Math 113: Introduction to abstract algebra.

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Fall 1995, MWF 11–12 a.m., 2 Evans.

Required text:

John B. Fraleigh, *A first course in abstract algebra*, fifth edition, Addison-Wesley Publishing Company, Reading, Mass., 1994.

Syllabus for the final examination, Monday, December 11, 12:30-3:30 p.m., 60 Evans:

Chapter 5; Chapter 6; Chapter 8, sections 1–3.

The exam is open book, open notes, no calculators. No blue books are required. A practice examination follows.

Name:

Note. You have to do **four** out of the five problems. Cross out the problem that you don't want to be graded. *Give complete proofs of the assertions you are making and of the correctness of your answers.* Theorems proved in the book or in class may be used without proof (but do give the formulation).

1	
2	
3	
4	
5	
Total	

Problem 1.
Let R be a commutative ring with 1, and let A be the subset

$$A = \{e \in R \,|\, e^2 = e\}$$

of R.
(a) Give an example to show that A is not necessarily a subring of R.
(b) Suppose that one has $1 + 1 = 0$ in R. Prove that A is a subring of R.

Solution:

Problem 2.

If R is a ring, we write $N(R)$ for the number of elements $a \in R$ with the property that $a^3 + a = a^2$.

(a) Prove that $N(R_1 \times R_2) = N(R_1) \cdot N(R_2)$ for any two rings R_1 and R_2. (If you wish you may assume that $N(R_1)$ and $N(R_2)$ are *finite*.)

(b) Compute $N(\mathbb{Z}_{1001})$. (You may use that $1001 = 7 \cdot 11 \cdot 13$.)

Solution:

Problem 3.
Let T be a set, and let $P(T)$ be the set of all subsets of T. You know from class that $P(T)$ is a ring with respect to the operations $+$ and $\cdot$ defined by

$$A + B = (A \cup B) - (A \cap B), \qquad A \cdot B = A \cap B,$$

for A, $B \subseteq T$. Fix an element $t_0 \in T$, and define $f\colon P(T) \to \mathbb{Z}_2$ by

$$f(A) = \begin{cases} 1 & \text{if } t_0 \in A; \\ 0 & \text{if } t_0 \notin A. \end{cases}$$

Prove that f is a ring homomorphism. Prove also that the kernel of f is a *principal* ideal of $P(T)$.

Solution:

Problem 4.

Let α be an element of an extension field of $\mathbb{Z}_5$ with the property that $\alpha^2 = 3$.

(a) How many elements does the field $\mathbb{Z}_5(\alpha)$ have?

(b) Find elements c_0 and c_1 of $\mathbb{Z}_5$ with the property that

$$(2 + 3\alpha)^{-1} = c_0 + c_1\alpha.$$

Solution:

Problem 5.
Let α denote the complex number $1 + \frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}$.
(a) Find the irreducible polynomial of α over $\mathbb{Q}$.
(b) Determine $[\mathbb{Q}(\alpha) : \mathbb{Q}]$.
(c) Find the irreducible polynomial of α over $\mathbb{Q}(i)$.

Solution: