Final Exam, Thursday, 12/11/03
8:10AM – 11:00AM
Math 32, Fall 2003
Instructor: Benjamin Johnson

Student's name: solution

GSI: ______________________

Do not open your exam until instructed to do so.
Please read all directions carefully.
Please simplify your answers as much as possible.
Please draw a box around all your final answers.
You may not use a calculator on this exam.

The exam consists of 15 questions, plus one bonus question. The point values for each question are indicated below, and also before the problem numbers.

You will have 2 hours and 50 minutes to complete this exam. Please work carefully, and check your answers when you are done. Remember not to spend too much time on any one problem. If you get stuck on a difficult problem, move on to a problem that you know how to do, and come back to the difficult problem later.

If you finish your exam before 10:50AM, you may turn in your exam and leave the room. Solution sheets will be distributed to all those who turn in their exams after 10:00AM. If you leave before 10:00AM, you will not receive a solution sheet. Once you leave the room and collect a solution sheet, you may not return until after the exam is over and all the exams have been collected. Also, you may not leave during the last 10 minutes of the exam.

Please do not write anything below this line.

Problem 1	_______	(out of 6)	Problem 10	_______	(out of 6)
Problem 2	_______	(out of 6)	Problem 11	_______	(out of 6)
Problem 3	_______	(out of 6)	Problem 12	_______	(out of 8)
Problem 4	_______	(out of 6)	Problem 13	_______	(out of 8)
Problem 5	_______	(out of 6)	Problem 14	_______	(out of 8)
Problem 6	_______	(out of 6)	Problem 15	_______	(out of 8)
Problem 7	_______	(out of 6)	Bonus Problem	_______	(out of 6)
Problem 8	_______	(out of 6)			
Problem 9	_______	(out of 6)	Total Score	_______	(out of 100)

Problem 1 (6 points)

Solve the following quadratic equation using any method you choose.

$$x^2+\frac{7}{2}x-2=0$$

$2x^2 + 7x - 4 = 0$

$(2x-1)(x+4) = 0$

$\boxed{x = 1/2 \text{ or } x = -4}$

Problem 2 (6 points)

Find an equation of the line passing through the points $(5,3)$ and $(7,-1)$. Please express your answer in slope – intercept form.

$m = \frac{3-(-1)}{5-7} = \frac{4}{-2} = -2$

$y - 3 = -2(x-5)$

$y - 3 = -2x + 10$

$\boxed{y = -2x + 13}$

Problem 3 (6 points)

Let $f(x)=x^2$. Let $g(x)=3x+1$. Find an explicit defining formula for the function $f\circ g^{-1}$.

$$g(x) = 3x+1$$

$$g(x)-1 = 3x$$

$$\frac{g(x)-1}{3} = x = g^{-1}(g(x))$$

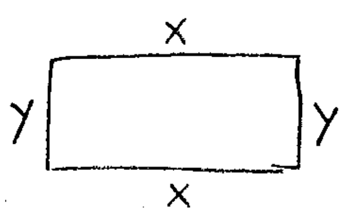
so $g^{-1}(x) = \frac{x-1}{3}$

so $f\circ g^{-1}(x) = f(g^{-1}(x)) = f\left(\frac{x-1}{3}\right) = \boxed{\left(\frac{x-1}{3}\right)^2}$

or $\boxed{\frac{x^2-2x+1}{9}}$

Problem 4 (6 points)

What is the largest possible area for a rectangle with perimeter 80 meters?



Given $2x+2y=80 \Rightarrow 2y = 80-2x$

$y = 40-x$

Maximize xy

$= x(40-x)$

$= -x^2+40x$

$= -1(x^2-40x+400-400)$

$= -1(x-20)^2+400$

max value of the quadratic function is 400

The maximum area is

$\boxed{400 \text{ meters}^2}$

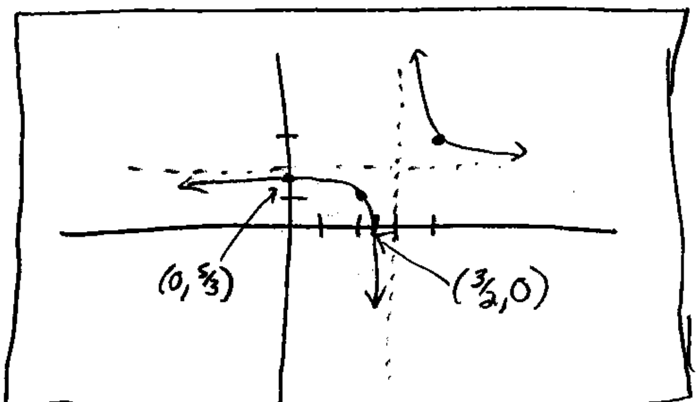
Problem 5 (6 points)

Sketch a graph of the function $g(x)=\dfrac{2x-5}{x-3}$. Be sure to include any important features in your graph including any vertical asymptotes, horizontal asymptotes, x-intercepts, or y-intercepts.

vertical asymptote: $x=3$
horizontal asymptote: $y=2$
x-intercept: $\frac{5}{2}$ $(5/2, 0)$
y-intercept: $5/3$ $(0, 5/3)$

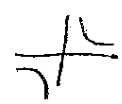
$$x-3\,\overline{)\,2x-5}\quad\text{quotient } 2,\quad \frac{2x-6}{1}$$

so $\dfrac{2x-5}{x-3} = 2 + \dfrac{1}{x-3}$



Graph of $y=\frac{2x-5}{x-3} = 2+\frac{1}{x-3}$ looks like graph of $y=\frac{1}{x}$ shifted 3 units to the right and two units up.

$y=\frac{1}{x}$



Problem 6 (6 points)

Solve the following logarithmic equation.

$$\log_{10}(x-6)+\log_{10}(x+3)=1$$

$\log_{10}((x-6)(x+3)) = 1$

$(x-6)(x+3) = 10^1$

$x^2-3x-18 = 10$

$x^2-3x-28 = 0$

$(x+4)(x-7) = 0$

$\cancel{x=-4}$ or $\boxed{x=7}$

(-4 is an extraneous solution since it is not in the domain of the original expression $\log_{10}(x-6)+\log_{10}(x+3)$)

Problem 7 (6 points)

Solve the following inequality.

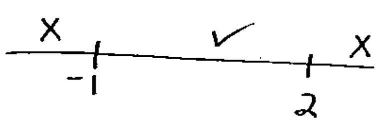
$$\log_2\left(\frac{2x-1}{x-2}\right)<0$$

$$\frac{2x-1}{x-2} < 2^0 = 1$$

$$\frac{2x-1}{x-2} - 1 < 0$$

$$\frac{2x-1-(x-2)}{x-2} < 0$$

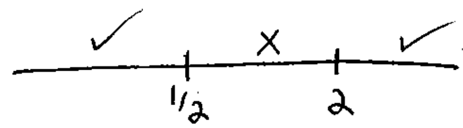
$$\frac{x+1}{x-2} < 0$$



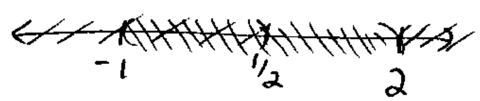
$(-1,2)$

Domain of expression
need

$$\frac{2x-1}{x-2} > 0$$



$(-\infty, 1/2) \cup (2,\infty)$



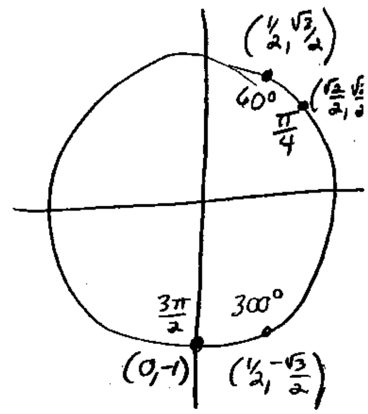
$(-1,2) \cap \left[(-\infty,1/2) \cup (2,\infty)\right]$

$= \boxed{(-1, 1/2)}$

Problem 8 (8 points)

Complete the following table.

θ	$60°$	$3\pi/2$	$300°$	$\pi/4$
$\sin\theta$	$\sqrt{3}/2$	-1	$-\sqrt{3}/2$	$\sqrt{2}/2$
$\cos\theta$	$1/2$	0	$1/2$	$\sqrt{2}/2$
$\tan\theta$	$\sqrt{3}$	undefined	$-\sqrt{3}$	1
$\csc\theta$	$2\sqrt{3}/3$	-1	$-2\sqrt{3}/3$	$\sqrt{2}$
$\sec\theta$	2	undefined	2	$\sqrt{2}$
$\cot\theta$	$\sqrt{3}/3$	0	$-\sqrt{3}/3$	1



Problem 9 (6 points)

A block attached to a spring moves up and down in simple harmonic motion governed by the equation $s = 2\sin\left(\frac{\pi t}{3}\right)$.

a. (3 points) State the Amplitude, Period and Phase Shift for this function.

$A\sin(Bx - C)$

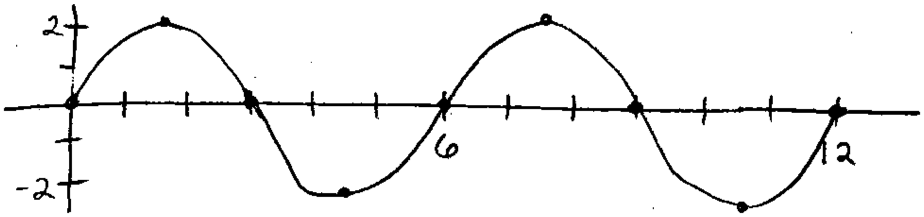
$A = 2, \quad B = \frac{\pi}{3}, \quad C = 0$

Amplitude $= |A| = \boxed{2}$

Period $= \frac{2\pi}{B} = \frac{2\pi}{\pi/3} = \boxed{6}$

Phase Shift $= C/B = \boxed{0}$

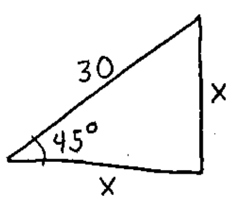
b. (3 points) Sketch a graph of the height of the block for two full periods, starting at $t = 0$.



Problem 10 (6 points)

A building contractor wants to put a fence around the perimeter of a lot that has the shape of a right triangle. One angle of the triangle is 45° and the hypotenuse is 30 meters.

a. (3 points) Find the length of fencing required.



$$\sin 45^\circ = \frac{x}{30}$$
$$\frac{\sqrt{2}}{2} = \frac{x}{30}$$
$$15\sqrt{2} = x$$

Perimeter
$$= 30 + 15\sqrt{2} + 15\sqrt{2}$$
$$= \boxed{30 + 30\sqrt{2} \text{ meters}}$$

b. (3 points) Find the area of the lot.

$$A = \frac{1}{2}bh = \frac{1}{2}x^2 = \frac{1}{2}(15\sqrt{2})^2 = \frac{1}{2}\cdot 225 \cdot 2$$
$$= \boxed{225 \text{ meters}^2}$$

Problem 11 (6 points)

Prove
$\sin(2\theta) = \dfrac{2\tan\theta}{1+\tan^2\theta}$.

(Hint: start with the right side, and simplify to obtain the left side).

$$\frac{2\tan\theta}{1+\tan^2\theta} = \frac{2\frac{\sin\theta}{\cos\theta}}{1+\frac{\sin^2\theta}{\cos^2\theta}}$$

$$= \frac{\left(\frac{2\sin\theta}{\cos\theta}\right)}{\left(\frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}\right)}$$

$$= \frac{\left(\frac{2\sin\theta}{\cos\theta}\right)}{\left(\frac{1}{\cos^2\theta}\right)}$$

$$= \frac{2\sin\theta\cos^2\theta}{\cos\theta}$$

$$= 2\sin\theta\cos\theta$$

$$= \sin(2\theta)$$

Problem 12 (8 points)

Let $z = 4-3i$, and let $w = 1+i$. Express each of the following in standard rectangular form, (i.e. in the form $a+bi$, where a and b are real numbers).

a. $\bar{z} = \boxed{4+3i}$

b. $z+w = 4-3i+(1+i) = \boxed{5-2i}$

c. $zw = (4-3i)(1+i) = 4+4i-3i-3i^2$
$= 4+i-3(-1)$
$= \boxed{7+i}$

d. $\frac{z}{w} = \frac{z\overline{w}}{w\overline{w}} = \frac{(4-3i)(1-i)}{(1+i)(1-i)} = \frac{4-4i-3i+3i^2}{1-i^2}$

$= \frac{1-7i}{2} = \boxed{\frac{1}{2} - \frac{7}{2}i}$

Problem 13 (8 points)

Use mathematical induction to prove that for every positive integer n, $\sum_{i=1}^{n} 2i-1 = n^2$.

Let $P(n)$ say $\sum_{i=1}^{n} 2i-1 = n^2$

Step 1: show $P(1)$ is true

$P(1)$ says $\sum_{i=1}^{1} 2i-1 = 1^2$

$\sum_{i=1}^{1} 2i-1 = 2(1)-1 = 1 = 1^2$ so $P(1)$ is true.

Step 2: Assume $P(k)$ is true. Deduce $P(k+1)$ is true.

$P(k)$ says $\sum_{i=1}^{k} 2i-1 = k^2$ ← call this the induction hypothesis

$P(k+1)$ says $\sum_{i=1}^{k+1} 2i-1 = (k+1)^2$

$\sum_{i=1}^{k+1} 2i-1$

basic algebra of sums

$\stackrel{\text{basic algebra of sums}}{=} \sum_{i=1}^{k} 2i-1 + 2(k+1)-1$

$\stackrel{\text{induction hypothesis}}{=} k^2 + 2k+1$

$\stackrel{\text{basic algebra}}{=} (k+1)^2$

so $P(k+1)$ is true.

Hence by the principle of mathematical induction, $P(n)$ is true for every positive integer n.

