its upper and lower bounds on that interval?
   iv. The number δ is supposed to be small, so it's reasonable to assume that $\delta < 1$. Given that assumption, what is another upper bound for $y = |x + 3|$ on $[3 - \delta, 3 + \delta]$?
   v. Now, how can you choose δ so that $|x^2 - 9| < \varepsilon$?

2. For each line in the following table, graph $f(x)$ and indicate the interval $|y - L| < \varepsilon$.

$f(x)$	a	L	ε	δ	δ'
$2x$	0	0	1		
$2x$	0	0	1/2		
$2x$	-1	-2	1		
$2x$	-1	-2	1/2		
x^2	0	0	1		
x^2	0	0	1/2		
x^2	-1	1	1		
x^2	-1	1	1/2		

   (a) Based on the graph, choose a value for δ so that $|f(x)-L| < \varepsilon$ whenever $|x-a| < \delta$.

   (b) Could you have chosen different values for δ? Go back and find a δ' for each group of $f(x)$, a, L, and ε, where $\delta' \neq \delta$.

   (c) Is there a maximum possible δ for each $f(x)$, a, L, and ε? Is there a minimum possible δ for each $f(x)$, a, L, and ε? Explain your answers.

3. You may have the impression from the previous exercise that it's always possible to find a δ, given values for a, L, ε, and a function $f(x)$. But this problem shows it is not.

(a) Graph the function $f(x) = \begin{cases} \dfrac{x^2-1}{x+1} & x \neq -1 \\ 2 & x = -1 \end{cases}$.

(b) Now, suppose that $a = -1$ and that L is chosen to be $f(-1) = L = 2$. If $\varepsilon = \frac{1}{2}$, show that there is *no* possible δ so that if $|x - (-1)| < \delta$, then $|f(x) - 2| < \frac{1}{2}$. (**Hint:** Given any $\delta > 0$, find a point x such that $|x - 2| < \delta$ but $|f(x) - 2| \not< \frac{1}{2}$).

(c) From the graph, make a better guess for what $L = \lim_{x \to -1} f(x)$ should be. With this new L and $\varepsilon = \frac{1}{2}$, find a value of δ so that if $|x - (-1)| < \delta$, then $|f(x) - L| < \frac{1}{2}$.

(d) In general, for an arbitrary value of ε, what δ would you choose so that if $|x - (-1)| < \delta$, then $|f(x) - L| < \varepsilon$?
(**Hint:** express δ as a function of ε.)

6. Continuity

Questions

1. There are at least three different types of discontinuities. Give a graphical example of each discontinuity: removable, jump, and infinite.

2. What three properties does a function $f(x)$ need if it's going to be continuous at a point a?

3. What does it mean for a function to be continuous on an interval?

4. (a) Continuous functions are quite common. What are two basic types of continuous functions?

   (b) What operations allow you to build more continuous functions from already-known continuous functions? List as many of the operations as you can.

5. When can you "pull the limit inside a function"? That is, what do you need to know about f and g, so that

$$\lim_{x\to a} f(g(x)) = f(b) = f(\lim_{x\to a} g(x))?$$

Problems

1. Let $f(x) = \begin{cases} 1-x^2 & \text{if } 0 \le x \le 1 \\ 1+\frac{x}{2} & \text{if } 1 < x \le 2 \end{cases}$

   (a) Show that f is not continuous on $[0, 2]$.

   (b) Show that f does not take on all values between $f(0)$ and $f(2)$, in other words that there's a number between $f(0)$ and $f(2)$ that is not a value of f on the interval $[0, 2]$.

2. (a) Show that if f is a continuous function on an interval, then so is $|f|$.

   (b) If $|f|$ is continuous, must f be continuous? If so, prove it. If not, find a counterexample.

3. Assume that $f(x)$ and $g(x)$ are continuous at a number a and that c is a constant.

   (a) Prove that $cf(x)$ is also continuous at a.

   (b) Prove that $f(x) \cdot g(x)$ is also continuous at a.

4. (a) What does the Intermediate Value Theorem say? Draw an example illustrating this theorem. Explain why it makes sense based on what you know about the graph of a continuous functions.
   (b) Does the Intermediate Value Theorem say that there is *only* one number c in the interval (a, b) with the property you want – where $f(c) = N$? Draw a picture with multiple c's, if you haven't done so already.

Additional Problems

1. Give an $\varepsilon - \delta$ definition of f being continuous at a. There is one that is quite similar to the $\varepsilon - \delta$ definition of $\lim_{x \to a} f(x) = L$.
2. (a) Find a function whose domain is all real numbers but which is continuous nowhere.
   (b) Find a function whose domain is all real numbers and is continuous at exactly one point.
3. A **fixed point** of a function f is a number c in its domain such that $f(c) = c$. (The function doesn't move c; it stays fixed.)
   (a) Sketch the graph of a continuous function with domain $[0, 1]$ whose range also lies in $[0, 1]$. Locate a fixed point of f.
   (b) Try to draw the graph of a continuous function with domain$[0, 1]$ and range $[0, 1]$ that does *not* have a fixed point. What is the obstacle?
   (c) Use the Intermediate Value Theorem to prove that any continuous function with domain $[0, 1]$ and range in $[0, 1]$ must have a fixed point.
   (**Hint:** Apply the intermediate value theorem to the function $g(x) = f(x) - x$.)

7. Limits at Infinity and Asymptotes

Questions

1. (a) Draw the graphs of three functions with vertical asymptotes at $x = a$ which are as different as possible. Describe these differences using limits, e.g., $\lim_{x\to a^+} f(x) = \infty$, but $\lim_{x\to a^+} g(x) = -\infty$. How many different vertical asymptotes can the graph of a function have?

   (b) Draw three different graphs of functions with horizontal asymptotes that are as different as possible. How many different horizontal asymptotes can the graph of a function have?

2. Let f, g, h, j, and k be functions. Assume that

   i. $\lim_{x\to\infty} f(x) = \infty$,
   ii. $\lim_{x\to\infty} g(x) = -\infty$,
   iii. $\lim_{x\to\infty} h(x) = c > 0$ (where c is a constant),
   iv. $\lim_{x\to\infty} j(x) = 0$,
   v. $\lim_{x\to\infty} k(x) = 0^+$.

   Have each person in the group explain to the group how to do *two* of the following problems. Simplify all expressions that you can. Indicate which limits you can't evaluate. Explain your reasoning and explain when you use (i) through (v) above.

   (a) $\lim_{x\to\infty}[f(x) + j(x)] =$
   (b) $\lim_{x\to\infty}[g(x) + h(x)] =$
   (c) $\lim_{x\to\infty}[f(x) + g(x)] =$
   (d) $\lim_{x\to\infty}[f(x) - g(x)] =$
   (e) $\lim_{x\to\infty}[h(x)j(x)] =$
   (f) $\lim_{x\to\infty}[h(x)g(x)] =$
   (g) $\lim_{x\to\infty}[k(x)g(x)] =$
   (h) $\lim_{x\to\infty}[f(x)g(x)] =$
   (i) $\lim_{x\to\infty}[j(x)/f(x)] =$
   (j) $\lim_{x\to\infty}[f(x)/j(x)] =$
   (k) $\lim_{x\to\infty}[f(x)/g(x)] =$
   (l) $\lim_{x\to\infty}[k(x)/j(x)] =$

Problems

1. Make a rough sketch of the curve $y = x^n$, where n is an integer, for the following five cases: (i) $n = 0$; (ii) $n > 0$, n odd; (iii) $n > 0$, n even; (iv) $n < 0$, n even; and (v) $n < 0$, n odd. In each case, find the following limits.

   (a) $\lim_{x \to 0^+} x^n$

   (b) $\lim_{x \to 0^-} x^n$

   (c) $\lim_{x \to \infty} x^n$

   (d) $\lim_{x \to -\infty} x^n$

2. Let P and Q be polynomials with leading coefficients a and b respectively. Find $\lim_{x \to \infty} \frac{P(x)}{Q(x)}$ if the degree of P is

   (a) less than the degree of Q.

   (b) equal to the degree of Q.

   (c) greater than the degree of Q.

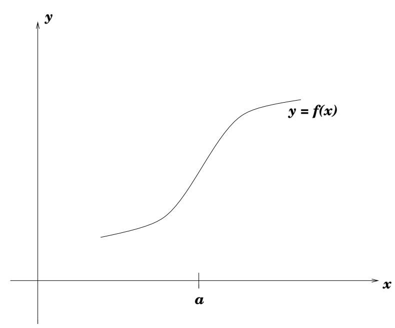
8. Derivatives

Questions

1. (a) Suppose h is a positive number. For one such h draw the line through $(a+h, f(a+h))$ and $(a, f(a))$ on the graph below. What happens to the line as h goes to 0? How is this expressed in terms of limits?

   (b) Now suppose h is a negative number. For one such h draw the line through $(a+h, f(a+h))$ and $(a, f(a))$ on the graph below. What happens to the line as h goes to 0? How is this expressed in terms of limits?

   (c) Now suppose x is simply a real number. For one such x draw the line through $(x, f(x))$ and $(a, f(a))$ on the graph below. What happens to the line as x goes to a? How is this expressed in terms of limits?



   (d) What is the equation of the line tangent to f at $(a, f(a))$?

2. This question centers around the fact that sometimes "derivative" means a number and sometimes it means a function.

   (a) Evaluate $\lim_{x \to 4} \dfrac{x^2 - 16}{x - 4}$. Write your answer in function notation.

   (b) Evaluate $\lim_{x \to a} \dfrac{x^2 - a^2}{x - a}$. Write your answer in function notation.

   (c) Evaluate $\lim_{h \to 0} \dfrac{(x+h)^2 - x^2}{h}$. Write your answer in function notation.

   (d) Let f be a differentiable function and let a be in the domain of f. Which of $f'(x)$ and $f'(a)$ is a number and which is a function?

3. True or false? If false, give a counterexample. If true, try to explain why using the limit definitions of continuous and differentiable.

   (a) If f is continuous at a, then f is differentiable at a.

   (b) If f is differentiable at a, then f is continuous at a.

Problems

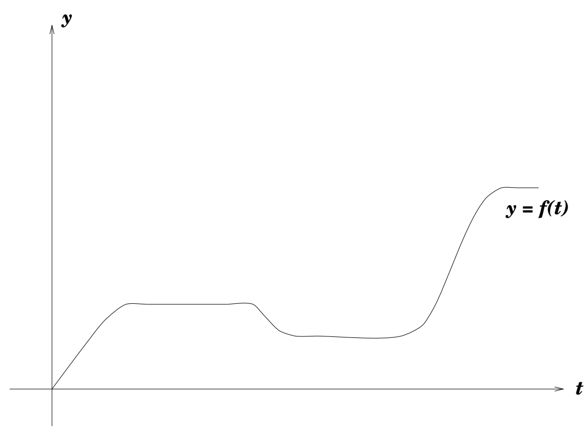
1. Let $f(t)$ be the function describing your distance from Berkeley at time t.

   (a) In each of these situations, graph $y = f'(t)$ and $y = f(t)$. When the graphs are positive or negative, increasing or decreasing, explain why they have these features.

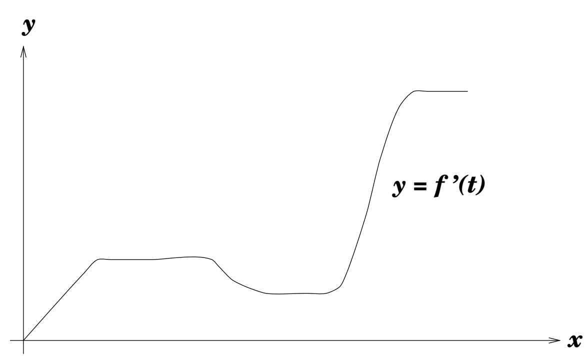
      i. You drive quickly to the Bay Bridge, get stuck in traffic on the bridge, and drive at a medium pace through San Francisco to your destination.

      ii. You drive to Emeryville but you have forgotten your books so you drive back to Berkeley, then back through Emeryville and over the bridge to San Francisco.

   (b) For the following graph $y = f(t)$, describe a driving situation similar to those above which corresponds to the graph. Then graph the corresponding $y = f'(t)$.



   (c) For the following graph $y = f'(t)$, describe a driving situation similar to those above which corresponds to the graph. Then graph the corresponding $y = f(t)$.



2. Let $f(x) = |x - 4|$.

   (a) Where is f continuous? Graph $y = f(x)$.

   (b) Find a formula for $f'(x)$ and sketch the graph of $y = f'(x)$.

3. What does it mean for a function $f(x)$ to be differentiable at a point a? What does it mean for $f(x)$ to be differentiable on an open interval?

Additional Problems

1. (a) In each part below, graph a cubic polynomial $y = ax^3 + bx^2 + cx + d$ which satisfies the given condition. You do not need to determine the values a, b, c and d.
    i. two horizontal tangents.
    ii. one horizontal tangent.
    iii. no horizontal tangents.

   (b) If you were going to choose coefficients, what conditions would a, b, c, and d have to satisfy in each case so that the graph of the polynomial $y = ax^3 + bx^2 + cx + d$ has precisely
    i. two horizontal tangents?
    ii. one horizontal tangent?
    iii. no horizontal tangents?

2. Let $h(x)$ be a function. Explain why it is not enough to know that $\lim_{x\to a^-} h'(x)$ and $\lim_{x\to a^+} h'(x)$ exist and are equal at the point a in order to know that the derivative $h'(a)$ exists.

3. A tangent line is drawn to the hyperbola $xy = c$ at a point P.

   (a) The coordinate axes cut off a line segment from the tangent line. Show that the midpoint of this line segment is P.

   (b) Show that the triangle formed by the tangent line and the coordinate axes always has the same area, not matter where P is located.

