

FINAL EXAMINATION

Open book, open notes. In your solutions, you may use any result from the lectures, from the parts of the textbook we covered, or from the homework exercises.

Each question is worth 10 points.

1. Prove that the set $G = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \neq x_2 \neq x_3 \neq x_1\}$ is open.
2. Let X be a metric space and x_0 a point of X . Let the function $f : X \setminus \{x_0\} \rightarrow \mathbb{R}$ be defined by $f(x) = 1/d(x, x_0)$. Prove f is continuous. (Note that x_0 is not in the domain of f .)
3. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ be continuous functions. Let the subset C of \mathbb{R}^2 be defined by $C = \{(f(x), g(x)) : x \in \mathbb{R}\}$.
 - (a) Prove C is connected.
 - (b) Is C necessarily closed? Give a proof or a counterexample.
4. Let S be the set of nondecreasing functions of \mathbb{Q} into the doubleton $\{0, 1\}$. Is S countable? Justify your answer.
5. Let $(s_n)_{n=1}^{\infty}$ be a bounded sequence in \mathbb{R} . Let $a = \liminf_{n \rightarrow \infty} s_n$ and $b = \limsup_{n \rightarrow \infty} s_n$. Prove $\limsup_{n \rightarrow \infty} s_n^2 = \max\{a^2, b^2\}$.
6. Let X and Y be metric spaces, and let $(f_n)_{n=1}^{\infty}$ be a uniformly convergent sequence of continuous functions from X to Y , with limit f . Let $(x_n)_{n=1}^{\infty}$ be a convergent sequence in X , with limit x_0 . Prove $\lim_{n \rightarrow \infty} f_n(x_n) = f(x_0)$.
7. Let $f : \mathbb{R}^k \rightarrow \mathbb{R}^m$ be a continuous function with the property that $f^{-1}(K)$ is compact whenever K is a compact subset of \mathbb{R}^m . Prove that $f(C)$ is closed whenever C is a closed subset of \mathbb{R}^k .
8. Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series, with $|a_n| < 1$ for all n . Let the function $f : (-1, 1) \rightarrow \mathbb{R}$ satisfy $f(0) = 0$ and be differentiable at 0. Prove the series $\sum_{n=1}^{\infty} f(a_n)$ converges absolutely.
9. Let the function $F : [a, b] \times [\alpha, \beta] \rightarrow \mathbb{R}$ be continuous, and let the function $f : [a, b] \rightarrow \mathbb{R}$ be defined by $f(x) = \int_{\alpha}^{\beta} F(x, \xi) d\xi$. Prove f is continuous.
10. Let $f : [a, b] \rightarrow \mathbb{R}$ be a bounded function, with upper and lower Darboux integrals $U(f)$ and $L(f)$, respectively. Suppose there is a Riemann-integrable function $g : [a, b] \rightarrow \mathbb{R}$ such that $|f(x) - g(x)| \leq \varepsilon$ for all x . Prove $U(f) - L(f) \leq 2\varepsilon(b - a)$.