Problem 14 (8 points)

What is the coefficient of x^{10} in the expansion of $(x+2)^{12}$? (Hint: The easiest way to answer this question is to apply the binomial theorem to the expression $(x+2)^{12}$, considering only the relevant term in the expansion. You may receive some partial credit for stating the binomial theorem correctly.)

The binomial theorem says (for $a,b \in \mathbb{R}$ and for $n \in \mathbb{N}$)

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Now $$(x+2)^{12} = \sum_{k=0}^{12} \binom{12}{k} x^{12-k} \cdot 2^k$$

The term in which x occurs to the 10^{th} power is the term in which $k=2$

i.e. $$\binom{12}{2} x^{12-2} \cdot 2^2$$

$$= \frac{12!}{2!\,10!} x^{10} \cdot 4$$

$$= \frac{12 \cdot 11 \cdot \cancel{10} \cdots}{2 \cdot 1 \cdot \cancel{10} \cdots} x^{10} \cdot 4$$

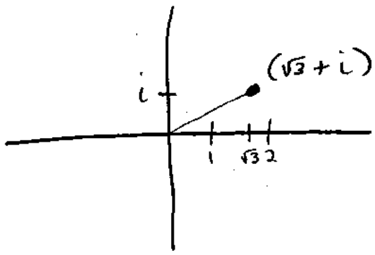
$$= 66 \cdot x^{10} \cdot 4$$

$$= 264 x^{10}$$

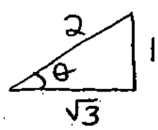
so the coefficient of x^{10} in this term is $\boxed{264}$

Problem 15 (8 points)

Compute $\left(\sqrt{3}+i\right)^6$. (Hint: the easy way to do this problem is to first convert $\sqrt{3}+i$ to trigonometric form, and then use DeMoivre's theorem).



$$r = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2$$

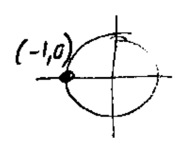


$\theta = 30°$ by observation (or can use $\theta = \sin^{-1}(1/2) = 30°$)

so $\sqrt{3} + i = 2(\cos 30° + i \sin 30°)$

so $(\sqrt{3}+i)^6 = \left[2(\cos 30° + i \sin 30°)\right]^6$

by DeMoivre's Theorem $= 2^6(\cos(6\cdot 30°) + i \sin(6 \cdot 30°))$



$= 64(\cos 180° + i \sin 180°)$

$= 64(-1 + i \cdot 0)$

$= \boxed{-64}$

Bonus (6 points)

Let $f(x)=2x^5+13x^4+50x^3+82x^2+56x+13$. Express $f(x)$ as the product of 5 linear factors.

By rational roots theorem, only possible rational roots are $\pm 1, \pm 13, \pm \frac{1}{2}, \pm \frac{13}{2}$

By Descartes' rule of signs, no positive real roots.
Let's try -1 first.

$$\begin{array}{r|rrrrrr} -1 & 2 & 13 & 50 & 82 & 56 & 13 \\ & & -2 & -11 & -39 & -43 & -13 \\ \hline & 2 & 11 & 39 & 43 & 13 & 0 \end{array}$$

so $f(x) = (x+1)(2x^4+11x^3+39x^2+43x+13)$
let's try -1 again

$$\begin{array}{r|rrrrr} -1 & 2 & 11 & 39 & 43 & 13 \\ & & -2 & -9 & -30 & -13 \\ \hline & 2 & 9 & 30 & 13 & 0 \end{array}$$

so $f(x) = (x+1)(x+1)(2x^3+9x^2+30x+13)$
let's try -1 again

$$\begin{array}{r|rrrr} -1 & 2 & 9 & 30 & 13 \\ & & -2 & -7 & -23 \\ \hline & 2 & 7 & 23 & -10 \end{array}$$

so -1 is not a root of $2x^3+9x^2+30x+13$.
let's try $-1/2$

$$\begin{array}{r|rrrr} -1/2 & 2 & 9 & 30 & 13 \\ & & -1 & -4 & -13 \\ \hline & 2 & 8 & 26 & 0 \end{array}$$

so $f(x) = (x+1)(x+1)(x+1/2)(2x^2+8x+26)$

Using the quadratic formula, the roots of $2x^2+8x+26$ are

$$x = \frac{-8 \pm \sqrt{8^2-4(2)(26)}}{2(2)} = \frac{-8 \pm \sqrt{64-208}}{4}$$

$$= \frac{-8 \pm \sqrt{-144}}{4} = \frac{-8 \pm 12i}{4} = -2 \pm 3i$$

so $\boxed{f(x) = 2(x+1)(x+1)(x+1/2)(x-(-2+3i))(x-(-2-3i))}$

or $\boxed{f(x) = 2(x+1)(x+1)(x+1/2)(x+2-3i)(x+2+3i)}$