4. Evaluate $\lim\limits_{x\to 1} \dfrac{x^{1000} - 1}{x - 1}$ in two different ways.

9. Differentiation

Questions

1. Explain how to use the product rule to find $\frac{d}{dx}(x^2)$. Now find this derivative using the limit definition of derivative. Which is easier? Would you rather find $\frac{d}{dx}(x^4)$ using the product rule or the definition? Why?

2. Evaluate the following limits. Explain how you derived your answers.

   (a) $\lim_{x\to\infty} \cos\left(\frac{1}{x}\right)$

   (b) $\lim_{x\to\infty} \left(x - x\cos\frac{1}{x}\right)$

   (c) $\lim_{x\to 0} \frac{\sin(\sin x)}{\sin x}$

   (d) $\lim_{x\to 0} \frac{\sin(\sin x)}{x}$

Problems

1. The general polynomial of degree n has the form $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x^1 + a_0$, where $a_n \neq 0$. Find the derivative of P. Which differentiation rules did you use?

2. Find $\frac{d}{dx}(x+2)$, $\frac{d}{dx}((x+2)^2)$ and $\frac{d}{dx}((x+2)^3)$ by using the product rule. What do you think $\frac{d}{dx}((x+2)^{100})$ would be? How would you calculate this using the power rule?

3. Prove the power rule $\frac{d}{dx}(x^n) = n\cdot x^{n-1}$ when n is a positive integer. First try using the $\lim_{x\to a}$ definition of derivative. How would the proof work with the $\lim_{h\to 0}$ definition instead? How would the proof go using the product rule?

4. (a) Let $\theta = 2x$, then the identity $\cos 2x = 1 - 2\sin^2 x$ becomes $\cos\theta = 1 - 2\sin^2(\theta/2)$. Use this fact and $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$ to prove that $\lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} = 0$.

   (b) Using the $\lim_{h\to 0}$ definition of a derivative and the result from part (a), show that $\frac{d}{dx}\cos x = -\sin x$.

   (c) Now use the result from part (a), the equation $\sin^2 x + \cos^2 x = 1$ and the product rule to prove that $\frac{d}{dx}\sin x = \cos x$.

Additional Problems

1. (a) What is the definition of an even function? What is the definition of an odd function?
   (b) Sketch the graphs of two different even functions and two different odd functions.
   (c) Prove that, given it has a derivative, the derivative of an even function is an odd function.
   (d) Prove that, given it has a derivative, the derivative of an odd function is an even function.
   (e) Let $g(x)$ be a differentiable even function. Find $g'(0)$.

2. Make a careful sketch of the graph of the sine function. Draw in the tangent lines at the points 0, $\pi/6$, $\pi/4$, $\pi/3$, and $\pi/2$. Now, using *only* information from this graph, sketch the graph of the derivative of sine. How does your graph compare to the graph of the function you know is the derivative of sine?

3. Prove the product rule for derivatives. (**Hint:** Let $F(x) = f(x)g(x)$. Find $F'(x)$ using the $\lim_{h\to 0}$ definition of a derivative. Adding and subtracting $f(x+h)g(x)$ in the numerator might help.)

10. The Chain Rule

Questions

1. Let $f(x)$ and $g(x)$ be differentiable functions. Decide which of the following statements are true. When a statement is false, give an example which contradicts the assertion. Then give a correct version of the statement and an example that illustrates it.

   (a) The derivative of a product fg is the product of the derivatives $f'g'$.

   (b) The derivative of a sum $f+g$ is the sum of the derivatives $f'+g'$.

   (c) The derivative of a quotient f/g is the quotient of the derivatives f'/g'.

   (d) The derivative of a composition $f(g(x))$ is the composition of the derivatives $f'(g'(x))$.

2. Before attempting to use the chain rule, it is important to understand composition of functions well. Suppose that $k(x) = f(g(h(x)))$. Fill in the blanks with the appropriate functions.

If $f(x) = x^2$,	$g(x) = \sin x$,	$h(x) = \cos 4x$,	then $k(x) =$ _____.
If $f(x) = 1/x$,	$g(x) =$ _____,	$h(x) = (12 - x^2)$,	then $k(x) = (12 - x^2)^{-2}$.
If $f(x) =$ _____,	$g(x) = x$,	$h(x) =$ _____,	then $k(x) = \cos(\tan x)$.
If $f(x) =$ _____,	$g(x) =$ _____,	$h(x) =$ _____,	then $k(x) = [1 - (3x-2)^3]^4$.

3. You've learned about composing two functions, e.g., if $f(x) = x^2$ and $g(x) = x+1$, then $f \circ g(x) = (x+1)^2$. Decomposing a function reverses this process. For instance, if $h(x) = (x+1)^2$ then $f \circ g(x)$, where f and g are as above, is called a decomposition of h.

   (a) $h(x) = (x+1)^2$ can be decomposed in several other ways. Find one.

   (b) Can a function *always* be decomposed in more than one way? For instance, can $h(x) = x$ be decomposed in more than one way? Give an example of a function that can't be decomposed in two ways, or explain why you think that all functions can be decomposed in more than one way.

4. (a) Consider the function $h(x) = (x+2)^{100}$. (See the previous worksheet, problem 4.) Find two functions f and g such that $h(x) = (f \circ g)(x)$.

   (b) Find $h'(x)$ using the chain rule.

   (c) Explain how the chain rule makes solving this problem much easier.

Problems

1. (a) Use the chain rule to find $\frac{d}{dx}(\frac{1}{g(x)})$.
   (b) Use part a, the chain rule, and the product rule to prove the quotient rule.
2. (a) Sketch the graph of the function $f(x) = |\sin x|$.
   (b) At what points is f not differentiable?
   (c) Give a formula for f' and sketch its graph. (**Hint:** $|x| = \sqrt{x^2}$.)
   (d) Now do parts (a)–(c) for $g(x) = \sin|x|$.
3. Use the Chain Rule to prove that the derivative of an even function is an odd function, and that the derivative of an odd function is an even function. (This is the same problem as Additional Problem 1 on the previous worksheet. Are these statements easier to prove using the definition of derivative or using the chain rule?)

Additional Problems

1. Use the Chain Rule to show that if θ is measured in degrees, then $\frac{d}{d\theta}(\sin\theta) = \frac{\pi}{180}\cos\theta$. (This gives one reason for the convention that radian measure is always used when dealing with trigonometric functions in calculus; the differentiation formulas would be much more complicated otherwise!)
2. (a) If n is a positive integer, prove that $\frac{d}{dx}(\sin^n x \cos nx) = n\sin^{n-1} x \cdot \cos(n+1)x$.
   (b) Find a similar formula for $\frac{d}{dx}(\cos^n x \cos nx)$.
3. Prove the chain rule $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ starting with $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$.
   (a) Case I: $\frac{du}{dx} \neq 0$.
   (b) Case II: $\frac{du}{dx} = 0$, where $\Delta u = 0$ for some values of Δx and $\Delta u \neq 0$ for other values of Δx. What happens if $\Delta x \to 0$ through values such that $\Delta u \neq 0$? What happens if instead $\Delta x \to 0$ through values such that $\Delta u = 0$?

11. Implicit differentiation and higher derivatives

Questions

1. Give an example of y as a function of x, $y = f(x)$. Now give an example of an implicit function describing a relation involving x and y, $F(x, y) = 0$, such as $\sin x - y^3 = 0$. Can your implicit function be rewritten in terms of explicit functions? Try to find an implicit function which no one else in your group can rewrite in terms of an explicit function.

2. A family of curves is described by a set of functions with one (or more) parameters. For example, the general formula of a circle $x^2 + y^2 = r^2$ is actually a family of curves because the parameter r is a stand-in for all different possible radii lengths. The goal of this question is to answer part (c) below but parts (a) and (b) are meant to serve as preliminary questions.

   (a) How can you tell if two lines are perpendicular? (Give graphical and algebraic answers.)

   (b) What does it mean for two curves to be orthogonal? (Give graphical and algebraic answers.)

   (c) What does it mean for two families of curves to have orthogonal trajectories? (Give graphical and algebraic answers.)

3. Find the derivatives by figuring out a general pattern:

   (a) $D^{99} \sin x$

   (b) $D^{100} \cos x$

   (c) $D^{50} \cos 2x$

   (d) $D^{35} x \sin x$

4. What does $n!$ mean?

Problems

1. (a) Show, using implicit differentiation, that the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_0, y_0) is $\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$.

   (b) Sketch a graph of an ellipse and draw in the tangent lines at the points of intersection with the coordinate axes.

   (c) Do the equations of these lines agree with what you expected from the first part?

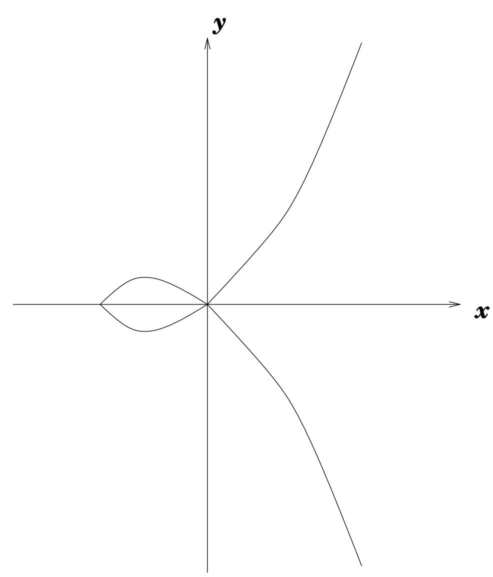
2. Is the following statement true or false? If it's true, explain why. If it's false, give a counterexample and correct the statement.

$$\text{If } P \text{ is a polynomial of degree 6, then } P^{(7)} = 0.$$

   (Recall that $P^{(7)}(x)$ is the seventh derivative of P.) Once you have a true preceding statement, come up with a more general statement that is also true.

3. Consider the curve defined implicitly by the equation $y^3 + xy + x^4 = 11$. Find $\frac{d^2y}{dx^2}$ at the point $(1, 2)$. Find a formula for $\frac{d^2y}{dx^2}$ in terms of only x and y (that is, without any $\frac{dy}{dx}$'s in it).

4. Consider the curve by the equation $y^2 = x^3 + x^2$ whose graph is



   (a) Explain what the tangent to the curve at $(0, 0)$ is. Is it possible for a curve to have two tangents at a single point? Why or why not?

   (b) Now find $\frac{dy}{dx}$ by implicit differentiation and find $\frac{dy}{dx}\,|_{(x,y)=(0,0)}$.

   (c) Explain how your answers to parts (b) and (c) are related. (There should be some relationship since the derivative is intuitively the slope of the tangent line.)

5. If $y = f(u)$ and $u = g(x)$, where f and g are twice differentiable functions, show that

$$\frac{d^2y}{dx^2} = \frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2 + \frac{dy}{du}\,\frac{d^2u}{dx^2}.$$

Additional Problems

1. Find a third-degree polynomial Q such that $Q(1) = 1$, $Q'(1) = 3$, $Q''(1) = 6$, and $Q'''(1) = 12$.

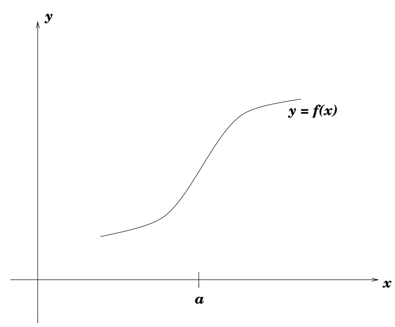
2. Use implicit differentiation to prove the binomial theorem. Let $f(x) = (1+x)^n$. f is obviously a nth degree polynomial, so it must look like $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n$. Now, $f(0) = (0+1)^n = 1$ so we know that $a_0 = 1$. Differentiating f tells us that $f'(x) = n(1+x)^{n-1} = a_1 + 2a_2x + \ldots + na_nx^{n-1}$ and since $f'(0) = n = a_1$, we know two coefficients already. Continue this process to find expressions for all the coefficients.

12. Using Differentiation to do Approximations

Questions

1. What is the equation of the tangent line to the curve $y = f(x)$ at $(a, f(a))$? Now express this equation in terms of differentials.

2. (a) What values can dx take on? How is dy defined? Which of dx and dy is independent and which is dependent?



   (b) On the graph above, label dy and Δy. (Recall $\Delta y = f(x + \Delta x) - f(x)$.) Show the difference between dy and Δy graphically.

   (c) When is $\Delta y \approx dy$? Which is easier to compute, Δy or dy?

3. For a function f, Newton's method approximates a number r such that $f(r) = 0$. (This is the only thing Newton's method does!) The first step of Newton's method is to pick a number x_1 close to r. Given that you don't know what r is, how do you pick a "good" x_1? (**Hint:** You want to pick x_1 so that an interval containing x_1 and r is small. What theorem guarantees you can find a root of a function in an interval?)

Problems

1. *Without* graphing the entire curve $\sin x$, graph the points on the curve $y = \sin x$ for $x = 0$, $\pi/2$, π, $3\pi/2$, and 2π. Now find the slopes of the tangent lines to $y = \sin x$ at these four points, and sketch these tangent lines.

2. (a) Apply Newton's method to the equation $x^2 - a = 0$ to derive the following square-root algorithm: $x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n})$.

   (b) Now compute $\sqrt{2}$ to six decimal places.

3. The area of a circle is a function of its radius, $A = \pi r^2$.

(a) Sketch two circles, one with radius r and the second with radius $r + \Delta r$ where Δr is reasonably small.

(b) Express ΔA and dA symbolically.

(c) Shade in the area representing ΔA on your sketch.

(d) Explain why dA is a good approximation to ΔA. (**Hint:** Recall that the inner circle has circumference $2\pi r$.)

4. *Without* graphing $\sin x$, sketch the quadratic approximation to $f(x) = \sin x$ for the points $a = \pi/2$ and $3\pi/2$. (To find the quadratic approximation $P(x)$ to the function $f(x)$ at $x = a$, start with $P(x) = A + B(x-a) + C(x-a)^2$ where $P(a) = f(a)$, $P'(a) = f'(a)$ and $P''(a) = f''(a)$. You'll have three equations with the variables A, B and C and you plug them into each other to find $P(x)$.) How much does this quadratic approximation resemble $\sin x$?

Additional Problems

1. If $f(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ -\sqrt{-x} & \text{if } x < 0 \end{cases}$, then the root of f is $x = 0$. Explain why Newton's method fails to find the root no matter what initial approximation $x_1 \neq 0$ is used. Illustrate your explanation with a sketch.

2. Prove the following rules for working with differentials, where c is a constant and u and v are functions of x.

   (a) $dc = 0$

   (b) $d(cu) = cdu$

   (c) $d(u + v) = du + dv$

   (d) $d(uv) = udv + vdu$

   (e) $d(\frac{u}{v}) = \frac{vdu+udv}{v^2}$

   (f) $d(x^n) = nx^{n-1}dx$

3. How many solutions are there to $x = \sin x$? Explain why your answer is correct. (**Hint:** It may be helpful to graph $y = x$ and $y = \sin x$ on the same graph.)

13. Exponential Functions

Questions

1. (a) Let n be a positive integer and a be a real number. Write an expression without exponential notation for a^n. In other words, how do you define a^n?

   (b) Show, by giving counterexamples, that the equations $a^{x+y} = a^x + a^y$ and $(a^x)^y = a^{(x^y)}$ do not always hold. What are the correct rules for a^{x+y} and $(a^x)^y$?

2. Sketch the graphs of $y = 1^x$, $y = 2^x$, $y = 4^x$, $y = (1/2)^x$, and $y = (1/4)^x$. Are there any points that all these graphs have in common? For each graph, find the approximate value of the slope of the tangent to the graph through its y-intercept.

3. Graph the exponential function, $f(x) = e^x$. What is the equation of the tangent line at the y-intercept?

Problems

1. (a) Using the $\lim_{h\to 0}$ definition of a derivative, show that the derivative of an exponential function is proportional to the function itself. In other words, show that if $f(x) = a^x$, then $f'(x) = k \cdot a^x = k\, f(x)$. What is the value of the constant k for each function a^x?

   (b) How is the function $f(x) = e^x$ different from other exponential functions? (**Hint:** how is the derivative of e^x different from 2^x or 3^x?)

2. Find the hundredth derivative of $f(x) = xe^{-x}$. (**Hint:** compute the first few derivatives. What is the pattern?)

3. (a) Use the Intermediate Value Theorem to show that there is a root of the equation $e^x + x = 0$.

   (b) Use Newton's Method to find that root, correct to six decimal places.

4. Sketch the graphs of the functions x^2 and 2^x. How many times do the graphs intersect? Find at least one of these points of intersection. Which function is growing the most quickly? How can you tell?

Additional Problems

1. Evaluate $\lim_{x\to\pi} \dfrac{e^{\sin x} - 1}{x - \pi}$.

2. For what value of a are the graphs of $y = a^x$ and $y = \log_a x$ tangent to each other?

14. Inverse Functions

Questions

1. A function is "one-to-one" if for any two distinct points $x_1 \neq x_2$ in the domain, $f(x_1) \neq f(x_2)$. Give an example of a one-to-one function. Now give an example of a function which is *not* one-to-one.

2. Given a graph, how can you tell geometrically if it's the graph of a function? How can you tell from the graph of a function whether it's one-to-one? Sketch some pictures to support your answers.

3. To find the inverse of a function $f(x)$: (i) Write down $x = f(y)$; (ii) Solve this equation for y; (iii) Then $f^{-1}(x) = y$. We show this for $f(x) = 3x + 4$:

$$x = 3y + 4 = f(y)$$

$$3y = x - 4$$

$$y = \frac{1}{3}x - \frac{4}{3}$$

$$f^{-1}(x) = \frac{1}{3}x - \frac{4}{3}$$

   Check by plugging in that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$ in the above case. Now find $g^{-1}(x)$ when $g(x) = \frac{x^3}{2} - 5$.

4. Suppose f is a function with inverse f^{-1}.

   (a) If $f^{-1}(x) = y$, what is $f(y)$? What is $f(x)$?

   (b) If $f(x) = y$, what is $f^{-1}(x)$? What is $f^{-1}(y)$?

   (c) Match up:

domain of f	domain of f^{-1}
range of f	range of f^{-1}

Problems

1. (a) If (a, b) is a point on the graph of $y = f(x)$, what point must lie on the graph of $y = f^{-1}(x)$?

   (b) Graph the function $f(x) = x^3$ and its inverse on the same axes.

   (c) Which line do the graph of a function and the graph of its inverse reflect through? (**Hint:** Turn your paper (or your head) $\pi/4$ radians.)

(d) What do you know about the points where the graphs of f and f^{-1} intersect?

2. (a) Why must a function be one-to-one in order to have an inverse? (**Hint:** Think about how the graphs of f and f^{-1} are related (Question 4(c)). Also think about the test for a graph to be a function, and the test for a function to be one-to-one (Question 2.))

   (b) Often a function is *not* one-to-one and we use the idea of restricting its domain to make it one-to-one so we can find its inverse.

      i. Is $\sin x$ one-to-one? Is $\sin x$ one-to-one when restricted to the interval $[-\pi/2, \pi/2]$? Explain how to define the well-known function $\sin^{-1} x$.

      ii. Explain how to restrict the domain of $f(x) = x^2$ so we can find that its inverse is $g(x) = \sqrt{x}$.

3. (a) Use implicit differentiation, the chain rule and $f(f^{-1}(x)) = x$ to find an expression for $(f^{-1}(x))'$.

   (b) Now try using $f^{-1}(f(x)) = x$ and $y = f(x)$ to find the derivative of $f^{-1}(y)$.

   (c) Does it matter which equality you use?

4. Suppose that f is a differentiable function, g is the inverse of f, and let $G(x) = 1/g(x)$. If $f(3) = 2$ and $f'(3) = \frac{1}{9}$, find $G'(2)$.

Additional Problems

1. Try sketching some examples and then explain why:

   (a) The inverse of a continuous function is continuous.

   (b) Except for points where there are horizontal tangents, the inverse of a differentiable function is also differentiable.

2. If f is a one-to-one, twice differentiable function with inverse function g, show that $g''(x) = -\frac{f''(g(x))}{[f'(g(x))]^3}$.

3. For which x does $f^{-1}(f(x)) = x$? For which x does $f(f^{-1}(x)) = x$? Using $f(x) = x^2$, where $x \geq 0$, and the technique of graphical function composition, show that these equalities work where they are defined.

   To find the composition $f(g(a))$ graphically, first graph both f and g on the same axes. Also draw in the line $y = x$. Now, starting at the point a on the x-axis, go up until you reach the point $(a, g(a))$. Next go over to the point $(g(a), g(a))$. From there, find the point $(g(a), f(g(a)))$. Finally, go back over to $(a, f(g(a)))$.

   Try graphical composition for a few points using $f(x) = x^2$, where $x \geq 0$, and $g = f^{-1}$. Where do you usually end up? If you add in a line connecting the points $(a, g(a))$ and $(a, f(g(a))$, what shape would you have drawn? What happens if you find $(a, g(f(a)))$ instead?

15. Logarithmic Functions and their Derivatives

Questions

1. For which real numbers a is the function $f(x) = a^x$ defined for all x?
   For which a does $f(x) = a^x$ have an inverse? Write a sentence explaining what $\log_a x$ means.

2. Write a sentence explaining what $\ln x$ means. What is $\ln e$? What is $\ln 1$? If $\ln x = y$, what is e^y? For which x is $\ln(e^x) = x$? For which x is $e^{\ln x} = x$?

3. (a) Graph $y = e^x$ and its tangent at the y-intercept. What is the equation of this tangent line?
   (b) Graph and describe in words the logarithmic function $y = \ln x$. What is the equation of the tangent line at the x-intercept?
   (c) How are the two graphs in parts a and b related?

4. Have each person in your group pick one of the functions below (each person should pick a different one). Then sketch the graph of the function.
   Explain to the group how the graph you've drawn is related to the graph of $y = \ln x$. For instance, the graph of $y = -\ln(x)$ is the reflection of the graph of $y = \ln(x)$ across the x-axis.

   (a) $y = -\ln(x)$
   (b) $y = \ln(-x)$
   (c) $y = -\ln(-x)$
   (d) $y = \ln|x|$
   (e) $y = \ln(e)$
   (f) $y = \ln(x^2)$
   (g) $y = \ln(1/x)$
   (h) $y = \ln(x + 3)$
   (i) $y = \ln|x + 3|$

Problems

1. Let $x, y > 0$. We can prove that $\log_a(xy) = \log_a x + \log_a y$ as follows:

   i. $xy = xy$
   ii. $xy = a^{\log_a x} \cdot a^{\log_a y}$

iii. $xy = a^{\log_a x + \log_a y}$

iv. $\log_a(xy) = \log_a \left[a^{\log_a x + \log_a y}\right]$

v. $\log_a(xy) = \log_a x + \log_a y$

Explain why each step of the proof above is true. Then write your own proofs for the other two algebraic properties of logarithms using the corresponding properties of exponents. Make sure all members of your group understand all proofs completely.

2. Solve the following. (**Hint:** Remember what the domain of $\ln x$ is.)
   (a) $\ln(x-1) - \ln x = 1$.
   (b) $\ln(x^2 - 2x - 2) \leq 0$.

3. (a) Explain why $g(x) = e^{\ln g(x)}$ for any function $g(x) > 0$.
   (b) Explain why $g(x)^{h(x)} = e^{h(x)\cdot \ln g(x)}$ for any functions $g > 0$ and h.

4. In general, the four possible combinations of constants and functions as exponents and bases are:

constant base with constant exponent	function base with constant exponent
constant base with function exponent	function base with function exponent

   For each case, differentiate the following functions and explain which rule you used for the differentiation.

   Suppose that a and b are constants with $a > 0$, and that $f(x)$ and $g(x)$ are functions with $f(x) > 0$. Find

   (a) $\frac{d}{dx}(a^b)$
   (b) $\frac{d}{dx}[f(x)]^b$
   (c) $\frac{d}{dx}[b^{g(x)}]$
   (d) $\frac{d}{dx}[f(x)^{g(x)}]$

Additional Problems

1. *Without* using a calculator, determine which of the numbers $\log_{10} 99$ or $\log_9 82$ is larger.

2. Prove that $\ln x$ is a differentiable function. (**Hint:** Use Theorem 7 on page 205.)

3. Use the (ε, N) definitions of limits from Chapter 1 to prove that $\lim_{x\to-\infty} e^x = 0$ and that $\lim_{x\to\infty} e^x = \infty$.

16. Inverse Trigonometric Functions

Questions

1. Explain why $\sin x$ does not have an inverse function. Is $\sin x$ one-to-one on $[0, \pi/4]$? Does $\sin x$ obtain all values on $[0, \pi/4]$? Why or why not? What is the length of the shortest interval on which $\sin x$ obtains all of its values? Is $\sin x$ one-to-one on an interval of this length?

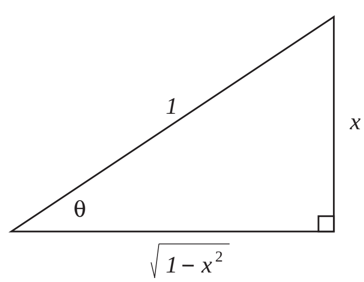
2. Graph $y = \cos x$ and $y = \tan x$. Find intervals on which they are one-to-one, but where the functions obtain all of their values. How many choices do you have in picking these intervals? For each function, what is the "standard" choice?

3. (a) Graph the restricted tangent function and its inverse. Describe the graph (in words). How are the asymptotes of $\tan x$ and $\tan^{-1}(x)$ related?
   (b) Evaluate $\lim_{x\to\infty} \tan^{-1}(\ln x)$.

4. Graph $y = \sin^{-1} x$. Now graph $y = 1/\sin x$. (If you have trouble with this, graph $y = \sin x$ first.) Are they equivalent for $-1 \le x \le 1$?

Problems

1. (a) Using the triangle below, find an expression for $\cos(\sin^{-1} x)$ that does not involve trigonometric functions. Then find $\sin(\cos^{-1} x)$.



   (b) Now find $\cos^{-1}(\sin x)$ and $\sin^{-1}(\cos x)$. (You may need to draw a new diagram.)
   (c) Sketch the graph of $h(x) = \cos^{-1}(\sin x)$, where x is any real number, and find its derivative.

2. Find the derivatives of arcsine, arccosine, and arctangent by implicit differentiation. What are the domains of the derivatives?

3. (a) Sketch the graph of $f(x) = \sin(\sin^{-1} x)$. What is its domain?

(b) Carefully sketch the graph of $g(x) = \sin^{-1}(\sin x)$, where x is any real number. (**Hint:** remember that $\sin^{-1} x$ is *not* an inverse for $\sin x$ for all values of x.)

(c) Show that $g'(x) = \dfrac{\cos x}{|\cos x|}$.

Additional Problems

1. (a) Prove that $\sin^{-1} x + \cos^{-1} x = \pi/2$.

   (b) Use this fact to calculate the derivative of $\cos^{-1} x$.

2. Let $f(x) = x \arctan(1/x)$ if $x \neq 0$ and $f(0) = 0$.

   (a) Is f continuous at 0?

   (b) Is f differentiable at 0?

3. When $xy \neq 1$, prove that $\arctan x + \arctan y = \arctan \frac{x+y}{1-xy}$. Assume that the left hand side lies between $-\pi/2$ and $\pi/2$.

17. Hyperbolic Functions

Questions

1. Are $\cosh x$ and $\cos x$ the same function? What is the definition of $\cosh x$? What is the definition of $\cos x$?

2. (a) Sketch the graphs of $\frac{1}{2}e^x$, $-\frac{1}{2}e^{-x}$, and $\sinh x$. How are they related? Describe the graph of $\sinh x$.

   (b) Sketch the graphs of $\frac{1}{2}e^x$, $\frac{1}{2}e^{-x}$, and $\cosh x$. How are they related? Describe the graph of $\cosh x$.

   (c) Sketch and describe the graph of $\tanh x$. How is $\tanh x$ related to $\sinh x$ and $\cosh x$?

3. Which hyperbolic functions are one-to-one? How is the inverse hyperbolic cosine function defined? Sketch and describe the graphs of $\sinh^{-1} x$, $\cosh^{-1} x$, and $\tanh^{-1} x$.

4. Show that $\sinh x$ is an odd function. Show that $\cosh x$ is an even function.

Problems

1. (a) Graph $\sinh x$ and $\cosh x$ on the same axes. Describe three different ways in which these two functions are related. (**Hint:** What happens if you add, subtract, differentiate, ...)

   (b) Prove algebraically that these relationships are correct.

2. Because $\frac{d^2}{dx^2}(\sin x) = -\sin x$ and $\frac{d^2}{dx^2}(\cos x) = -\cos x$, we say that $y = \sin x$ and $y = \cos x$ are solutions to the differential equation $y'' = -y$.

   (a) What differential equation do you think $y = \cosh x$ and $y = \sinh x$ satisfy?

   (b) What is the hundredth derivative of $\sinh x$?

   (c) Find the 12345th derivative of $\cosh x$.

3. (a) How do we know that the point $P = (\cos t, \sin t)$ lies on the unit circle? Draw a picture illustrating this fact. What does t represent on this picture?

   (b) On what type of graph does the point $Q = (\cosh t, \sinh t)$ lie? Why is this true? Draw a picture illustrating this fact. Does t represent an angle in this case? If not, what does t represent?

4. (a) Show that $\cosh^{-1}(x) = \ln(x + \sqrt{x^2 - 1})$. (**Hint:** There are at least two ways to do this. One is to start with the defining equation for $\cosh x$. Interchange x and y, and multiply by $2e^y$. Rearrange to get a quadratic equation in e^y. Solve using the quadratic formula.)
   (b) Calculate the derivative of $\cosh^{-1}$. Check that your answer is correct by implicitly differentiating $y = \cosh^{-1} x$ to find $\frac{dy}{dx}$.
5. (a) Find $\lim_{x\to\infty} \tanh(\ln x)$.
   (b) Show that $\tanh(\ln x) = \frac{x^2-1}{x^2+1}$. Was your answer in part (a) correct?

Additional Problems

1. Show that $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$, where n is any real number.
2. Show that $\sin^{-1}(\tanh x) = \tan^{-1}(\sinh x)$.
3. Write down the hyperbolic angle sum formulas. For example, $\sinh(x + y) = \ldots$ Explain how they are different from the regular trigonometric angle sum formulas. What is $\sinh(x - y)$? What is $\cosh(x - y)$?

18. Indeterminate Forms and l'Hospital's Rule

Questions

1. It's very important to know when you *can* and *cannot* use a rule like l'Hospital's.
   (a) Evaluate $\lim_{x\to 0} \frac{x}{x+1}$. Did you need to use l'Hospital's Rule?
   (b) What value would you have gotten if you had tried to use l'Hospital's Rule? Why does l'Hospital's rule not apply in this situation?
   (c) What conditions does a limit need to fulfill before you can apply l'Hospital's Rule?

2. True or false? (When the statement is false, give a counterexample):
   (a) L'Hospital's Rule says that the limit of a quotient is *always* equal to the limit of the quotient of their derivatives.
   (b) L'Hospital's Rule also applies to one-sided limits and for limits at positive or negative infinity.
   (c) In order to be able to apply l'Hospital's Rule to $\lim_{x\to a^+} \frac{f(x)}{g(x)}$ or $\lim_{x\to a} \frac{f(x)}{g(x)}$, $g'(a)$ and $f'(a)$ have to exist.

Problems

1. Consider, but don't evaluate, the following limits.

   i. $\lim_{x\to 1} \frac{x^s-1}{x^t-1}$ for the cases $s > t$, $t > s$ and $s = t$
   ii. $\lim_{x\to\infty} xe^{-x}$
   iii. $\lim_{x\to 0^+} x^{1/x^2}$
   iv. $\lim_{x\to 1^+} (\frac{1}{x})^{-\ln(x-1)}$
   v. $\lim_{x\to 0^+} \ln(x)e^{1/x^2}$
   vi. $\lim_{x\to\infty} x\sin(\frac{1}{x})$
   vii. $\lim_{x\to\infty} \ln(x) - x$
   viii. $\lim_{x\to 0} \frac{\tan(x)}{\tanh(x)}$
   ix. $\lim_{x\to 0^+} [\sin(x)]^{e^{-1/x^2}}$
   x. $\lim_{x\to 0^+} \frac{1}{x} + \ln(x)$
   xi. $\lim_{x\to 0} (1 + ax)^{1/x}$
   xii. $\lim_{x\to 0} \frac{s^x-1}{t^x-1}$ when $s > t > 0$, $t > s > 0$ and $s = t > 0$

(a) From the above list, pick an indeterminate form of the type $0 \cdot \infty$. Can you convert it to the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$? If so, apply l'Hospital's rule after the conversion and determine the limit.

(b) Pick an indeterminate form of the type $\infty - \infty$ from the above list. Can you convert it to the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$? If you can, apply l'Hospital's rule after the conversion and determine the limit.

(c) Pick an indeterminate form of each of the following types: 0^0, 0^∞ and 1^∞. How would you convert these types of indeterminate forms into those of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$? By making the appropriate conversions and using l'Hospital's rule, determine the limits.

(d) Pick two more limits from the above list and determine their values.

2. Sketch the graphs of $y = \ln x$ and $y = x$, $y = x^2$, $y = x^{(1/2)}$, $y = x^{(1/3)}$, etc. It's a fact that the logarithm approaches infinity more slowly than any positive power of x. Use l'Hospital's Rule to prove this by showing that $\lim_{x\to\infty} \frac{\ln x}{x^p} = 0$ for any number $p > 0$.

3. Suppose that $f(a) = g(a) = 0$, $f'(x)$ and $g'(x)$ are continuous, and $g'(a) \neq 0$. Starting with $\lim_{x\to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}$ and using the $\lim_{x\to a}$ definition of a derivative, show that l'Hospital's Rule works. In which step did we use the continuity of $f'(x)$ and $g'(x)$?

4. Show that the exponential function approaches infinity faster than any power of x. To do this, first sketch $y = e^x$, $y = x$, $y = x^2$ and $y = x^3$ on the same axes. Then, use l'Hospital's Rule to prove that $\lim_{x\to\infty} \frac{e^x}{x^n} = \infty$, where n is any integer. (**Hint:** use induction. Start with $n = 1$, $n = 2$, $n = 3$, ... Explain as clearly as you can what the pattern is.) Another approach to this problem is to use the result from problem 2 and think about the inverse functions. If you choose this route, make sure to explain your reasoning clearly.

Additional Problems

1. (a) Draw a diagram and interpret the quotient $\frac{f(x+h)-f(x-h)}{2h}$ as the slope of a secant line.

   (b) If f is differentiable at x, show that $\lim_{h\to 0} \frac{f(x+h)-f(x-h)}{2h} = f'(x)$.
   (**Hint:** try adding and subtracting the same quantity in the numerator.)

   (c) Show that it is possible for the limit $\lim_{h\to 0} \frac{f(x+h)-f(x-h)}{2h} = f'(x)$ to exist, but for $f'(x)$ not to exist. (**Hint:** consider $f(x) = |x|$.)

   (d) If f' is continuous, use l'Hospital's Rule to show that $\lim_{h\to 0} \frac{f(x+h)-f(x-h)}{2h} = f'(x)$.

   (e) Explain in your own words what conclusion you can draw from parts (b), (c) and (d) above. Why do those three results not contradict each other?

2. If f'' is continuous, show that $\lim_{h\to 0} \frac{f(x+h)-2f(x)+f(x-h)}{h^2} = f''(x)$. Why do you need to know that the second derivative is continuous?

19. Falling Objects and Limits

A great controversy developed during the 1996 Major League Baseball season. Batters were hitting more home runs than in previous seasons. Different explanations were given for this "Power Surge of 1996." Were the batters suddenly stronger? Were the pitchers suddenly worse? Or, was it something to do with the baseball itself? If the baseball actually flew through the air with greater ease, could this be enough to account for the Power Surge? A reporter for the *Santa Rosa Press-Democrat* came to campus to discuss the question with Mechanical Engineering Professor George Johnson. To investigate the question of whether the 1996 balls were traveling through the air faster, Professor Johnson and two of his graduate students dropped a 1996 baseball and a 1994 baseball from the Campanile which is 64 meters high.

Questions

1. Suppose you drop a baseball off the edge of a very high building that, like most buildings, is surrounded by air rather than a vacuum. On the axes below sketch a possible graph of its downward velocity versus time. Does the ball keep speeding up? How is this indicated on your graph?

   You may not know everything you'd like to know about the motion of the baseball. In this and the next question, sketch what you do know and write down any questions that you have. (They are likely to be answered in the next part of this worksheet.)

2. Sketch a graph that shows how far the baseball has fallen versus time. Indicate where it hits the ground.

On the same axes sketch the graph of a falling object that has more air resistance.

Problems

You can use calculus and the laws of physics to investigate the speed of the baseballs that Professor Johnson and his students dropped. Assume that the frictional force on a moving baseball is proportional to the square of its speed. (That is, frictional force is bv^2 where b is a coefficient of friction representing air resistance.)

This assumption and Newton's Law, $F = ma$ (force equals mass times acceleration), imply that the speed v of a falling object satisfies the differential equation

$$\frac{dv}{dt} = g - bv^2.$$

where g is the acceleration due to gravity. The constant g is known to be approximately 9.8 meters/sec^2, and the friction coefficient b is to be measured in the experiment.

In Math 1B, you will learn how to solve differential equations like the one above. Here is the solution:

$$v = \sqrt{g/b}\left(\frac{e^{2t\sqrt{bg}} - 1}{e^{2t\sqrt{bg}} + 1}\right).$$

1. (a) Show that v is a solution of the differential equation.
   (b) According to this formula for v, what happens to the velocity of the ball if it falls for a really long time? Does its speed keep getting larger and larger? Is there some maximum speed that it does not exceed? If there is a maximum speed, find it in terms of g and b.
   (c) Sketch the graph of v. In this situation the maximum speed that you calculated has a special name: terminal velocity. How does terminal velocity show up on your graph of velocity? What aspect of the physical situation does it correspond to?

2. To find the position x of the baseball at time t, one needs to solve the differential equation $dx/dt = v$. Finding the function x requires techniques of integration which you will learn later in this course and in Math 1B. Here is the solution:

   $$x = \frac{1}{b}\ln(e^{2t\sqrt{bg}} + 1) - t\sqrt{g/b} + C,$$

   where C is a constant.

   (a) Check that x satisfies the differential equation $dx/dt = v$.
   (b) Does the graph you sketched for the position of the baseball look as if it could be the graph of x?

(c) Show that x is quite well approximated by the simple expression $t\sqrt{g/b}$ as t becomes very large. First do this by "dropping relatively insignificant terms" in the formula for x as a function of t. Next show it by calculating that $\lim_{t\to\infty} \frac{x(t)}{t\sqrt{g/b}} = 1$.

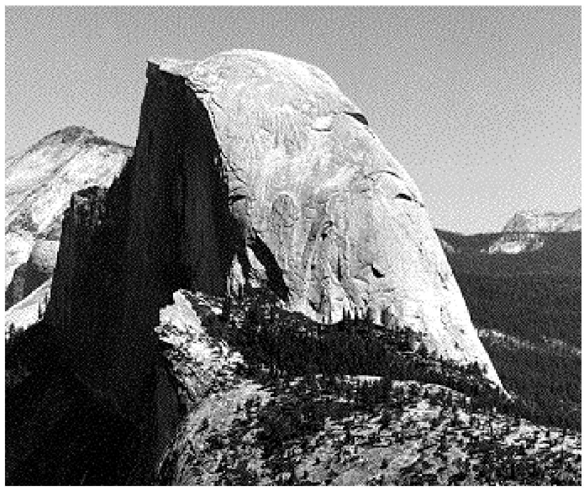
(d) In one of Professor Johnson's experiments, the two balls were dropped simultaneously from the Campanile. When the 1996 ball reached the ground, the 1994 ball was still 1.7 meters away. What does this say about the friction coefficients b for the two balls?

Reference

Adair, Robert K. *The Physics of Baseball.* Harper Perennial. 1994.

20. Maximum and Minimum Values

Standing at the top of Half Dome in Yosemite or at the bottom of the Grand Canyon, you can appreciate nature's spectacular maxima and minima. Although much less dramatic, the graphs of functions also have their peaks and basins. We can often find these maximum and minimum points using the derivative.



Questions

1. Let A be a function whose domain (input) is points on the UC Berkeley campus and whose range (output) is the real number of meters that point is above (or below) sea level.

   (a) Assume the campus boundaries are Oxford, Hearst, Bancroft, and Piedmont. At what point on campus does A attain its absolute maximum value?

   (b) Where does A attain an absolute minimum?
   (By expanding the domain and range of A (i.e. to all points on the surface of the earth), one can specify other interesting geographic questions.)

2. (a) When does a function f have an **absolute maximum** at a point c?

   (b) When does a function f have an **absolute minimum** at a point d?

   (c) How many different **extreme values** can a function f have?

   (d) When does a function f have a **local maximum** at a point c?

   (e) When does a function f have a **local minimum** at a point d?

   (f) How many different local maximum or minimum values can a function f have?

Problems

1. True or false? (When false, give a counterexample):

   (a) If f has a local extremum at c, then c is a critical number of f. [Note: The word "extremum" means the same thing as the phrase "maximum or minimum."]

   (b) If c is a critical number of f, then f has a local extremum at c.

2. What does the Extreme Value Theorem say? Be sure to specify the correct hypotheses and the correct conclusion(s). Sketch a graph illustrating what the Extreme Value Theorem says.

3. Sketch the graph of each of these functions.
   Match each function with an appropriate description.
   Explain why each function does *not* contradict the Extreme Value Theorem.

$f(x) = x^3 - 3x^2 + 2x,\ 0 < x < 2$	a. continuous function with max./min.
$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/x & \text{if } 0 < x \leq 1 \\ x - 1 & \text{if } 1 < x \leq 2 \end{cases}$	b. function on a closed interval with max./min.
$f(x) = x^2,\ 0 < x < 2$	c. continuous function with no max./min.
$f(x) = \begin{cases} x^2 & \text{if } 0 \leq x < 1 \\ x - 1 & \text{if } 1 \leq x \leq 2 \end{cases}$	d. function on a closed interval with no max./min.

4. (a) What does Fermat's Theorem say? Be sure to specify the correct hypotheses and the correct conclusion(s). If a function f does satisfy these hypotheses, what can you conclude about f? Sketch a graph illustrating what Fermat's theorem says.

   (b) Explain why the functions $f(x) = x^{2/3}$ and $f(x) = x^3$ do *not* contradict Fermat's Theorem.

5. Find the largest and smallest possible values of

   (a) $\sin x - \cos x$

   (b) $\sin x + \cos x$

   (c) $\sin x + \cos^2 x$

   (d) $\sin^2 x + \cos^2 x$

Additional Problems

1. Prove Fermat's Theorem for the case when f has a local minimum at c.

(a) First, assume that the case when f has a local maximum at c has already been proved. Show that if f has a maximum value at c, then the function $g(x) = -f(x)$ has a minimum value at c. What can you conclude?

(b) Now try to prove it directly, beginning from $f(c) \leq f(c+h)$, where h is sufficiently close to 0, either positive or negative.

2. Find a function with infinitely many critical points but no local maxima or minima. Even if you can't find a formula, draw the graph of such a function.

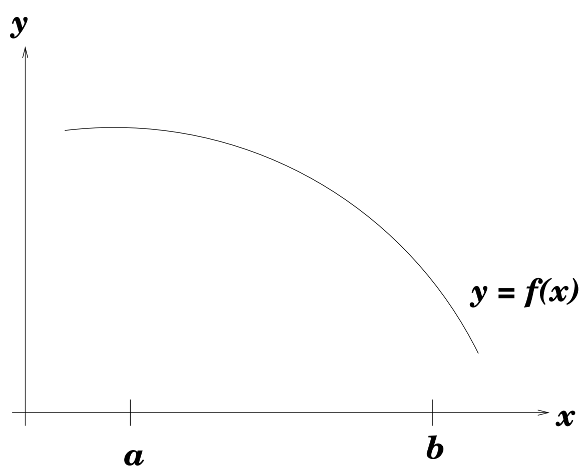
21. The Mean Value Theorem

Questions

1. What are the three hypotheses of Rolle's theorem? If these are all true, what can you conclude? Draw and label a picture to support your statements.

2. Let $f(x) = 1 - |x|$ and consider the interval $[-1, 1]$. Is there a number c such that $f'(c) = 0$? Why doesn't Rolle's Theorem work?

3. Which two hypotheses from Rolle's theorem also make an appearance in the Mean Value Theorem? Explain in words what the conclusion of the Mean Value Theorem is and how it is different from the conclusion of Rolle's theorem. Label the following picture with all the information from the Mean Value Theorem.



4. Let $f(x) = \frac{x^3 - x^2}{x - 1}$ on $[0, 2]$. Show that there is no value of c such that $f(2) - f(0) = f'(c)(2 - 0)$. Why doesn't the Mean Value Theorem work?

5. (a) If a function has "at most two roots," could it have just one root? How about no roots at all? Could it have three roots?

   (b) If a function has "at least two roots," could it have just one root? How about two roots? Could it have four roots?

   (c) If a function has "at most two roots" and "at least two roots," then how many possible roots does it have?

Problems

1. (a) Suppose that f and g are continuous on $[a,b]$ and differentiable on (a,b). Suppose also that $f(a)=g(a)$ and $f'(x)<g'(x)$ for $a<x<b$. Prove that $f(b)<g(b)$. (**Hint:** apply the Mean Value Theorem to the function $h=f-g$.)

   (b) Sketch a picture and explain, in terms of the derivative as a rate of change, why this is so.

2. (a) Show that a polynomial of degree 3 has at most three real roots.

   (b) Show that a polynomial of degree n has at most n real roots.
   (**Hint:** Use mathematical induction. That is, assume that the statement is true for a polynomial of degree $n-1$ and using that prove it true for a polynomial of degree n.)

3. Let $f(x)=[x]$ be the greatest integer function. Sketch the graph of f and use it to find a formula for $f'(x)$. Is $f(x)$ a constant function? Why *isn't* this a counterexample to the idea that "if $f'(x)=0$, then f is constant."

4. Sometimes the Mean Value Theorem can be used to show things that are not obviously related to the Mean Value Theorem. Show that $\sqrt{1+x}<1+\frac{1}{2}x$ if $x>0$.
   (**Hint:** let $f(x)=1+\frac{1}{2}x-\sqrt{1+x}$. What is $f(0)$? If you assume that there exists a number $b>0$ such that $f(b)=0$ also, is there a problem?)

5. (a) Suppose that f is differentiable on $\mathbf{R}$ and has two roots. Show that f' has at least one root.

   (b) Suppose that f is twice differentiable on $\mathbf{R}$ and has three roots. Show that f'' has at least one real root.

   (c) What is the pattern? Try to express the general statement.

Additional Problems

1. Show that $|\sin a-\sin b|\leq|a-b|$ for all a and b.

2. Let $f(x)=\frac{1}{x}$ and $g(x)=\frac{1}{x}-\frac{x}{|x|}$. Show that $\frac{d}{dx}[f(x)-g(x)]=0$ but that $f(x)-g(x)\neq c$, where c is a constant. Why *isn't* this a counterexample to the idea that "If $f'(x)=g'(x)$, then $f-g$ is constant."

3. Prove the Mean Value Theorem.
   (**Hint:** consider the function $y=f(x)+\frac{f(b)-f(a)}{b-a}(x-a)$. Looking at a sketch illustrating the Mean Value Theorem, what does this function represent? Can you apply Rolle's Theorem to it? If so, what can you conclude?)

22. Monotonicity and Concavity

Questions

1. (a) Draw the graph of a differentiable function f with domain $[1, 5]$ that is:
      i. increasing on $[1, 2]$.
      ii. decreasing on $[2, 3]$.
      iii. neither increasing nor decreasing on $[3, 4]$.
      iv. monotonic on $[4, 5]$.

   (b) What can you say about *any* function that satisfies the conditions in part a?

   (c) What are the critical points of your function f? Does it have a local maximum or minimum on $(1, 5)$? Must *any* function satisfying the conditions in part a have a local maximum or minimum on $(1, 5)$?

2. Sketch four graphs illustrating the four cases covered by the first derivative test. (**Hint:** These cases can be described by the notation $+/+$, $+/-$, $-/+$, and $-/-$.) On each graph label the local maximum or local minimum or say that the graph has neither at the point in question.

3. (a) Explain in your own words, in terms of tangent lines relating to the curve, what it means for a function to be concave down. Draw a picture illustrating this definition. What's the easiest way to tell if a twice differentiable function is concave down on an interval I?

   (b) Explain in your own words, in terms of tangent lines relating to the curve, what it means for a function to be concave up. Draw a picture illustrating this definition. What's the easiest way to tell if a twice differentiable function is concave up on an interval I?

   (c) Explain in your own word what a point of inflection is. Draw a picture illustrating this definition. What's the easiest way to tell if a twice differentiable function has a point of inflection on an interval I?

4. (a) State the second derivative test for critical points.

   (b) Draw two graphs illustrating the two situations to which the second derivative test applies. On each graph label the local minimum or local maximum.

   (c) Give an example of f and c where $f(c)$ is a local maximum and $f''(c) = 0$. Can you use the second derivative test in this situation?

   (d) Give an example of f and c where $f(c)$ is a local minimum and $f''(c)$ does not exist. Can you use the second derivative test in this situation?

Problems

1. (a) Show that a quadratic polynomial $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers, always has one critical point and no points of inflection. When is f concave up? When is f concave down?

   (b) How can you tell if a quadratic polynomial has two roots? One root? No roots?

   (c) Suppose that f has two real roots, r and s. Show that $f'(r) + f'(s) = 0$. Also show that the critical point of f is midway between the two roots.

2. (a) Show that $e^x \geq 1 + x$ for $x \geq 0$.

   (b) Show that $e^x \geq 1 + x + \frac{1}{2}x^2$ for $x \geq 0$.

Additional Problems

1. Suppose that f and g are both concave upwards on $(-\infty, \infty)$. Under what conditions on f will the composite function $h(x) = f(g(x))$ be concave upward?

2. Suppose that f and g are increasing functions on an interval I.

   (a) Show that $f + g$ is increasing on I.

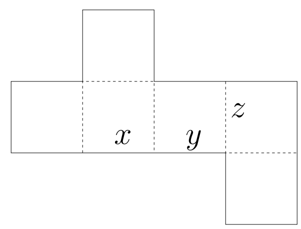
   (b) Suppose also that f and g are positive functions, and show that fg is increasing on I.

   (c) Why did you need to know that the functions were positive in the second part?

23. Applied Optimization

Making Boxes Efficiently

Small boxes, e.g. children's building blocks, are sometimes made by folding a shape like this (all corners are right angles and we ignore the small flaps):



1. Although this picture shows a box with square faces, we can build a rectangular box in the same way. There are edges labeled x, y, and z in the figure above. Each of the unlabeled edges must have the same length as one of these edges, so label them appropriately.

2. If this shape is to be cut from a rectangle of cardboard with edges parallel to the edges in the shape, what is the area of this rectangle in terms of x, y, and z?

3. Suppose that we want to cut this shape out of a rectangle of cardboard and fold it into a box with volume $1\,\mathrm{m}^3$. In addition, we want the box to have some square faces, so either: $x = y$, $y = z$, or $x = z$. In each case, eliminate one variable and express the area of the rectangle in terms of the other two.

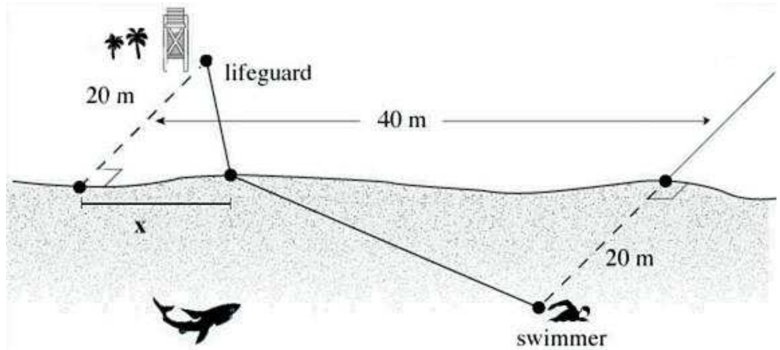
4. In each of these cases, what dimensions minimize the amount of cardboard used?

   (**Note:** Instead of finding x, y, and z exactly, you can use a graphing calculator to estimate the zeroes of a polynomial. In one of the three cases, this will be much faster.)

5. What values of x, y, and z give a box with unit volume, some square faces, and minimal use of cardboard? Note that not *all* of the faces must be square!

The Fastest Route from Here to There

Sophie Snell, a lifeguard, sees a swimmer in trouble some distance down the beach and wants to reach him in minimum time. On land, Sophie can run at speed $v_L = 6$ meters per second, while in the water she can swim at speed $v_W = 2$ meters per second.



1. Without doing any computations, sketch the path that you guess Sophie should take to reach the swimmer. Think about whether she should enter the water more or less than half-way down the beach.

2. Write a formula for the time required for Sophie to reach the swimmer if she enters the water at a point located x units down the beach from her station.

3. Find an equation for x to minimize the time. The equation for x can be solved numerically, but don't try that yet. Instead, use the equation to find a relation between the cosines of the angles between the shoreline and each of the two legs of Sophie's path. Which angle is larger? Use this relation to make a rough sketch of the shortest path, and compare it with your original guess.

4. Using a calculator if necessary, solve for x to an accuracy of 1 meter. Compare with your original guess and with the sketch made using the cosine relation.

(**Note:** This situation is just like that for light traveling from one medium to another. Fermat postulated that light would travel along the fastest path between two points. From this postulate, you can obtain Snell's law of refraction.)

Greenhouse

Your parents are going to knock out the bottom of the entire length of the south wall of their house, and turn it into a greenhouse by replacing some bottom portion of the wall by a huge sloped piece of glass (which is expensive). They have already decided they are going to spend a certain fixed amount. The triangular ends of the greenhouse will be made of various materials they already have lying around.

The floor space in the greenhouse is only considered usable if they can both stand up in it, so part of it will be unusable, but they don't know how much. Of course this depends on how they configure the greenhouse. They want to choose the dimensions of the greenhouse to get the most usable floor space in it, but they are at a real loss to know what the dimensions should be and how much usable space they will get. Fortunately they know you are taking calculus. Amaze them!

24. Antiderivatives

Questions

1. Assume that $F'(x) = f(x)$, $G'(x) = g(x)$, and c is a constant. Fill in the following table.

Function	**Particular Antiderivative**
x^n $(n \neq -1)$	
$1/x$	
e^x	
$\cos x$	
$\sin x$	
$\sec^2 x$	
$cf(x)$	
$f(x) + g(x)$	
$1/\sqrt{1-x^2}$	
$1/(1+x^2)$	

2. (a) Sketch the graph of $f(x) = e^x$. Then sketch the graph of the antiderivative F of f based *only* on the graph of e^x, where $F(0) = 0$.

   (b) Sketch the graph of $f(x) = \cos^2 x$. Then sketch the graph of the antiderivative F of f based *only* on the graph of $\cos^2 x$, where $F(0) = 0$.

Problems

1. (a) How many specific antiderivatives can one function f have? How are all the antiderivatives related? In other words, if $G'(x) = f(x)$ and $F'(x) = f(x)$, what is known about the relationship between G and F? How do we write the general antiderivative of f on an interval I?

   (b) Find a function f such that $f'(x) = x^3$ and the line $x + y = 0$ is tangent to the graph of f.

2. In each case, sketch a function $f(x)$ such that

   (a) $f(x) > 0$, $f'(x) > 0$, and $f''(x) > 0$ for all x.

   (b) $f'(x) < 0$ and $f''(x) < 0$ for all x.

   (c) Is it possible to define $f(x)$ so that for all real numbers x, $f(x) > 0$, $f'(x) < 0$, and $f''(x) < 0$? Explain.

In each case above, try to find an algebraic expression for a function f which satisfies the properties listed.

3. Show that for motion in a straight line with constant acceleration a, initial velocity v_0, and initial displacement s_0, the displacement after time t is $s = \frac{1}{2}at^2 + v_0 t + s_0$.

4. (a) Draw a direction field for the function $f(x) = 1/x^2$ and use it to sketch several members of the family of antiderivatives.

   (b) Compute the general antiderivative explicitly and sketch several particular antiderivatives. Compare these sketches with your previous one.

5. An object is projected upward with initial velocity v_0 meters per second from a point s_0 meters above the ground. Show that $[v(t)]^2 = v_0^2 - 19.6[s(t) - s_0]$.

Additional Problems

1. Find all functions f such that $f''' = f''$.

2. Use a direction field to graph the antiderivative that satisfies $F(0) = 0$.

   (a) $f(x) = \frac{\sin x}{x}$, $0 < x < 2\pi$

   (b) $f(x) = x \tan x$, $-\pi/2 < x < \pi/2$

25. Sigma Notation and Mathematical Induction

Questions

1. (a) Write a sentence or two explaining in your own words what $\sum_{k=1}^{n} a_k$ means.

   (b) Are $\sum_{k=1}^{n} a_k$ and $\sum_{j=1}^{n} a_j$ the same sums? Explain why or why not.

   (c) Suppose that m and n integers such that $m \leq n$, and that a_m, a_{m+1}, $\ldots$, a_n are real numbers. Write $a_m + a_{m+1} + \ldots + a_{n-1} + a_n$ in sigma notation.

   (d) In part (c), which letter did you choose for the index of summation? What values does it take on? Could you have chosen a different letter? Would it have made any difference? Can you think of a few letters which you should *not* use?

   (e) Write out (but don't add up): $\sum_{k=0}^{4} 1 + 2k$

2. Express in sigma notation

   (a) $a^5 + a^6 + a^7 + \ldots + a^{10}$

   (b) $3 - 5 + 7 - 9 + 11 - 13$

   (c) $1/4 - 1/6 + 1/8 - 1/10 + 1/12 - 1/14$

   (d) $3 + 9 + 27 + 81$

   (e) $1 + 4 + 27 + 256$

3. Assume that c is a constant and that m and n are positive integers. Fill in each blank with the correct formula.

   (a) $c \sum_{i=m}^{n} a_i = \sum_{i=m}^{n}$ ____________

   (b) $\sum_{i=m}^{n} a_i + \sum_{j=m}^{n} b_j = \sum_{k=m}^{n}$ ____________

   (c) $\sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_{i+1} = \sum_{i=1}^{n}$ ____________

   (d) $\sum_{i=1}^{n} a_i + \sum_{i=0}^{n} b_i = (\sum_{i=1}^{n}$ ____________$)$ + ____________

Problems

1. (a) Express the sum of the first n even integers in sigma notation.

   (b) Express the sum of the first n odd integers in sigma notation.

2. Assume that $r \neq 1$ and prove the formula for the sum of a finite geometric series as follows:

(a) Write the expression

$$a + ar + ar^2 + ... + ar^{n-1}$$

in sigma notation.

(b) Prove that the above sum equals $\frac{a(r^n-1)}{r-1}$.
(**Hint:** There are two approaches. One is to use induction. The other is to multiply the expression by $r-1$ and figure out which terms cancel.)

3. (a) Express the sum $1/2 + 1/6 + 1/12 + 1/20$ in sigma notation. What is its value?

   (b) Suppose the sum in part (a) is $\sum_{k=1}^{4} a_k$. What does a_k equal? Using this same formula for a_k, use mathematical induction to prove as follows that $\sum_{k=1}^{n} a_k = \frac{n}{n+1}$.

      i. Check that $\sum_{k=1}^{1} a_k = \frac{1}{1+1}$.

      ii. *Assume* that $\sum_{k=1}^{n-1} a_k = \frac{n-1}{(n-1)+1}$.

      iii. Using the assumption in part (ii), prove that $\sum_{k=1}^{n} a_k = \frac{n}{n+1}$.

4. Prove that $\sum_{i=1}^{n} i^3 = [\frac{n(n+1)}{2}]^2$ as follows:

   (a) First, evaluate $\sum_{i=1}^{n}[(1+i)^4 - i^4]$ as a telescoping sum, and use the summation rules for 1, i, and i^2 to find the value of $\sum_{i=1}^{n} i^3$.

   (b) Now prove the equality using induction.

5. Evaluate $\sum_{i=1}^{n}[\sum_{j=1}^{n}(i+j)]$.

6. Using induction, prove the generalized triangle inequality $|\sum_{i=1}^{n} a_i| \leq \sum_{i=1}^{n} |a_i|$.

Additional Problems

1. (a) Use the formulas for $\sin(x+y)$ and $\sin(x-y)$ to show that

$$2\sin\frac{1}{2}x\cos ix = \sin\left(i+\frac{1}{2}\right)x - \sin\left(i-\frac{1}{2}\right)x$$

   for any integer i.

   (b) Use this identity and telescoping sums to prove the formula

$$\sum_{i=1}^{n}\cos ix = \frac{\sin(n+\frac{1}{2})x - \sin\frac{1}{2}x}{2\sin\frac{1}{2}x}$$

   where x is not an integer multiple of 2π. Deduce that

$$\sum_{i=1}^{n}\cos ix = \frac{\sin\frac{1}{2}nx\cos\frac{1}{2}(n+1)x}{\sin\frac{1}{2}x}$$

2. Use the method of the previous problem to prove the formula

$$\sum_{i=1}^{n} \sin ix = \frac{\sin \frac{1}{2}nx \sin \frac{1}{2}(n+1)x}{\sin \frac{1}{2}x}$$

26. Area

Questions

1. (a) Sketch the interval $[0,1]$. Give an example of a partition of $[0,1]$ with four subintervals. What are the subintervals in your example?

   (b) In your example label the endpoints of the subintervals x_0, x_1, x_2, x_3, and x_4. If you were to partition $[0,1]$ into n subintervals how many x_i would you need?

   (c) What is the right endpoint of the ith subinterval? What is the left endpoint of the ith subinterval? What is always the left endpoint of the first subinterval? What is always the right endpoint of the nth subinterval?

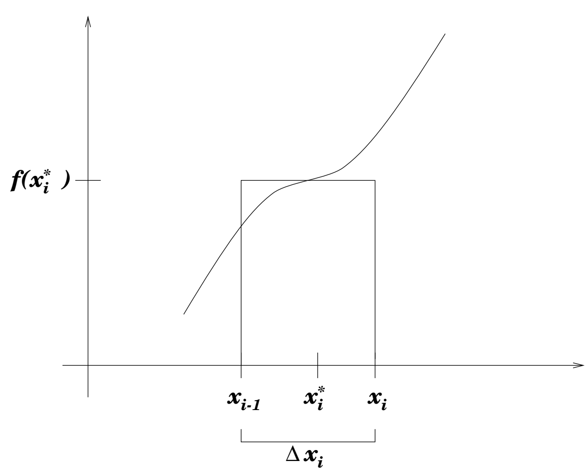
   (d) What does the notation Δx_i mean? What are Δx_1, Δx_2, Δx_3, and Δx_4 in your example?

   (e) What do we call a partition where all the Δx_i's are the same size? Does this describe your partition?

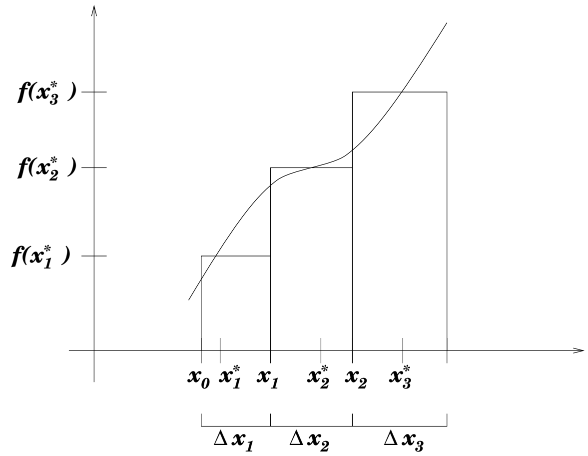
   (f) The **norm** of P is denoted $\|P\|$. What do we mean by the norm of P? What is the norm of your partition?

2. (a) What is the formula for the area A of a rectangle in terms of its length ℓ and height h?

   (b) What is the area of the rectangle in the following picture?



   (c) Write the total area of the following rectangles in sigma notation.



Problems

1. (a) Find the area under the curve $f(x) = x^3$ from $x = 0$ to $x = 1$ in three different ways: taking x_i^* to be either (a) the left endpoint, (b) the right endpoint, or (c) the midpoint of the ith subinterval. In each case, use subintervals of equal length and be sure to sketch the approximating rectangles.

   (b) Looking at the approximating rectangles, which choices of x_i^* give you an overestimate of the area? Which give you an underestimate?

   (c) Did your three answers agree?

2. (a) By choosing a suitable partition of $[1, 2]$, show that the area under $f(x) = \frac{1}{x}$ lies between $(1/4)(4/5 + 2/3 + 4/7 + 1/2)$ and $(1/4)(1 + 4/5 + 2/3 + 4/7)$.

   (b) Suppose that to evaluate the area under a particular curve $f(x)$, you partition $[a, b]$ into regular subintervals and average the upper and lower sums for f. If $f''(x) > 0$ on $[a, b]$, what can you say about your estimate?
   (**Hint:** does choosing x_i^* to be the left endpoints of the subintervals give you an upper or lower sum? What about the right endpoints? Is the upper sum more or less of an overestimate than the lower sum is an underestimate? Sketch the approximating rectangles.)

   (c) If $f''(x) < 0$ on $[a, b]$, what could you say about the estimate?

3. In the definition of area, we took the limit as $||P|| \to 0$. You might ask why did we not just let the number of subintervals approach ∞. To figure out why we can't do this, find a sequence $\{P_j\}$ of partitions of $[0, 1]$ such that each P_j has j subintervals but $||P_j|| = 1/2$ for all values of j. First sketch the partitions and then try to find a formula. Explain why these partitions do not approximate the area under $y = f(x)$ well.

4. Find the area under the curve $f(x) = \frac{1}{x}$ from $x = 1$ to $x = 2$ using the partition $P = [1, a], [a, a^2], \ldots, [a^{n-1}, a^n]$, where $a^n = 2$.

(a) Does $\|P\| \to 0$? (**Hint:** is a the same for each n?)

(b) Find upper and lower bounds for the area under the curve. Sketch the approximating rectangles.

(c) Show that $n(a-1) \to \ln 2$ as $n \to \infty$.

(d) What is the area under the curve?

(e) Why do you think that you *weren't* asked to find the area in the interval $[0,1]$?

Additional Problems

1. Give the $\varepsilon - \delta$ statement of the fact that the area under a curve can be approximated by a sum of areas of rectangles to within an arbitrary degree of accuracy by taking the norm of the partition sufficiently small.

2. (a) Let A_n be the area of a polygon with n equal sides inscribed in a circle with radius r. By dividing the polygon into n congruent triangles with central angle $2\pi/n$, show that $A_n = \frac{1}{2}nr^2 \sin(2\pi/n)$.

   (b) Show that $\lim_{n\to\infty} A_n = \pi r^2$. (**Hint:** recall that $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$.)

3. Find the area under the curve $y = \sin x$ from 0 to π.
   (**Hint:** use equal subintervals, right endpoints, and the last problem from the previous worksheet which said that $\sum_{i=1}^{n} \sin ix = \frac{\sin \frac{1}{2}nx \sin \frac{1}{2}(n+1)x}{\sin \frac{1}{2}x}$ where x is not an integer multiple of 2π.)

27. The Definite Integral

Questions

1. What sort of functions are guaranteed to be integrable? What does it mean for a function to be integrable?

2. Does $\int_a^b f(x)\,dx$ depend on x? What is x (number, function, etc.)? What is the integral sign? The integrand? The lower limit of integration? The upper limit of integration? Does dx mean anything?

3. (a) When $f(x) \geq 0$ and $a < b$, what is the interpretation of $\int_a^b f(x)\,dx$? Sketch a picture illustrating this interpretation.

   (b) When $a > b$, what does $\int_a^b f(x)\,dx$ mean? If $a = b$, what does $\int_a^a f(x)\,dx$ equal?

4. Sketch $f(x) = x^3$. Write $\int_{-2}^1 x^3\,dx$ as a difference of areas.

5. (a) Draw an example of a function $f(x)$ that is bounded on an interval $[a, b]$. Indicate upper and lower bounds.

   (b) Draw an example of a function $f(x)$ that is unbounded on an interval $[a, b]$.

   (c) True or false? If true, explain why. If false, give a counterexample:

      i. If f is bounded on $[a, b]$, then f is integrable on $[a, b]$.

      ii. If f is integrable on $[a, b]$, then f is bounded on $[a, b]$.

      iii. If f is not bounded on $[a, b]$, then f is not integrable on $[a, b]$.

      iv. If f is not integrable on $[a, b]$, then f is not bounded on $[a, b]$.

   (d) What two pairs of the previous four statements actually mean the same thing?

6. (a) If f is integrable, as long as $\|P\| \to 0$, does it matter what sort of partition or x_i^* are chosen?

   (b) If you just want an *approximation* to $\int_a^b f(x)\,dx$, which rule could you use? In that case, what is meant by Δx and $\bar{x}_i$?

Problems

1. Suppose that f is integrable and let P be a regular partition of $[a, b]$ into n subintervals.

   (a) Express the following in terms of a, b, n, and i:

      i. Δx_i for $i = 1$ to n.

      ii. $\|P\|$

iii. x_0, x_1, x_i, and x_n.

(b) If x_i^* is chosen to be the right endpoint, then $x_i^* = ?$

(c) As $n \to \infty$, what happens to $\|P\|$?
Starting from $\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{i=1}^n f(x_i^*)\Delta x_i$ use the previous expressions to find a simpler formula for calculating integrals.

2. Express the following limits as definite integrals.

   (a) $\displaystyle\lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left[3(1+\frac{2i}{n}) - 6\right]$.

   (b) $\displaystyle\lim_{n\to\infty} \frac{\sqrt{1}+\sqrt{2}+\sqrt{3}+\ldots+\sqrt{n}}{n^{3/2}}$.

3. Show geometrically and explain why:

   (a) If $f(x)$ is an even function, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)\,dx$.

   (b) If $f(x)$ is an odd function, then $\int_{-a}^{a} f(x)dx = 0$.

4. Let $f(x) = \begin{cases} \frac{1}{x} & \text{if } 0 < x \le 1 \\ 0 & \text{if } x = 0 \end{cases}$

   (a) Show that f is not continuous on $[0, 1]$.

   (b) Show that f is unbounded on $[0, 1]$.

   (c) Show that $\int_0^1 f(x)\,dx$ does not exist, that f is not integrable on $[0, 1]$.
   (**Hint:** show that the first term in the Riemann sum, $f(x_i^*)\Delta x_i$ can be made arbitrarily large.)

Additional Problems

1. Give the (ε, δ) statement of the fact that a definite integral can be approximated to within any desired degree of accuracy by a Riemann sum.

2. Evaluate $\int_0^1 e^x\,dx$.
   (**Hint:** use the fact that e^x is continuous, and so therefore integrable. How does this help you? Recall the sum of a geometric series, and l'Hospital's Rule.)

3. Assume that the following integrals exist and that $a \le b$.

   (a) Prove that if $f(x) \ge 0$ for $a \le x \le b$, the $\int_a^b f(x)\,dx \ge 0$.

   (b) Show that if $f(x) \ge g(x)$ for $a \le x \le b$, then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$.

   (c) Prove that if $m \le f(x) \le M$ for $a \le x \le b$, then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$.

   (d) Show that $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$.

4. (a) Suppose that we defined $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^n f(a + i\frac{b-a}{n})\frac{b-a}{n}$.

   Let $f(x) = \begin{cases} 1 & x \text{ irrational} \\ 0 & x \text{ rational.} \end{cases}$

   Calculate $\int_0^1 f(x)\,dx$ and $\int_{\sqrt{2}+1}^{\sqrt{2}} f(x)\,dx$ using the above "definition". Geometrically, you would expect both integrals to be the same; the way that $f(x)$ is defined the interval $[0,1]$ shouldn't "look" any different from $[\sqrt{2}+1, \sqrt{2}]$. Were the integrals the same? Why is this definition of a definite integral *not* as good as the more general one?

   (b) Let $f(x) = \begin{cases} 1 & x \text{ irrational} \\ 0 & x \text{ rational} \end{cases}$

   Show that f is bounded but not integrable on $[a,b]$.

   (**Hint:** show that, no matter how small $\|P\|$ is, some Riemann sums are 0 whereas others are equal to $b-a$.)

28. The Fundamental Theorem of Calculus

Questions

1. Make triples by choosing an appropriate term from each column:

definite integral	function	$\int_a^b f(x)\,dx$
indefinite integral	number	$\int f(x)\,dx$

2. Suppose that f is continuous on $[a, b]$ and define $g(x) = \int_a^x f(t)\,dt$.
   (a) What does the Fundamental Theorem of Calculus say about $g(x)$? What sort of properties does g have?
   (b) Let $a = 1$, $b = 2$, and $f(t) = 1/t$. Sketch a picture of $g(x)$.

3. (a) Suppose that f is continuous on $[a, b]$, then the Fundamental Theorem of Calculus tells us that $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is an antiderivative of f.
      i. What does it mean for F to be an antiderivative of f?
      ii. Let $g(x) = \int_a^x f(t)\,dt$. Is $g' = f$?
      iii. What is $F - g$?
   (b) Give an example of F, a, and b where $\int_a^b F'(x)\,dx \neq F(a) - F(b)$. Why *didn't* the Fundamental Theorem of Calculus apply to this case?
   (**Hint:** what happens if you try to integrate $f(x) = \frac{1}{x^2}$ over an interval containing 0?)

4. Find the following. Once you have finished, compare answers to make sure that all members of the group agree.
   (a) $\int x^n\,dx =$ $\qquad (n \neq -1)$
   (b) $\int \frac{1}{x}\,dx =$
   (c) $\int e^x\,dx =$
   (d) $\int a^x\,dx =$
   (e) $\int \sin x\,dx =$
   (f) $\int \cos x\,dx =$
   (g) $\int \sec^2 x\,dx =$
   (h) $\int \csc^2 x\,dx =$
   (i) $\int \sec x \tan x\,dx =$

(j) $\int \csc x \cot x \, dx =$

(k) $\int \frac{1}{x^2+1} \, dx =$

(l) $\int \frac{1}{\sqrt{1-x^2}} \, dx =$

Problems

1. (a) Explain what $\frac{d}{dx} \int_a^x f(t)\, dt = f(x)$ means in words.

   (b) Explain what $\int_a^b F'(x)\, dx = F(b) - F(a)$ means in words.

   (c) What do these two statements say about the relationship between integration and differentiation?

2. Suppose that a particle is moving in a straight line with position function $s(t)$, velocity function $v(t)$, and acceleration function $a(t)$.

   (a) What is the difference between speed and velocity?

   (b) Explain how "displacement" differs from "distance traveled."

   (c) Use the Fundamental Theorem of Calculus to express both displacement and distance traveled as functions of velocity.

   (d) Explain how both displacement and distance traveled can be interpreted in terms of areas under a velocity curve. Illustrate this with a picture.

   (e) What direction is the particle traveling when $v \geq 0$? When $v \leq 0$?

3. Let $g(x) = \int_{-1}^x |t|\, dt$.

   (a) Find $g(x)$ for $-1 < x < 0$ and $0 < x < 1$.

   (b) Plot $g(x)$ for $-1 < x < 1$.

   (c) Find $g'(x)$ and $g'(0)$.

   (d) Does $g''(0)$ exist?

4. Show that $\int x^2 \sin x\, dx = -x^2 \cos x + 2 \int x \cos x\, dx$.
   (**Hint:** Differentiate both sides.)

5. (a) Show that $1 \leq \sqrt{1+x^3} \leq 1 + x^3$ for $x \geq 0$.

   (b) Show that $1 \leq \int_0^1 \sqrt{1+x^3}\, dx \leq 1.25$.

6. (a) Find $\frac{dy}{dx}$ if $y = \int_{\cos x}^{x^2} \cos(u^2)\, du$.

   (b) If f is continuous, and g and h are differentiable functions, find a formula for $\frac{d}{dx} \int_{g(x)}^{h(x)} f(t)\, dt$.

Additional Problems

1. If $w'(t)$ is the rate of growth of a child in pounds per year, what does $\int_5^{10} w'(t)\,dt$ represent?
2. Find a function f such that $f(1) = 0$ and $f'(x) = 2^x/x$.

29. The Substitution Rule

Questions

1. If $F' = f$, show that $\int f(g(x))g'(x)\,dx = F(g(x)) + C$ by differentiating both sides.

2. Applying the substitution rule involves *guessing* an appropriate substitution. Often there is more than one substitution that will work! Find two different possible substitutions for $\int \sqrt{5x+2}\,dx$ and integrate to show that both substitutions give you the same answer.

3. Evaluate the following definite integrals (if they exist) by making a substitution of the form $u =$ a function of x. Make sure to write down explicitly what u and du are equal to in terms of x.

   (a) $\int_0^a x\sqrt{a^2 - x^2}\,dx$

   (b) $\int_0^a x\sqrt{x^2 + a^2}\,dx$, where $a > 0$

   (c) $\int_{-a}^a x\sqrt{x^2 + a^2}\,dx$

4. Evaluate $\int \frac{e^x+1}{e^x}\,dx$ and $\int \frac{e^x}{e^x+1}\,dx$.

Problems

1. Use the Fundamental Theorem of Calculus to prove $\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$, where g' is continuous on $[a, b]$ and f is continuous on the range of g. (**Hint:** Using $F' = f$, show that both sides of the equality are equal to a common middle.)

2. Evaluate $\int_{-2}^2 (x+3)\sqrt{4 - x^2}\,dx$. (**Hint:** remember that integrals can be interpreted in terms of area.)

3. Assume that f is continuous on $[-a, a]$ where $a > 0$, and prove that:

   (a) If f is an even function, then $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$.

   (b) If f is an odd function, then $\int_{-a}^a f(x)\,dx = 0$.

   (c) Sketch graphs which illustrate both of these points.

4. (a) If f is continuous on $\mathbf{R}$, prove by making a substitution that $\int_a^b f(-x)\,dx = \int_{-b}^{-a} f(x)\,dx$. For the case where $f(x) \geq 0$, draw a diagram to interpret this equation geometrically as an equality of areas.

   (b) If f is continuous on $\mathbf{R}$, prove that $\int_a^b f(x+c)\,dx = \int_{a+c}^{b+c} f(x)\,dx$. For the case where $f(x) \geq 0$, draw a diagram to interpret this equation geometrically as another equality of areas.

Additional Problems

1. Evaluate $\int xe^{-x^2}\,dx$. Can you evaluate $\int e^{-x^2}\,dx$?
2. (a) Evaluate $\int (x^3+1)^{-1/2}\,x^5\,dx$.
   (b) If a and b are positive numbers, show that $\int_0^1 x^a(1-x)^b\,dx = \int_0^1 x^b(1-x)^a\,dx$.
3. Evaluate the integral $\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx$.
   (**Hint:** use the substitution $u=\pi-x$ to show that $\int_0^\pi xf(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx$.)

30. The Logarithm defined as an Integral

Questions

1. Sketch the graphs of $\frac{1}{x}$ and $\ln x$, both for $x > 0$.
   (a) How are the graphs related to one another?
   (b) Why is $\ln 1 = 0$? Use the definition of $\ln x$.
   (c) If $0 < x < 1$, why is $\ln x < 0$?
   (d) Why is $\frac{d}{dx} \ln x = \frac{1}{x}$?
2. (a) What condition must a function satisfy in order to have an inverse?
   (b) How do you know that $\ln x$ has an inverse function (based only on the integral definition of $\ln x$)?
   (c) If $\exp(x)$ is defined to be the inverse of $\ln x$, what is $\exp(0)$? What is $\exp(1)$?
   (d) Sketch the graph of $\exp(x)$ based on the fact that it is the inverse of $\ln x$.
3. Write a^x in terms of $\ln x$ and e^x.

Problems

1. (a) Use the Intermediate Value Theorem to show that there is a solution to the equation $\ln x = 1$.
   (b) Use the Mean Value Theorem to show that this solution is unique.
   (c) This solution has a name. What is it?
2. Prove the laws of exponents for e^x (i.e. $e^{x+y} = \ldots$ and $(e^x)^y = \ldots$) using the laws of logarithms for $\ln x$.

Additional Problems

1. Show geometrically that $\frac{1}{2} < \int_1^2 \frac{dt}{t} < 1$. Remember that $\ln(a^n) = n \ln a$ and that $(\ln x)' > 0$. Argue that if $x > 2^n$, then $\ln x > n \ln 2 > \frac{n}{2}$. What does this tell you about $\ln x$ as $x \to \infty$?
2. Show geometrically that if $x < 2^{-n}$, then $\ln x < -n \ln 2$. Use this fact to show that $\ln x \to -\infty$ as $x \to 0^+$.
3. Use the integral definition of $\ln(ab)$ to show that $\ln(ab) = \ln a + \ln b$.
   (**Hint:** how does the substitution $u = t/a$ help you?)

31. Areas Between Curves

Questions

1. (a) Sketch the graphs of two continous functions f and g on an interval $[a, b]$ where $f > g$ on some subinterval $[c, d]$ of $[a, b]$ and $g > f$ on some other subinterval $[c', d']$ of $[a, b]$.
   (b) If x is in $[c, d]$, how can you simplify $|f(x) - g(x)|$?
   (c) If x is in $[c', d']$, how can you simplify $|f(x) - g(x)|$?
   (d) How is the area bounded by f, g, $x = a$, and $x = b$ defined?

2. Let $k(x)$ be x^2 and let $h(x)$ be x.
   (a) Sketch the region bounded by these two functions. Find the coordinates of the intersection points algebraically. Which function has greater y-values between the intersection points? Which function has smaller y-values between the intersection points?
   (b) Partition the interval between the two points of intersection into four subintervals. What values did you choose for your five partition points?
   (c) What are the lengths of your four subintervals? Find a formula to express the length of the ith subinterval.
   (d) Pick a x_i^* for each subinterval. Which values did you choose?
   (e) Sketch the four approximating rectangles. Find a formula to express the base b and the height h of the ith rectangle.
   (f) Are your rectangles tall and narrow or short and wide? As $\|P\| \to 0$, would the approximating rectangles become skinnier or flatter?
   (g) Find a formula which expresses the area of the ith rectangle. What variable appears in this expression?
   (h) Write the whole area as a definite integral and evaluate it.

Problems

1. (a) Sketch the region bounded by the curves $x + y^2 = 2$ and $x + y = 0$. What are the points of intersection?
   (b) Think of y as a function of x and sketch four (tall and narrow) approximating rectangles.
   (c) Express the base of the ith rectangle as change in a variable. Which variable appears in that formula?

(d) Find a formula for the height of the rectangles to the *left* of the line $x = 1$. Find a formula for the height of the rectangles to the *right* of the line $x = 1$. Which variable appears in both of these formulas?

(e) Can you use just one formula for the height of the ith approximating rectangle? Why or why not?

(f) Write down formulas for the area of the ith approximating rectangle, *including* the intervals on which such formulas are valid.

(g) Express the area of the whole region as a sum of two definite integrals. Evaluate the integrals to find a numerical value for the area.

2. (a) Again sketch the region bounded by the curves $x + y^2 = 2$ and $x + y = 0$. This time, though, think of x as a function of y. Which function is further to the right? This function has greater x-values. Which function is further to the left? This function has lesser x-values.

   (b) What interval will your partition to get the approximating rectangles? Express the length of the ith subinterval as a change in a variable. What variable should you use? (**Hint:** When y was a function of x, we partitioned an interval along the x-axis. Now, though, y is our *independent* variable and x is our *dependent* variable!)

   (c) Sketch four (shorter and wider) approximating rectangles. As $\|P\| \to 0$, will these rectangles get skinnier or flatter?

   (d) The height of the ith rectangle is the same as the length of the ith subinterval. Now express the width of the ith rectangle. What variable should you use? (**Hint:** the width is a difference of x-values corresponding to y_i^*. Subtract the greater x-value from the lesser.)

   (e) Express the area as another definite integral. Is this a simpler expression than the one from the previous problem? Evaluate the integral to find a numerical value for the area. Did you get the same answer as before? (You should!)

3. (a) Find a number a such that the line $x = a$ bisects the area under the curve $y = 1/x^2$, $1 \le x \le 4$.

   (b) Find a number b such that the line $y = b$ also bisects the same area.

4. Suppose that $0 < c < \pi/2$. For what values of c is the area of the region enclosed by the curves $y = \cos x$, $y = \cos(x - c)$, and $x = 0$ equal to the area of the region enclosed by the curves $y = \cos(x - c)$, $x = \pi$, and $y = 0$?

Additional Problems

1. (a) Find the area of the region above the x-axis bounded by $y = 1/\sqrt{x}$, $x = 1$, and $x = b$ where $b > 1$. Your answer will depend on b. What happens as $b \to \infty$?

   (b) Repeat the first part using the curve $y = 1/x^2$.

(c) What would happen as $b \to \infty$ if you used the curve $y = 1/x^p$ instead, where p was a fixed number greater than 1?

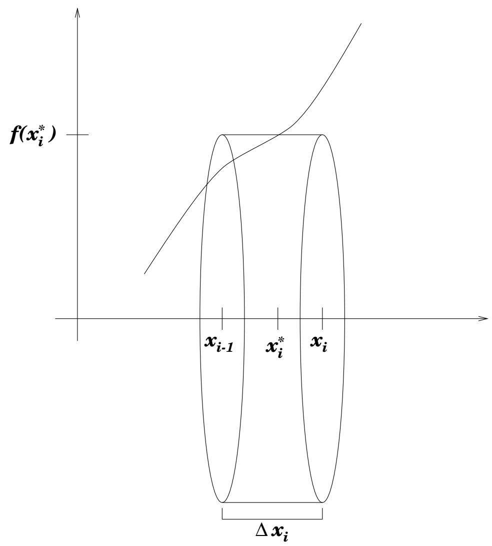
(d) What would happen as $b \to \infty$ if p were a fixed number less than 1?

2. Sketch a rectangle with sides parallel to the coordinate axes that has one vertex at the origin and the diagonally opposite vertex on the curve $y = ax^n$ at $x = b$. a, n, and b are all assumed to be positive values. Show that the fraction of the rectangle's area below the curve depends on n, but not on a or b.

32. Volume

Questions

1. In the picture below, find the volume of the disk.



2. Answer the Questions listed below for each of these three solids of revolution:

   i. $y = x^2$, $x = 0$, $x = 1$, $y = 0$ rotated around the x-axis.
   ii. $y = x^2$, $y = x$ rotated around the y-axis.
   iii. $y = x^2$, $y = x$ rotated around the line $x = -1$.

   (a) Graph the (x, y) region. Then sketch the resulting solid of revolution when the region is revolved around the indicated axis.
   (b) Sketch a typical approximating disk or washer. The volume of such a disk is $V = \pi r^2 h$. Express the side length h in terms of a change in a variable. Which variable did you use?
   (c) Is Δx (or Δy) parallel or perpendicular to the axis of rotation?
   (d) Indicate the radius r on your disk/washer. Express the radius as a function. Which variable should you use?
   (e) Write down a formula for the volume of the ith approximating disk/washer.
   (f) As $\|P\| \to 0$, does the washer become skinnier or flatter? How does this correspond to the variables you used before?
   (g) Express the volume as a definite integral. Explain how you chose the limits of integration, the variable of integration, and the function being integrated.
   (h) Compute the volume.

Problems

1. For each of the following solids, sketch the solid and find its volume.

   (a) A sphere of radius r.

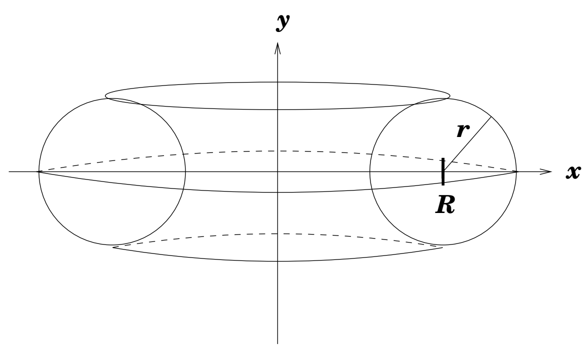
   (b) A right circular cone with height h and base radius r.

   (c) A pyramid with height h and base that is an equilateral triangle with sides a.

   (d) The cap of a sphere with radius r. The cap extends from the top of the sphere down to a height h.

2. Find the volume of the solid generated by rotating the region bounded by $y = e^x$, $y = 0$, $x = 0$, and $x = 1$ around the x-axis. Now write down the integral that represents the volume of the solid generated by rotating that region around the y-axis. In each case, make sure to draw a picture!

3. (a) Set up an integral for the volume of a solid torus (a doughnut shaped solid as shown below) with radii r and R.



   (b) By interpreting the integral as an area, find the volume of the torus.

Additional Problems

1. Some of the pioneers of calculus, such as Kepler and Newton, were inspired by the problem of finding the volumes of wine barrels. (In fact, Kepler published a book *Stereometria doliorum* in 1715 devoted to methods for finding volumes of barrels.) They often approximated the shape of the sides by parabolas.

   (a) A barrel with height h and maximum radius R is constructed by rotating about the x-axis the parabola $y = R - cx^2$, $-h/2 \leq x \leq h/2$, where c is a positive constant. Show that the radius of each end of the barrel is $r = R - d$, where $d = ch^2/4$.

   (b) Show that the volume enclosed by the barrel is

$$V = \frac{1}{3}\pi h(2R^2 + r^2 - \frac{2}{5}d^2)$$

2. Water in an open bowl evaporates as a rate proportional to the area of the surface of the water. (This means that the rate of decrease of the volume is proportional to the area of the surface.) Show that the depth of the water decreases at a constant rate, regardless of the shape of the bowl.

3. Evaluate the integral in the second part of Problem 2.

33. Volumes by Cylindrical Shells

Questions

1. Answer the Questions below for each of these three solids of revolution:

   i. $y = x^2$, $x = 0$, $x = 1$, $y = 0$ rotated around the x-axis.

   ii. $y = x^2$, $y = x$ rotated around the y-axis.

   iii. $y = x^2$, $y = x$ rotated around the line $x = -1$.

   (a) Graph the (x, y) region. Then sketch the resulting solid of revolution when the region is revolved around the indicated axis.

   (b) Sketch a typical approximating shell. The volume of such a cylinder is $V = 2\pi rh\,\Delta r$. Express the Δr in terms of a change in a variable. Which variable did you use?

   (c) Is Δx (or Δy) parallel or perpendicular to the axis of rotation?

   (d) Indicate the height h on your cylinder. Express the height as a function. Which variable should you use?

   (e) Indicate the circumference $2\pi r$ and the radius r on your cylinder. Express the radius as a function. Which variable should you use?

   (f) Write down a formula for the volume of the ith approximating cylinder.

   (g) As $\|P\| \to 0$, what happens to the shell? How does this correspond to the variables you used before?

   (h) Express the volume as a definite integral. Explain how you chose the limits of integration, the variable of integration, and the function being integrated.

   (i) Compute the volume.

Problems

1. Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified axis. Sketch the region and the solid of revolution. Indicate why you chose the method (disks or cylinders) that you used.

   (a) $y = \tan x$, $y = 0$, $x = \pi$, $x = 3\pi/2$ about the y-axis.

   (b) $x = \sin y$, $x = 0$, $y = 0$, $y = \pi$ about the y-axis.

   (c) $y = \cosh x$, $y = 0$, $x = 0$, $x = 1$ about the y-axis.

(d) $y = \ln x$, $y = 0$, $x = 1$, $x = 10$ about the x-axis.

(e) $y = e^x$, $x = 0$, $y = e^2$ about the x-axis.

2. Find the volumes of the described solids using the method of cylindrical shells. (**Hint:** always begin by sketching a picture.)

   (a) A sphere of radius r.

   (b) A right circular cone with height h and base radius r.

3. Using the cylindrical method, set up the integral representing the volume of the solid generated by rotating the region bounded by $y = e^x$, $y = 0$, $x = 0$, and $x = 1$ around the y-axis. Now set up the integral representing the volume obtained by rotating that region around the x-axis. In each case, make sure to draw a picture!

Additional Problems

1. The formula $V = \int_a^b 2\pi f(x)\,dx$ is valid only if $b > a \geq 0$. Show that if $a < b \leq 0$, then $V = -\int_a^b 2\pi f(x)\,dx$.

2. Find a formula for the volume of a solid obtained when the region under the graph of f from a to b is rotated about the vertical line $x = c$ in the cases where (i) $c \leq a < b$ and (ii) $a < b \leq c$.

34. Integration and Optimization

Production Cycles

A factory produces P flat panel displays per day at a cost of C dollars per display. Unfortunately, the demand for these displays is $D < P$ per day, and the cost of holding a display in inventory is H dollars per day. Because of this holding cost, it doesn't make sense to run the factory continuously, since this would lead to a large, and costly, excess of displays. It is more cost effective to run the factory for N days, then stop production for M days until all of the displays are purchased. When production resumes for the next cycle, of period $T = N + M$, a start-up cost S dollars must be paid. (Note that we are considering the process to be essentially continuous, so that N, M and T need not be integers.)

(a) The number of displays $W(t)$ in the warehouse (in inventory) at any time t is given by

$$\begin{aligned} W(t) &= (P - D)t, && 0 \le t \le N \\ W(t) &= (P - D)N - D(t - N), && N \le t \le T \end{aligned}$$

Sketch the inventory $W(t)$ over a full production cycle ($0 \le t \le T$). Be able to explain why $W(t)$ has this form.

(b) Show that the production time N and the total period T are related by the equation $NP = TD$. What is NP?

(c) The total inventory cost $I(T)$ over a production cycle is given as

$$I(T) = H \int_0^T W(s)ds.$$

Evaluate this integral [hint: try doing it geometrically] to show that

$$I(T) = \frac{1}{2}HDT^2(1 - D/P).$$

(d) The total number of displays produced during a production cycle is $NP = TD$, so that the total cost Z of this production is the sum of three terms: the start up cost, the production cost, and the inventory cost

$$Z = S + CDT + \frac{1}{2}HDT^2(1 - D/P)$$

Use this to obtain an equation in terms of the time T for the *overall cost per display.* This *overall cost per display* is greater than the production cost per display, C, due to the start up and inventory costs.

(e) Find the value of T which will minimize the *overall cost per display.* Find the numerical value of T for the conditions given in the following table.

Variable	**Significance**	**Value**
P	Production Rate in Units per Day	1000
C	Production Cost per Unit	\$250
D	Demand in Units per Day	200
H	Holding Cost per Unit per Day	\$1.00
S	Start up Cost	\$8,000

(f) For the optimal value of T, find the contributions to the overall cost per display of the three elements of the production cycle: start up, production, and inventory. What is interesting about this result? Could you have predicted it without using calculus?

(g) What production schedule should you use if the start up cost could be reduced to zero?